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Nontrivial solutions of second-order singular Dirichlet systems

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Abstract
We study the existence of nontrivial solutions for second-order singular Dirichlet systems. The proof is based on a well-known fixed point theorem in cones and the Leray-Schauder nonlinear alternative principle. We consider a very general singularity and generalize some recent results.

MSC: 34B15

Keywords: nontrivial solutions; singular Dirichlet systems; Leray-Schauder alternative principle; fixed point theorem in cones

1 Introduction

We devote this paper to the study of the existence of nontrivial solutions for the following second-order Dirichlet system:

$$\begin{cases} \ddot{u} + q(t)f(t,u) + e(t) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $q \in \mathbb{C}((0,1),\mathbb{R})$, $e = (e_1,\ldots,e_N)^{\mathrm{T}} \in \mathbb{C}((0,1),\mathbb{R}^N)$, $N \geq 1$, and the nonlinear term $f(t,u) \in \mathbb{C}((0,1) \times \mathbb{R}^N \setminus \{0\},\mathbb{R}^N)$. We are mainly motivated by the recent excellent works [1–4], in which singular periodic systems were extensively studied. Let $\mathbb{R}^N_+$ denote the set of vectors of $\mathbb{R}^N$ with positive components. For a fixed vector $v \in \mathbb{R}^N_+$, we say that system (1.1) presents a singularity at the origin if

$$\lim_{u\to 0, u\in\mathbb{R}^N_+} \langle v, f(t,u)\rangle = +\infty \quad \text{uniformly in } t.$$

However, the word 'singularity' has a more general meaning in our case because we do not need all components of the nonlinear term $f(t,u)$ to be singular at the origin as those in [5, 6]. A nontrivial solution of (1.1) is a function $u = (u_1,\ldots,u_N)^{\mathrm{T}} \in \mathbb{C}([0,1],\mathbb{R}^N) \cap \mathbb{C}^2((0,1),\mathbb{R}^N)$ that satisfies (1.1) and $\langle v, u(t)\rangle \not\equiv 0$ for all $t \in (0,1)$.

Singular differential equations arise from different applied sciences. For example, the singular problem (1.1) occurs in chemical reactor theory [7, 8], boundary layer theory [9], and the transport of coal slurries down conveyor belts [10]. Because of these wide applications, during the last few decades, different types of singular differential equations have been considered. Among those, the problem of looking for nontrivial solutions becomes one of the central topics, and so it has drawn the attention of many researchers. See, for example, [11–18] for one-dimensional Dirichlet problems, [19, 20] for one-dimensional p-Laplacian problems, [21–25] for problems of partial differential equations, and [2, 6, 26] for periodic problems. For instance, Agarwal and O'Regan [11] showed that the scalar singular system

$$\begin{cases} \ddot{u} + q(t)f(t,u) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = 0, \end{cases}$$

has at least two nontrivial solutions in some reasonable cases by a well-known fixed point theorem in cones and the Leray-Schauder alternative principle. The result of [11] was extended in [5] to systems.

In this work, we establish existence results for system (1.1). Our aim is to generalize and improve the results in [5] in the following direction: we do not need each component of the nonlinear term $f(t,u)$ to be singular at the origin, so that we can work out some systems that cannot be dealt with in [5]. To illustrate our new results, we consider two systems

$$\begin{cases} \ddot{u} + \sqrt{(u^2 + w^2)^{-\alpha}} + \mu\sqrt{(u^2 + w^2)^{\beta}} + e_1(t) = 0, \\ \ddot{w} + \sqrt{(u^2 + w^2)^{-\alpha}} + \mu\sqrt{(u^2 + w^2)^{\beta}} + e_2(t) = 0, \\ u(0) = u(1) = 0, \qquad w(0) = w(1) = 0, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \ddot{u} + \sqrt{(u^2 + w^2)^{-\alpha}} + e_1(t) = 0, \\ \ddot{w} + \mu\sqrt{(u^2 + w^2)^{\beta}} + e_2(t) = 0, \\ u(0) = u(1) = 0, \qquad w(0) = w(1) = 0, \end{cases} \tag{1.3}$$

in which $\alpha, \beta > 0$ and $\mu \in \mathbb{R}$ is a parameter. Note that (1.3) cannot be dealt with the results used in the literature.

Finally, we give some notation used in this paper. Given $u, w \in \mathbb{R}^N$, their inner product is denoted by

$$\langle u, w\rangle = \sum_{i=1}^{N} u_i w_i.$$

Let $|u|_\nu$ denote the usual ν-norm, that is,

$$|u|_\nu = \sum_{i=1}^{N} \nu_i |u_i|,$$

where $\nu \in \mathbb{R}_+^N$ is a fixed vector. We will denote by $\|\cdot\|$ the supremum norm of $\mathbb{C}([0,1],\mathbb{R})$ and take $X = \mathbb{C}([0,1],\mathbb{R}) \times \cdots \times \mathbb{C}([0,1],\mathbb{R})$ (N times). For any $u = (u_1,\ldots,u_N) \in X$, the ν-norm becomes

$$|u|_\nu = \sum_{i=1}^{N} \nu_i \|u_i\| = \sum_{i=1}^{N} \nu_i \cdot \max_t \bigl|u_i(t)\bigr|.$$

Obviously, X is a Banach space.

2 Preliminaries

Let us first recall the following inequality, which can be found in [11].

Lemma 2.1 *Let*

$$\mathcal{A} = \{u \in \mathbb{C}([0,1],\mathbb{R}) : u(t) \geq 0, t \in [0,1], \textit{and } u(t) \textit{ is concave on } [0,1]\}.$$

Then for all $u \in \mathcal{A}$,

$$u(t) \geq t(1-t)\|u\|, \quad 0 \leq t \leq 1.$$

To prove our main results, we shall apply the following two well-known results.

Lemma 2.2 ([27]) *Assume that* Ω *is an open subset of a convex set* K *in a normed linear space* X *and* $p \in \Omega$. *Let* $T : \overline{\Omega} \to K$ *be a compact continuous map. Then one of the following two conclusions holds*:

(I) T *has at least one fixed point in* $\overline{\Omega}$.

(II) *There exists* $u \in \partial\Omega$ *and* $0 < \lambda < 1$ *such that* $u = \lambda Tu + (1-\lambda)p$.

Let K be a cone in X, and let D be a subset of X. We set $D_K = D \cap K$ and $\partial_K D = (\partial D) \cap K$.

Lemma 2.3 ([28]) *Let* X *be a Banach space, and let* K *be a cone in* X. *Assume that* Ω^1, Ω^2 *are open bounded subsets of* X *with* $\Omega^1_K \neq \emptyset$, $\overline{\Omega}^1_K \subset \Omega^2_K$. *Let*

$$S : \overline{\Omega}^2_K \to K$$

be a continuous and completely continuous operator such that

(i) $u \neq \lambda Su$ *for* $\lambda \in [0,1)$ *and* $u \in \partial_K \Omega^1$, *and*

(ii) *there exists* $w \in K \setminus \{0\}$ *such that* $u \neq Su + \lambda w$ *for all* $u \in \partial_K \Omega^2$ *and all* $\lambda > 0$.

Then S *has a fixed point in* $\overline{\Omega}^2_K \setminus \Omega^1_K$.

The following three restricted conditions need to be required throughout this paper. For a given vector $v \in \mathbb{R}^N_+$,

(D_1) $\langle v, f(t,u)\rangle : [0,1] \times \mathbb{R}^N \setminus \{0\} \to \mathbb{R}_+$ is continuous;

(D_2) $q(t) \in \mathbb{C}(0,1)$, $q(t) > 0$ on (0,1), and $\int_0^1 t(1-t)q(t)\,\mathrm{d}t < \infty$;

(D_3) $\langle v, e(t)\rangle : [0,1] \to \mathbb{R}$ is continuous, and $\int_0^1 t(1-t)|\langle v, e(t)\rangle|\,\mathrm{d}t < \infty$.

By condition (D_3) we get that the linear system

$$\begin{cases} \ddot{u} + e(t) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = 0, \end{cases}$$

has a unique solution $\gamma(t)$, which can be given as

$$\gamma(t) = \int_0^1 G(t,s)e(s)\,\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ (1-s)t, & 0 \le t < s \le 1, \end{cases}$$

is the Green's function. To simplify the notation, let

$$\Gamma(t) = \langle \nu, \gamma(t) \rangle, \qquad \Lambda(t) = \left|\gamma(t)\right|_\nu = \sum_{i=1}^N \nu_i \left|\gamma_i(t)\right|,$$

and

$$\Gamma_* = \min_t \Gamma(t), \qquad \Lambda^* = \max_t \Lambda(t).$$

It is obvious that $\Gamma_* \le 0$.

3 Main results

In this section, we always assume that (D_1)-(D_3) are satisfied and $\Gamma_* = 0$.

Theorem 3.1 *Given a vector $\nu \in \mathbb{R}_+^N$, suppose that there exists a constant $r > 0$ such that*

(H_1) *there exists a continuous nonnegative function $\phi_{r+\Lambda^*}(t)$ on $[0,1]$ such that*

$$\langle \nu, f(t,u) \rangle \ge \phi_{r+\Lambda^*}(t)$$

for all $t \in (0,1)$ and $u \in \mathbb{R}_+^N$ with $0 < |u|_\nu \le r + \Lambda^*$;

(H_2) *there exist two continuous nonnegative functions $g(\cdot)$ and $h(\cdot)$ on $(0,\infty)$ such that*

$$0 \le \langle \nu, f(t,u) \rangle \le g\left(|u|_\nu\right) + h\left(|u|_\nu\right)$$

for all $t \in (0,1)$ and $u \in \mathbb{R}_+^N$ with $0 < |u|_\nu \le r + \Lambda^*$, where $g(\cdot) > 0$ is nonincreasing and $h(\cdot)/g(\cdot)$ is nondecreasing;

(H_3) *the following inequality is satisfied:*

$$\left\{1 + \frac{h(r+\Lambda^*)}{g(r+\Lambda^*)}\right\} b < \int_0^r \frac{1}{g(x)}\,\mathrm{d}x,$$

where

$$b = \max\left\{2\int_0^{1/2} t(1-t)q(t)\,\mathrm{d}t, 2\int_{1/2}^1 t(1-t)q(t)\,\mathrm{d}t\right\}.$$

Then (1.1) *has at least one nontrivial solution u with $0 < |u - \gamma|_\nu < r$.*

Proof First, we show that the system

$$\begin{cases} \ddot{u} + q(t)f(t,u(t) + \gamma(t)) = 0, & 0 < t < 1, \\ u(0) = 0, \qquad u(1) = 0, \end{cases} \tag{3.1}$$

has a nontrivial solution u satisfying $|u(t) + \gamma(t)|_\nu > 0$ for $t \in (0,1)$ and $0 < |u|_\nu < r$. If this is true, by calculating we get

$$\ddot{u} + \ddot{\gamma} + q(t)f\bigl(t, u(t) + \gamma(t)\bigr) + e(t) = 0,$$

that is, $y(t) = u(t) + \gamma(t)$ is a nontrivial solution of (1.1) with $0 < |y - \gamma|_\nu < r$.

Since (H_3) holds, we can choose a positive constant ϵ with $\epsilon < r$ such that

$$\left\{1 + \frac{h(r + \Lambda^*)}{g(r + \Lambda^*)}\right\} b < \int_\epsilon^r \frac{1}{g(x)}\,\mathrm{d}x. \tag{3.2}$$

Choose a positive integer $n_0 \in \{1, 2, \ldots\}$ such that $\frac{1}{n_0} < \frac{\epsilon}{2}$. Next, we set $N_0 = \{n_0, n_0 + 1, \ldots\}$ and fix $n \in N_0$. To this end, we consider the family of systems

$$\begin{cases} \ddot{u} + \lambda q(t) f^n(t, u(t) + \gamma(t)) = 0, & 0 < t < 1, \\ u(0) = \frac{\mathbf{1}}{\mathbf{n}}, & u(1) = \frac{\mathbf{1}}{\mathbf{n}}, \end{cases} \tag{3.3}$$

where $\lambda \in [0,1]$, $\frac{\mathbf{1}}{\mathbf{n}} = (\frac{1}{n}, \ldots, \frac{1}{n}) \in \mathbb{R}_+^N$, and

$$f^n(t,u) = \begin{cases} f(t,u) & \text{if } |u|_\nu \geq \frac{1}{n}, \\ f(t, u_1, \ldots, u_{i-1}, \frac{1}{n}, u_{i+1}, \ldots, u_N) & \text{if } |u|_\nu < \frac{1}{n}. \end{cases}$$

It is immediate that a nontrivial solution of (3.3) is exactly a fixed point of the operator equation

$$u = \lambda T u + (1 - \lambda) p, \tag{3.4}$$

where $p = \frac{\mathbf{1}}{\mathbf{n}}$, and T stands for the operator

$$(Tu)(t) = \int_0^1 G(t,s) q(s) f^n\bigl(s, u(s) + \gamma(s)\bigr)\,\mathrm{d}s + p.$$

Next, we show that any fixed point u of (3.4) for all $\lambda \in [0,1]$ must satisfy

$$|u|_\nu \neq r. \tag{3.5}$$

Assume on the contrary that there exists $\lambda \in [0,1]$ such that u, a fixed point of (3.4), satisfies $|u|_\nu = r$. We conclude from (3.3) that, for all $t \in [0,1]$,

$$\bigl\langle \nu, \ddot{u}(t) \bigr\rangle = \bigl\langle \nu, -\lambda q(t) f^n\bigl(t, u(t) + \gamma(t)\bigr)\bigr\rangle \leq 0.$$

Then we have $\langle \nu, u(t)\rangle \geq \frac{1}{n}$ for $0 \leq t \leq 1$. Furthermore, from Lemma 2.1 we have

$$\bigl\langle \nu, u(t) \bigr\rangle \geq t(1-t)|u|_\nu, \quad 0 \leq t \leq 1.$$

It is obvious that there exists $t_n \in (0,1)$ such that $\langle \nu, \dot{u}(t)\rangle \geq 0$ on $(0, t_n)$, $\langle \nu, \dot{u}(t)\rangle \leq 0$ on $(t_n, 1)$, and $\langle \nu, u(t_n)\rangle = |u|_\nu = r$. Hence, for all $z \in (0,1)$, we have

$$\begin{aligned}\langle \nu, -\ddot{u}(z)\rangle &= \langle \nu, \lambda q(z) f^n(z, u(z) + \gamma(z))\rangle \\ &= \lambda q(z)\langle \nu, f(z, u(z) + \gamma(z))\rangle \\ &\leq q(z)\langle \nu, f(z, u(z) + \gamma(z))\rangle \\ &\leq q(z) g(|u(z) + \gamma(z)|_\nu)\left\{1 + \frac{h(|u(z) + \gamma(z)|_\nu)}{g(|u(z) + \gamma(z)|_\nu)}\right\}.\end{aligned} \tag{3.6}$$

Since $\Gamma_* = 0$, we have

$$\langle \nu, u(t)\rangle \leq \langle \nu, u(t) + \gamma(t)\rangle \leq |u(t) + \gamma(t)|_\nu \leq |u(t)|_\nu + |\gamma(t)|_\nu \leq r + \Lambda^*.$$

Calculating the integral for (3.6) from t ($t \leq t_n$) to t_n, we have

$$\begin{aligned}\langle \nu, \dot{u}(t)\rangle &\leq g(|u(z) + \gamma(z)|_\nu)\left\{1 + \frac{h(|u(z) + \gamma(z)|_\nu)}{g(|u(z) + \gamma(z)|_\nu)}\right\}\int_t^{t_n} q(z)\,\mathrm{d}z \\ &\leq g(\langle \nu, u(t)\rangle)\left\{1 + \frac{h(r + \Lambda^*)}{g(r + \Lambda^*)}\right\}\int_t^{t_n} q(z)\,\mathrm{d}z.\end{aligned}$$

Thus, for $t \leq t_n$, we have

$$\frac{\langle \nu, \dot{u}(t)\rangle}{g(\langle \nu, u(t)\rangle)} \leq \left\{1 + \frac{h(r + \Lambda^*)}{g(r + \Lambda^*)}\right\}\int_t^{t_n} q(z)\,\mathrm{d}z. \tag{3.7}$$

Integrating (3.7) from 0 to t_n, we have

$$\int_{\frac{1}{n}}^{r} \frac{\mathrm{d}x}{g(x)} \leq \left\{1 + \frac{h(r + \Lambda^*)}{g(r + \Lambda^*)}\right\}\int_0^{t_n} t q(t)\,\mathrm{d}t.$$

Accordingly,

$$\int_{\epsilon}^{r} \frac{\mathrm{d}x}{g(x)} \leq \left\{1 + \frac{h(r + \Lambda^*)}{g(r + \Lambda^*)}\right\}\frac{1}{1 - t_n}\int_0^{t_n} t(1-t) q(t)\,\mathrm{d}t. \tag{3.8}$$

Applying this calculation method again and integrating (3.6) from t_n to t ($t \geq t_n$) and then from t_n to 1, we get

$$\int_{\epsilon}^{r} \frac{\mathrm{d}x}{g(x)} \leq \left\{1 + \frac{h(r + \Lambda^*)}{g(r + \Lambda^*)}\right\}\frac{1}{t_n}\int_{t_n}^{1} t(1-t) q(t)\,\mathrm{d}t. \tag{3.9}$$

According to (3.8) and (3.9), we have

$$\int_{\epsilon}^{r} \frac{\mathrm{d}x}{g(x)} \leq b\left\{1 + \frac{h(r + \Lambda^*)}{g(r + \Lambda^*)}\right\},$$

which is a contradiction to (3.2), and so the assertion is proved.

Under the assertion above, using Lemma 2.2, we get that

$$u = Tu \tag{3.10}$$

has a fixed point denoted by u_n. In other words, the system

$$\begin{cases} \ddot{u} + q(t)f^n(t, u(t) + \gamma(t)) = 0, & 0 < t < 1, \\ u(0) = \frac{\mathbf{1}}{\mathbf{n}}, \qquad u(1) = \frac{\mathbf{1}}{\mathbf{n}}, \end{cases} \tag{3.11}$$

has a solution u_n satisfying $|u_n|_\nu < r$. For all $t \in [0,1]$, since $\langle \nu, u_n(t)\rangle \geq \frac{1}{n} > 0$, u_n is certainly a nontrivial solution of (3.11).

Next, we claim that $\langle \nu, u_n(t) + \gamma(t)\rangle$ has a uniform positive lower bound. To get the claim above, we need to prove that there exists a constant $\delta > 0$, independent of $n \in N_0$, such that, for any $t \in [0,1]$,

$$\bigl\langle \nu, u_n(t) + \gamma(t)\bigr\rangle \geq \delta t(1-t).$$

Since $\Gamma_* = 0$, we only need to show that

$$\bigl\langle \nu, u_n(t)\bigr\rangle \geq \delta t(1-t) \tag{3.12}$$

for all $n \in N_0$ and $t \in [0,1]$. Since (H_1) holds, there exists a continuous nonnegative function $\phi_{r+\Lambda^*}$ such that

$$\bigl\langle \nu, f(t,u)\bigr\rangle \geq \phi_{r+\Lambda^*}(t)$$

for all $t \in (0,1)$ and u with $0 < |u|_\nu \leq r + \Lambda^*$. Let $u^{r+\Lambda^*}$ be the unique solution of the problem

$$\begin{cases} \ddot{u} + q(t)\Phi(t) = 0, & 0 < t < 1, \\ u(0) = 0, \qquad u(1) = 0, \end{cases}$$

with $\Phi = (\phi_{r+\Lambda^*}, \ldots, \phi_{r+\Lambda^*})^T$. Then we have

$$\bigl\langle \nu, u^{r+\Lambda^*}(t)\bigr\rangle = \int_0^1 G(t,s)q(s)\bigl\langle \nu, \phi_{r+\Lambda^*}(s)\bigr\rangle \,\mathrm{d}s.$$

Moreover, for $t \in [0,1]$,

$$\begin{aligned} \bigl\langle \nu, \dot{u}^{r+\Lambda^*}(t)\bigr\rangle = {} & \int_t^1 (1-s)q(s)\bigl\langle \nu, \phi_{r+\Lambda^*}(s)\bigr\rangle \,\mathrm{d}s \\ & - \int_0^t sq(s)\bigl\langle \nu, \phi_{r+\Lambda^*}(s)\bigr\rangle \,\mathrm{d}s \end{aligned}$$

and

$$\bigl\langle \nu, u^{r+\Lambda^*}(0)\bigr\rangle = 0, \qquad \bigl\langle \nu, u^{r+\Lambda^*}(1)\bigr\rangle = 0.$$

Assume that there exists a constant $k_0 = \langle \nu, \dot{u}^{r+\Lambda^*}(0)\rangle = \int_0^1 (1-s)q(s)\langle \nu, \phi_{r+\Lambda^*}(s)\rangle\,\mathrm{d}s$, and if not, then $\langle \nu, \dot{u}^{r+\Lambda^*}(0)\rangle = \infty$. Regardless of the two cases above, there exists a positive constant δ_1 independent of n such that $\langle \nu, \dot{u}^{r+\Lambda^*}(0)\rangle \geq 2\delta_1$. Hence, there exists a positive constant ϵ_1 such that $\langle \nu, u^{r+\Lambda^*}(t)\rangle \geq \delta_1 t(1-t)$ for all $t \in [0, \epsilon_1]$. Analogously, there exists a positive constant δ_2, independent of n, and $\epsilon_2 > 0$ such that $\langle \nu, u^{r+\Lambda^*}(t)\rangle \geq \delta_2 t(1-t)$ for all $t \in [1-\epsilon_2, 1]$.

Besides, for $t \in [\epsilon_1, 1-\epsilon_2]$, it is easily seen that

$$\frac{\langle \nu, u^{r+\Lambda^*}(t)\rangle}{t(1-t)} \text{ is continuous.}$$

Then there exists a positive constant δ_3, independent of n, such that

$$\bigl\langle \nu, u^{r+\Lambda^*}(t)\bigr\rangle \geq \delta_3 t(1-t).$$

So, if we choose a positive constant $\delta = \min\{\delta_1, \ldots, \delta_N\}$, then (3.12) is true.

To pass from the solution u_n of (3.11) to that of (3.1), it is necessary to prove that

$$\{u_n\}_{n\in N_0} \text{ is bounded and equicontinuous on } [0,1]. \tag{3.13}$$

Recalling the argument to establish (3.7) and applying it again with u replaced by u_n, we obtain the inequalities

$$\frac{\langle \nu, \dot{u}_n(t)\rangle}{g(\langle \nu, u_n(t)\rangle)} \leq \left\{1 + \frac{h(r+\Lambda^*)}{g(r+\Lambda^*)}\right\} \int_t^{t_n} q(z)\,\mathrm{d}z \tag{3.14}$$

and

$$-\frac{\langle \nu, \dot{u}_n(t)\rangle}{g(\langle \nu, u_n(t)\rangle)} \geq \left\{1 + \frac{h(r+\Lambda^*)}{g(r+\Lambda^*)}\right\} \int_{t_n}^{t} q(z)\,\mathrm{d}z.$$

Accordingly,

$$\frac{|\langle \nu, \dot{u}_n(t)\rangle|}{g(u_n(t))} \leq \left\{1 + \frac{h(r+\Lambda^*)}{g(r+\Lambda^*)}\right\} \left| \int_t^{t_n} q(z)\,\mathrm{d}z \right|. \tag{3.15}$$

Under this claim, we have to show that there exist two constants a, b satisfying $0 < a < b < 1$ such that

$$a < \inf\{t_n, n \in N_0\} \leq \sup\{t_n, n \in N_0\} < b. \tag{3.16}$$

Hence, we just need to prove the following two inequalities: $\inf\{t_n, n \in N_0\} > 0$ and $\sup\{t_n, n \in N_0\} < 1$. First, assume that the inequality $\inf\{t_n, n \in N_0\} > 0$ is incorrect. Let A be a subsequence of N_0 with $t_n \to 0$ as $n \to \infty$ in A. Integrating (3.14) from 0 to t_n, we have

$$\int_0^{\langle \nu, u_n(t_n)\rangle} \frac{\mathrm{d}x}{g(x)} \leq \left\{1 + \frac{h(r+\Lambda^*)}{g(r+\Lambda^*)}\right\} \int_0^{t_n} t q(t)\,\mathrm{d}t + \int_0^{\frac{1}{n}} \frac{\mathrm{d}x}{g(x)}$$

for $n \in A$. Since $t_n \to 0$ as $n \to \infty$ in A, from this inequality we get that $u_n(t_n) \to 0$ as $n \to \infty$ in A. Furthermore, $\langle v, \dot{u}(t_n)\rangle = 0$, and u_n has a local maximum at t_n. Then we obtain that $u_n \to 0$ in $\mathbb{C}[0,1]$ as $n \to \infty$ in A, which contradicts our claim. So, $\inf\{t_n, n \in N_0\} > 0$. Analogously, we can also prove that $\sup\{t_n, n \in N_0\} < 1$.

According to (3.15) and (3.16), we obtain that

$$\frac{|\langle v, \dot{u}_n(t)\rangle|}{g(\langle v, u_n(t)\rangle)} \le \left\{1 + \frac{h(r+\Lambda^*)}{g(r+\Lambda^*)}\right\} V(t),$$

where

$$V(t) = \int_{\min\{t,a\}}^{\max\{t,b\}} q(z)\,\mathrm{d}z.$$

Obviously, $V \in L^1[0,1]$. Let us define $I : [0,\infty) \to [0,\infty)$ by

$$I(z) = \int_0^z \frac{1}{g(x)}\,\mathrm{d}x.$$

Note that $g(x) > 0$ is nonincreasing on $(0,\infty)$. Then the map $I : [0,\infty) \to [0,\infty)$ is increasing, and $I(\infty) = \infty$. Analogously, for any $D > 0$, the map I is continuous. Furthermore, we have

$$\begin{aligned}\bigl|I\bigl(u_n(t)\bigr) - I\bigl(u_n(s)\bigr)\bigr| &= \left|\int_s^t \frac{\langle v, \dot{u}_n(z)\rangle}{g(\langle v, u_n(z)\rangle)}\,\mathrm{d}z\right| \\ &\le \left\{1 + \frac{h(r+\Lambda^*)}{g(r+\Lambda^*)}\right\}\left|\int_s^t V(z)\,\mathrm{d}z\right|,\end{aligned}$$

which implies that

$$\bigl\{I(u_n)\bigr\}_{n\in N_0} \text{ is bounded and equicontinuous on } [0,1].$$

Due to the uniform continuity of the inverse map I^{-1} on $[0, I(r+\Lambda^*)]$ and the equality

$$\bigl|u_n(t) - u_n(s)\bigr| = \bigl|I^{-1}\bigl(I\bigl(u_n(t)\bigr)\bigr) - I^{-1}\bigl(I\bigl(u_n(s)\bigr)\bigr)\bigr|,$$

we have that (3.13) is certainly true.

Now the Arzelà-Ascoli theorem guarantees that $\{u_n\}_{n\in N_0}$ has a subsequence that converges uniformly on $[0,1]$ to a function $u \in \mathbb{C}[0,1]$. It is easy to verify that

$$\ddot{u}(t) + q(t) f\bigl(t, u(t) + \gamma(t)\bigr) = 0.$$

Moreover, we have $\langle v, u(0)\rangle = \langle v, u(1)\rangle = 0$, $0 < |u|_v \le r$, and $\langle v, u(t)\rangle \ge \delta t(1-t)$ for all $0 \le t \le 1$. Then u is a nontrivial solution of (3.1) satisfying $0 < |u|_v < r$. □

Theorem 3.2 *Suppose that* (H_2)-(H_3) *hold. Assume further that*

(H_4) *there exist two continuous nonnegative functions* $g_1(\cdot)$, $h_1(\cdot)$ *on* $(0,\infty)$ *such that*

$$\bigl\langle v, f(t,u)\bigr\rangle \ge g_1\bigl(|u|_v\bigr) + h_1\bigl(|u|_v\bigr)$$

for all $t \in (0,1)$ and $u \in \mathbb{R}_+^N$, where $g_1(\cdot) > 0$ is nonincreasing, and $h_1(\cdot)/g_1(\cdot)$ is nondecreasing;

(H_5) *there exists a positive constant $R > r$ such that*

$$\frac{R}{g_1(R+\Lambda^*)(1+\frac{h_1(\sigma R)}{g_1(\sigma R)})} \leq \int_a^{1-a} G(\xi,s)q(s)\,\mathrm{d}s,$$

where $a \in (0,\frac{1}{2})$ is fixed, $\sigma = a(1-a)$, and $0 \leq \xi \leq 1$ is such that

$$\int_a^{1-a} G(\xi,s)q(s)\,\mathrm{d}s = \sup_{0\leq t\leq 1} \int_a^{1-a} G(t,s)q(s)\,\mathrm{d}s.$$

Then (1.1) *has a nontrivial solution u with $r < |u-\gamma|_\nu \leq R$.*

Proof First, we return to the beginning of the proof of Theorem 3.1. Similarly, we only need to prove that (3.1) has a nontrivial solution u, which satisfies $r < |u|_\nu \leq R$ and $\langle \nu, u(t)+\gamma(t)\rangle > 0$ for all $t \in (0,1)$.

Since (H_3) holds, we can choose a positive constant ϵ with $\epsilon < r$ such that inequality (3.2) holds. Obviously, there exists a positive integer $n_1 \in \{1,2,\ldots\}$ such that

$$\frac{1}{n_1} < \min\left\{\frac{\epsilon}{2}, \sigma R\right\}.$$

Let $N_1 = \{n_1, n_1+1,\ldots\}$. Fix $n \in N_1$. Let us reconsider system (3.11) and define the set

$$K = \left\{u \in X : \langle \nu, u(t)\rangle \geq t(1-t)|u|_\nu \text{ for } t \in [0,1]\right\}.$$

We can easily see that K is a cone in X. Set

$$\Omega^1 = \left\{u \in X : |u|_\nu < r\right\}, \qquad \Omega^2 = \left\{u \in X : |u|_\nu < R\right\}.$$

Define the operator $S : \overline{\Omega}_K^2 \setminus \Omega_K^1 \to K$ as

$$(Su)(t) = \int_0^1 G(t,s)q(s)f^n\big(s, u(s)+\gamma(s)\big)\,\mathrm{d}s + p.$$

A standard argument shows that the operator $S : \overline{\Omega}_K^2 \setminus \Omega_K^1 \to X$ is continuous and completely continuous. It is easily seen that the operator $S : \overline{\Omega}_K^2 \setminus \Omega_K^1 \to K$ is well defined by Lemma 2.1. To get the desired result, we need to make the following two assertions:

(i) $u \neq \lambda Su$ for $\lambda \in [0,1]$ and $u \in \partial_K \Omega^1$, and

(ii) there exists a vector $w \in K \setminus \{0\}$ such that $u \neq Su + \lambda w$ for all $\lambda > 0$ and all $u \in \partial_K \Omega^2$.

We start with (i). Assume that there exiss $\lambda \in [0,1]$ and $u \in \partial_K \Omega^1$ such that $u = \lambda Su$. Suppose that $\lambda \neq 0$. Now $u = \lambda Su$ can lead to a contradiction following the same ideas in proving (3.5), and so (i) holds. We omit the details.

Next, we consider assertion (ii). Let $w(t) = (1,1,\ldots,1)^{\mathrm{T}}$. Then $w \in K \setminus \{0\}$. Let us prove that $u \neq Su + \lambda w$ for all $u \in \partial_K \Omega^2$ and $\lambda > 0$. If not, there would exist $u \in \partial_K \Omega^2$ and $\lambda > 0$

such that $u = Su + \lambda w$. Now since $u \in \partial_K \Omega^2$, we have that $|u|_\nu = R$. It is obvious that $\langle \nu, u(t)\rangle$ is concave on $[0,1]$. By Lemma 2.1, for all $t \in [0,1]$, we have

$$\langle \nu, u(t)\rangle \geq t(1-t)R.$$

We suppose that there exists $t \in [a, 1-a]$ such that

$$\sigma R = a(1-a)R \leq \langle \nu, u(t)\rangle \leq R.$$

Hence, for $t \in [a, 1-a]$, we have

$$\sigma R \leq \langle \nu, u(t) + \gamma(t)\rangle \leq R + \Lambda^*.$$

Therefore, for $t \in [a, 1-a]$, we obtain $f^n(u(s) + \gamma(s)) = f(u(s) + \gamma(s))$. Consequently, from (H_4) we have

$$\begin{aligned}
R &\geq \langle \nu, u(\xi)\rangle = \langle \nu, (Su)(\xi)\rangle + \langle \nu, \lambda w\rangle \\
&= \int_0^1 G(\xi, s)q(s)\langle \nu, f^n(s, u(s) + \gamma(s))\rangle \,\mathrm{d}s + \langle \nu, p\rangle + \langle \nu, \lambda w\rangle \\
&\geq \int_0^1 G(\xi, s)q(s)\langle \nu, f(s, u(s) + \gamma(s))\rangle \,\mathrm{d}s \\
&\geq \int_0^1 G(\xi, s)q(s)\bigl[g_1(|u(s) + \gamma(s)|_\nu) + h_1(|u(s) + \gamma(s)|_\nu)\bigr] \,\mathrm{d}s \\
&\geq \int_0^1 G(\xi, s)q(s)g_1(|u(s) + \gamma(s)|_\nu)\left\{1 + \frac{h_1(|u(s) + \gamma(s)|_\nu)}{g_1(|u(s) + \gamma(s)|_\nu)}\right\} \mathrm{d}s \\
&\geq g_1(R + \Lambda^*)\left\{1 + \frac{h_1(\sigma R)}{g_1(\sigma R)}\right\} \int_a^{1-a} G(\xi, s)q(s) \,\mathrm{d}s,
\end{aligned}$$

which is a contradiction to (H_5). So assertion (ii) is proved.

Now it follows from Lemma 2.3 that S has at least one fixed point $u_n \in \overline{\Omega}_K^2 \setminus \Omega_K^1$ with $r \leq |u_n|_\nu \leq R$. By assertion (i) we can further get that $|u_n|_\nu > r$. Therefore, system (3.11) has a solution u_n with $\langle \nu, u_n(t)\rangle \geq \frac{1}{n}$ for all $t \in [0,1]$, which implies that system (3.1) has a nontrivial solution u_n with

$$\langle \nu, u_n(t)\rangle \geq \frac{1}{n}, \quad 0 \leq t \leq 1, \qquad r < |u_n|_\nu \leq R,$$

and

$$\langle \nu, u_n(t)\rangle \geq t(1-t)r, \quad 0 \leq t \leq 1.$$

Now, using a similar argument as in the proof of Theorem 3.1, we can show that

$$\{u_n\}_{n \in N_0} \text{ is bounded and equicontinuous on } [0,1],$$

and the Arzelà-Ascoli theorem guarantees that $\{u_n\}_{n\in N_0}$ has a subsequence that converges uniformly on $[0,1]$ to a function $u \in \mathbb{C}[0,1]$, which is a nontrivial solution of

$$\ddot{u}(t) + q(t)f\big(t, u(t) + \gamma(t)\big) = 0$$

and satisfies $r < |u|_v \le R$. □

The following multiplicity result is a direct consequence of Theorems 3.1 and 3.2.

Theorem 3.3 *Assume that* (H_2)-(H_5) *are satisfied. Then* (1.1) *has at least two nontrivial solutions* u, $\tilde{u}$ *with* $\langle v, u(t)\rangle > 0$, $\langle v, \tilde{u}(t)\rangle > 0$ *for* $t \in (0,1)$ *and* $|u - \gamma|_v < r < |\tilde{u} - \gamma|_v \le R$.

Corollary 3.4 *Suppose* $\alpha > 0$, $\beta \ge 0$, $\Gamma_* = 0$, *and* $e_1, e_2 \in \mathbb{C}([0,1], \mathbb{R})$.

(i) *For each* $\mu > 0$, *system* (1.2) *has at least one nontrivial solution if* $\beta < 1$.
(ii) *For each* $0 < \mu < \mu_1$, *system* (1.2) *has at least one nontrivial solution if* $\beta \ge 1$, *where* μ_1 *is a positive constant.*
(iii) *For each* $0 < \mu < \mu_1$, *system* (1.2) *has at least two nontrivial solutions if* $\beta > 1$.

Proof We will apply Theorem 3.3. Let $v = (1,1)^{\mathrm{T}}$. Let

$$g(y) = 2^{1+\frac{\alpha}{2}} y^{-\alpha}, \qquad h(y) = 2\mu y^{\beta}, \qquad g_1(y) = 2y^{-\alpha}, \qquad h_1(y) = 2^{1-\frac{\beta}{2}} \mu y^{\beta}.$$

Then it is easily seen that (H_2) and (H_4) are satisfied by using the inequalities

$$\frac{(|u| + |w|)^2}{2} \le u^2 + w^2 \le \big(|u| + |w|\big)^2 \quad \text{for all } u, w \in \mathbb{R}.$$

Note that

$$b = \max\left\{2\int_0^{1/2} t(1-t)\,\mathrm{d}t, 2\int_{1/2}^{1} t(1-t)\,\mathrm{d}t\right\} = \frac{1}{6}.$$

Now condition (H_3) holds if there exists a positive constant r such that

$$\mu < \frac{3r^{\alpha+1} - (\alpha+1)\cdot 2^{\frac{\alpha}{2}}}{(\alpha+1)(r + \Lambda^*)^{\alpha+\beta}},$$

which can be deduced to

$$0 < \mu < \mu_1 := \sup_{r>0} \frac{3r^{\alpha+1} - (\alpha+1)\cdot 2^{\frac{\alpha}{2}}}{(\alpha+1)(r + \Lambda^*)^{\alpha+\beta}}.$$

Notice that $\mu_1 = \infty$ since $\beta < 1$ and $\mu_1 < \infty$ since $\beta \ge 1$. We have (i) and (ii). The other existence condition (H_5) becomes

$$\mu \ge \frac{R(R + \Lambda^*)^{\alpha} - 2L}{L \cdot 2^{1-\frac{\beta}{2}} (\sigma R)^{\alpha+\beta}}, \tag{3.17}$$

where

$$L = \max_{0\le t\le 1} \int_{\frac{1}{5}}^{\frac{4}{5}} G(t,s)\,\mathrm{d}s.$$

Since $\beta > 1$, we obtain that the right-hand side of (3.17) tends to zero as $R \to +\infty$. Therefore, for any $0 < \mu < \mu_1$, we can find R large enough such that inequality (3.17) is satisfied. Therefore, system (1.2) has another nontrivial solution. □

Similarly, we can prove the following result for system (1.3).

Corollary 3.5 *Suppose that* $\alpha > 0$, $\beta > 1$, $\Gamma_* = 0$, *and* $e_1, e_2 \in \mathbb{C}([0,1], \mathbb{R})$. *Then there exists a positive constant* μ_2 *such that system* (1.3) *has at least two nontrivial solutions for each* $0 < \mu < \mu_2$.

4 Conclusions

In this paper, we established the multiplicity of nontrivial solutions for a second-order Dirichlet system by a well-known fixed point theorem in cones and the Leray-Schauder alternative principle. Some recent results in the literature are generalized and improved. We do not need each component of the nonlinear term $f(t,u)$ to be singular at the origin, and therefore we can deal with some new systems.

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Competing interests
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Authors' contributions
Both authors contributed to each part of this study equally and read and approved the final version of the manuscript.

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