Du *Boundary Value Problems* (2017) 2017:185
https://doi.org/10.1186/s13661-017-0916-4

Boundary Value Problems
a SpringerOpen Journal

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Approximate controllability of the coupled degenerate system with two boundary controls

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Abstract

In this paper, we investigate the approximate controllability of the coupled system with boundary degeneracy. The control functions act on the degenerate boundary. We prove the Carleman estimate and the unique continuation of the adjoint system. Then we get the approximate controllability by constructing the control functions.

MSC: 93B05; 93C20; 35K65

Keywords: approximate controllability; the coupled degenerate system; boundary control

1 Introduction

In this paper, we investigate the approximate controllability of the coupled degenerate system

$$u_t - \left(x^p u_x\right)_x + \lambda_1 u + \lambda_2 v = 0, \quad (x,t) \in (0,1) \times (0,T), \tag{1.1}$$

$$v_t - \left(x^p v_x\right)_x + \lambda_3 u + \lambda_4 v = 0, \quad (x,t) \in (0,1) \times (0,T), \tag{1.2}$$

$$u(0,t) = g_1 \chi_{[T_1,T_2]}, \qquad u(1,t) = 0, \quad t \in (0,T), \tag{1.3}$$

$$v(0,t) = g_2 \chi_{[T_1,T_2]}, \qquad v(1,t) = 0, \quad t \in (0,T), \tag{1.4}$$

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in (0,1), \tag{1.5}$$

where $0 < p < 1$, $\lambda_i \in L^\infty((0,1) \times (0,T))$, $i = 1,2,3,4$, $u_0, v_0 \in L^2(0,1)$, $g_1, g_2 \in L^2(0,T)$ are the control functions, χ is the characteristic function, $0 < T_1 < T_2 < T$.

Recently, the controllability of the following degenerate parabolic equation has been investigated; see references [1–5]:

$$u_t - \left(x^p u_x\right)_x + c(x,t)u = h\chi_\omega, \quad (x,t) \in (0,1) \times (0,T), \tag{1.6}$$

where $c \in L^\infty((0,1) \times (0,T))$. The degenerate equation (1.6) can be obtained by suitable transformations of the Prandtl equations; see [6]. The equation (1.6) is divided into two cases, the weak degenerate case $0 < p < 1$ and the strong degenerate case $p \geq 2$. Different boundary conditions are proposed in two cases. When $0 < p < 1$, the boundary condition is

$$u(0,t) = u(1,t) = 0, \quad t \in (0,T). \tag{1.7}$$

When $p \geq 1$, the boundary condition is

$$x^p u_x(0,t) = u(1,t) = 0, \quad t \in (0,T). \tag{1.8}$$

In both cases, the initial value condition is

$$u(x,0) = u_0(x), \quad x \in (0,1), \tag{1.9}$$

where $u_0 \in L^2(0,1)$; see [2].

The authors prove that the problem (1.6), (1.7) or (1.8) and (1.9) is null controllable if $0 < p < 2$, and the problem is not null controllable if $p \geq 2$, see the references [1–5]. On the other hand, it is shown that, for every $p > 0$, the problem (1.6), (1.7) or (1.8) and (1.9) is approximate controllability; see [7, 8]. In [9], the author investigated the null controllability of the coupled system with internal control.

Moreover, it is considered whether the degenerate problem is controllable if the control function acts on the degenerate boundary. The following problem is studied; see [10–12]:

$$u_t - \left(x^p u_x\right)_x + c(x,t)u = 0, \quad (x,t) \in (0,1) \times (0,T), \tag{1.10}$$

$$u(0,t) = h\chi_{[T_1,T_2]}, \qquad u(1,t) = 0, \quad t \in (0,T), \tag{1.11}$$

$$u(x,0) = u_0(x), \quad x \in (0,1), \tag{1.12}$$

where $0 < p < 1$. Note that it is not necessary that we propose the boundary condition on the degenerate boundary when $1 \leq p < 2$; see [13]. Further, there are a lot of work on the controllability; see references [14–17] and so on.

The degenerate parabolic system (1.1)-(1.5) is the mathematical model coming from mathematical biology and physical phenomena; see [18, 19]. In the present paper, we prove the approximate controllability for the system (1.1)-(1.5). That is to say, for any $\varepsilon > 0$ and $u_0, v_0, u_1, v_1 \in L^2(0,1)$, there exist $g_1, g_2 \in L^2(0,T)$, such that the solution (u,v) to the system (1.1)-(1.5) satisfies

$$\left\| u(x,T) - u_1(x) \right\|^2_{L^2(0,1)} + \left\| v(x,T) - v_1(x) \right\|^2_{L^2(0,1)} \leq \varepsilon^2.$$

First, we prove the Carleman estimate for the adjoint system. Next, the unique continuation can be derived from the Carleman estimate. Then, by constructing the functional, we show the functional has a unique minimum point. Finally, we construct the control functions by the minimum point of the functional and get the approximate controllability.

The paper is organized as follows. In Section 2, we prove the Carleman estimate and the unique continuation for the adjoint system. In Section 3, we prove the approximate controllability of the coupled system (1.1)-(1.5).

2 Unique continuation for the adjoint system

In this section, we prove the unique continuation for the adjoint system by Carleman estimate.

First, we study the well-posedness of the adjoint system

$$-y_t - (x^p y_x)_x + \lambda_1 y + \lambda_3 z = f_1, \quad (x,t) \in (0,1) \times (0,T), \tag{2.1}$$

$$-z_t - (x^p z_x)_x + \lambda_2 y + \lambda_4 z = f_2, \quad (x,t) \in (0,1) \times (0,T), \tag{2.2}$$

$$y(0,t) = 0, \qquad y(1,t) = 0, \quad t \in (0,T), \tag{2.3}$$

$$z(0,t) = 0, \qquad z(1,t) = 0, \quad t \in (0,T), \tag{2.4}$$

$$y(x,T) = y_T(x), \qquad z(x,T) = z_T(x), \quad x \in (0,1), \tag{2.5}$$

where $f_1, f_2 \in L^2((0,1) \times (0,T))$, $y_T, z_T \in L^2(0,1)$.

Define $H_p^1(0,1)$, $H_p^2(0,1)$ are the closures of $C_0^\infty(0,1)$ with respect to the norms; see [1],

$$\|u\|_{H_p^1(0,1)} = \left(\int_0^1 \left(u^2 + x^p u_x^2\right) dx \right)^{1/2}, \quad u \in H_p^1(0,1),$$

$$\|u\|_{H_p^2(0,1)} = \left(\int_0^1 \left(u^2 + x^p u_x^2 + \left(x^p u_x\right)_x^2\right) dx \right)^{1/2}, \quad u \in H_p^2(0,1),$$

respectively. Denote $\mathbb{B} = L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H_p^1(0,1))$ and $\mathbb{D} = L^2(0,T;H_p^2(0,1)) \cap H^1(0,T;L^2(0,1))$ with respect to the norms

$$\|u\|_{\mathbb{B}} = \left(\sup_{t \in (0,T)} \int_0^1 \left(u(x,t)\right)^2 dx + \int_0^T \int_0^1 \left(u^2 + x^p u_x^2\right) dx\, dt \right)^{1/2}, \quad u \in \mathbb{B},$$

$$\|u\|_{\mathbb{D}} = \left(\int_0^T \int_0^1 u_t^2\, dx + \int_0^T \int_0^1 \left(u^2 + x^p u_x^2 + \left(x^p u_x\right)_x^2\right) dx\, dt \right)^{1/2}, \quad u \in \mathbb{D},$$

respectively.

Definition 2.1 A pair of functions $(y,z) \in \mathbb{B} \times \mathbb{B}$ is called a solution to the system (2.1)-(2.5), if for any $\varphi, \psi \in \mathbb{B}$ with $\varphi_t, \psi_t \in L^2((0,1) \times (0,T))$ and $\varphi(x,0) = 0$, $\psi(x,0) = 0$, $x \in (0,1)$, the following integral equalities hold:

$$\int_0^T \int_0^1 \left(y\varphi_t + x^p y_x \varphi_x + \lambda_1 y\varphi + \lambda_3 z\varphi\right) dx\, dt = \int_0^T \int_0^1 f_1 \varphi\, dx\, dt + \int_0^1 y_T(x)\varphi(x,T)\, dx,$$

$$\int_0^T \int_0^1 \left(z\psi_t + x^p z_x \psi_x + \lambda_2 y\psi + \lambda_4 z\psi\right) dx\, dt = \int_0^T \int_0^1 f_2 \psi\, dx\, dt + \int_0^1 z_T(x)\psi(x,T)\, dx.$$

By energy estimates, one can prove the well-posedness as the case of the single equations.

Theorem 2.1 *There exists a unique solution* $(y,z) \in \mathbb{B} \times \mathbb{B}$ *to the problem* (2.1)-(2.5) *satisfying*

$$\begin{aligned} &\|y\|_{\mathbb{B}} + \|z\|_{\mathbb{B}} + \left\|x^p y_x(0,t)\right\|_{L^2(T_1,T_2)} + \left\|x^p y_x(0,t)\right\|_{L^2(T_1,T_2)} + \left\|x^p z_x(0,t)\right\|_{L^2(T_1,T_2)} \\ &\quad \le C_1 \left(\|f_1\|_{L^2((0,1)\times(0,T))} + \|f_2\|_{L^2((0,1)\times(0,T))} + \|y_T\|_{L^2(0,1)} + \|z_T\|_{L^2(0,1)}\right), \end{aligned}$$

where C_1 is depending only on T, T_1, $\|\lambda_i\|_{L^\infty((0,1)\times(0,T))}$, $i = 1,2,3,4$. Further, if $(y_T,z_T) \in H_p^1(0,1) \times H_p^1(0,1)$, then there exists a constant C_2 depending only on T, T_1, $\|\lambda_i\|_{L^\infty((0,1)\times(0,T))}$, $i = 1,2,3,4$, such that

$$\|y\|_{\mathbb{D}} + \|z\|_{\mathbb{D}} \le C_2\big(\|f_1\|_{L^2((0,1)\times(0,T))} + \|f_2\|_{L^2((0,1)\times(0,T))} + \|y_T\|_{H_p^1(0,1)} + \|z_T\|_{H_p^1(0,1)}\big).$$

The proof is similar to Proposition 2.1 in [11] and Proposition 2.1 in [12].

Next, we prove the unique continuation. Consider the problem

$$w_t + \big(x^p w_x\big)_x = F, \quad (x,t) \in (0,1) \times (0,T), \tag{2.6}$$

$$w(0,t) = w(1,t) = 0, \quad t \in (0,T), \tag{2.7}$$

where $F \in L^2((0,1) \times (0,T))$. Then we have the following two lemmas.

Lemma 2.1 (Theorem 2.3 [10]) *Let $w \in \mathbb{D}$ be the solution to the problem* (2.6) *and* (2.7) *and satisfying*

$$\big(x^p w_x\big)(0,t) = \big(x^p w_x\big)(1,t) = 0.$$

Then, for fixed $q \in (1-p, 1-p/2)$, there exist two positive constants C and s_0 such that, for all $s \ge s_0$,

$$\begin{aligned}&\int_0^T \int_0^1 s^3 l^3(t) x^{2p+3q-4} w^2 e^{-2sx^q l(t)}\,dx\,dt + \int_0^T \int_0^1 sl(t) x^{2p+q-2} w_x^2 e^{-2sx^q l(t)}\,dx\,dt \\ &\quad\le C \int_0^T \int_0^1 F^2 e^{-2sx^q l(t)}\,dx\,dt,\end{aligned}$$

where $l(t) = \frac{1}{t(T-t)}$.

From Lemma 2.1, we can prove the Carleman estimate for the system (2.1)-(2.5).

Theorem 2.2 *Let $(y,z) \in \mathbb{D} \times \mathbb{D}$ be the solution to the system* (2.1)-(2.4) *and suppose that, for a.e. $t \in (0,T)$,*

$$\big(x^p y_x\big)(0,t) = \big(x^p y_x\big)(1,t) = \big(x^p z_x\big)(0,t) = \big(x^p z_x\big)(1,t) = 0.$$

Then, for fixed $q \in (1-p, 1-p/2)$, there exist positive constants C_1 and s_1 such that, for all $s \ge s_1$,

$$\begin{aligned}&\int_0^T \int_0^1 s^3 l^3(t) x^{2p+3q-4} y^2 e^{-2sx^q l(t)}\,dx\,dt + \int_0^T \int_0^1 sl(t) x^{2p+q-2} y_x^2 e^{-2sx^q l(t)}\,dx\,dt \\ &\qquad + \int_0^T \int_0^1 s^3 l^3(t) x^{2p+3q-4} z^2 e^{-2sx^q l(t)}\,dx\,dt + \int_0^T \int_0^1 sl(t) x^{2p+q-2} z_x^2 e^{-2sx^q l(t)}\,dx\,dt \\ &\quad\le C_1\left(\int_0^T \int_0^1 |f_1|^2 e^{-2sx^q l(t)}\,dx\,dt + \int_0^T \int_0^1 |f_2|^2 e^{-2sx^q l(t)}\,dx\,dt\right).\end{aligned}$$

Proof It follows from Lemma 2.1 that there exist C and s_0 such that, for all $s \geq s_0$,

$$\begin{aligned}
&\int_0^T \int_0^1 s^3 l^3(t) x^{2p+3q-4} y^2 e^{-2sx^q l(t)}\,dx\,dt + \int_0^T \int_0^1 sl(t) x^{2p+q-2} y_x^2 e^{-2sx^q l(t)}\,dx\,dt \\
&\quad + \int_0^T \int_0^1 s^3 l^3(t) x^{2p+3q-4} z^2 e^{-2sx^q l(t)}\,dx\,dt + \int_0^T \int_0^1 sl(t) x^{2p+q-2} z_x^2 e^{-2sx^q l(t)}\,dx\,dt \\
&\leq C\biggl(\int_0^T \int_0^1 |f_1|^2 e^{-2sx^q l(t)}\,dx\,dt + \int_0^T \int_0^1 |f_2|^2 e^{-2sx^q l(t)}\,dx\,dt \\
&\quad + \bigl(\|\lambda_1\|^2_{L^\infty(0,1)\times(0,T)} + \|\lambda_2\|^2_{L^\infty(0,1)\times(0,T)}\bigr) \int_0^T \int_0^1 |y|^2 e^{-2sx^q l(t)}\,dx\,dt \\
&\quad + \bigl(\|\lambda_3\|^2_{L^\infty(0,1)\times(0,T)} + \|\lambda_4\|^2_{L^\infty(0,1)\times(0,T)}\bigr) \int_0^T \int_0^1 |z|^2 e^{-2sx^q l(t)}\,dx\,dt\biggr).
\end{aligned}$$

Note that $2p + 3q - 4 < 0$ due to $q \in (1 - p, 1 - p/2)$. Take

$$s_1 = \max\left\{s_0, 2^{-5/3} T^2 C^{1/3} \left(\sum_{i=1}^4 \|\lambda_i\|^2_{L^\infty(0,1)\times(0,T)}\right)^{1/3}\right\}.$$

Then, for $s > s_1$, we have

$$\begin{aligned}
&\int_0^T \int_0^1 s^3 l^3(t) x^{2p+3q-4} y^2 e^{-2sx^q l(t)}\,dx\,dt + \int_0^T \int_0^1 sl(t) x^{2p+q-2} y_x^2 e^{-2sx^q l(t)}\,dx\,dt \\
&\quad + \int_0^T \int_0^1 s^3 l^3(t) x^{2p+3q-4} z^2 e^{-2sx^q l(t)}\,dx\,dt + \int_0^T \int_0^1 sl(t) x^{2p+q-2} z_x^2 e^{-2sx^q l(t)}\,dx\,dt \\
&\leq 2C\biggl(\int_0^T \int_0^1 |f_1|^2 e^{-2sx^q l(t)}\,dx\,dt + \int_0^T \int_0^1 |f_2|^2 e^{-2sx^q l(t)}\,dx\,dt\biggr).
\end{aligned}$$

The proof is complete. □

Similar to the proof of Theorem 3.1 [10] and Proposition 4.2 [12], one can prove the following unique continuation properties.

Theorem 2.3 *Let $(y,z) \in \mathbb{D} \times \mathbb{D}$ be the solution to the system* (2.1)-(2.4) *and suppose that, for almost every $t \in (0,T)$,*

$$\bigl(x^p y_x\bigr)(0,t) = \bigl(x^p z_x\bigr)(0,t) = 0.$$

If $f_1(x,t) = f_2(x,t) = 0$, then $y(x,t) = 0$, $z(x,t) = 0$, where $(x,t) \in (0,1) \times (0,T)$.

Theorem 2.4 *Let $(y,z) \in \mathbb{B} \times \mathbb{B}$ be the solution to the system* (2.1)-(2.5) *and suppose that, for almost every $t \in (0,T)$,*

$$\bigl(x^p y_x\bigr)(0,t) = \bigl(x^p z_x\bigr)(0,t) = 0.$$

If $f_1(x,t) = f_2(x,t) = 0$, then $y(x,t) = 0$, $z(x,t) = 0$, where $(x,t) \in (0,1) \times (0,T)$.

3 Approximate controllability for the control system

In this section, we prove the approximate controllability for the control system (1.1)-(1.5).

Define the mapping

$$\mathcal{L}:\mathbb{X}\to\mathbb{T},\qquad (y_T,z_T)\longmapsto \bigl(x^p y_x(0,t)\chi_{\omega_1},x^p z_x(0,t)\chi_{\omega_1}\bigr),$$

where $\mathbb{X}=L^2(0,1)\times L^2(0,1)$ with the norm

$$\bigl\|(w_1,w_2)\bigr\|_{\mathbb{X}}=\bigl(\|w_1\|^2_{L^2(0,1)}+\|w_2\|^2_{L^2(0,1)}\bigr)^{1/2},\quad (w_1,w_2)\in\mathbb{X}$$

and $\mathbb{T}=L^2(T_1,T_2)\times L^2(T_1,T_2)$ with the norm

$$\bigl\|(w_1,w_2)\bigr\|_{\mathbb{T}}=\bigl(\|w_1\|^2_{L^2(T_1,T_2)}+\|w_2\|^2_{L^2(T_1,T_2)}\bigr)^{1/2},\quad (w_1,w_2)\in\mathbb{T}.$$

For any $(u_1,v_1)\in\mathbb{X}$, define the functional

$$J\bigl((y_T,z_T)\bigr)=\frac{1}{2}\bigl\|\bigl(x^p y_x(0,t),x^p z_x(0,t)\bigr)\bigr\|^2_{\mathbb{T}}+\varepsilon\bigl\|(y_T,z_T)\bigr\|_{\mathbb{X}}-\bigl\langle(u_1,v_1),(y_T,z_T)\bigr\rangle_{\mathbb{X}},$$

where $(y_T,z_T)\in\mathbb{X}$ and $\langle\cdot,\cdot\rangle_{\mathbb{X}}$ is the inner product in $\mathbb{X}$.

Proposition 3.1 *$J(\cdot)$ is strictly convex and satisfies*

$$\liminf_{\|(y_T,z_T)\|_{\mathbb{X}}\to+\infty}\frac{J((y_T,z_T))}{\|(y_T,z_T)\|_{\mathbb{X}}}\geq\varepsilon.\tag{3.1}$$

Furthermore, $J(\cdot)$ achieves its minimum at a unique point $(\hat{y}_T,\hat{z}_T)$ in $\mathbb{X}$ and

$$(\hat{y}_T,\hat{z}_T)=0\ \textit{a.e. in}\ \Omega\quad\Longleftrightarrow\quad\bigl\|(u_1,v_1)\bigr\|_{\mathbb{X}}\leq\varepsilon.\tag{3.2}$$

The proof is similar to the proof of Proposition 3.1 in [7].

Now, we are ready to prove the approximate controllability of the system (1.1)-(1.5).

Theorem 3.1 *The system* (1.1)-(1.5) *is approximately controllable. That is to say, for any given $u_0,v_0,u_1,v_1\in L^2(0,1)$ and $\varepsilon>0$, there exist $g_1,g_2\in L^2(T_1,T_2)$ such that the weak solution (u,v) to the system* (1.1)-(1.5) *satisfies*

$$\bigl\|\bigl(u(x,T)-u_1,v(x,T)-v_1\bigr)\bigr\|_{\mathbb{X}}\leq\varepsilon.\tag{3.3}$$

Proof Without loss of generality, we assume

$$u_0(x)=0,\qquad v_0(x)=0,\quad \text{a.e. } x\in(0,1).\tag{3.4}$$

If $\|(u_1,v_1)\|_{\mathbb{X}}\leq\varepsilon$, (3.3) holds by taking $g_1,g_2=0$. Now we suppose $\|(u_1,v_1)\|_{\mathbb{X}}>\varepsilon$.

In this case, Proposition 3.1 yields $(\hat{y}_T,\hat{z}_T)\neq(0,0)$. For any $(\theta_0,\psi_0)\in\mathbb{X}$, denote (θ,ψ) to be the solution of the coupled system (2.1)-(2.5) with $(y_T,z_T)=(\theta_0,\psi_0)$. Since $(\hat{y}_T,\hat{z}_T)$ is the unique point of minimum of $J(\cdot)$, one gets

$$\begin{aligned}&\big\langle\big(x^p\hat{y}_x(0,t),x^p\hat{z}_x(0,t)\big),\big(x^p\theta_x(0,t),x^p\psi_x(0,t)\big)\big\rangle_{\mathbb{T}}+\varepsilon\frac{\langle(\hat{y}_T,\hat{z}_T),(\theta_0,\psi_0)\rangle_{\mathbb{X}}}{\|(\hat{y}_T,\hat{z}_T)\|_{\mathbb{X}}}\\&\quad-\big\langle(u_1,v_1),(\theta_0,\psi_0)\big\rangle_{\mathbb{X}}=0.\end{aligned}\tag{3.5}$$

It follows from the definition of the weak solution (u,v) to the system (1.1)-(1.4) and (3.4) that

$$\int_0^T\int_0^1\big(u_t\theta\,dx\,dt+x^pu_x\theta_x+\lambda_1u\theta+\lambda_2v\theta\big)\,dx\,dt=0,\tag{3.6}$$

$$\int_0^T\int_0^1\big(v_t\psi\,dx\,dt+x^pv_x\psi_x+\lambda_3u\psi+\lambda_4v\psi\big)\,dx\,dt=0.\tag{3.7}$$

Additionally, the definition of the weak solution (θ,ψ) to the system (2.1)-(2.5) with $(y_T,z_T)=(\theta_0,\psi_0)$ gives

$$\begin{aligned}&\int_0^T\int_0^1\big(\theta u_t+x^p\theta_xu_x+\lambda_1\theta u+\lambda_3\psi u\big)\,dx\,dt\\&\quad=\int_0^1\theta_0(x)u(x,T)\,dx-\int_{T_1}^{T_2}x^p\theta_x(0,t)g_1\,dt,\end{aligned}\tag{3.8}$$

$$\begin{aligned}&\int_0^T\int_0^1\big(\psi v_t+x^p\psi_xv_x+\lambda_2\theta v+\lambda_4\psi v\big)\,dx\,dt\\&\quad=\int_0^1\psi_0(x)v(x,T)\,dx-\int_{T_1}^{T_2}x^p\psi_x(0,t)g_2\,dt.\end{aligned}\tag{3.9}$$

From (3.6)-(3.9), one can get

$$\begin{aligned}&\int_{T_1}^{T_2}\big(x^p\theta_x(0,t)x^p\hat{y}_x(0,t)+x^p\psi_x(0,t)x^p\hat{z}_x(0,t)\big)\,dt\\&\quad=\int_0^1\theta_0(x)u(x,T)\,dx+\int_0^1\psi_0(x)v(x,T)\,dx\end{aligned}\tag{3.10}$$

by taking

$$g_1(t)=\begin{cases}x^p\hat{y}_x(0,t),& t\in[T_1,T_2],\\0,& t\in[0,T_1)\cup(T_2,T],\end{cases}$$

$$g_2(t)=\begin{cases}x^p\hat{z}_x(0,t),& t\in[T_1,T_2],\\0,& t\in[0,T_1)\cup(T_2,T].\end{cases}$$

Combining (3.10) with (3.5) yields

$$\big\langle\big(u_1-u(x,T),v_1-v(x,T)\big),(\theta_0,\psi_0)\big\rangle_{\mathbb{X}}=\varepsilon\frac{\langle(\hat{y}_T,\hat{z}_T),(\theta_0,\psi_0)\rangle_{\mathbb{X}}}{\|(\hat{y}_T,\hat{z}_T)\|_{\mathbb{X}}},$$

which implies (3.3) due to the arbitrariness of $(\theta_0,\psi_0)\in\mathbb{X}$. □

Acknowledgements
This work was supported by the Natural Science Foundation for Young Scientists of Jilin Province (20170520048JH), the Scientific and Technological Project of Jilin Province's Education Department in Thirteenth Five-Year (JJKH20170534KJ) and the National Natural Science Foundation of China (11401049).

Competing interests
The author declares to have no competing interests.

Authors' contributions
Author read and approved the final manuscript.

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Received: 29 July 2017 Accepted: 7 December 2017 Published online: 16 December 2017

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