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New approximation methods for solving elliptic boundary value problems via Picard-Mann iterative processes with mixed errors

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Abstract

In this paper, we introduce and study a class of new Picard-Mann iterative methods with mixed errors for common fixed points of two different nonexpansive and contraction operators. We also give convergence and stability analysis of the new Picard-Mann iterative approximation and propose numerical examples to show that the new Picard-Mann iteration converges more effectively than the Picard iterative process, Mann iterative process, Picard-Mann iterative process due to Khan and other related iterative processes. Furthermore, as an application, we explore iterative approximation of solutions for an elliptic boundary value problem in Hilbert spaces by using the new Picard-Mann iterative methods with mixed errors for contraction operators.

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Keywords: new Picard-Mann approximation; common fixed point; nonexpansion and contraction; elliptic boundary value problem; convergence and stability

1 Introduction

In order to find a weak solution of the following elliptic boundary value problem (so-called Dirichlet problem):

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, Ayadi et al. [1] proved a new global minimization theorem in Hilbert spaces by using the notion of a nonexpansive potential operator. As we all know, multidimensional dynamical systems are frequently formulated by partial differential equations, which generally depend on space and time, i.e., parabolic or evolutionary type equations, and are treated with emphasis on various real-world applications in (thermo)mechanics of solids and fluids, electrical devices, engineering, chemistry, biology, etc. (see [2, 3]). But under some suitable conditions, the time-dependent form of partial differential equations can be rewritten as a time-independent form (see [4, Example 4, p. 161]), and some special cases of the Dirichlet problem (1.1) represent elliptic variational forms of second order physician, physicist and anatomist equation (see [2, 5]). Thus, the nonlinear elliptic problem (1.1) has been studied via fixed point index theory, critical point theory, Morse theory, variational inequality theory and so on. See, for example, [4, 5] and the references therein. The boundary value problem is widely used in physics. In 1997, Marin [6] established necessary and sufficient conditions for the existence and uniqueness of the weak solution to the mixed boundary value problem in the domain of dipolar bodies with voids. Later, Marin and Vlase [7] showed that the existence of internal state variables has no effect on the uniqueness of the solution associated with the mixed initial boundary value problem in thermoelasticity of microstretch bodies (see [7, Theorems 1-3, p. 248]), the proof of the uniqueness of the solution and some useful estimations are also contained.

In particular, many problems in physics and other applications cannot be formulated as equations but have some more complicated structure, and usually the so-called complementarity problem, which is equivalent to a variational inequality. Further, the applicability of variational inequality theory, which was initially developed to cope with equilibrium problems (e.g., the Signorini problem, which was first posed by Antonio Signorini in 1959), has been extended to involve problems in economics, finance, electrodynamics, mechanics, engineering science, optimization and game theory. Hence, the variational method is very important in optimal control theory, and such generalization is often needed in optimal-control theory of elliptic problems. In fact, optimal control problems in control theory are searching for a kind of control mode which can transform the initial state of the control object to the terminal state and make sure that the objective function can reach the maximum or minimum. For more details on variational inequalities in the context of their optimal control, one can refer to [2–5] and the references therein, and the following examples.

Example 1.1 ([8]) Consider the following optimal boundary control problem of elliptic equation constraints:

$$\min J(u), \tag{1.2}$$

where state variable $y(u) \in V = H^1(\Omega)$, state space, and control variable $u \in U = L^2(\partial\Omega)$, control space, satisfy

$$\begin{cases} -\Delta y = f, & x \in \Omega, \\ \frac{\partial y}{\partial \vec{n}} = u + g, & x \in \Gamma_N, \\ y = y_d, & x \in \Gamma_D, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded convex region with smooth boundary $\partial\Omega$, Γ_N and Γ_D are respectively Neumann boundary and Dirichlet boundary, $\partial\Omega = \Gamma_N \cup \Gamma_D$, $\Gamma_N \cap \Gamma_D = \emptyset$, $\vec{n}$ is the unit normal vector of $\partial\Omega, f, g$ and y_d are given functions.

Define the objective function in (1.2) by

$$J(u) = \frac{1}{2}\left\{\int_\Omega \gamma\left|y(x) - y_0(x)\right|^2 dx + \int_{\Gamma_N} \alpha\left|u(x)\right|^2 ds\right\}, \tag{1.4}$$

where $y_0(x)$ is a given target state variable, $\gamma, \alpha > 0$ are two constants, which play a role of balance to the state variable y and the control variable u. By optimal control theory [9] and the definition of directional derivative for functional, now we know that solving the optimal problem (1.2) on a convex set U is equivalent to finding the control variable $u \in U$ such that the following variational inequality holds:

$$\langle J'(u), v - u\rangle = \int_{\Gamma_N} (\alpha u + p)(v - u)\,ds \geq 0, \quad \forall v \in U, \tag{1.5}$$

where p is the dual state variable of y and satisfies the following state equation:

$$\begin{cases} -\Delta p = \gamma(y - y_0), & x \in \Omega, \\ \frac{\partial p}{\partial \vec{n}} = 0, & x \in \Gamma_N, \\ p = 0, & x \in \Gamma_D, \end{cases} \tag{1.6}$$

which is the dual problem of (1.3). Inequality (1.5) is called optimality condition for the optimal control problems (1.2) and (1.3), which is equivalent to the equation system composed of (1.3), (1.5) and (1.6).

Based on the above analysis of dualization and optimality condition for the optimal control problems (1.2) and (1.3), Liu and Sun [8] introduced and studied an iterative nonoverlapping domain decomposition method for (1.3)-(1.6) and proved convergence of the sequence generated by the iterative method. Furthermore, by using an iterative algorithm due to the penalized gradient projection method, adaptive finite element method, edge stabilization Galerkin method, variational iteration method, etc., such kind of problems as (1.2) or (1.3) were considered by many authors and researchers. See, for example, [9–16] and the references therein. Especially noteworthy, Zhou and Li [17] pointed out 'though much achievement has been achieved, application of the variational iteration method to Cauchy problems has not yet been dealt with'.

On the other hand, in order to compare to Picard, Mann and Ishikawa iterations for approximating fixed points and to solve equation systems, Khan [18] introduced and studied a Picard-Mann hybrid iterative process and showed that the Picard-Mann hybrid iterative process converges faster than all of Picard, Mann and Ishikawa iterative processes for contractions. Following on the works of Khan [18], by using an up-to-date method for approximating common fixed points of countable families of nonlinear operators, Deng [19] introduced a modified Picard-Mann hybrid iterative algorithm for a sequence of nonexpansive mappings and established strong convergence and weak convergence of the iterative sequence generated by the modified hybrid iterative algorithm in a convex Banach space. Okeke and Abbas [20] introduced and studied Picard-Krasnoselskii hybrid iterations, which converge faster than Picard, and gave an application to delay differential equations Mann, Krasnoselskii and Ishikawa iterative processes for contractive nonlinear operators. However, one can know that the Picard-Krasnoselskii hybrid iteration is a special case of the Picard-Mann hybrid iterative process due to Khan [18], it is because $\alpha_n \in (0,1)$ of (1.4) in [18] includes $\lambda \in (0,1)$ in (1.7) of [20] (see [20, Example 2.2, p. 25]). Jiang et al. [21] proved convergence of Mann iterative sequences for approximating solutions of a higher order nonlinear neutral delay differential equation and proposed advantages of the presented results through three extraordinary examples. However, how to establish the error estimates between the approximate solutions and the exact solutions for partial differential equations is not reported in the literature.

Moreover, Roussel [22] pointed out that equilibria are not always stable. Since stable and unstable equilibria play quite different roles in the dynamics of a system, it is useful to be able to classify equilibrium points based on their stability. Thus, there are many scholars and researchers who have discussed stability of the iterative sequence generated by the algorithm for solving the investigated problems. See, for example, [23–27] and the references therein. Especially, stimulated by the work of Bosede and Rhoades [28], Akewe and Okeke [27] obtained stability results for the Picard-Mann hybrid iterative scheme due to Khan [18] for a general class of contractive-like operators introduced by Bosede and Rhoades [28]. However, how does one obtain stability analysis when the Picard-Mann hybrid iterative scheme due to Khan [18] is generalized for two different nonexpansive and contraction operators and one involves errors or mixed errors? This is a significant and challenging research work.

Motivated and inspired by the above works, we aim in this paper to introduce and study a class of new Picard-Mann iterative methods with mixed errors for common fixed points of two different nonexpansive and contraction operators. Then convergence and stability analysis of the new Picard-Mann iterative approximation are given. Finally, two numerical examples to verify effectiveness of the new Picard-Mann iteration are presented, and a new iterative approximation of solutions for an elliptic boundary value problem in Hilbert spaces is investigated by using the new Picard-Mann iterative methods with mixed errors for nonexpansive operators, which are different from the method proposed in [1].

2 New Picard-Mann approximation methods

In this section, we shall introduce and study a class of new Picard-Mann iterative methods with mixed errors for common fixed points of two different nonexpansive and contraction operators and prove convergence and stability of the new Picard-Mann iterative approximation.

We need the following definitions and lemmas for our main results.

Definition 2.1 Let X be a normed space and $K \subset X$ be a nonempty subset. Then an operator $T : K \to K$ is said to be

(i) nonexpansive if

$$\|Tu - Tv\| \leq \|u - v\|, \quad \forall u, v \in K; \tag{2.1}$$

(ii) contraction if there exists a constant $k \in [0, 1)$ such that

$$\|Tu - Tv\| \leq k\|u - v\|, \quad \forall u, v \in K. \tag{2.2}$$

Remark 2.1 The constant k in Definition 2.1(ii) is called the Lipschitz constant of T. Contractive operators are sometimes called Lipschitzian operators. If the above condition is instead satisfied for $k \leq 1$, then the operator T is said to be nonexpansive.

Definition 2.2 Let S be a selfmap of the normed space X, $x_0 \in X$, and let $x_{n+1} = h(S, x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\} \subset X$. Suppose that

$\{x \in X : Sx = x\} \neq \emptyset$ and $\{x_n\}$ converges to a fixed point x^* of S. Let $\{w_n\} \subset X$ and let $\varepsilon_n = \|w_{n+1} - h(S, w_n)\|$. If $\lim \epsilon_n = 0$ implies that $w_n \to x^*$, then the iteration procedure defined by $x_{n+1} = h(S, x_n)$ is said to be S-stable or stable with respect to S.

Lemma 2.1 ([29]) *Let X be a normed space and C be a nonempty closed convex bounded subset of X. Then each nonexpansive operator $T : C \to C$ has a fixed point in C.*

Lemma 2.2 ([30]) *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + t_n b_n + c_n, \tag{2.3}$$

where $t_n \in [0,1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$, $\sum_{n=0}^{\infty} c_n < \infty$. Then $a_n \to 0$ $(n \to \infty)$.

Now, we establish a class of new Picard-Mann iterations with mixed errors for common fixed points of two different nonlinear operators (in short, (PMMD)) as follows.

Algorithm 2.1 Step 1. Choose x_0 in a normed space X.

Step 2. Let

$$\begin{cases} x_{n+1} = T_1 y_n + h_n, \\ y_n = (1 - \alpha_n)x_n + \alpha_n T_2 x_n + \alpha_n d_n + e_n, \end{cases} \tag{2.4}$$

where $T_1, T_2 : X \to X$ are two nonlinear operators, and $h_n, d_n, e_n \in X$ are errors to take into account a possible inexact computation of the operator points.

Step 3. Choose sequences $\{\alpha_n\}$, $\{h_n\}$, $\{d_n\}$ and $\{e_n\}$ such that for $n \geq 0$, $\{\alpha_n\} \subset [0,1]$ and $\{h_n\}$, $\{d_n\}$, $\{e_n\}$ are three sequences in X satisfying the following conditions $\mathcal{P}$:

(i) $d_n = d'_n + d''_n$;

(ii) $\lim_{n\to\infty} \|d'_n\| = 0$;

(iii) $\sum_{n=0}^{\infty} \|h_n\| < \infty$, $\sum_{n=0}^{\infty} \|d''_n\| < \infty$, $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Step 4. If x_{n+1}, y_n, α_n, h_n, d_n and e_n satisfy (2.4) to sufficient accuracy, go to Step 5; otherwise, set $n := n + 1$ and return to Step 2.

Step 5. Let $\{w_n\}$ be any sequence in X and define $\{\varepsilon_n\}$ by

$$\begin{cases} \varepsilon_n = \|w_{n+1} - (T_1 \xi_n + h_n)\|, \\ \xi_n = (1 - \alpha_n)w_n + \alpha_n T_2 w_n + \alpha_n d_n + e_n. \end{cases} \tag{2.5}$$

Step 6. If ε_n, w_{n+1}, ξ_n, α_n, h_n, d_n and e_n satisfy (2.5) to sufficient accuracy, stop; otherwise, set $n := n + 1$ and return to Step 3.

Remark 2.2 For special choices of the operators T_1 and T_2, the space X, and the errors h_n, d_n and e_n in (2.4), one can obtain a large number of Picard iterative process, Mann iterative process, Picard-Mann iterative process due to Khan [18] and other related iterations. Now we list some special cases of iteration (2.4) as follows.

Special Case I If $h_n = d_n = e_n = 0$, the iterative process (2.4) becomes the following Picard-Mann iteration for two different operators (in short, (PMD)): For any given

$x_0 \in X$,

$$\begin{cases} x_{n+1} = T_1 y_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n T_2 x_n. \end{cases} \tag{2.6}$$

Special Case II When $T_1 = T_2 = T$, for any given $x_0 \in X$, iteration (2.4) reduces to the sequence $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} = Ty_n + h_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n Tx_n + \alpha_n d_n + e_n. \end{cases} \tag{2.7}$$

We note that the iterative processes (PMD) and the Picard-Mann iteration with mixed errors (2.7) (in short, (PMM)) are new and not studied in the literature.
Special Case III If $T_1 = T_2 = T$, then (2.6) reduces to

$$\begin{cases} x_{n+1} = Ty_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \end{cases} \tag{2.8}$$

which was the Picard-Mann iterative process (in short, (PM)) studied by Khan [18] when $\alpha_n \in (0,1)$. We note that (PM) can be obtained from (2.7) if $h_n = d_n = e_n = 0$ for all $n \geq 0$. Further, the iterative process (2.8) reduces to the Picard-Krasnoselskii hybrid iterations studied by Okeke and Abbas [20] when $\alpha_n = \lambda \in (0,1)$. As Khan [18] pointed out, the iteration (2.8) is independent of all Picard and Mann iterative processes if $\{\alpha_n\} \subset (0,1)$. But one can easily see that the iterative process (2.8) will reduce to Picard and a special case of Ishikawa iterative process when $\alpha_n = 0$ and $\alpha_n = 1$, respectively.
Special Case IV When $T_1 = I$, the identity operator, for any given $x_0 \in X$, the iteration (PMD) defined by (2.6) can be written as

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_2 x_n, \tag{2.9}$$

which is the Mann iterative process (in short, (MI)) for $\alpha_n \in [0,1]$.

Based on Lemma 2.1 and the existence of fixed point for a contraction operator, in the sequel, we will prove convergence and stability of the new Picard-Mann iterative processes with mixed errors generated by Algorithm 2.1.

Theorem 2.1 *Let X be a normed space and $C \subset X$ be a nonempty closed convex bounded subset. Let $T_1 : C \to C$ be nonexpansive and $T_2 : C \to C$ be a contraction operator with constant $\theta \in [0,1)$. Suppose that $F(T_1 \cap T_2) := \{x \in C : T_i x = x, i = 1,2\} \neq \emptyset$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then*

(i) *the iterative sequence $\{x_n\}$ generated by* (*PMMD*) *in Algorithm* 2.1 *converges to $x^* \in F(T_1 \cap T_2)$ with convergence rate*

$$\vartheta = 1 - \hat{\alpha}(1-\theta) < 1, \tag{2.10}$$

where $\hat{\alpha} = \limsup_{n\to\infty} \alpha_n \in (0,1]$;

(ii) *if, in addition, for any sequence* $\{z_n\} \subset X$, *there exists* $\alpha > 0$ *such that* $\alpha_n \geq \alpha$ *for all* $n \geq 0$, *then*

$$\lim_{n\to\infty} w_n = x^* \quad \textit{if and only if} \quad \lim_{n\to\infty} \varepsilon_n = 0, \tag{2.11}$$

where ε_n *is defined by* (2.5).

Proof It follows from (2.4) that

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &\quad\leq \|y_n - x^*\| + \|h_n\| \\ &\quad\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|T_2x_n - x^*\| \\ &\qquad + \alpha_n(\|d'_n\| + \|d''_n\|) + \|e_n\| + \|h_n\| \\ &\quad\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta\|x_n - x^*\| \\ &\qquad + \alpha_n\|d'_n\| + (\|d''_n\| + \|e_n\| + \|h_n\|) \\ &\quad= \vartheta_n\|x_n - x^*\| + (1-\theta)\alpha_n \cdot \frac{1}{1-\theta}\|d'_n\| \\ &\qquad + (\|d''_n\| + \|e_n\| + \|h_n\|), \end{aligned} \tag{2.12}$$

where $\vartheta_n = 1 - (1-\theta)\alpha_n$. Since $\sum_{n=0}^{\infty} \alpha_n = \infty$, by Lemma 2.2 and (2.12), now we know that $\|x_n - x^*\| \to 0$ ($n \to \infty$). Thus, the sequence $\{x_n\}$ converges to x^* for ϑ_n.

Further, by (2.12), we have

$$\limsup_{n\to\infty} \vartheta_n = 1 - \hat{\alpha}(1-\theta), \tag{2.13}$$

where $\hat{\alpha} = \limsup_{n\to\infty} \alpha_n$.

Next, we prove the conclusion (ii). Since $0 < \alpha \leq \alpha_n$, it follows from the proof of inequality (2.12) and (2.5) that

$$\begin{aligned} &\|T_1\xi_n + h_n - x^*\| \\ &\quad\leq [1 - (1-\theta)\alpha_n]\|w_n - x^*\| + \alpha_n\|d'_n\| + (\|d''_n\| + \|e_n\| + \|h_n\|), \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &\|w_{n+1} - x^*\| \\ &\quad\leq \|T_1\xi_n + h_n - x^*\| + \varepsilon_n \\ &\quad\leq [1 - (1-\theta)\alpha_n]\|w_n - x^*\| + \alpha_n\|d'_n\| + \varepsilon_n \\ &\qquad + (\|d''_n\| + \|e_n\| + \|h_n\|) \\ &\quad\leq [1 - (1-\theta)\alpha_n]\|w_n - x^*\| + (1-\theta)\alpha_n \cdot \frac{1}{1-\theta}\left(\|d'_n\| + \frac{\varepsilon_n}{\alpha}\right) \\ &\qquad + (\|d''_n\| + \|e_n\| + \|h_n\|). \end{aligned} \tag{2.15}$$

Let $\lim_{n\to\infty}\varepsilon_n = 0$. Then, by $\sum_{n=0}^{\infty}\alpha_n = \infty$, Lemma 2.2 and (2.15), we know that $\lim_{n\to\infty} w_n = x^*$.

Conversely, if $\lim_{n\to\infty} w_n = x^*$, then it follows from (2.14) and $\alpha_n \le 1$ that, for all $n \ge 0$,

$$\begin{aligned}
\varepsilon_n &= \left\| w_{n+1} - (T_1\xi_n + h_n)\right\| \\
&\le \left\| w_{n+1} - x^*\right\| + \left\| T_1\xi_n + h_n - x^*\right\| \\
&\le \left\| w_{n+1} - x^*\right\| + \left[1 - (1-\theta)\alpha_n\right]\left\| w_n - x^*\right\| \\
&\quad + \alpha_n\left\| d'_n\right\| + \left(\left\| d''_n\right\| + \|e_n\| + \|h_n\|\right) \\
&\le \left\| w_{n+1} - x^*\right\| + \left\| w_n - x^*\right\| + \left(\left\| d'_n\right\| + \left\| d''_n\right\| + \|e_n\| + \|h_n\|\right),
\end{aligned} \tag{2.16}$$

this implies that $\varepsilon_n \to 0$ as $n \to \infty$. This completes the proof. □

Remark 2.3 (i) Since the errors in Algorithm 2.1 exist objectively when the inexact calculation of operator points is considered, the iterative process (2.4) (i.e., (PMMD)) is more truthful than the Picard iteration, Mann iteration, Picard-Mann iteration due to Khan [18] and so on. One can easily observe in the next numerical simulations visually.

(ii) We note that the stability analysis in Theorem 2.1 is little discussed in the literature. Akewe and Okeke [27] gave the stability theorems for the Picard-Mann hybrid iterative scheme for a general class of contractive-like operators. However, comparing with the stability analysis in [27], we use a different method to analyze the stability and also extend the application of stability for iterations.

(iii) According to inequality (2.12), one can obtain

$$\begin{aligned}
&\left\| x_{n+1} - x^*\right\| \\
&\quad\le \vartheta_n\left\| x_n - x^*\right\| + \left(\|d_n\| + \|e_n\| + \|h_n\|\right) \\
&\quad\le \vartheta_n\vartheta_{n-1}\left\| x_{n-1} - x^*\right\| + \vartheta_n\left(\|d_{n-1}\| + \|e_{n-1}\| + \|h_{n-1}\|\right) \\
&\quad\quad + \left(\|d_n\| + \|e_n\| + \|h_n\|\right) \\
&\quad\le \cdots \\
&\quad\le \prod_{i=1}^{n}\vartheta_i\left\| x_1 - x^*\right\| + \sum_{k=1}^{n-1}\prod_{i=k+1}^{n}\vartheta_i\left(\|d_k\| + \|e_k\| + \|h_k\|\right) \\
&\quad\quad + \left(\|d_n\| + \|e_n\| + \|h_n\|\right),
\end{aligned} \tag{2.17}$$

where $\prod_{i=1}^{n}\vartheta_i = \vartheta_1 \cdot \vartheta_2 \cdot \cdots \cdot \vartheta_n$ and ϑ_i is the same as in (2.12) for all $i = 1, 2, \ldots, n$. As a matter of fact, $\sum_{k=1}^{n-1}\prod_{i=k+1}^{n}\vartheta_i(\|d_k\| + \|e_k\| + \|h_k\|) + (\|d_n\| + \|e_n\| + \|h_n\|) = o(\|d_n\| + \|e_n\| + \|h_n\|)$. Hence, these errors in (2.4) can help to adjust the iteration results to improve the algorithms by using this infinitesimal of a higher order sequence.

From Theorem 2.1 and Remark 2.1, we have the following result.

Theorem 2.2 *Let $C \subset X$ be a nonempty closed convex bounded subset of a normed space X, and let $T : C \to C$ be a contraction operator with constant $\theta \in [0, 1)$. If $\{\alpha_n\} \subset [0, 1]$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$ and $\{h_n\}$, $\{d_n\}$, $\{e_n\}$ are three sequences in X satisfying the conditions $\mathcal{P}$, then*

(i) *the iterative sequence* $\{x_n\}$ *generated by* (2.7) (*that is*, (*PMM*)) *converges to* $p \in F(T) := \{x \in C : Tx = x\}$ *with convergence rate* $\vartheta = 1 - \hat{\alpha}(1-\theta) < 1$, *where* $\hat{\alpha} = \limsup_{n\to\infty} \alpha_n \in (0,1]$;

(ii) *if, in addition, for any sequence* $\{z_n\} \subset X$, *there exists* $\alpha > 0$ *such that* $\alpha_n \geq \alpha$ *for all* $n \geq 0$, *then*

$$\lim_{n\to\infty} z_n = p \quad \Longleftrightarrow \quad \lim_{n\to\infty} \epsilon_n = 0, \tag{2.18}$$

where ϵ_n *is defined by*

$$\begin{cases} \epsilon_n = \|z_{n+1} - (Ts_n + h_n)\|, \\ s_n = (1-\alpha_n)z_n + \alpha_n Tz_n + \alpha_n d_n + e_n. \end{cases} \tag{2.19}$$

3 Numerical simulations and an application

In order to verify our main results presented in the above section, in this section, we give some numerical simulations and consider approximation of the elliptic boundary value problem (1.1) by using the new Picard-Mann iterative methods with mixed errors for contractive operators.

3.1 Numerical examples

We first give the following examples and their numerical simulations to show verification of Theorem 2.1 and Remark 2.3(iii) and to display effectiveness of the new Picard-Mann iterative methods with mixed errors.

Example 3.1 Let $X = \mathbb{R}$, $1 < k \leq \frac{86}{49}$, $C = [-1, \frac{5}{2} + \frac{1}{2}\sqrt{\frac{135}{k-1}}]$, $T_1 x = \frac{1}{\pi}\sin(\pi x) + 8$ and $T_2 x = \sqrt{x^2 - 5x + 40}$ for all $x \in C$, and $h_n = -\frac{5}{407^n}$, $\alpha_n = \frac{1}{n}$, $d_n = \frac{1}{n^2} + \frac{1}{n^3}$ and $e_n = -\frac{14}{n^7}$ for $n \geq 1$. It is easy to see that T_1 is nonexpansive and T_2 is a contraction operator with constant $\frac{1}{\sqrt{k}}$. In fact, for all $x, y \in C$, we have

$$\begin{aligned} \|T_1 x - T_1 y\| &= \frac{1}{\pi}\big\|\sin(\pi x) - \sin(\pi y)\big\| \\ &\leq \frac{1}{\pi}\|\pi x - \pi y\| = \|x - y\| \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \|T_2 x - T_2 y\| &= \left\| \frac{(x-2.5)^2 - (y-2.5)^2}{\sqrt{(x-2.5)^2 + 33.75} + \sqrt{(y-2.5)^2 + 33.75}} \right\| \\ &= \left\| \frac{(x-y)[(x-2.5)+(y-2.5)]}{\|x-2.5\| + \|y-2.5\|} \right\| \\ &\quad \cdot \frac{\|x-2.5\| + \|y-2.5\|}{\sqrt{(x-2.5)^2 + 33.75} + \sqrt{(y-2.5)^2 + 33.75}} \\ &\leq \frac{1}{\sqrt{k}}\|x - y\|. \end{aligned} \tag{3.2}$$

Further, one can see that T_1 is nonexpansive but not a contraction, and $F(T_1 \cap T_2) = \{8\} \neq \emptyset$. Hence, the conditions in Theorem 2.1 and Algorithm 2.1 hold and the sequence

$\{x_n\}$ generated by (PMMD) can be rewritten as follows:

$$(\text{PMMD}) \quad \begin{cases} x_{n+1} = \frac{1}{\pi}\sin(\pi y) + 8 - \frac{5}{407^n}, \\ y_n = (1-\frac{1}{n})x_n + \frac{1}{n}\sqrt{x_n^2 - 5x_n + 40} + \frac{1}{n}(\frac{1}{n^2} + \frac{1}{n^3}) - \frac{14}{n^7}. \end{cases}$$

Moreover, the corresponding two special cases are listed as well.

$$(\text{PMD}) \quad \begin{cases} x_{n+1} = \frac{1}{\pi}\sin(\pi y) + 8, \\ y_n = (1-\frac{1}{n})x_n + \frac{1}{n}\sqrt{x_n^2 - 5x_n + 40}, \end{cases}$$

$$(\text{MI}) \quad x_{n+1} = \left(1 - \frac{1}{n}\right)x_n + \frac{1}{n}\sqrt{x_n^2 - 5x_n + 40}.$$

By Theorem 2.1, now we know that $\{x_n\}$ generated by (PMMD) converges to $x^* = 8$. Further, in order to show the availability of the New Picard-Mann iterative methods with mixed errors, by using software Matlab 7.0, the numerical simulation results for the sequences $\{x_n\}$ generated by (PMMD), (PMD) and (MI) are given with 70, more than 200 and more than 200 iterations in Figure 1 and Table 1, respectively.

Remark 3.1 If these mixed errors can be used properly, the property of (2.4) will be better than the other algorithms. From Figure 1 and Table 1, it is easy to see that the iterative process (PMMD) is effective and the sequence $\{x_n\}$ generated by (PMMD) converges much faster.

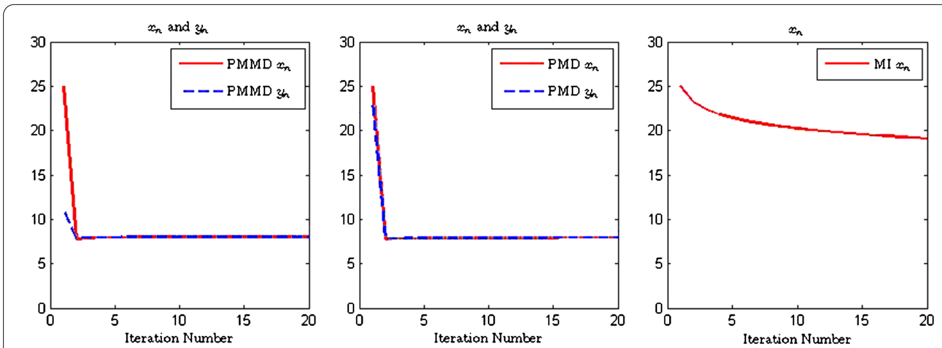

Figure 1 Iterative solutions of (PMMD), (PMD) and (MI).

Table 1 A comparison of the iterative processes (PMMD), (PMD) and (MI)

Iteration number	(PMMD)	(PMD)	(MI)	Iteration number	(PMMD)	(PMD)	(MI)
0	25.0000	25.0000	25.0000	55	7.9999	7.9585	17.5743
5	7.9781	7.8794	21.0937	60	7.9999	7.9602	17.4521
10	7.9944	7.9102	20.0644	65	7.9999	7.9617	17.3403
15	7.9975	7.9249	19.4604	70	8.0000	7.9630	17.2373
20	7.9987	7.9340	19.0346	75	8.0000	7.9642	17.1418
25	7.9992	7.9403	18.7069	80	8.0000	7.9652	17.0529
30	7.9995	7.9451	18.4413	85	8.0000	7.9662	16.9697
35	7.9996	7.9489	18.2183	90	8.0000	7.9671	16.8915
40	7.9997	7.9519	18.0264	95	8.0000	7.9679	16.8179
45	7.9998	7.9545	17.8583	100	8.0000	7.9687	16.7483
50	7.9999	7.9567	17.7088	105	8.0000	7.9694	16.6823

Next, we verify Theorem 2.2 by the following numerical example.

Example 3.2 Let $X = \mathbb{R}$, constant $1 < l \leq \frac{49}{25}$, $C = [-1, 4 + 2\sqrt{\frac{6}{l-1}}]$, $Tx = \sqrt{x^2 - 8x + 40}$ for all $x \in C$, and $h_n = \frac{1}{10^n}$, $\alpha_n = \frac{1}{2^n}$, $d_n = \frac{1}{n} + \frac{1}{n^2}$ and $e_n = -\frac{16}{n^5}$ for $n \geq 1$. Then, for all $x, y \in C$, we have

$$\begin{aligned} \|Tx - Ty\| &= \left\| \frac{(x-4)^2 - (y-4)^2}{\sqrt{(x-4)^2 + 24} + \sqrt{(y-4)^2 + 24}} \right\| \\ &= \left\| \frac{(x-y)[(x-4) + (y-4)]}{\|x-4\| + \|y-4\|} \right\| \\ &\quad \cdot \frac{\|x-4\| + \|y-4\|}{\sqrt{(x-4)^2 + 24} + \sqrt{(y-4)^2 + 24}} \\ &\leq \frac{1}{\sqrt{l}} \|x - y\|, \end{aligned} \tag{3.3}$$

and so T is a contraction operator. Thus, we obtain the following iterative processes as two special cases of (PMMD):

$$\text{(PMM)} \quad \begin{cases} x_{n+1} = \sqrt{y_n^2 - 8y_n + 40} + \frac{1}{10^n}, \\ y_n = (1 - \frac{1}{2^n})x_n + \frac{1}{2^n}\sqrt{x_n^2 - 8x_n + 40} \\ \qquad + \frac{1}{2^n}(\frac{1}{n} + \frac{1}{n^2}) - \frac{16}{n^5}, \end{cases}$$

$$\text{(PM)} \quad \begin{cases} x_{n+1} = \sqrt{y_n^2 - 8y_n + 40}, \\ y_n = (1 - \frac{1}{2^n})x_n + \frac{1}{2^n}\sqrt{x_n^2 - 8x_n + 40}. \end{cases}$$

It follows from Theorem 2.2 that x_n generated by (PMM) converges to $p = 5$, which is the unique fixed point of T. Similarly, in order to compare (PMM) to (PM), the numerical simulations are displayed with 9 and 13 iterations in Figure 2 and Table 2, respectively. One can clearly see that the acceleration efficiency is 44.44%.

Remark 3.2 Figure 2 and Table 2 show that the iterative process (PMM) is effective and the sequence $\{x_n\}$ generated by (PMM) converges faster than that produced by (PM).

3.2 An application to the elliptic boundary value problem

In 2013, by using the notion of a nonexpansive potential operator, Ayadi et al. [1] proved a new global minimization theorem in Hilbert spaces to find a weak solution of the following

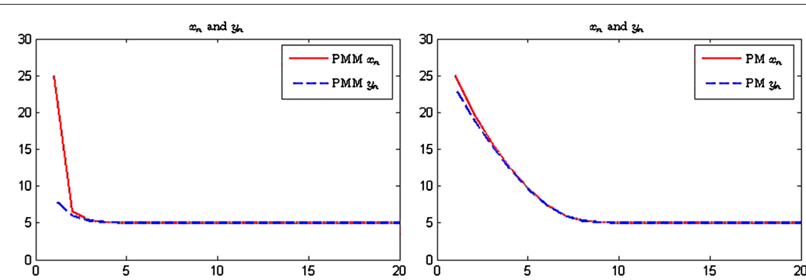

Figure 2 Iterative solutions of (PMM) and (PM).

Table 2 A comparison of the iterative processes (PMM) and (PM)

Iteration number	(PMM)	(PM)	Iteration number	(PMM)	(PM)
0	25.0000	25.0000	7	5.0002	5.2762
1	6.6065	19.8945	8	5.0001	5.0623
2	5.3218	15.8549	9	5.0000	5.0128
3	5.0625	12.4783	10	5.0000	5.0026
4	5.0131	9.6469	11	5.0000	5.0005
5	5.0031	7.4247	12	5.0000	5.0001
6	5.0008	5.9644	13	5.0000	5.0000

elliptic boundary value problem:

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.4}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain in an n-dimensional real space. $f, f' : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, here f' is the derivative of f with respect to its second variable.

One can know that a weak solution of (3.4) is a solution of the following variational problem:

$$\begin{cases} \int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega f(x,u) \cdot v \, dx = 0, & \forall v \in H_0^1(\Omega), \\ u(x) \in H_0^1(\Omega). \end{cases} \tag{3.5}$$

Let $\phi : H_0^1(\Omega) \to \mathbb{R}$ be a nonlinear operator such that

$$\phi(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)\, dx, \quad F(x,u) = \int_0^u f(x,\zeta)\, d\zeta. \tag{3.6}$$

From Theorem 2.2, we have the following existence results of solutions for problem (3.4).

Theorem 3.1 *Let $\mathbb{R}^n$ be an n-dimensional real space and $\Omega \subset \mathbb{R}^n$ be a nonempty bounded domain. Define $T : C \to C$ by $T = I - \phi'$, where ϕ is determined by* (3.6), *$C = [v, \omega] = \{u \in H_0^1(\Omega) : v(x) \le u(x) \le \omega(x), \forall x \in \Omega\}$, here $v, \omega \in H_0^1(\Omega)$ are a subsolution and a supersolution of problem* (3.5), *respectively. If $F(T) := \{u \in C : Tu = u\} \neq \emptyset$ and $\sum_{n=0}^\infty \alpha_n = \infty$, then*

(i) *the iterative sequence $\{u_n\}$ generated by* (2.7) *converges to a weak solution $u^* \in F(T)$ of problem* (3.4) *with convergence rate $\vartheta = 1 - \hat{\alpha}(1-\theta) < 1$, where $\hat{\alpha} = \limsup_{n\to\infty} \alpha_n \in (0,1]$ and $\theta = \sup_{u\in C} \|(I' - \phi'')u\|$;*

(ii) *if, in addition, there exists $\alpha > 0$ such that $\alpha_n \ge \alpha$ for all $n \ge 0$, then*

$$\lim_{n\to\infty} z_n = u^* \iff \lim_{n\to\infty} \epsilon_n = 0, \tag{3.7}$$

where ϵ_n is defined by (2.19) *and $\{z_n\}$ is any sequence.*

Proof From the proof of [1, Theorem 6], it follows that $C \subset H_0^1(\Omega)$ is a closed convex and bounded subset, and $\|(I' - \phi'')u\| < 1$ for some $u \in C$. By the proof of Theorem 4 in [1], we know that T is a contraction operator. Since a contraction operator has fixed points, the results hold from Theorem 2.2. This completes the proof. □

Remark 3.3 In the proof of Theorem 3.1, we employ the new Picard-Mann iterative approximation with mixed errors for contraction operators, which differs from the method proposed in Ayadi et al. [1] for showing that problem (3.4) has a weak solution.

4 Concluding remarks

In this paper, we introduced a class of new Picard-Mann iterative methods with mixed errors for two different nonlinear operators as follows:

$$\begin{cases} x_{n+1} = T_1 y_n + h_n, \\ y_n = (1 - \alpha_n)x_n + \alpha_n T_2 x_n + \alpha_n d_n + e_n, \end{cases} \tag{4.1}$$

where $T_1, T_2 : X \to X$ are respectively nonexpansive and contraction operators, $\alpha_n \in [0, 1]$ and $h_n, d_n, e_n \in X$ are errors to take into account a possible inexact computation of the operator points. Iteration (4.1) includes the Picard-Mann iterative process due to Khan [18], Picard iterative process, Mann iterative process and other related iterative processes as special cases.

Then we gave convergence and stability analysis of the new Picard-Mann iterative approximation and proposed two numerical examples to show that the new Picard-Mann iteration converges more effectively than the Picard iterative process, Mann iterative process, Picard-Mann iterative process due to Khan and other related iterative processes. Furthermore, as an application of the new Picard-Mann iterative methods with mixed errors for contractive operators, which are different from the method proposed in [1], we explore iterative approximation of solutions for the following elliptic boundary value problem:

$$\begin{cases} -\Delta u = f(x, u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

However, can our results be obtained when T is only nonexpansive in Theorem 2.2 or T_2 is also nonexpansive in Theorem 2.1? These are still *open questions* that are worth further studying.

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Abbreviations
(PMMD), new Picard-Mann iterations with mixed errors for two different nonlinear operators; (PMD), Picard-Mann iteration for two different operators; (PMM), Picard-Mann iteration with mixed errors; (PM), Picard-Mann iterative process; (MI), Mann iterative process.

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Ethics approval and consent to participate
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Competing interests
The authors declare that they have no competing interests.

Consent for publication
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Authors' contributions
T-FL carried out the proof of the theorems and gave some numerical simulations to show the main results. H-YL conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

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