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A class of extremising sphere-valued maps with inherent maximal tori symmetries in $\mathbf{SO}(n)$

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Abstract

In this paper we consider an energy functional depending on the norm of the gradient and seek to extremise it over an admissible class of Sobolev maps defined on an annulus and taking values on the unit sphere whilst satisfying suitable boundary conditions. We establish the existence of an infinite family of solutions with certain symmetries to the associated nonlinear Euler-Lagrange system in even dimensions and discuss the stability of such extremisers by way of examining the positivity of the second variation of the energy at these solutions.

Keywords: spherical whirls; symmetries; generalised sphere-valued harmonic maps; **SO**(*n*); maximal tori and conjugacy; second energy variations

1 Introduction

In this paper we address questions on existence, multiplicity and stability for a certain class of symmetric extremisers of the variational integral

$$\mathbb{F}\big[u;\mathbb{X}^n\big] := \int_{\mathbb{X}^n} F\big(|x|,|\nabla u|^2\big)\,dx. \tag{1.1}$$

Here $\mathbb{X}^n = \mathbb{X}^n[a,b] = \{x \in \mathbb{R}^n : a < |x| < b\}$ with fixed $b > a > 0$ is an annulus, u is a unit vector field on $\mathbb{X}^n$ (see below), and $|\nabla u|$ denotes the Hilbert-Schmidt norm of the gradient of u. We assume that $F \in \mathscr{C}^{1,2}([a,b]\times\mathbb{R})$, that is, $\mathscr{C}^1$ with respect to the first variable and $\mathscr{C}^2$ with respect to the second. Furthermore we assume that there exist $c_1, c_2 > 0$ and $c_0 \in \mathbb{R}$ such that

$$\big|F'\big(r,\zeta^2\big)\zeta\big| \le c_2|\zeta|^{p-1}, \quad \forall a \le r \le b, \forall \zeta \in \mathbb{R}, \tag{1.2}$$

$$c_0 + c_1|\zeta|^p \le F\big(r,\zeta^2\big) \le c_2|\zeta|^p, \quad \forall a \le r \le b, \forall \zeta \in \mathbb{R}, \tag{1.3}$$

with $1 < p < \infty$. (Here F' denotes the derivative of F with respect to its second variable.) As a result, $\mathbb{F}$ is well-defined, finite and coercive on $\mathscr{W}^{1,p}(\mathbb{X}^n, \mathbb{S}^{n-1})$. As for convexity, we further assume that $F' > 0$ and $F'' \ge 0$ on $[a,b] \times (0,\infty)$ and that the twice continuously differentiable function $\zeta \mapsto F(r,\zeta^2)$ is uniformly convex in ζ for all $a \le r \le b$ and $\zeta \in \mathbb{R}$.

The competing vector fields u for the energy integral $\mathbb{F}$ are restricted to the space of admissible sphere-valued maps given by

$$\mathscr{A}^p_\varphi(\mathbb{X}^n) := \{u \in \mathscr{W}^{1,p}(\mathbb{X}^n, \mathbb{S}^{n-1}) : u = \varphi \text{ on } \partial\mathbb{X}^n\} \tag{1.4}$$

for some suitable and fixed boundary map $\varphi \in \mathscr{C}^\infty(\partial\mathbb{X}^n, \mathbb{S}^{n-1})$. Note that here we can write the boundary as a union of its two inner and outer spherical components, that is, $\partial\mathbb{X}^n = \partial\mathbb{X}^n_a \cup \partial\mathbb{X}^n_b$ and as customary[a]

$$\mathscr{W}^{1,p}(\mathbb{X}^n, \mathbb{S}^{n-1}) := \{u \in \mathscr{W}^{1,p}(\mathbb{X}^n, \mathbb{R}^n) : |u| = 1 \text{ a.e. in } \mathbb{X}^n\}. \tag{1.5}$$

Moving forward, we consider the first-order condition $d/d\varepsilon(\mathbb{F}[u_\varepsilon; \mathbb{X}^n])|_{\varepsilon=0} = 0$ with $u_\varepsilon = (u + \varepsilon\psi)/|u + \varepsilon\psi|$ where $u \in \mathscr{A}^p_\varphi(\mathbb{X}^n)$, $\psi \in \mathscr{C}^\infty_0(\mathbb{X}^n, \mathbb{R}^n)$, and $\varepsilon \in \mathbb{R}$ is sufficiently small. This leads to the Euler-Lagrange equation associated with the energy integral $\mathbb{F}$ on $\mathscr{A}^p_\varphi(\mathbb{X}^n)$ as the nonlinear system

$$\begin{cases} \mathscr{L}[u] = \operatorname{div}(F'\nabla u) + F'|\nabla u|^2 u = 0 & \text{in } \mathbb{X}^n, \\ |u| = 1 & \text{in } \mathbb{X}^n, \\ u = \varphi & \text{on } \partial\mathbb{X}^n, \end{cases} \tag{1.6}$$

where for brevity we have written $F' \equiv F'(|x|, |\nabla u|^2)$ (recall that the '*prime*' over F stands for the derivative with respect to the second variable). Additionally we point out that the divergence operator on the first line in $\mathscr{L}[u]$ is understood to act row-wise on the tensor field $F'(|x|, |\nabla u|^2)\nabla u$.

In this paper we examine as solutions to the nonlinear system (1.6) a class of geometrically motivated and suitably symmetric maps referred to hereafter as spherical whirls (or whirls for simplicity). These by definition, and in their general form, are maps $u \in \mathscr{C}(\overline{\mathbb{X}}^n, \mathbb{S}^{n-1})$ admitting the representation

$$u : x \mapsto \mathbf{Q}(\rho_1, \ldots, \rho_N)\theta, \quad x \in \overline{\mathbb{X}}^n, \tag{1.7}$$

$$\rho = \rho(x) = (\rho_1, \ldots, \rho_N), \quad \theta = x|x|^{-1}, \tag{1.8}$$

where $\mathbf{Q} = \mathbf{Q}(\rho_1, \ldots, \rho_N)$ is a continuous $\mathbf{SO}(n)$-valued map depending on the spatial variable $x = (x_1, \ldots, x_n)$ through the auxiliary 2-plane radial variables $\rho = (\rho_1, \ldots, \rho_N)$. Here, depending on the spatial dimension $n \geq 2$ being even or odd, we have introduced the 2-plane variables $\rho = (\rho_1, \ldots, \rho_N)$ and $y = (y_1, \ldots, y_N)$ with $\|\rho\| = \sqrt{\rho_1^2 + \cdots + \rho_N^2}$ and the integer $N = N(n) \geq 1$ as:

- (n even) put $N = n/2$ and set $\rho_j = |y_j| = \sqrt{x_{2j-1}^2 + x_{2j}^2}$ (with $1 \leq j \leq N$) where

$$x = (x_1, \ldots, x_n) = (y_1, \ldots, y_N), \quad y_j = (x_{2j-1}, x_{2j}) \text{ for } 1 \leq j \leq N, \tag{1.9}$$

- (n odd) put $N = (n+1)/2$ and set $\rho_j = |y_j|$ (with $1 \leq j \leq N-1$) and $\rho_N = y_N$ where

$$x = (x_1, \ldots, x_n) = (y_1, \ldots, y_N), \quad y_j := \begin{cases} (x_{2j-1}, x_{2j}), & 1 \leq j \leq N-1, \\ x_n, & j = N. \end{cases} \tag{1.10}$$

From the above formulation it is clear that $\rho = (\rho_1, \dots, \rho_N)$ sits in the semi-annular domain $\mathbb{A}_N \subset \mathbb{R}^N$ where $\mathbb{A}_N = \{\rho \in \mathbb{R}^N_+ : a < \|\rho\| < b\}$ for $n = 2N$ and $\mathbb{A}_N = \{\rho \in \mathbb{R}^{N-1}_+ \times \mathbb{R} : a < \|\rho\| < b\}$ for $n = 2N - 1$. With this notation in place and in accordance with earlier discussion, we now set $\mathbf{Q} \in \mathscr{C}(\overline{\mathbb{A}}_N, \mathbf{SO}(n))$. In fact, for considerations of symmetry, as will become clear later on, the map $\mathbf{Q}$ will have to take values on a maximal torus of $\mathbf{SO}(n)$, hereafter, and without loss of generality the maximal torus of block diagonal matrices consisting of 2×2 rotation blocks. Next, for the sake of convenience, in notation let us agree to write $k = N$ when $n = 2N$ and $k = N - 1$ when $n = 2N - 1$. Then the map $\mathbf{Q}$ can be given the explicit representation

$$\mathbf{Q}(\rho_1, \dots, \rho_N) = \begin{cases} \operatorname{diag}(\mathcal{R}[f_1], \dots, \mathcal{R}[f_k]) & \text{for } n = 2k, \\ \operatorname{diag}(\mathcal{R}[f_1], \dots, \mathcal{R}[f_k], 1) & \text{for } n = 2k + 1, \end{cases} \tag{1.11}$$

where for each $1 \le l \le k$, $f_l \in \mathscr{C}(\overline{\mathbb{A}}_N, \mathbb{R})$ satisfies $f_l \equiv 0$ when $\|\rho\| = a$ and $f_l \equiv 2\pi m$ for some integer m when $\|\rho\| = b$. Here and below, $\mathcal{R}[f]$ stands for the 2×2 matrix of counterclockwise rotation by angle f (see, *e.g.*, (2.7)).

Now for the space of admissible maps $\mathscr{A}^p_\varphi(\mathbb{X}^n)$ to contain spherical whirls $u = \mathbf{Q}(\rho_1, \dots, \rho_N)x|x|^{-1}$, it is evident that further differentiability assumptions on the map $\mathbf{Q}$ (hence $f_1, \dots, f_k$) and further restrictions on the boundary map φ must be imposed. Indeed, for the latter, we must have

$$\varphi(x) = \begin{cases} \mathbf{R}_a x|x|^{-1} & \text{on } \partial\mathbb{X}^n_a = \{x : |x| = a\}, \\ \mathbf{R}_b x|x|^{-1} & \text{on } \partial\mathbb{X}^n_b = \{x : |x| = b\} \end{cases} \tag{1.12}$$

for suitable $\mathbf{R}_a$, $\mathbf{R}_b$ in $\mathbf{SO}(n)$ and subsequently in the level of u that $\mathbf{Q}(\rho) = \mathbf{R}_a$ for $\|\rho\| = a$ and $\mathbf{Q}(\rho) = \mathbf{R}_b$ for $\|\rho\| = b$. Hence, in view of the $\mathbf{SO}(n)$ invariance of system (1.6) (notice that, for any $\mathbf{R} \in \mathbf{SO}(n)$, we have $\mathscr{L}[\mathbf{R}u] = \mathbf{R}\mathscr{L}[u]$, and so $\mathscr{L}[\mathbf{R}u] = 0 \iff \mathscr{L}[u] = 0$), we may assume without loss of generality that $\mathbf{R}_a = \mathbf{I}_n$ and write

$$\mathbf{R} := \mathbf{R}_b = \begin{cases} \operatorname{diag}(\mathcal{R}[z_1], \dots, \mathcal{R}[z_k]) & \text{if } n = 2k, \\ \operatorname{diag}(\mathcal{R}[z_1], \dots, \mathcal{R}[z_k], 1) & \text{if } n = 2k + 1, \end{cases} \tag{1.13}$$

where $\mathsf{z} = (z_1, \dots, z_k) \in \mathbb{T}^k = [-\pi, \pi)^k$. Now with the above notation and differentiability assumptions in place, for a spherical whirl $u = \mathbf{Q}(\rho_1, \dots, \rho_N)x|x|^{-1}$, the square of the Hilbert-Schmidt norm of the gradient ∇u is seen to be

$$|\nabla u|^2 = \left| \frac{\mathbf{Q}(\mathbf{I}_n - \theta \otimes \theta)}{r} + \sum_{l=1}^N \mathbf{Q}_{,l}\theta \otimes \nabla\rho_l \right|^2 = \frac{n-1}{r^2} + \frac{1}{r^2} \sum_{l=1}^k \rho_l^2 |\nabla f_l|^2, \tag{1.14}$$

where $a \le r = \sqrt{\rho_1^2 + \cdots + \rho_N^2} \le b$ and $\rho = (\rho_1, \dots, \rho_N) \in \overline{\mathbb{A}}_N$. Therefore the restriction of the $\mathbb{F}$-energy (1.1) to the class of such spherical whirls simplifies to (see also (2.14)-(2.15))

$$\mathbb{F}\big[\mathbf{Q}(\rho_1, \dots, \rho_N)x|x|^{-1}; \mathbb{X}^n\big] = \int_{\mathbb{X}^n} F\left(r, \frac{n-1}{r^2} + \frac{1}{r^2} \sum_{l=1}^k \rho_l^2 |\nabla f_l|^2\right) dx. \tag{1.15}$$

In particular, the sufficiently regular extremisers of the $\mathbb{F}$-energy in the form of spherical whirls $u = \mathbf{Q}(\rho_1, \ldots, \rho_N)x|x|^{-1}$ with $\mathbf{Q}$ as in (1.11) should satisfy - for the unknown vector field $f = (f_1, \ldots, f_k)$ in the semi-annular domain $\mathbb{A}_N \subset \mathbb{R}^N$ - the nonlinear system of equations (see Section 2 for further details and notation):

$$\begin{cases} \operatorname{div}\{F'(r, \frac{n-1}{r^2} + \frac{1}{r^2}\sum_{l=1}^k \rho_l^2|\nabla f_l|^2)\frac{\rho_\alpha^2}{r^2}\nabla f_\alpha \prod_{j=1}^k \rho_j\} = 0 & \text{in } \mathbb{A}_N, \\ F'(r, \frac{n-1}{r^2} + \frac{1}{r^2}\sum_{l=1}^k \rho_l^2|\nabla f_l|^2)\frac{\rho_\alpha^2}{r^2}\partial_\nu f_\alpha \prod_{j=1}^k \rho_j = 0 & \text{on } \Gamma_N, \\ f = (f_1, \ldots, f_k) \equiv 0 & \text{on } (\partial\mathbb{A}_N)_a, \\ f = (f_1, \ldots, f_k) \equiv 2\pi \mathsf{m} + \mathsf{z} & \text{on } (\partial\mathbb{A}_N)_b, \end{cases} \tag{1.16}$$

where $1 \le \alpha \le k$, $\mathsf{m} = (m_1, \ldots, m_k)$ and $\mathsf{z} = (z_1, \ldots, z_k)$. While for given m and z, system (1.16) can be shown to have a unique solution, the cases of particular interest and significance here in relation to the original system (1.6), correspond to when m and z are '*linear*', that is, $m_1 = \cdots = m_k = m$ for $m \in \mathbb{Z}$ and $z_1 = \cdots = z_k = z$ for $z \in \mathbb{T}$. Most notably in the case where $n \ge 2$ is *even* and m and z are linear the solution $f = (f_1, \ldots, f_N)$ will be shown to be a function of the radial variable $r = \|\rho\|$ only, that is,

$$f_\alpha(\rho_1, \ldots, \rho_N; m) = \mathscr{G}\big(\|\rho\|; m\big), \quad 1 \le \alpha \le N = n/2, \tag{1.17}$$

where the *monotone* function $\mathscr{G} = \mathscr{G}(r; m) \in \mathscr{C}^2([a, b], \mathbb{R})$ is in turn the solution to an associated two-point boundary value problem:

$$\begin{cases} \frac{d}{dr}[F'(r, \frac{n-1}{r^2} + \dot{\mathscr{G}}^2)r^{n-1}\dot{\mathscr{G}}] = 0, \\ \mathscr{G}(a) = 0, \\ \mathscr{G}(b) = 2\pi m + z. \end{cases} \tag{1.18}$$

The task then is to pass on from (1.16) to the original Euler-Lagrange system (1.6) that in the case of a spherical whirl $u = \mathbf{Q}(\rho_1, \ldots, \rho_N)x|x|^{-1}$ with $\mathbf{Q}$ as in (1.11) can be formulated as[b]

$$\begin{aligned} &\operatorname{div}\left\{F'\left(|x|, \frac{n-1}{r^2} + \frac{1}{r^2}\sum_{l=1}^k \rho_l^2|\nabla f_l|^2\right)\left(\frac{1}{r}(\mathbf{Q} - \mathbf{Q}\theta \otimes \theta) + \sum_{l=1}^N \mathbf{Q}_{,l}\theta \otimes \nabla\rho_l\right)\right\} \\ &\quad + F'\left(|x|, \frac{n-1}{r^2} + \frac{1}{r^2}\sum_{l=1}^k \rho_l^2|\nabla f_l|^2\right)\left[\frac{n-1}{r^2} + \frac{1}{r^2}\sum_{l=1}^k \rho_l^2|\nabla f_l|^2\right]\mathbf{Q}\theta = 0. \end{aligned} \tag{1.19}$$

A corresponding analysis of this system for solutions thus obtained leads to the following multiplicity result. See also Theorem 3.1 for more qualitative features.

Main Theorem *Consider the nonlinear system* (1.6) *where φ is as in* (1.12) *with $\mathbf{R}_a = \mathbf{I}_n$ and $\mathbf{R}_b$ as in* (1.13) *with $z_1 = \cdots = z_k = z$. Then, when n is even, there is an infinite family of spherical whirls serving as solutions to* (1.6); *specifically, for each $m \in \mathbb{Z}$, we have the solution*

$$\begin{aligned} u(x; m) &= \mathbf{Q}(\rho_1, \ldots, \rho_N; m)x|x|^{-1}, \quad x \in \overline{\mathbb{X}}^n, \rho = (\rho_1, \ldots, \rho_N) \in \overline{\mathbb{A}}_N, \\ &= \operatorname{diag}\big(\mathcal{R}[f_1](\rho; m), \ldots, \mathcal{R}[f_k](\rho; m)\big)x|x|^{-1}, \end{aligned} \tag{1.20}$$

where $f_\alpha(\rho) = \mathscr{G}(\|\rho\|; m)$ for $1 \le \alpha \le k$, and $\mathscr{G} = \mathscr{G}(r; m)$ is the solution to the two-point boundary value problem (1.18).[c]

In the final section of the paper we take a further step and discuss the *stability* of these solutions by way of computing the second variation of the $\mathbb{F}$-energy over $\mathscr{A}_\varphi^p(\mathbb{X}^n)$ and examining its positivity at the spherical twist solutions to (1.16).

2 Spherical whirls as extremisers of the $\mathbb{F}$-energy

The aim of this section is to introduce and examine spherical whirls as potential extremisers of the energy integral $\mathbb{F}$, that is, as solutions to the nonlinear system of Euler-Lagrange equations (1.6).

To this end, recall first the 2-plane radial variables $\rho = (\rho_1, \dots, \rho_N)$ from the previous section, defined as functions of the spatial variable $x = (x_1, \dots, x_n)$ on $\mathbb{X}^n$ for n even and odd by [a] (n even) $\rho_j = (x_{2j-1}^2 + x_{2j}^2)^{1/2}$ for $1 \le j \le k = n/2$ and likewise [b] (n odd) $\rho_j = (x_{2j-1}^2 + x_{2j}^2)^{1/2}$ for $1 \le j \le k = (n-1)/2$ and $\rho_N = x_n$ (*i.e.*, $j = (n+1)/2 = N$), respectively. Note that, as indicated earlier, here in order to simplify notation, we write $N = N(n)$ as $N = k$ when $n = 2k$ and $N = k + 1$ when $n = 2k + 1$. Now $\rho = (\rho_1, \dots, \rho_N)$ lies in the semi-annulus $\mathbb{A}_N \subset \mathbb{R}^N$ given by (a) (n even) $\mathbb{A}_N = \{\rho \in \mathbb{R}_+^k : a < \|\rho\| < b\}$ with $n = 2k$, and (b) (n odd) $\{\rho \in \mathbb{R}_+^k \times \mathbb{R} : a < \|\rho\| < b\}$ with $n = 2k + 1$, respectively. We write $(\partial\mathbb{A}_N)_a = \{\rho \in \partial\mathbb{A}_N : \|\rho\| = a\}$, $(\partial\mathbb{A}_N)_b = \{\rho \in \partial\mathbb{A}_N : \|\rho\| = b\}$ and $\Gamma_N = \partial\mathbb{A}_N \setminus \{\rho \in \partial\mathbb{A}_N : \|\rho\| = a \text{ or } \|\rho\| = b\}$ to denote the three components of the boundary $\partial\mathbb{A}_N$. Note that $x = (x_1, \dots, x_n) \in \partial\mathbb{X}_a^n \iff \rho = (\rho_1, \dots, \rho_N) \in (\partial\mathbb{A}_N)_a$, and likewise $x \in \partial\mathbb{X}_b^n \iff \rho \in (\partial\mathbb{A}_N)_b$.

With this notation in place, we now define a *spherical whirl* as a map $u \in \mathscr{C}(\overline{\mathbb{X}}^n, \mathbb{S}^{n-1})$ having the form

$$u : x \mapsto \mathbf{Q}(\rho)\theta = \mathbf{Q}(\rho_1, \dots, \rho_N)x|x|^{-1}, \quad x \in \overline{\mathbb{X}}^n, \tag{2.1}$$

where $\rho = (\rho_1, \dots, \rho_N)(x)$ is the vector of 2-plane variables as described above and $\mathbf{Q} \in \mathscr{C}(\overline{\mathbb{A}}_N, \mathbf{SO}(n))$. Later on, especially in studying the extremising properties of spherical whirls, we may need to improve the regularity of u to $\mathscr{C}^2$, but for the sake of this general definition, continuity is enough.

Generally we think of a $u \in \mathscr{C}(\overline{\mathbb{X}}^n, \mathbb{S}^{n-1})$ as being *rotationally* symmetric iff it is invariant under all rotations R, that is, iff it satisfies $u(x) = Ru(R^t x)$ for all $x \in \mathbb{X}^n$ and $R \in \mathbf{SO}(n)$. For the sake of this paper, however, we think of weakening this condition and referring to u as being symmetric iff u is invariant under all rotations $R \in \mathbb{T} \subset \mathbf{SO}(n)$, that is, $u(x) = Ru(R^t x)$ for all $x \in \mathbb{X}^n$ and $R \in \mathbb{T}$ where $\mathbb{T}$ is a fixed maximal torus in $\mathbf{SO}(n)$, that is, a maximal commutative subgroup in $\mathbf{SO}(n)$. Now we demand any spherical whirl u to be invariant under the subgroup $\mathbb{T} \subset \mathbf{SO}(n)$ of all planar rotations in the (x_{2j-1}, x_{2j})-planes with j ranging as described above. It is well known that here $\mathbb{T}$ is a maximal torus in $\mathbf{SO}(n)$ and as such is maximally commutative. This therefore fixes the range of $\mathbf{Q}$ and gives $\mathbf{Q} \in \mathscr{C}(\overline{\mathbb{A}}_N, \mathbb{T})$, since if u is invariant under $\mathbb{T}$, then

$$\begin{aligned} Ru(R^t x) &= R\mathbf{Q}(\rho_1, \dots, \rho_N)R^t x|R^t x|^{-1} \\ &= R\mathbf{Q}(\rho_1, \dots, \rho_N)R^t x|x|^{-1} \\ &= \mathbf{Q}(\rho_1, \dots, \rho_N)x|x|^{-1} = u(x), \quad \forall x \in \overline{\mathbb{X}}^n, \forall \rho \in \overline{\mathbb{A}}_N, \forall R \in \mathbb{T}, \end{aligned} \tag{2.2}$$

and so for each $\rho \in \overline{\mathbb{A}}_N$, $\mathbf{Q}(\rho)$ commutes with $\mathbb{T}$, which by definition of $\mathbb{T}$ being maximal commutative implies that $\mathbf{Q}(\rho) \in \mathbb{T}$. Note that in the above we have taken advantage of the fact that $\rho(Rx) = \rho$ for all $x \in \mathbb{X}^n$, $R \in \mathbb{T}$. In conclusion, for the outlined reasons of commutativity and symmetry, the spherical whirls must take the form

$$u(x) = \mathbf{Q}(\rho_1, \ldots, \rho_N)x|x|^{-1}, \quad \rho = \rho(x) = (\rho_1, \ldots, \rho_N) \in \overline{\mathbb{A}}_N, x \in \overline{\mathbb{X}}^n,$$

where the mapping $\mathbf{Q} = \mathbf{Q}(\rho_1, \ldots, \rho_N)$ admits the specific block diagonal matrix form

$$\mathbf{Q}(\rho_1, \ldots, \rho_N) = \begin{cases} \mathrm{diag}(\mathcal{R}[f_1], \ldots, \mathcal{R}[f_k]) & \text{for } n = 2k, \\ \mathrm{diag}(\mathcal{R}[f_1], \ldots, \mathcal{R}[f_k], 1) & \text{for } n = 2k+1. \end{cases} \tag{2.3}$$

Here, for $1 \le l \le k$, $f_l \in \mathscr{C}(\overline{\mathbb{A}}_N, \mathbb{R})$ satisfy $f_l \equiv 0$ on $(\partial\mathbb{A}_N)_a$ and $f_l = 2\pi m_l + z_l$ on $(\partial\mathbb{A}_N)_b$. The latter will ensure in view of (2.1)-(2.3) that $u = \varphi$ on $\partial\mathbb{X}^n$. We start by calculating some of the quantities associated with spherical whirls.

Lemma 2.1 *For a spherical whirl $u = \mathbf{Q}(\rho_1, \ldots, \rho_N)x|x|^{-1}$ with $x \in \overline{\mathbb{X}}^n$ and $(\rho_1, \ldots, \rho_N) \in \overline{\mathbb{A}}_N$ and subject to $\mathbf{Q} \in \mathscr{C}(\overline{\mathbb{A}}_N, \mathbf{SO}(n)) \cap \mathscr{C}^1(\mathbb{A}_N, \mathbf{SO}(n))$, we have*

- $\nabla u = \nabla\big(\mathbf{Q}(\rho_1, \ldots, \rho_N)x|x|^{-1}\big) = \dfrac{\mathbf{Q}(\mathbf{I}_n - \theta \otimes \theta)}{r} + \displaystyle\sum_{l=1}^{N} \mathbf{Q}_{,l}\theta \otimes \nabla\rho_l,$
- $|\nabla u|^2 = \mathrm{tr}\big\{[\nabla u][\nabla u]^t\big\} = \dfrac{n-1}{r^2} + \displaystyle\sum_{l=1}^{N} |\mathbf{Q}_{,l}\theta|^2.$

If additionally $\mathbf{Q} \in \mathscr{C}^2(\mathbb{A}_N, \mathbf{SO}(n))$, that is, $\mathbf{Q}$ is twice continuously differentiable on $\mathbb{A}_N$, then

- $\Delta u = \displaystyle\sum_{l=1}^{N} \bigg[\mathbf{Q}_{,ll}\theta + \frac{2}{r}\mathbf{Q}_{,l}\nabla\rho_l + \mathbf{Q}_{,l}\theta\Big(\Delta\rho_l - \frac{2\rho_l}{r^2}\Big)\bigg] - \frac{n-1}{r^2}\mathbf{Q}\theta.$

Here $\mathbf{Q}_{,l}$ and $\mathbf{Q}_{,ll}$ denote the first- and second-order derivatives of $\mathbf{Q}$ with respect to ρ_l respectively, whereas $\nabla\rho_l$ and $\Delta\rho_l$ denote the gradient and Laplacian of ρ_l with respect to the spatial variable $x = (x_1, \ldots, x_n)$.

Proof Firstly a straightforward differentiation using the given formulation of the map $u = \mathbf{Q}(\rho_1, \ldots, \rho_N)x|x|^{-1}$ gives

$$\begin{aligned} \nabla u &= \mathbf{Q}\nabla\theta + \sum_{l=1}^{N} \mathbf{Q}_{,l}\theta \otimes \nabla\rho_l \\ &= \frac{1}{r}(\mathbf{Q} - \mathbf{Q}\theta \otimes \theta) + \sum_{l=1}^{N} \mathbf{Q}_{,l}\theta \otimes \nabla\rho_l, \end{aligned} \tag{2.4}$$

where $r = |x| = \sqrt{\rho_1^2 + \cdots + \rho_N^2}$. With the aid of this we can then calculate the Hilbert-Schmidt norm of the gradient ∇u by writing

$$\begin{aligned} |\nabla u|^2 &= \mathrm{tr}\big\{[\nabla u][\nabla u]^t\big\} \\ &= \mathrm{tr}\left\{\frac{1}{r^2}(\mathbf{I}_n - \mathbf{Q}\theta \otimes \mathbf{Q}\theta) + \frac{1}{r}\sum_{l=1}^{N}(\mathbf{Q} - \mathbf{Q}\theta \otimes \theta)(\nabla\rho_l \otimes \mathbf{Q}_{,l}\theta) + \sum_{l=1}^{N} \mathbf{Q}_{,l}\theta \otimes \mathbf{Q}_{,l}\theta\right\} \end{aligned}$$

$$= \frac{n-1}{r^2} + \frac{1}{r}\sum_{l=1}^{N}\left\{\langle \mathbf{Q}\nabla\rho_l, \mathbf{Q}_{,l}\theta\rangle - \langle \mathbf{Q}\theta, \mathbf{Q}_{,l}\theta\rangle\langle\theta,\nabla\rho_l\rangle + r|\mathbf{Q}_{,l}\theta|^2\right\}$$

$$= \frac{n-1}{r^2} + \frac{1}{r}\sum_{l=1}^{N}\left\{\langle \mathbf{Q}\nabla\rho_l, \mathbf{Q}_{,l}\theta\rangle + r|\mathbf{Q}_{,l}\theta|^2\right\}, \tag{2.5}$$

where in deriving the last line we used the fact that $(\mathbf{Q}^t\mathbf{Q})_{,l} = 0$ implying in turn that the matrix product $\mathbf{Q}^t\mathbf{Q}_{,l}$ is skew-symmetric and subsequently $\langle \mathbf{Q}\theta, \mathbf{Q}_{,l}\theta\rangle = 0$. We now move on to the inner product term $\langle \mathbf{Q}\nabla\rho_l, \mathbf{Q}_{,l}\theta\rangle$. First, upon recalling that $\mathbf{Q}$ is of the form (2.3), a straightforward differentiation gives

$$\mathbf{Q}^t\mathbf{Q}_{,l} = \begin{cases} \mathrm{diag}(\partial_l f_1\mathbf{J}, \dots, \partial_l f_k\mathbf{J}) & \text{if } n = 2k, \\ \mathrm{diag}(\partial_l f_1\mathbf{J}, \dots, \partial_l f_k\mathbf{J}, 0) & \text{if } n = 2k+1, \end{cases} \tag{2.6}$$

where recalling and referring to (2.3) and (2.6), $\mathbf{J}$ and $\mathcal{R}[f] = \exp(f\mathbf{J})$ are the 2×2 matrices

$$\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{R}[f] = \begin{bmatrix} \cos f & -\sin f \\ \sin f & \cos f \end{bmatrix}, \tag{2.7}$$

respectively. If we write $y_l = (x_{2l-1}, x_{2l})$ for $1 \le l \le k$ if $n = 2k$ and additionally $y_{2k+1} = x_{2k+1}$ if $n = 2k+1$, then we have

$$\mathbf{Q}^t\mathbf{Q}_{,l}\theta = \begin{cases} \frac{1}{|x|}(\partial_l f_1\mathbf{J}y_1, \dots, \partial_l f_k\mathbf{J}y_k) & \text{if } n = 2k, \\ \frac{1}{|x|}(\partial_l f_1\mathbf{J}y_1, \dots, \partial_l f_k\mathbf{J}y_k, 0) & \text{if } n = 2k+1. \end{cases} \tag{2.8}$$

Furthermore, differentiating ρ_j and using identities (1.9) and (1.10), we see that

$$\nabla\rho_j = \frac{1}{\rho_j}(0, \dots, y_j, \dots, 0), \quad 1 \le j \le N. \tag{2.9}$$

Therefore, by substitution, the following inner product identity is seen to hold (note that here there is *no* summation over $1 \le j, l \le N$ and the penultimate equality excludes the relatively simpler case $j = N$ for n odd in which the identity trivially holds):

$$\langle \mathbf{Q}\nabla\rho_j, \mathbf{Q}_{,l}\theta\rangle = \left\langle \nabla\rho_j, \mathbf{Q}^t\mathbf{Q}_{,l}\theta\right\rangle = \frac{\partial_l f_j}{|x|\rho_j}\langle y_j, \mathbf{J}y_j\rangle = 0. \tag{2.10}$$

It now follows, upon referring to (2.5), that

$$|\nabla u|^2 = \frac{n-1}{r^2} + \frac{1}{r}\sum_{l=1}^{N}\left\{\langle \mathbf{Q}\nabla\rho_l, \mathbf{Q}_{,l}\theta\rangle + r|\mathbf{Q}_{,l}\theta|^2\right\} = \frac{n-1}{r^2} + \sum_{l=1}^{N}|\mathbf{Q}_{,l}\theta|^2. \tag{2.11}$$

Finally, the Laplacian of u is obtained by using $\Delta u = \mathrm{div}\,\nabla u$ and noting the identities $\nabla\rho_l \cdot \nabla\rho_k = \delta_{lk}$, $\nabla\rho_l \cdot x = \rho_l$ and $\Delta\rho_l = 1/\rho_l$ except of course for n odd and $l = N$, where we have $\Delta\rho_N = 0$. □

In the specific case $f_\alpha(\rho) = \mathscr{G}(\|\rho\|)$ for $\alpha = 1, \dots, k$ and with $\mathscr{G}$ sufficiently regular, the above quantities simplify and can be expressed as in the following lemma. Hereafter, with

$\mathbf{J}$ as in (2.7), we write $\mathbf{H}$ for the constant $n \times n$ skew-symmetric matrix

$$\mathbf{H} = \begin{cases} \mathrm{diag}(\mathbf{J}, \ldots, \mathbf{J}) & \text{if } n = 2k, \\ \mathrm{diag}(\mathbf{J}, \ldots, \mathbf{J}, 0) & \text{if } n = 2k+1. \end{cases} \tag{2.12}$$

Lemma 2.2 *Let $u = \mathbf{Q}(\rho_1, \ldots, \rho_N)x|x|^{-1}$ with $x \in \overline{\mathbb{X}}^n$ and $(\rho_1, \ldots, \rho_N) \in \overline{\mathbb{A}}_N$ be a spherical whirl where $\mathbf{Q}$ is as given by* (2.3). *Assume furthermore that*

$$f_\alpha(\rho) = \mathscr{G}\big(\|\rho\|\big), \quad \rho = (\rho_1, \ldots, \rho_N) \in \overline{\mathbb{A}}_N, 1 \le \alpha \le k, \tag{2.13}$$

where we have set $\|\rho\| = \sqrt{\rho_1^2 + \cdots + \rho_N^2} = \sqrt{x_1^2 + \cdots + x_n^2} = r$ with $a \le r \le b$ and $\mathscr{G} \in \mathscr{C}([a,b], \mathbb{R}) \cap \mathscr{C}^1((a,b), \mathbb{R})$. Then

- $\nabla u = \mathbf{Q}\dfrac{\mathbf{I}_n + (r\dot{\mathscr{G}}\mathbf{H} - \mathbf{I}_n)\theta \otimes \theta}{r}$,
- $|\nabla u|^2 = \dfrac{n-1}{r^2} + \dot{\mathscr{G}}^2 |\mathbf{H}\theta|^2$.

Additionally, if $\mathscr{G} \in \mathscr{C}^2((a,b), \mathbb{R})$, then we also have

- $\Delta u = \dfrac{(n-1)(r\dot{\mathscr{G}}\mathbf{H} - \mathbf{I}_n) + r^2(\ddot{\mathscr{G}}\mathbf{H} - \dot{\mathscr{G}}^2\mathbf{I}_n)}{r^2}\mathbf{Q}\theta$.

Proof This follows easily from Lemma 2.1 upon substituting from (2.3), (2.13) and direct differentiation. Note that in the second identity when $n \ge 2$ is even, we have $|\mathbf{H}\theta|^2 = 1$, whilst for n odd, we have $|\mathbf{H}\theta|^2 = 1 - \theta_n^2$. □

Using the description of $|\nabla u|^2$ in Lemma 2.1, we can proceed by writing the $\mathbb{F}$-energy of a spherical whirl u as

$$\begin{aligned} \mathbb{F}\big[u; \mathbb{X}^n\big] &= \int_{\mathbb{X}^n} F\big(r, |\nabla u|^2\big)\, dx = \int_{\mathbb{X}^n} F\big(r, \big|\nabla\big[\mathbf{Q}(\rho_1, \ldots, \rho_N)x|x|^{-1}\big]\big|^2\big)\, dx \\ &= \int_{\mathbb{X}^n} F\left(r, \frac{n-1}{r^2} + \sum_{l=1}^{N} |\mathbf{Q}_{,l}\theta|^2\right) dx \\ &= \int_{\mathbb{X}^n} F\left(r, \frac{n-1}{r^2} + \sum_{l=1}^{k} \frac{\rho_l^2}{r^2}|\nabla f_l|^2\right) dx \qquad (2.14) \\ &= (2\pi)^k \int_{\mathbb{A}_N} F\left(r, \frac{n-1}{r^2} + \sum_{l=1}^{k} \frac{\rho_l^2}{r^2}|\nabla f_l|^2\right) \prod_{j=1}^{k} \rho_j \, d\rho =: (2\pi)^k \mathbb{H}[f; \mathbb{A}_N], \qquad (2.15) \end{aligned}$$

where the penultimate equality is obtained after a basic change of variables, and for the energy integral $\mathbb{H}[f; \mathbb{A}_N]$ in (2.15), we have $f = (f_1, \ldots, f_k)$. Indeed, the admissible vector field $f = f(\rho)$ with $\rho = (\rho_1, \ldots, \rho_N)$ here is assumed to lie in the space

$$\begin{aligned} \mathscr{B}^p_{\mathsf{m}}(\mathbb{A}_N) := \big\{ & f = (f_1, \ldots, f_k) \in \mathscr{W}^{1,p}\big(\mathbb{A}_N, \mathbb{R}^k\big) : f_l \equiv 0 \text{ on } (\partial\mathbb{A}_N)_a, \\ & f_l \equiv 2\pi m_l + z_l \text{ on } (\partial\mathbb{A}_N)_b \text{ for all } 1 \le l \le k \big\}, \qquad (2.16) \\ \mathsf{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k, & \qquad \mathsf{z} = (z_1, \ldots, z_k) \in \mathbb{T}^k. \end{aligned}$$

The Euler-Lagrange equation associated with the energy integral $\mathbb{H}[f;\mathbb{A}_N]$ from (2.15) over the space $\mathscr{B}^p_{\mathsf{m}}(\mathbb{A}_N)$ is seen to be the nonlinear system

$$\begin{cases} \operatorname{div}\{F'(r,\frac{n-1}{r^2}+\sum_{l=1}^k\frac{\rho_l^2}{r^2}|\nabla f_l|^2)\frac{\rho_\alpha^2}{r^2}\nabla f_\alpha\prod_{j=1}^k\rho_j\}=0 & \text{in } \mathbb{A}_N, \\ F'(r,\frac{n-1}{r^2}+\sum_{l=1}^k\frac{\rho_l^2}{r^2}|\nabla f_l|^2)\frac{\rho_\alpha^2}{r^2}\partial_\nu f_\alpha\prod_{j=1}^k\rho_j=0 & \text{on } \Gamma_N, \\ f=(f_1,\dots,f_k)\equiv 0 & \text{on } (\partial\mathbb{A}_N)_a, \\ f=(f_1,\dots,f_k)\equiv 2\pi\mathsf{m}+\mathsf{z} & \text{on } (\partial\mathbb{A}_N)_b, \end{cases} \tag{2.17}$$

where $\alpha=1,\dots,k$, $\mathsf{m}=(m_1,\dots,m_k)$ and $\mathsf{z}=(z_1,\dots,z_k)$. Note that ∂_ν is the partial derivative in the outward pointing normal direction to Γ_N.

Proposition 2.1 *For each* $\mathsf{m}=(m_1,\dots,m_k)\in\mathbb{Z}^k$ *and* $\mathsf{z}=(z_1,\dots,z_k)\in\mathbb{T}^k$, *the solution* $f=f(\rho;\mathsf{m})\in\mathscr{C}^1(\overline{\mathbb{A}}_N,\mathbb{R}^k)\cap\mathscr{C}^2(\mathbb{A}_N,\mathbb{R}^k)$ *to system* (2.17) *is unique. This solution is also the unique minimiser of* $\mathbb{H}$ *with respect to its own boundary condition.*

Proof This is a result of a standard convexity argument. Indeed, in view of the growth assumption on F', minimisers of $\mathbb{H}$ are solutions to the Euler-Lagrange system (2.17) and conversely by the uniform convexity of the integrand solutions to (2.17) are minimisers of $\mathbb{H}$ with respect to their own boundary conditions. As a matter of fact, let f as described be a solution to (2.17) and pick $g\in\mathscr{B}^p_{\mathsf{m}}(\mathbb{A}_N)$. Put $\psi=g-f$ where $\psi\equiv 0$ on $(\partial\mathbb{A}_N)_a\cup(\partial\mathbb{A}_N)_b$. Then a standard convexity argument followed by an application of the divergence theorem gives

$$\begin{aligned} \Delta\mathbb{H} &= \mathbb{H}[g;\mathbb{A}_N]-\mathbb{H}[f;\mathbb{A}_N] \\ &\geq \int_{\mathbb{A}_N} F'\left(r,\frac{n-1}{r^2}+\sum_{l=1}^k\frac{\rho_l^2}{r^2}|\nabla f_l|^2\right)\sum_{\alpha=1}^k\frac{\rho_\alpha^2}{r^2}\left(|\nabla g_\alpha|^2-|\nabla f_\alpha|^2\right)\prod_{j=1}^k\rho_j\,d\rho \\ &\geq -2\sum_{\alpha=1}^k\int_{\mathbb{A}_N}\operatorname{div}\left[F'\left(r,\frac{n-1}{r^2}+\sum_{l=1}^k\frac{\rho_l^2}{r^2}|\nabla f_l|^2\right)\frac{\rho_\alpha^2}{r^2}\nabla f_\alpha\prod_{j=1}^k\rho_j\right]\psi_\alpha\,d\rho \\ &\quad +2\sum_{\alpha=1}^k\int_{\Gamma_N}\left[F'\left(r,\frac{n-1}{r^2}+\sum_{l=1}^k\frac{\rho_l^2}{r^2}|\nabla f_l|^2\right)\frac{\rho_\alpha^2}{r^2}\partial_\nu f_\alpha\prod_{j=1}^k\rho_j\right]\psi_\alpha\,d\rho \\ &\quad +\int_{\mathbb{A}_N}F'\left(r,\frac{n-1}{r^2}+\sum_{l=1}^k\frac{\rho_l^2}{r^2}|\nabla f_l|^2\right)\sum_{\alpha=1}^k\frac{\rho_\alpha^2}{r^2}|\nabla\psi_\alpha|^2\prod_{j=1}^k\rho_j\,d\rho\geq 0, \end{aligned}$$

where in deducing the last inequality we have noted that the first and second integrals on the left vanish due to f being a solution to (2.17). The uniqueness assertion now follows by observing that the last inequality is strict for nonzero ψ. □

As an instructive example, in case of the Dirichlet energy (with $F(r,t)\equiv t$), the above system decouples, and we can compute explicitly the unique solution $f=(f_1,\dots,f_k)=f(\rho;\mathsf{m})$ to (2.17). This is then seen to be given by ($n\geq 3$)

$$f_\alpha(\rho_1,\dots,\rho_N;\mathsf{m})=(2\pi m_\alpha+z_\alpha)\frac{\|\rho\|^{2-n}-a^{2-n}}{b^{2-n}-a^{2-n}},\quad 1\leq\alpha\leq k. \tag{2.18}$$

Moreover, the spherical whirl associated with the above f is a solution to (1.6) iff in the even case $m_1 = \cdots = m_k$, $z_1 = \cdots = z_k$ and so $f_1 = \cdots = f_k$, and in the odd case $z_1 = \cdots = z_k = 0$, $m_1 = \cdots = m_k = 0$ and so $f_1 = \cdots = f_k = 0$. Motivated by this observation, we now focus on the n even case with $z_1 = \cdots = z_k = z$ for $z \in \mathbb{T}$ and $m_1 = \cdots = m_k = m$ for $m \in \mathbb{Z}$. In this situation, as is stated below, the solution $f = f(\rho_1, \dots, \rho_N; \mathsf{m})$ depends solely on $\|\rho\| = \sqrt{x_1^2 + \cdots + x_{2N}^2}$.

Proposition 2.2 *For $n \geq 2$ even and $m \in \mathbb{Z}$, system* (2.17) *admits a unique solution $f = f(\rho; \mathsf{m})$ in $\mathscr{C}^2(\overline{\mathbb{A}}_N, \mathbb{R}^k)$ where $\mathsf{m} = (m, \dots, m)$ and $\mathsf{z} = (z, \dots, z)$ with $z \in \mathbb{T}$. Moreover, this solution $f = (f_1, \dots, f_k) = f(\rho; \mathsf{m})$ has components given explicitly by*

$$f_\alpha(\rho_1, \dots, \rho_N; \mathsf{m}) = \mathscr{G}\big(\|\rho\|; m\big), \quad 1 \leq \alpha \leq k, \tag{2.19}$$

where the function $\mathscr{G} = \mathscr{G}(r; m) \in \mathscr{C}^2([a, b], \mathbb{R})$ is the solution to the boundary value problem

$$\begin{cases} \frac{d}{dr}[F'(r, \frac{n-1}{r^2} + \dot{\mathscr{G}}^2) r^{n-1} \dot{\mathscr{G}}] = 0, \\ \mathscr{G}(a) = 0, \\ \mathscr{G}(b) = 2\pi m + z. \end{cases} \tag{2.20}$$

Proof That the boundary value problem (2.20) has a unique solution with the given degree of regularity follows by using variational methods. Indeed, thanks to the monotonicity and convexity assumptions on F, the energy integral

$$\mathscr{G} \mapsto \mathbb{F}\big[\exp\big(\mathscr{G}(r)\mathbf{H}\big)x|x|^{-1}; \mathbb{X}^n\big] = n\omega_n \int_a^b F\left(r, \frac{n-1}{r^2} + \dot{\mathscr{G}}^2\right) r^{n-1}\, dr \tag{2.21}$$

on $\mathscr{B}_m^p = \{\mathscr{G} \in W^{1,p}(a, b) : \mathscr{G}(a) = 0, \mathscr{G}(b) = 2\pi m + z\}$ is sequentially weakly lower semi-continuous and coercive, and so the existence of a minimiser follows from an application of the direct methods. The $\mathscr{C}^2$-regularity and uniqueness of the minimiser $\mathscr{G}$ then follows from standard convexity arguments and Hilbert's differentiability theorem (*cf.*, *e.g.*, [1], pp. 57-61). Note also that from (2.20) it follows upon noting $F' > 0$ that the solution $\mathscr{G}$ is *monotone* in r, that is, increasing when $2\pi m + z > 0$ and decreasing when $2\pi m + z < 0$. It thus remains to show that $f = (f_1, \dots, f_k)$ as given satisfies (2.17). Indeed, f is easily seen to satisfy the boundary conditions on $(\partial\mathbb{A}_N)_a$ and $(\partial\mathbb{A}_N)_b$ and the flat parts of $\partial\mathbb{A}_N$. Next, for $1 \leq \alpha \leq k$ and $1 \leq i \leq N$, a basic differentiation yields

$$\frac{\partial f_\alpha}{\partial \rho_i} = \frac{\rho_i}{r} \dot{\mathscr{G}}. \tag{2.22}$$

Furthermore, as $n = 2N$ and $k = N$, we have

$$|\nabla f_\alpha|^2 = \sum_{i=1}^k \frac{\rho_i^2}{r^2} \dot{\mathscr{G}}^2 = \dot{\mathscr{G}}^2 \quad \Longrightarrow \quad \frac{1}{r^2} \sum_{l=1}^k \rho_l^2 |\nabla f_\alpha|^2 = \dot{\mathscr{G}}^2. \tag{2.23}$$

We can now verify that f is a solution to (2.17). To save space, we will from now on write $\mathscr{H}(r) = (n-1)/r^2 + \dot{\mathscr{G}}^2$. Then proceeding directly and using the ODE for $\mathscr{G}$, we have

$$
\begin{aligned}
&\operatorname{div}\left[F'\left(r, \frac{n-1}{r^2} + \frac{1}{r^2}\sum_{l=1}^{k} \rho_l^2 |\nabla f_l|^2\right) \frac{\rho_\alpha^2}{r^2} \nabla f_\alpha \prod_{j=1}^{k} \rho_j\right] \\
&= \sum_{i=1}^{k} \frac{\partial}{\partial \rho_i}\left[F'\left(r, \frac{n-1}{r^2} + \dot{\mathscr{G}}^2\right) \frac{\rho_i}{r^3} \dot{\mathscr{G}} \rho_\alpha^2 \prod_{j=1}^{k} \rho_j\right] \\
&= \sum_{i=1}^{k} \Bigg\{ \frac{d}{dr} F'(r,\mathscr{H}) \frac{\rho_i^2}{r^4} \dot{\mathscr{G}} \rho_\alpha^2 \prod_{j=1}^{k} \rho_j + F'(r,\mathscr{H}) \frac{\rho_i^2}{r^4} \ddot{\mathscr{G}} \rho_\alpha^2 \prod_{j=1}^{k} \rho_j \\
&\quad - 3F'(r,\mathscr{H}) \frac{\rho_i^2}{r^5} \dot{\mathscr{G}} \rho_\alpha^2 \prod_{j=1}^{k} \rho_j + F'(r,\mathscr{H}) \frac{1}{r^3} \dot{\mathscr{G}} \rho_\alpha^2 \prod_{j=1}^{k} \rho_j \\
&\quad + F'(r,\mathscr{H}) \frac{\rho_i}{r^3} \dot{\mathscr{G}} 2\rho_\alpha \delta_i^\alpha \prod_{j=1}^{k} \rho_j + F'(r,\mathscr{H}) \frac{\rho_i}{r^3} \dot{\mathscr{G}} \rho_\alpha^2 \prod_{j=1, j\neq i}^{k} \rho_j \Bigg\} \\
&= \frac{\rho_\alpha^2}{r^2}\left(\prod_{j=1}^{k} \rho_j\right)\Bigg\{ \frac{d}{dr} F'(r,\mathscr{H}) \dot{\mathscr{G}} + F'(r,\mathscr{H}) \ddot{\mathscr{G}} - 3F'(r,\mathscr{H}) \frac{\dot{\mathscr{G}}}{r} \\
&\quad + kF'(r,\mathscr{H}) \frac{\dot{\mathscr{G}}}{r} + 2F'(r,\mathscr{H}) \frac{\dot{\mathscr{G}}}{r} + kF'(r,\mathscr{H}) \frac{\dot{\mathscr{G}}}{r} \Bigg\} \\
&= \frac{\rho_\alpha^2}{r^2}\left(\prod_{j=1}^{k} \rho_j\right)\Bigg\{ \frac{d}{dr} F'(r,\mathscr{H}) \dot{\mathscr{G}} + F'(r,\mathscr{H}) \ddot{\mathscr{G}} + \frac{n-1}{r} F'(r,\mathscr{H}) \dot{\mathscr{G}} \Bigg\} = 0.
\end{aligned}
\tag{2.24}
$$

The uniqueness of the solution f and the remaining minimality assertions follow from the previous proposition. □

From the description of the solution $f = f(\rho_1, \dots, \rho_N; \mathsf{m})$ it follows that f is solely a function of the radial variable $r = \|\rho\|$. Hence, with a slight abuse of notation, the associated spherical whirl has the form $u = \mathbf{Q}(r)x|x|^{-1}$ where $\mathbf{Q} \in \mathscr{C}^2([a,b], \mathbf{SO}(n))$; indeed, $\mathbf{Q}(r) = \exp(\mathscr{G}(r)\mathbf{H})$ where $\mathscr{G} = \mathscr{G}(r)$ is as in Proposition 2.2 and $\mathbf{H}$ is the constant $n \times n$ skew-symmetric matrix from (2.12). It therefore follows from similar results in [2] (see also [3]) that the spherical whirl $u = \mathbf{Q}(\rho_1, \dots, \rho_N; \mathsf{m})x|x|^{-1}$ with $\mathbf{Q}$ as in (1.11) and f from Proposition 2.2 is a classical solution to the nonlinear system (1.6) when n is even. Alternatively and more directly, referring to (1.6), Proposition 2.2, the explicit form of $u = \exp(\mathscr{G}(r)\mathbf{H})x|x|^{-1}$ and the ODE (2.20) satisfied by $\mathscr{G} = \mathscr{G}(r)$, we can write, with $\mathscr{H}(r) = (n-1)/r^2 + \dot{\mathscr{G}}^2$ as before and starting from $\mathscr{L}[u]$:

$$
\begin{aligned}
\mathscr{L}[u] &= \operatorname{div}\big[F'\big(r, |\nabla u|^2\big)\nabla u\big] + F'\big(r, |\nabla u|^2\big)|\nabla u|^2 u \\
&= 2F''(r,\mathscr{H})\big(\dot{\mathscr{G}}\ddot{\mathscr{G}} - (n-1)r^{-3}\big)\dot{\mathbf{Q}}\theta + \partial_r F'(r,\mathscr{H})\dot{\mathbf{Q}}\theta + F'(r,\mathscr{H}) \\
&\quad \times \big(\ddot{\mathbf{Q}} + (n-1)r^{-2}(r\dot{\mathbf{Q}} - \mathbf{Q})\big)\theta + F'(r,\mathscr{H})\big(\dot{\mathscr{G}}^2 + (n-1)r^{-2}\big)\mathbf{Q}\theta \\
&= \big\{2F''(r,\mathscr{H})\big(\dot{\mathscr{G}}\ddot{\mathscr{G}} - (n-1)r^{-3}\big)\dot{\mathscr{G}} + \partial_r F'(r,\mathscr{H})\dot{\mathscr{G}} \\
&\quad + F'(r,\mathscr{H})\big(\ddot{\mathscr{G}} + (n-1)r^{-1}\dot{\mathscr{G}}\big)\big\}\mathbf{H}\mathbf{Q}\theta = 0.
\end{aligned}
\tag{2.25}
$$

This proves that u is a solution to the nonlinear system (1.6) and hence justifies the Main Theorem stated in the Introduction. As a remark, this also shows that for spherical whirls u with f as in Proposition 2.2 (*cf.* (2.19)), the reduced system (2.17) is equivalent to the original *full* system (1.6), a conclusion that is *not* in general true for spherical whirls with unequal components of f merely solving (2.17). As an example, see (2.18) and the accompanying discussion.

3 The second variation of $\mathbb{F}$ at spherical whirls

In this section we compute the second variation of the energy integral $\mathbb{F}$ and discuss conditions under which this second variation at an extremising spherical whirl is positive definite. Towards this end, let $u \in \mathscr{A}^p_\varphi(\mathbb{X})$ be an admissible map of class $\mathscr{C}^1$ and pick $\phi \in \mathscr{C}^\infty_0(\mathbb{X}^n, \mathbb{R}^n)$ and, for $\varepsilon \in \mathbb{R}$ sufficiently small, set

$$u_\varepsilon = \frac{u + \varepsilon\phi}{|u + \varepsilon\phi|}. \tag{3.1}$$

It is evident that the one-parameter family (u_ε) is well-defined and lies in $\mathscr{A}^p_\varphi(\mathbb{X})$ and agrees with u for $\varepsilon = 0$. A straightforward computation now gives

$$\frac{d}{d\varepsilon} u_\varepsilon \bigg|_{\varepsilon=0} = \phi - \langle u, \phi\rangle u, \qquad \frac{d^2}{d\varepsilon^2} u_\varepsilon \bigg|_{\varepsilon=0} = 3\langle u, \phi\rangle^2 u - |\phi|^2 u - 2\langle u, \phi\rangle \phi. \tag{3.2}$$

For convenience, we hereafter use the notation $\widehat{\phi} = \phi - \langle u, \phi\rangle u$ that denotes the part of ϕ tangential to u. We can now proceed by writing

$$\begin{aligned}
\frac{d}{d\varepsilon}\mathbb{F}[u_\varepsilon; \mathbb{X}^n]\bigg|_{\varepsilon=0} &= 2\int_{\mathbb{X}^n} F'\big(r, |\nabla u|^2\big)\Big\langle \nabla u_\varepsilon, \nabla \frac{d}{d\varepsilon} u_\varepsilon\Big\rangle \, dx\bigg|_{\varepsilon=0} \\
&= 2\int_{\mathbb{X}^n} F'\big(r, |\nabla u|^2\big)\big[\langle \nabla u, \nabla\phi\rangle - |\nabla u|^2 \langle u, \phi\rangle\big]\, dx,
\end{aligned} \tag{3.3}$$

where we have used $\langle u \otimes \nabla\langle u, \phi\rangle, \nabla u\rangle = 0$ by virtue of $\langle u, u\rangle = 1$. If u is such that the first-order condition $d/d\varepsilon(\mathbb{F}[u_\varepsilon; \mathbb{X}^n])|_{\varepsilon=0} = 0$ holds for all $\phi \in \mathscr{C}^\infty_0(\mathbb{X}^n, \mathbb{R}^n)$, then the second variation can be computed by a further differentiation and use of (3.3) as

$$\begin{aligned}
\frac{d^2}{d\varepsilon^2}\mathbb{F}[u_\varepsilon; \mathbb{X}^n]\bigg|_{\varepsilon=0} &= \int_{\mathbb{X}^n} \bigg\{ 4F''\big(r, |\nabla u_\varepsilon|^2\big)\Big\langle \nabla u_\varepsilon, \nabla \frac{d}{d\varepsilon} u_\varepsilon\Big\rangle^2 + 2F'\big(r, |\nabla u_\varepsilon|^2\big)\bigg|\nabla \frac{d}{d\varepsilon} u_\varepsilon\bigg|^2 \\
&\quad + 2F'\big(r, |\nabla u_\varepsilon|^2\big)\Big\langle \nabla u_\varepsilon, \nabla \frac{d^2}{d\varepsilon^2} u_\varepsilon\Big\rangle \bigg\} \, dx\bigg|_{\varepsilon=0} \\
&= \int_{\mathbb{X}^n} \bigg\{ 4F''\big(r, |\nabla u|^2\big)\langle \nabla u, \nabla\widehat{\phi}\rangle^2 + 2F'\big(r, |\nabla u|^2\big)|\nabla\widehat{\phi}|^2 \\
&\quad + 2F'\big(r, |\nabla u|^2\big)\Big\langle u|\nabla u|^2, \frac{d^2}{d\varepsilon^2} u_\varepsilon\Big|_{\varepsilon=0}\Big\rangle \bigg\} \, dx \\
&= \int_{\mathbb{X}^n} \big\{ 4F''\big(r, |\nabla u|^2\big)\langle \nabla u, \nabla\widehat{\phi}\rangle^2 + 2F'\big(r, |\nabla u|^2\big)|\nabla\widehat{\phi}|^2 \\
&\quad - 2F'\big(r, |\nabla u|^2\big)|\nabla u|^2 |\widehat{\phi}|^2 \big\} \, dx.
\end{aligned} \tag{3.4}$$

Before proceeding further and discussing the positivity of this second variation at an extremising spherical whirl, it is instructive to note that on the level of the $\mathbb{H}$-energy the second variation at $f = (f_1, \dots, f_k)$ and with $f_\varepsilon = f + \varepsilon\varphi$ with $\varepsilon \in \mathbb{R}$ and $\varphi \in \mathscr{C}_0^\infty(\mathbb{A}_N, \mathbb{R}^k)$ is given by

$$\begin{aligned}
\delta^2\mathbb{H}[f](\varphi,\varphi) &:= \frac{d^2}{d\varepsilon^2}\mathbb{H}[f_\varepsilon]\bigg|_{\varepsilon=0} \\
&= 2\int_{\mathbb{A}_N}\Bigg[2F''\Bigg(r, \frac{n-1}{r^2} + \sum_{l=1}^{k}\frac{\rho_l^2}{r^2}|\nabla f_l|^2\Bigg)\Bigg(\sum_{l=1}^{k}\frac{\rho_l^2}{r^2}\langle\nabla f_l, \nabla\varphi_l\rangle\Bigg)^2 \\
&\quad + F'\Bigg(r, \frac{n-1}{r^2} + \sum_{l=1}^{k}\frac{\rho_l^2}{r^2}|\nabla f_l|^2\Bigg)\sum_{l=1}^{k}\frac{\rho_l^2}{r^2}|\nabla\varphi_l|^2\Bigg]\prod_{j=1}^{k}\rho_j\,d\rho.
\end{aligned} \tag{3.5}$$

Additionally, when f is given by Proposition 2.2 so that $f_l(\rho) = \mathscr{G}(\|\rho\|)$ for all l and n even, the above simplifies to

$$\begin{aligned}
\delta^2\mathbb{H}[f](\varphi,\varphi) &= 2\int_{\mathbb{A}_N}\Bigg[2F''\Bigg(r, \frac{n-1}{r^2} + \dot{\mathscr{G}}^2\Bigg)\Bigg(\sum_{l=1}^{k}\dot{\mathscr{G}}\frac{\rho_l^2}{r^3}\langle\rho, \nabla\varphi_l\rangle\Bigg)^2 \\
&\quad + F'\Bigg(r, \frac{n-1}{r^2} + \dot{\mathscr{G}}^2\Bigg)\sum_{l=1}^{k}\frac{\rho_l^2}{r^2}|\nabla\varphi_l|^2\Bigg]\prod_{j=1}^{k}\rho_j\,d\rho.
\end{aligned} \tag{3.6}$$

Both these second variations are seen to be uniformly positive *everywhere* in view of the uniform convexity of the integrand F. However, this is far from true for the second variation of $\mathbb{F}$.

Restricting now to the case $n \geq 2$ even, by using Lemma 2.2 we have that if $f = f(\rho; \mathsf{m}) = (f_1, \dots, f_k)$ is given by Proposition 2.2, then the corresponding spherical whirl $u = \mathbf{Q}(\rho)x|x|^{-1}$ with $\mathbf{Q}(\rho) = \mathbf{Q}(\rho_1, \dots, \rho_N) = \exp(\mathscr{G}(r)\mathbf{H}) = \mathrm{diag}(\mathcal{R}[\mathscr{G}](r), \dots, \mathcal{R}[\mathscr{G}](r))$ satisfies

$$\nabla u = \mathbf{Q}\frac{\mathbf{I}_n + (r\dot{\mathscr{G}}\mathbf{H} - \mathbf{I}_n)\theta\otimes\theta}{r}, \qquad |\nabla u|^2 = \left[\frac{n-1}{r^2} + \dot{\mathscr{G}}^2\right]. \tag{3.7}$$

Therefore referring to the calculations from earlier in the section leading to (3.4), we have that the second variation of the $\mathbb{F}$-energy at the extremising spherical whirl u in the direction of nonzero ϕ satisfying $\langle\phi, u\rangle = 0$ simplifies to

$$\begin{aligned}
\mathscr{J}[\mathscr{G}](\phi,\phi) &= \int_{\mathbb{X}^n}\Bigg\{4F''\Bigg(r, \frac{n-1}{r^2} + \dot{\mathscr{G}}^2\Bigg)\Bigg\langle\mathbf{Q}\frac{\mathbf{I}_n + (r\dot{\mathscr{G}}\mathbf{H} - \mathbf{I}_n)\theta\otimes\theta}{r}, \nabla\phi\Bigg\rangle^2 \\
&\quad + 2F'\Bigg(r, \frac{n-1}{r^2} + \dot{\mathscr{G}}^2\Bigg)|\nabla\phi|^2 \\
&\quad - 2F'\Bigg(r, \frac{n-1}{r^2} + \dot{\mathscr{G}}^2\Bigg)\Bigg[\frac{n-1}{r^2} + \dot{\mathscr{G}}^2\Bigg]|\phi|^2\Bigg\}\,dx.
\end{aligned} \tag{3.8}$$

Now focusing on the right-hand side of (3.8), in view of the convexity of the integrand F with respect to the second argument, that is, $F''(r,t) \geq 0$, it is evident that for any such ϕ, we have the inequality

$$\int_{\mathbb{X}^n} F''\Bigg(r, \frac{n-1}{r^2} + \dot{\mathscr{G}}^2\Bigg)\Bigg\langle\mathbf{Q}\frac{\mathbf{I}_n + (r\dot{\mathscr{G}}\mathbf{H} - \mathbf{I}_n)\theta\otimes\theta}{r}, \nabla\phi\Bigg\rangle^2 dx \geq 0. \tag{3.9}$$

Hence in particular it follows that on the level of the second variation of energy we have the inequality and lower bound

$$\mathscr{I}[\mathscr{G}](\phi,\phi) \geq \int_{\mathbb{X}^n} 2F'\left(r, \frac{n-1}{r^2} + \dot{\mathscr{G}}^2\right)\left(|\nabla\phi|^2 - \left[\frac{n-1}{r^2} + \dot{\mathscr{G}}^2\right]|\phi|^2\right) dx. \tag{3.10}$$

Next put $\gamma = \overline{d}/\underline{d} \geq 1$ where $\overline{d} \geq \underline{d} > 0$ denote the supremum and infimum of the continuous function $F'(|x|, |\nabla u|^2)$ on $\overline{\mathbb{X}}^n$, respectively, that is,

$$\overline{d} = \sup_{a \leq r \leq b} F'\left(r, \frac{n-1}{r^2} + \dot{\mathscr{G}}^2(r)\right), \qquad \underline{d} = \inf_{a \leq r \leq b} F'\left(r, \frac{n-1}{r^2} + \dot{\mathscr{G}}^2(r)\right). \tag{3.11}$$

(Note that $\underline{d} > 0$ follows from $F' > 0$ on $[a,b] \times (0,\infty)$ by using a compactness argument.) As an initial attempt, we can obtain a lower bound on the second variation by ignoring the orthogonality $\langle \phi, \mathbf{Q}\theta \rangle = 0$ satisfied by ϕ and bringing in the first Dirichlet eigenvalue. Setting $c = \sup[(n-1)/r^2 + \dot{\mathscr{G}}^2(r)]$ on $a \leq r \leq b$, from the formulation (3.8) it follows that

$$\begin{aligned}
\mathscr{I}[\mathscr{G}](\phi,\phi) &\geq 2\underline{d} \int_{\mathbb{X}^n} \left(|\nabla\phi|^2 - \gamma\left[\frac{n-1}{r^2} + \dot{\mathscr{G}}^2\right]|\phi|^2\right) dx \\
&\geq 2\underline{d} \int_{\mathbb{X}^n} \left(|\nabla\phi|^2 - \gamma c|\phi|^2\right) dx \\
&\geq 2\underline{d}\left(1 - \frac{\gamma c}{\lambda_1^{\mathsf{D}}(\mathbb{X}^n)}\right) \int_{\mathbb{X}^n} |\nabla\phi|^2\, dx,
\end{aligned} \tag{3.12}$$

where $\lambda_1^{\mathsf{D}}(\mathbb{X}^n) > 0$ is the first Dirichlet eigenvalue of $-\Delta$ in $\mathbb{X}^n$. As a result, this leads to the following conclusions:

- If $\gamma c < \lambda_1^{\mathsf{D}}(\mathbb{X}^n)$, then for all $\phi \in \mathscr{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n)$ nonzero, $\mathscr{I}[\mathscr{G}](\phi,\phi) > 0$.
- There exists $m = m(\gamma c) > 0$ such that for all $\phi \in \mathscr{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n)$ nonzero, if $|\operatorname{supp}\phi| \leq m$, then $\mathscr{I}[\mathscr{G}](\phi,\phi) > 0$.[d]

A more refined estimate taking into account the orthogonality condition on ϕ is given in the next theorem. For consequences of the positivity of the second variation to the extremiser u furnishing a strong local minimiser, see [4, 5].

Theorem 3.1 *Let $n \geq 4$ be even, and let $u = \mathbf{Q}(\rho)x|x|^{-1}$ be an extremising spherical whirl with $\mathbf{Q}(\rho) = \exp(\mathscr{G}(r)\mathbf{H}) = \operatorname{diag}(\mathcal{R}[\mathscr{G}](r), \dots, \mathcal{R}[\mathscr{G}](r))$ where $r = \|\rho\|$ and $\mathscr{G} \in \mathscr{C}^2([a,b],\mathbb{R})$ is a solution to* (2.20). *Assume the smallness condition*

$$\gamma\left(n - 1 + r^2\dot{\mathscr{G}}^2(r)\right) \leq \frac{(n-2)^2}{4} + 2, \quad a \leq r \leq b, \tag{3.13}$$

with equality holding at most on a null set in (a,b). Then the second variation of $\mathbb{F}$ at u is positive, that is, $\mathscr{I}[\mathscr{G}](\phi,\phi) > 0$ for all nonzero $\phi \in \mathscr{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n)$ satisfying $\langle \phi, u \rangle = 0$ a.e. in $\mathbb{X}^n$.

Proof Referring to the formulation of the second variation, the inequality given by (3.10) and the notation introduced above, we can write

$$\begin{aligned}\mathscr{J}[\mathscr{G}](\phi,\phi) &= \int_{\mathbb{X}^n} 4F''\left(r, \frac{n-1}{r^2} + \dot{\mathscr{G}}^2\right)\left\langle \mathbf{Q}\frac{\mathbf{I}_n + (r\dot{\mathscr{G}}\mathbf{H} - \mathbf{I}_n)\theta\otimes\theta}{r}, \nabla\phi\right\rangle^2 \\ &\quad + 2F'\left(r, \frac{n-1}{r^2} + \dot{\mathscr{G}}^2\right)\left(|\nabla\phi|^2 - \left[\frac{n-1}{r^2} + \dot{\mathscr{G}}^2\right]|\phi|^2\right)dx \\ &\geq \int_{\mathbb{X}^n} 2F'\left(r, \frac{n-1}{r^2} + \dot{\mathscr{G}}^2\right)\left(|\nabla\phi|^2 - \left[\frac{n-1}{r^2} + \dot{\mathscr{G}}^2\right]|\phi|^2\right)dx \\ &\geq 2\underline{d}\int_{\mathbb{X}^n}\left(|\nabla\phi|^2 - \gamma\left[(n-1) + r^2\dot{\mathscr{G}}^2\right]\frac{|\phi|^2}{r^2}\right)dx. \end{aligned} \tag{3.14}$$

Next, to proceed further, we recall an estimate from [4] (*cf.* Lemma 3.1 and Theorem 3.1): Suppose $n \geq 3$ and let $\mathbf{Q} = \mathbf{Q}(r)$ in $\mathscr{C}([a,b];\mathbf{SO}(n))$ be an arbitrary twist path. Then

$$\int_{\mathbb{X}^n}\left(|\nabla\phi|^2 - \frac{n-1}{r^2}|\phi|^2\right)dx \geq \frac{(n-4)^2}{4}\int_{\mathbb{X}^n}\frac{|\phi|^2}{r^2}\,dx \tag{3.15}$$

for all $\phi \in \mathscr{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n)$ satisfying $\langle\phi, \mathbf{Q}(r)x|x|^{-1}\rangle = 0$ a.e. in $\mathbb{X}^n$. Using this, combined with the previous lower bound on $\mathscr{J}$, we can therefore write

$$\begin{aligned}\mathscr{J}[\mathscr{G}](\phi,\phi) &\geq 2\underline{d}\int_{\mathbb{X}^n}\left(|\nabla\phi|^2 - \gamma\left[(n-1) + r^2\dot{\mathscr{G}}^2\right]\frac{|\phi|^2}{r^2}\right)dx \\ &\geq 2\underline{d}\int_{\mathbb{X}^n}\left(\frac{(n-4)^2}{4} - (n-1)(\gamma-1) - \gamma r^2\dot{\mathscr{G}}^2(r)\right)\frac{|\phi|^2}{r^2}\,dx > 0 \end{aligned} \tag{3.16}$$

for nonzero ϕ as described and subject to the smallness condition set in the theorem. This leads to the desired conclusion and thus completes the proof. □

- Note that upon setting $s = \sup r^2\dot{\mathscr{G}}^2(r)$ for $a \leq r \leq b$, a sufficient condition implying (3.13) is the *strict* inequality

$$\gamma(n-1+s) < \frac{(n-2)^2}{4} + 2. \tag{3.17}$$

- In case of the Dirichlet energy (*i.e.*, $F(r,t) \equiv t$) it is easily seen that $\overline{d} = \underline{d}$ and so $\gamma = 1$. Moreover, the ODE for $\mathscr{G}$, that is, (2.20) can be integrated to give the solution $\mathscr{G}(r) = (2\pi m + z)\mathscr{N}(r)$ for $a \leq r \leq b$ where the *profile* $\mathscr{N}$ is described by (*cf.* (2.18))

$$\mathscr{N}(r) = \frac{(r/a)^{2-n} - 1}{(b/a)^{2-n} - 1}, \quad n \geq 3. \tag{3.18}$$

As a result, the smallness condition here becomes

$$|2\pi m + z| \leq \frac{(n-4)}{2(n-2)}\left(1 - (a/b)^{n-2}\right), \tag{3.19}$$

and so under this hypothesis, we have $\mathscr{J}[\mathscr{G}](\phi,\phi) > 0$ for all nonzero $\phi \in \mathscr{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n)$ satisfying $\langle\phi, \mathbf{Q}x|x|^{-1}\rangle = 0$ a.e. in $\mathbb{X}^n$.

- The radial projection $u = x|x|^{-1}$ is a solution to system (1.6) (for all even or odd $n \geq 2$). It follows by a similar reasoning as in the proof of Theorem 3.1 (with $\mathscr{G}(r) \equiv 0$ in (3.14) and (3.16)) that subject to

$$0 \leq \gamma - 1 < \frac{(n-4)^2}{4(n-1)}, \quad n \geq 3, \tag{3.20}$$

the second variation of the energy at u is positive, that is, $\mathscr{J}[\mathscr{G}](\phi,\phi) > 0$ for all nonzero $\phi \in \mathscr{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n)$ satisfying $\langle \phi, \theta \rangle = 0$ a.e. in $\mathbb{X}^n$.

4 Conclusions

Considering a class of energy functionals depending on the norm of the gradient, we have established, under suitable boundary conditions, the existence of an infinite family of sphere-valued whirling solutions to the associated nonlinear Euler-Lagrange system in even dimensions. In sharp contrast, we have that in odd dimensions there can be one or no such whirling solutions. We have proved that in the former case, subject to a smallness condition, the second variation of the energy is strictly positive at these whirling solutions and hence that the latter solutions are stable.

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Endnotes

[a] For more on the structure of the Sobolev spaces of maps between Riemannian manifolds, see [6–10] and the references therein, and for applications and further discussion and reading relating to the work presented here, see [8, 11–16] as well as [3, 10, 17–19, 22–24].

[b] The divergence here is taken with respect to the x-variables whilst $\rho = \rho(x_1, \dots, x_n)$ and $r = \|\rho\| = (\rho_1^2 + \cdots + \rho_N^2)^{1/2}$. In (1.16) the divergence is taken with respect to the ρ-variables.

[c] The radial projection $u = x|x|^{-1}$ is always a solution to system (1.6) with $\varphi = x|x|^{-1}$ on $\partial\mathbb{X}^n$ (regardless of n even or odd). In fact here $\mathscr{L}[u] = \nabla u \nabla F' + F'(\Delta u + |\nabla u|^2 u) = 0$ as a consequence of $\nabla u = (\mathbf{I}_n - \theta \otimes \theta)/r$, $F'(r, |\nabla u|^2)$ depending only on r and $\Delta u + |\nabla u|^2 u = 0$.

[d] For a proof of inequalities of this type following on from the lower bound in the second line of (3.12) and more, see [5, 20, 21], in particular, Lemma 3.3 on pp. 224 in [20].

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