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Existence of solutions for fractional Sturm-Liouville boundary value problems with $p(t)$-Laplacian operator

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Abstract

This paper is concerned with the solvability for fractional Sturm-Liouville boundary value problems with $p(t)$-Laplacian operator at resonance using Mawhin's continuation theorem. Sufficient conditions for the existence of solutions have been acquired, and they would extend the existing results. Furthermore, an example is provided to illustrate the main result.

MSC: 34A08; 34B15

Keywords: fractional differential equation; boundary value problems; $p(t)$-Laplacian operator; continuation theory

1 Introduction

The last two decades have witnessed a wide application of fractional differential equations in various fields of natural science and engineering technology (see [1–7]). Introduced by Bagley and Torvik [1], the famous fractional differential model is used to describe radial vibration of a rigid plate connected to a massless spring immersing vertical in the ideal fluid:

$$ay''(t) + bD_t^{\frac{3}{2}}y(t) + cy(t) = f(t),$$

where $a, b, c > 0$, and the fractional derivative represents damping. With some theoretical discussions conducted regarding boundary value problem (BVP for short) of differential equations so far, valuable results have been obtained for BVP of fractional differential equations (see [8–18]). For instance, Kosmatov [12] studied the existence of solution for the following BVP of fractional differential equations using coincidence degree theory:

$$\begin{cases} D^\alpha u(t) = f(t, u(t), u'(t)), & a \cdot e \cdot t \in (0,1), \\ D_{0+}^{\alpha-2}u(0) = 0, & \eta u(\xi) = u(1), \end{cases}$$

where D^α is a Caputo fractional derivative, and $1 < \alpha \leq 2$.

It is generally known that the p-Laplacian equations normally derive from nonlinear elastic mechanics and non-Newtonian fluid theory. However, in view of their significance in theory and practice, more and more attention is being paid to the existence of solutions for fractional p-Laplacian BVP. Consequently, important results have been achieved in this regard by some researchers (see [19–25]). Chen and Liu [21] discussed the solvability of the following anti-periodic BVP:

$$\begin{cases} {}_0^C D_t^\beta \phi_p({}_0^C D_t^\alpha u) = f(t,u), & t \in [0,1], \\ u(0) = -u(1), & {}_0^C D_t^\alpha u(0) = -{}_0^C D_t^\alpha u(1), \end{cases}$$

where $0 < \alpha$, $\beta \le 1$, $\phi_p(\cdot)$ is a p-Laplacian operator defined by $\phi_p(s) = |s|^{p-2}s$ $(s \neq 0, p > 1)$, $\phi_p(0) = 0$. With Schaefer's fixed point theorem, the existence of solutions for BVP was obtained.

Mahmudov and Unul [25] studied the BVP

$$\begin{cases} D_{0+}^\beta \varphi_p(D_{0+}^\alpha x(t)) = f(t, x(t), D_{0+}^\gamma x(t)), & t \in [0,1], \\ x(0) + \mu_1 x(1) = \sigma_1 \int_0^1 g(s, x(s))\,\mathrm{d}s, \\ x'(0) + \mu_2 x'(1) = \sigma_2 \int_0^1 h(s, x(s))\,\mathrm{d}s, \\ D_{0+}^\alpha x(0) = 0, \qquad D_{0+}^\alpha x(1) = \nu D_{0+}^\alpha x(\eta), \end{cases}$$

where $1 < \alpha \le 2$, $0 < \beta$, $\gamma \le 1$, $0 < \eta < 1$, $\nu, \mu_i, \sigma_i > 0$ $(i = 1,2)$, D_{0+}^α is a Caputo fractional derivative, $\varphi_p(\cdot)$ is a p-Laplacian operator, f, g, h are continuous functions. By constructing the Green's functions of BVP and by using the fixed point theory, the existence and uniqueness of the solutions were obtained under suitable conditions.

As far as we are concerned, the $p(t)$-Laplacian operator is a non-standard growth operator by nature, and it mainly derives from elasticity theory, nonlinear electrorheological fluids and image restoration. A lot of research regarding BVP of fractional differential equations with $p(t)$-Laplacian operator have been quite limited so far (see [26–30]). Specifically, Shen and Liu [26] studied the existence of solutions for the following BVP with $p(t)$-Laplacian operator at nonresonance and resonance by using Schaefer's fixed point theorem and Mawhin's continuation theorem:

$$\begin{cases} D_{0+}^\beta \varphi_{p(t)}(D_{0+}^\alpha x(t)) + f(t, x(t)) = 0, & t \in (0,1), \\ x(0) = 0, \qquad D_{0+}^{\alpha-1} x(1) = \gamma I_{0+}^{\alpha-1} x(\eta), \qquad D_{0+}^\alpha x(0) = 0, \end{cases}$$

where $\dim \operatorname{Ker} L = 1$, $1 < \alpha \le 2$, $0 < \beta \le 1$, $\gamma > 1$, $0 < \eta < 1$, $\varphi_{p(t)}(\cdot)$ is a $p(t)$-Laplacian operator, $p(t) > 1$, $p(t) \in C^1[0,1]$.

Inspired by the above findings, this paper studies the BVP subjected to Sturm-Liouville type integral boundary conditions for fractional differential equations with $p(t)$-Laplacian operator:

$$\begin{cases} {}^C D_{0+}^\beta \phi_{p(t)}({}^C D_{0+}^\alpha x(t)) = f(t, x(t), x'(t)), \\ x(0) + bx'(0) = \gamma \int_0^\xi x(t)\,\mathrm{d}t, \\ x(1) - mx'(1) = \sigma \int_0^\eta x(t)\,\mathrm{d}t, \\ {}^C D_{0+}^\alpha x(0) = 0, \end{cases} \tag{1.1}$$

where $0 < \beta \leq 1, 1 < \alpha \leq 2$, ${}^{C}D_{0+}^{\beta}$, ${}^{C}D_{0+}^{\alpha}$ are Caputo fractional derivatives, $b, m, \xi, \eta \in (0,1)$, $\sigma > 0$, $\gamma > 0$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. $\phi_{p(t)}(\cdot)$ is the $p(t)$-Laplacian operator, $p(t) > 1$, $p(t) \in C^1[0,1]$, and

$$\gamma \xi = 1, \qquad \sigma \eta = 1, \qquad b = \frac{1}{2}\xi, \qquad m = 1 - \frac{1}{2}\eta, \tag{1.2}$$

which leads to BVP (1.1) is resonant. It is also assumed that

$$C = (\alpha + 2)\bigl[1 - m(\alpha + 1)\bigr] + \xi(\alpha + 1)(m\alpha - 1) + \eta^{\alpha}(\xi - \eta) \neq 0. \tag{1.3}$$

It is worth noting that $p(t) = p$ herein, meaning it could be the famous p-Laplacian operator. Since the $p(t)$-Laplacian operator is a nonlinear operator, it is more difficult to construct a projection operator. So our results serve as a further development for the previous findings in this sense. Furthermore, we also observe that few scholars have ever considered fractional Sturm-Liouville BVP with $p(t)$-Laplacian operator before. The kernel space herein is extended to higher dimensions as well. To be specific, it is assumed that $\dim \operatorname{Ker} L = 2$ in the article. In comparison with the case when $\dim \operatorname{Ker} L = 1$, the system is more complex.

2 Preliminaries

To facilitate understanding, we would firstly make a brief introduction about the concepts and lemmas regarding fractional derivatives and integrals in the article. For more details, please refer to the references hereunder (see [31–33]).

Definition 2.1 ([31]) Let X, Y be real Banach spaces and $L : \operatorname{dom} L \subset X \to Y$ be a linear map. If $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is a closed subset in Y, then the map L is a Fredholm operator with index zero. If there exist such continuous projections as $P : X \to X$ and $Q : Y \to Y$, which meet the conditions that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$, then $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is reversible. We denote the inverse map by K_P, set $K_P = L_P^{-1}$ and $K_{P,Q} = K_P(I - Q)$. If Ω is an open bounded subset of X and $\operatorname{dom} L \cap \Omega \neq \varnothing$, the map N is L-compact on $\overline{\Omega}$ when $QN : \overline{\Omega} \to Y$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact.

Theorem 2.1 ([31]) *Let L be a Fredholm operator of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied*:

(a_1) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$.
(a_2) $Nx \notin \operatorname{Im} L$ *for every* $x \in \operatorname{Ker} L \cap \partial\Omega$.
(a_3) $\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \Omega$.

Definition 2.2 ([33]) The Riemann-Liouville fractional integral of order α ($\alpha > 0$) for the function $x : (0, +\infty) \to \mathbb{R}$: is defined as

$$I_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} x(s)\, ds,$$

assume that the right-hand side integral is defined on $(0, +\infty)$.

Definition 2.3 ([33]) The Caputo fractional integral of order α ($\alpha > 0$) for the function $x:(0,+\infty)\to\mathbb{R}$: is defined as

$$^{C}D_{0+}^{\alpha}x(t)=I_{0+}^{n-\alpha}\frac{d^{n}x(t)}{dt^{n}}=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-s)^{n-\alpha-1}x^{(n)}(s)\,\mathrm{d}s,$$

where $n=[\alpha]+1$, provided that the right-hand side integral is defined on $(0,+\infty)$.

Lemma 2.1 ([33]) *Let $n-1<\alpha\le n$, if $^{C}D_{0+}^{\alpha}x(t)\in C[0,1]$, then*

$$I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}x(t)=x(t)+c_{0}+c_{1}t+c_{2}t^{2}+\cdots+c_{n-1}t^{n-1},$$

where $c_i\in\mathbb{R}$, $i=0,1,\ldots,n-1$, $n=[\alpha]+1$.

Lemma 2.2 ([33]) *Let $n-1<\alpha\le n$, then the fractional differential $^{C}D_{0+}^{\alpha}x(t)=0$ has the following form*:

$$x(t)=c_{0}+c_{1}t+c_{2}t^{2}+\cdots+c_{n-1}t^{n-1},$$

where $c_i\in\mathbb{R}$, $i=0,1,\ldots,n-1$, $n=[\alpha]+1$.

Lemma 2.3 ([29]) *For any $(t,x)\in[0,1]\times\mathbb{R}$, $\varphi_{p(t)}(x)=|x|^{p(t)-2}x$ is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$ and strictly monotone increasing for any fixed t. Moreover, its inverse operator $\varphi_{p(t)}^{-1}(\cdot)$ is defined by*

$$\begin{cases}\varphi_{p(t)}^{-1}(x)=|x|^{\frac{2-p(t)}{p(t)-1}}x, & x\in\mathbb{R}\backslash\{0\},\\ \varphi_{p(t)}^{-1}(0)=0, & x=0,\end{cases}$$

which is continuous and sends bounded sets into bounded sets.

Since Mawhin's continuation theorem is applicable to linear operators, the following lemma needs to be introduced in this paper.

Lemma 2.4 *BVP* (1.1) *is equivalent to the following problem*:

$$\begin{cases}D_{0+}^{\alpha}x(t)=\phi_{p(t)}^{-1}(I_{0+}^{\beta}f(t,x(t),x'(t))), & t\in(0,1),\\ x(0)+bx'(0)=\gamma\int_{0}^{\xi}x(t)\,\mathrm{d}t, \qquad x(1)-mx'(1)=\sigma\int_{0}^{\eta}x(t)\,\mathrm{d}t.\end{cases}\tag{2.1}$$

Proof On the one hand, by Definition 2.2, we have

$$\phi_{p(t)}\bigl(D_{0+}^{\alpha}x(t)\bigr)=I_{0+}^{\beta}f\bigl(t,x(t),x'(t)\bigr)+c,\quad c\in\mathbb{R}.$$

Based on the boundary condition $^{C}D_{0+}^{\alpha}x(0)=0$, we get $c=0$. Thus,

$$\phi_{p(t)}\bigl(D_{0+}^{\alpha}x(t)\bigr)=I_{0+}^{\beta}f\bigl(t,x(t),x'(t)\bigr),$$
$$D_{0+}^{\alpha}x(t)=\phi_{p(t)}^{-1}\bigl(I_{0+}^{\beta}f\bigl(t,x(t),x'(t)\bigr)\bigr).$$

On the other hand, if $D_{0+}^{\alpha}x(t) = \phi_{p(t)}^{-1}(I_{0+}^{\beta}f(t,x(t),x'(t)))$, for $t = 0$, we have $D_{0+}^{\alpha}x(0) = 0$. Multiplying both sides of the equation by the operator $\phi_{p(t)}$ and D_{0+}^{β}, we get

$$D_{0+}^{\beta}\phi_{p(t)}\big(D_{0+}^{\alpha}x(t)\big) = f\big(t,x(t),x'(t)\big).$$

The proof is complete. □

3 Main result

Let $X = C^1[0,1]$, $Y = C[0,1]$ with the norm $\|x\|_X = \max_{t\in[0,1]}\{\|x\|_\infty, \|x'\|_\infty\}$, $\|y\|_Y = \|y\|_\infty$, where $\|x\|_\infty = \max_{t\in[0,1]}|x(t)|$. By Lemma 2.4, BVP (1.1) is equivalent to the following problems:

$$\begin{cases} D_{0+}^{\alpha}x(t) = \phi_{p(t)}^{-1}(I_{0+}^{\beta}f(t,x(t),x'(t))), & t\in(0,1), \\ x(0) + bx'(0) = \gamma\int_0^{\xi}x(t)\,\mathrm{d}t, & x(1) - mx'(1) = \sigma\int_0^{\eta}x(t)\,\mathrm{d}t. \end{cases} \tag{3.1}$$

Define the operator $L : \operatorname{dom} L \subset X \to Y$ by

$$Lx = D_{0+}^{\alpha}x(t), \tag{3.2}$$

where

$$\begin{aligned} \operatorname{dom} L = \bigg\{ x\in X \,\bigg|\, & D_{0+}^{\alpha}x(t)\in Y, x(0) + bx'(0) = \gamma\int_0^{\xi}x(t)\,\mathrm{d}t, \\ & x(1) - mx'(1) = \sigma\int_0^{\eta}x(t)\,\mathrm{d}t \bigg\}. \end{aligned}$$

Let $N : X \to Y$ as the Nemytskii operator

$$Nx(t) = \phi_{p(t)}^{-1}\big(I_{0+}^{\beta}f\big(t,x(t),x'(t)\big)\big), \quad \forall t\in[0,1].$$

Then BVP (1.1) is equivalent to the following operator equation:

$$Lx = Nx, \quad x\in\operatorname{dom} L.$$

For convenience, define the operators $T_1, T_2, Q_1, Q_2 : Y \to Y$:

$$\begin{aligned} T_1y &= \int_0^{\xi}(\xi - s)^{\alpha}y(s)\,\mathrm{d}s, \\ T_2y &= \alpha\int_0^1(1-s)^{\alpha-1}y(s)\,\mathrm{d}s - m\alpha(\alpha-1)\int_0^1(1-s)^{\alpha-2}y(s)\,\mathrm{d}s - \sigma\int_0^{\eta}(\eta - s)^{\alpha}y(s)\,\mathrm{d}s, \\ Q_1y &= \frac{1}{\Lambda}\big(\Lambda_4T_1y(t) - \Lambda_3T_2y(t)\big), \\ Q_2y &= \frac{1}{\Lambda}\big(-\Lambda_2T_1y(t) + \Lambda_1T_2y(t)\big), \end{aligned}$$

where

$$\Lambda_1 = \frac{\xi^{\alpha+1}}{\alpha+1}, \qquad \Lambda_2 = 1 - m\alpha - \frac{\eta^{\alpha}}{\alpha+1}, \qquad \Lambda_3 = \frac{\xi^{\alpha+2}}{(\alpha+1)(\alpha+2)},$$

$$\begin{aligned}
\Lambda_4 &= \frac{1}{\alpha+1} - m - \frac{\eta^{\alpha+1}}{(\alpha+1)(\alpha+2)}, \\
\Lambda &= \begin{vmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{vmatrix} \\
&= \frac{\xi^{\alpha+1}}{(\alpha+1)^2(\alpha+2)} \bigl\{(\alpha+2)\bigl[1-m(\alpha+1)\bigr] + \xi(\alpha+1)(m\alpha-1) + \eta^{\alpha}(\xi-\eta)\bigr\} \\
&= \frac{\xi^{\alpha+1}}{(\alpha+1)^2(\alpha+2)} C.
\end{aligned}$$

The following theorem is the main result of this paper.

Theorem 3.1 *Assume that the following conditions hold.*

(H_1) *If the function* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous, and there exist nonnegative functions* $a, b, c \in C[0,1]$ *such that*

$$\bigl|f(t,u,v)\bigr| \le a(t) + b(t)|u|^{\theta-1} + c(t)|v|^{\theta-1}, \quad \forall t \in [0,1], (u,v) \in \mathbb{R}^2, 1 < \theta \le P_L,$$

where $a_1 = \|a\|_\infty$, $b_1 = \|b\|_\infty$, $c_1 = \|c\|_\infty$, $P_L = \min_{t\in[0,1]} p(t)$.

(H_2) *There exists a constant* $\mathrm{B} > 0$ *such that for* $u \in \mathbb{R}$, *if* $|u| > \mathrm{B}$, *one has either*

$$u \cdot T_1 N u > 0 \quad \text{or} \quad u \cdot T_1 N u < 0.$$

(H_3) *There exists a constant* $\mathrm{D} > 0$ *such that for* $v \in \mathbb{R}$, *if* $|v| > D$, *one has either*

$$v \cdot T_2 N u > 0 \quad \text{or} \quad v \cdot T_2 N u < 0.$$

Then BVP (1.1) *has at least one solution provided that*

$$\frac{4^{\theta-1}(b_1(\alpha+1)^{\theta-1} + c_1\alpha^{\theta-1})}{\Gamma(\beta+1)(\Gamma(\alpha+1))^{\theta-1}} < \frac{1}{2}. \tag{3.3}$$

In order to prove the above theorem, it is necessary to introduce more relevant lemmas, as shown hereunder.

Lemma 3.1 *Let* L *be defined by* (3.2), *then*

$$\operatorname{Ker} L = \bigl\{x \in X | x(t) = c_0 + c_1 t, c_0, c_1 \in \mathbb{R}, \forall t \in [0,1]\bigr\}, \tag{3.4}$$

$$\operatorname{Im} L = \{y \in Y | T_1 y = T_2 y = 0\}. \tag{3.5}$$

Proof By Lemma 2.2, $D^{\alpha}_{0+} x(t) = 0$ has a solution, i.e.,

$$x(t) = c_0 + c_1 t, \quad c_0, c_1 \in \mathbb{R}.$$

From (3.1), we can obtain (3.4).

Next, we prove $\operatorname{Im} L = \{y \in Y | T_1 y = T_2 y = 0\}$.

If $y \in \operatorname{Im} L$, there exists $x \in \operatorname{dom} L$ such that $y = Lx \in Y$. By (2.1) ,we get

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,\mathrm{d}s + c_0 + c_1 t.$$

In view of the conditions of (3.1), we have

$$\begin{aligned} &\int_0^\xi (\xi - s)^\alpha y(s)\,\mathrm{d}s = 0, \\ &\alpha \int_0^1 (1-s)^{\alpha-1} y(s)\,\mathrm{d}s - m\alpha(\alpha-1) \int_0^1 (1-s)^{\alpha-2} y(s)\,\mathrm{d}s - \sigma \int_0^\eta (\eta - s)^\alpha y(s)\,\mathrm{d}s = 0, \end{aligned}$$

i.e., $T_1 y = T_2 y = 0$. On the other hand, if $T_1 y = T_2 y = 0$ for $y \in Y$, let $x(t) = I_{0+}^\alpha y(t)$, then $x \in \operatorname{dom} L$ and $D_{0+}^\alpha x(t) = y(t)$. Thus, $y \in \operatorname{Im} L$. □

Lemma 3.2 *Let L be defined by* (3.2), *then L is a Fredholm operator of index zero. The linear projection operators $P : X \to X$ and $Q : Y \to Y$ can be defined as follows*:

$$\begin{aligned} &Px(t) = x(0) + x'(0)t, \quad \forall t \in [0,1], \\ &Qy(t) = Q_1 y + Q_2 y \cdot t, \quad \forall t \in [0,1]. \end{aligned}$$

In addition, $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ *is defined as*

$$K_P y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,\mathrm{d}s, \quad \forall t \in [0,1].$$

Proof Clearly, $\operatorname{Im} P = \operatorname{Ker} L$ and $P^2 x = Px$. By $x = (x - Px) + Px$, we obtain $x = \operatorname{Ker} P + \operatorname{Ker} L$. After a simple calculation, we get $\operatorname{Ker} L \cap \operatorname{Ker} P = \{0\}$. Thus, we have

$$x = \operatorname{Ker} L \oplus \operatorname{Ker} P.$$

The next step is to prove $\operatorname{Ker} Q = \operatorname{Im} L$. It is clear that $\operatorname{Im} L \subset \operatorname{Ker} Q$. On the other hand, if $y \in \operatorname{Ker} Q \subset Y$, then $Q_1 y = Q_2 y = 0$, i.e.,

$$\begin{cases} \frac{1}{\Lambda}(\Lambda_4 T_1 y - \Lambda_3 T_2 y) = 0, \\ \frac{1}{\Lambda}(-\Lambda_2 T_1 y + \Lambda_1 T_2 y) = 0. \end{cases}$$

By (1.3), we have $\Lambda \neq 0$. Hence, $T_1 y = T_2 y = 0$. Thus we get $y \in \operatorname{Im} L$ and $\operatorname{Ker} Q \subset \operatorname{Im} L$. For $y \in Y$, we get

$$\begin{aligned} &Q_1^2 y = \frac{1}{\Lambda}\big(\Lambda_4 T_1(Q_1 y) - \Lambda_3 T_2(Q_1 y)\big) = \frac{1}{\Lambda}(\Lambda_4 \Lambda_1 - \Lambda_3 \Lambda_2) Q_1 y = Q_1 y, \\ &Q_2(Q_1 y) = \frac{1}{\Lambda}\big(-\Lambda_2 T_1(Q_1 y) + \Lambda_1 T_2(Q_1 y)\big) = \frac{1}{\Lambda}(-\Lambda_2 \Lambda_1 + \Lambda_1 \Lambda_2) Q_1 y = 0, \\ &Q_1(Q_2 y \cdot t) = \frac{1}{\Lambda}\big(\Lambda_4 T_1(Q_2 y \cdot t) - \Lambda_3 T_2(Q_2 y \cdot t)\big) = \frac{1}{\Lambda}(\Lambda_4 \Lambda_3 - \Lambda_3 \Lambda_4) Q_2 y = 0, \\ &Q_2(Q_2 y \cdot t) = \frac{1}{\Lambda}\big(-\Lambda_2 T_1(Q_2 y \cdot t) + \Lambda_1 T_2(Q_2 y \cdot t)\big) = \frac{1}{\Lambda}(-\Lambda_2 \Lambda_3 + \Lambda_1 \Lambda_4) Q_2 y = Q_2 y. \end{aligned}$$

Therefore, we have

$$Q^2y = Q_1(Q_1y + Q_2y \cdot t) + Q_2(Q_1y + Q_2y \cdot t)t = Q_1y + Q_2y \cdot t = Qy.$$

If $y \in Y$, let $y = (y - Qy) + Qy$, where $y - Qy \in \operatorname{Ker} Q = \operatorname{Im} L$, $Qy \in \operatorname{Im} Q$. It follows from $\operatorname{Ker} Q = \operatorname{Im} L$ and $Q^2y = Qy$ that $\operatorname{Im} Q \cap \operatorname{Im} L = \{0\}$. Then we get $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. Thus,

$$\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L = 2.$$

It implies that L is a Fredholm operator of index zero.

The last step is to prove that K_P is the inverse operator of $L|_{\operatorname{dom} L \cap \operatorname{Ker} P}$. In fact, for $y \in \operatorname{Im} L$, we have

$$LK_Py = D^\alpha_{0+} I^\alpha_{0+} y = y. \tag{3.6}$$

Additionally, for $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, we have $x(0) = x'(0) = 0$ and $K_PLx(t) = I^\alpha_{0+} D^\alpha_{0+} x(t) = x(t) + c_0 + c_1 t$. With the boundary condition $x(0) = x'(0) = 0$, we get

$$K_PLx = x. \tag{3.7}$$

Combining (3.6) with (3.7), we obtain $K_P = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$. The proof is complete. □

Theorem 3.1 is proved by the following three steps.

Step 1. Let

$$\Omega_1 = \{x \in \operatorname{dom} L \setminus \operatorname{Ker} L | Lx = \lambda Nx, \lambda \in (0,1)\}.$$

For any $x \in \Omega_1$, $x \notin \operatorname{Ker} L$, we have $Nx \in \operatorname{Im} L = \operatorname{Ker} Q$, then $QNx = 0$. By (3.5), we get

$$\begin{aligned} T_1Nx &= \int_0^\xi (\xi - s)^\alpha \phi^{-1}_{p(s)} \bigl(I^\beta_{0+} f\bigl(s, x(s), x'(s)\bigr)\bigr)\, ds = 0, \\ T_2Nx &= \alpha \int_0^1 (1 - s)^{\alpha-1} \phi^{-1}_{p(s)} \bigl(I^\beta_{0+} f\bigl(s, x(s), x'(s)\bigr)\bigr)\, ds \\ &\quad - m\alpha(\alpha - 1) \int_0^1 (1 - s)^{\alpha-2} \phi^{-1}_{p(s)} \bigl(I^\beta_{0+} f\bigl(s, x(s), x'(s)\bigr)\bigr)\, ds \\ &\quad - \sigma \int_0^\eta (\eta - s)^\alpha \phi^{-1}_{p(s)} \bigl(I^\beta_{0+} f\bigl(s, x(s), x'(s)\bigr)\bigr)\, ds = 0. \end{aligned}$$

From (H_2) and (H_3), there exist two constants $\varepsilon_1, \varepsilon_2 \in (0,1)$ such that $|x(\varepsilon_1)| \le \mathrm{B}$ and $|x'(\varepsilon_2)| \le \mathrm{D}$. Furthermore, by $x(t) = I^\alpha_{0+} D^\alpha_{0+} x(t) + c_0 + c_1 t$, we get

$$x'(t) = I^{\alpha-1}_{0+} D^\alpha_{0+} x(t) + c_1 = \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t - s)^{\alpha-2} D^\alpha_{0+} x(s)\, ds + c_1.$$

Let $t = \varepsilon_2$, then

$$x'(\varepsilon_2) = \frac{1}{\Gamma(\alpha - 1)} \int_0^{\varepsilon_2} (\varepsilon_2 - s)^{\alpha-2} D^\alpha_{0+} x(s)\, ds + c_1.$$

Since $|x'(\varepsilon_2)| \leq \mathrm{D}$, we have

$$\begin{aligned}
|c_1| &\leq \left|x'(\varepsilon_2)\right| + \frac{1}{\Gamma(\alpha-1)} \int_0^{\varepsilon_2} (\varepsilon_2 - s)^{\alpha-2} \left|D_{0+}^{\alpha} x(s)\right| \mathrm{d}s \\
&\leq \mathrm{D} + \frac{\varepsilon_2^{\alpha-1}}{\Gamma(\alpha)} \left\|D_{0+}^{\alpha} x\right\|_{\infty} \\
&\leq \mathrm{D} + \frac{1}{\Gamma(\alpha)} \left\|D_{0+}^{\alpha} x\right\|_{\infty}.
\end{aligned}$$

Then

$$\begin{aligned}
\left\|x'\right\|_{\infty} &\leq \frac{1}{\Gamma(\alpha-1)} \int_0^{t} (t-s)^{\alpha-2} \left|D_{0+}^{\alpha} x(s)\right| \mathrm{d}s + |c_1| \\
&\leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} \left\|D_{0+}^{\alpha} x\right\|_{\infty} + \mathrm{D} + \frac{1}{\Gamma(\alpha)} \left\|D_{0+}^{\alpha} x\right\|_{\infty} \\
&\leq \frac{2}{\Gamma(\alpha)} \left\|D_{0+}^{\alpha} x\right\|_{\infty} + \mathrm{D}.
\end{aligned}$$

Let $t = \varepsilon_1$, then

$$x(\varepsilon_1) = \frac{1}{\Gamma(\alpha)} \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\alpha-1} D_{0+}^{\alpha} x(s) \,\mathrm{d}s + c_0 + c_1 \varepsilon_1.$$

From $|x(\varepsilon_1)| \leq \mathrm{B}$, we have

$$\begin{aligned}
|c_0| &\leq \left|x(\varepsilon_1)\right| + \frac{1}{\Gamma(\alpha)} \int_0^{\varepsilon_1} (\varepsilon_1 - s)^{\alpha-1} \left|D_{0+}^{\alpha} x(s)\right| \mathrm{d}s + |c_1| \\
&\leq \mathrm{B} + \frac{\varepsilon_1^{\alpha}}{\Gamma(\alpha+1)} \left\|D_{0+}^{\alpha} x\right\|_{\infty} + \mathrm{D} + \frac{1}{\Gamma(\alpha)} \left\|D_{0+}^{\alpha} x\right\|_{\infty} \\
&\leq \mathrm{B} + \mathrm{D} + \frac{1+\alpha}{\Gamma(\alpha+1)} \left\|D_{0+}^{\alpha} x\right\|_{\infty}.
\end{aligned}$$

Then

$$\begin{aligned}
\|x\|_{\infty} &\leq \frac{1}{\Gamma(\alpha)} \int_0^{t} (t-s)^{\alpha-1} \left|D_{0+}^{\alpha} x(s)\right| \mathrm{d}s + |c_0| + |c_1| \\
&\leq \frac{t^{\alpha}}{\Gamma(\alpha+1)} \left\|D_{0+}^{\alpha} x\right\|_{\infty} + \mathrm{B} + 2\mathrm{D} + \frac{1}{\Gamma(\alpha)} \left\|D_{0+}^{\alpha} x\right\|_{\infty} + \frac{1+\alpha}{\Gamma(\alpha+1)} \left\|D_{0+}^{\alpha} x\right\|_{\infty} \\
&\leq \mathrm{B} + 2\mathrm{D} + \frac{2(\alpha+1)}{\Gamma(\alpha+1)} \left\|D_{0+}^{\alpha} x\right\|_{\infty}.
\end{aligned}$$

Furthermore, by $Lx = \lambda Nx$, we have

$$\begin{aligned}
&D_{0+}^{\alpha} x(t) = \lambda \phi_{p(t)}^{-1} \left(I_{0+}^{\beta} f\left(t, x(t), x'(t)\right)\right), \\
&\phi_{p(t)}\left(D_{0+}^{\alpha} x(t)\right) = \phi_{p(t)}\left(\lambda \phi_{p(t)}^{-1} \left(I_{0+}^{\beta} f\left(t, x(t), x'(t)\right)\right)\right) \\
&\qquad = \phi_{p(t)}(\lambda) I_{0+}^{\beta} f\left(t, x(t), x'(t)\right) \\
&\qquad = \lambda^{p(t)-1} I_{0+}^{\beta} f\left(t, x(t), x'(t)\right).
\end{aligned}$$

Combining (H_1) and $\lambda \in (0,1)$, we have

$$\begin{aligned}\left|D_{0+}^{\alpha}x(t)\right|^{p(t)-1} &\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\left|f\bigl(t,x(t),x'(t)\bigr)\right|\,\mathrm{d}s \\ &\leq \frac{1}{\Gamma(\beta+1)}\bigl(a_1 + b_1\|x\|_\infty^{\theta-1} + c_1\left\|x'\right\|_\infty^{\theta-1}\bigr) \\ &\leq \frac{1}{\Gamma(\beta+1)}\left[a_1 + b_1\left(\mathrm{B} + 2\mathrm{D} + \frac{2(\alpha+1)}{\Gamma(\alpha+1)}\left\|D_{0+}^{\alpha}x\right\|_\infty\right)^{\theta-1}\right. \\ &\quad \left.+ c_1\left(\frac{2}{\Gamma(\alpha)}\left\|D_{0+}^{\alpha}x\right\|_\infty + \mathrm{D}\right)^{\theta-1}\right].\end{aligned}$$

According to $(|a|+|b|)^p \leq 2^p(|a|^p + |b|^p)$, $p > 0$, we get

$$\left|D_{0+}^{\alpha}x(t)\right|^{p(t)-1} \leq A_1 + A_2\left\|D_{0+}^{\alpha}x\right\|_\infty^{\theta-1},$$

where

$$A_1 = \frac{a_1 + b_1(2\mathrm{B}+4\mathrm{D})^{\theta-1} + c_1(2\mathrm{D})^{\theta-1}}{\Gamma(\beta+1)}, \qquad A_2 = \frac{b_1(\frac{4(\alpha+1)}{\Gamma(\alpha+1)})^{\theta-1} + c_1(\frac{4}{\Gamma(\alpha)})^{\theta-1}}{\Gamma(\beta+1)}.$$

Hence, we have

$$\left\|D_{0+}^{\alpha}x\right\|_\infty \leq 2^{\frac{1}{p(t)-1}}\bigl(A_1^{\frac{1}{p(t)-1}} + A_2^{\frac{1}{p(t)-1}}\left\|D_{0+}^{\alpha}x\right\|_\infty^{\frac{\theta-1}{p(t)-1}}\bigr).$$

It follows from $\frac{\theta-1}{p(t)-1} \in (0,1]$ and $x^k \leq x+1$, $x > 0$, $k \in (0,1]$ that

$$\left\|D_{0+}^{\alpha}x\right\|_\infty \leq (2A_1)^{\frac{1}{p(t)-1}} + (2A_2)^{\frac{1}{p(t)-1}}\bigl(\left\|D_{0+}^{\alpha}x\right\|_\infty + 1\bigr).$$

By (3.3), there exists a constant $\mathrm{M}_1 > 0$ such that $\|D_{0+}^{\alpha}x\|_\infty \leq \mathrm{M}_1$. Thus,

$$\|x\|_\infty \leq \mathrm{B} + 2\mathrm{D} + \frac{2(\alpha+1)}{\Gamma(\alpha+1)}\mathrm{M}_1 := \mathrm{M}_2, \qquad \left\|x'\right\|_\infty \leq \mathrm{D} + \frac{2}{\Gamma(\alpha)}\mathrm{M}_1 := \mathrm{M}_3,$$

this proves that Ω_1 is bounded.

Step 2. Let

$$\Omega_2 = \{x \mid x \in \operatorname{Ker} L, Nx \in \operatorname{Im} L\}.$$

If $x(t) \in \Omega_2$, then $x(t) = c_0 + c_1 t$, $c_0, c_1 \in \mathbb{R}$ and $Nx \in \operatorname{Im} L$. Thus, we have $T_1 Nx = T_2 Nx = 0$. When it is combined with (H_3), we get $|x'(t)| = |c_1| \leq \mathrm{D}$. According to ($H_2$), there exists $\varepsilon_1 \in (0,1)$ such that $|x(\varepsilon_1)| = |c_0 + c_1\varepsilon_1| \leq \mathrm{B}$. It is clear that $|c_0| \leq \mathrm{B} + \mathrm{D}$. So $\|x\|_\infty \leq \mathrm{B} + 2\mathrm{D} := \mathrm{M}_3$. Thus, Ω_2 is bounded.

Step 3. Let

$$\Omega_3 = \bigl\{x \in \operatorname{Ker} L, \lambda Jx + (1-\lambda)QNx = 0, \lambda \in [0,1]\bigr\},$$

where $J : \operatorname{Ker} L \to \operatorname{Im} Q$ is a homeomorphism mapping:

$$J(c_0 + c_1 t) = \frac{1}{\Lambda}(\Lambda_4 c_0 - \Lambda_3 c_1) + \frac{1}{\Lambda}(-\Lambda_2 c_0 + \Lambda_1 c_1)t, \quad c_0, c_1 \in \mathbb{R}.$$

Let $x \in \Omega_3$, then $x(t) = c_0 + c_1 t$, $c_0, c_1 \in \mathbb{R}$ and $\lambda J(c_0 + c_1 t) + (1 - \lambda)QN(c_0 + c_1 t) = 0$, i.e.,

$$\lambda J(c_0 + c_1 t) + (1 - \lambda)\bigl[Q_1 N(c_0 + c_1 t) + Q_2 N(c_0 + c_1 t)t\bigr] = 0.$$

Then

$$\lambda(\Lambda_4 c_0 - \Lambda_3 c_1) + (1 - \lambda)\bigl[\Lambda_4 T_1 N(c_0 + c_1 t) - \Lambda_3 T_2 N(c_0 + c_1 t)\bigr] = 0,$$
$$\lambda(-\Lambda_2 c_0 + \Lambda_1 c_1) + (1 - \lambda)\bigl[-\Lambda_2 T_1 N(c_0 + c_1 t) + \Lambda_1 T_2 N(c_0 + c_1 t)\bigr] = 0.$$

Thus,

$$\lambda c_0 + (1 - \lambda)T_1 N(c_0 + c_1 t) = 0, \tag{3.8}$$
$$\lambda c_1 + (1 - \lambda)T_2 N(c_0 + c_1 t) = 0. \tag{3.9}$$

According to (3.8) and the first part of (H_2), we have $|c_0| \le B$. Otherwise, if $|c_0| > B$, by the first part of (H_2), we have

$$\lambda c_0^2 + (1 - \lambda)c_0 T_1 N(c_0 + c_1 t) > 0,$$

which is contradictory to (3.8). Similarly, by (3.9) and the first part of (H_3), we have $|c_1| \le D$. Otherwise, if $|c_1| > D$, by the first part of (H_3), we have

$$\lambda c_1^2 + (1 - \lambda)c_1 T_2 N(c_0 + c_1 t) > 0,$$

which is contradictory to (3.9). Hence, Ω_3 is bounded.

Let

$$\Omega = \bigl\{x \in X, \|x\|_\infty < \max\{M_2, M_3\} + 1\bigr\}.$$

As indicated by Lemma 3.2, L is a Fredholm operator of index zero. Based on the Arzela-Ascoli theorem, we obtain that N is L-compact on $\overline{\Omega}$. Then, by Step 1 and Step 2, we get

(a_1) $Lx \neq \lambda Nx$, $(x, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0, 1)$.
(a_2) $Nx \notin \operatorname{Im} L$, $x \in \operatorname{Ker} L \cap \partial\Omega$.

Let

$$H(x, \lambda) = \lambda J(x) + (1 - \lambda)QNx.$$

According to Step 3, we have $H(x, \lambda) \neq 0$ for $x \in \operatorname{Ker} L \cap \partial\Omega$, then

$$\begin{aligned}
\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= \deg\bigl(H(\cdot, 0), \Omega \cap \operatorname{Ker} L, 0\bigr) \\
&= \deg\bigl(H(\cdot, 1), \Omega \cap \operatorname{Ker} L, 0\bigr) \\
&= \deg(I, \Omega \cap \operatorname{Ker} L, 0) \neq 0.
\end{aligned}$$

Condition (a_3) of Theorem 2.1 is thus met. Through Theorem 2.1, we get that $Lx = Nx$ has at least one fixed point in $\operatorname{dom} L \cap \overline{\Omega}$. Hence, BVP (1.1) has at least one solution.

Remark 3.1 The proof process would be similar to that of Step 3 if the second inequality of both (H_2) and (H_3), or the first of (H_2) and the second of (H_3), or the first of (H_3) and the second of (H_2) hold. It is hence omitted herein.

Corollary 3.1 *Assume that the conditions of Theorem* 3.1 *hold. If* $\eta \le \xi$, *BVP* (1.1) *has at least one solution.*

Proof Here we just need to verify the condition $C \neq 0$.

By (1.2), we have

$$\begin{aligned} C &= (\alpha+2)\bigl[1-m(\alpha+1)\bigr] + \xi(\alpha+1)(m\alpha-1) + \eta^\alpha(\xi-\eta) \\ &= (\alpha+2) + (\alpha+1)\left(\xi\alpha - \alpha - 2 - \frac{1}{2}\xi\alpha\eta + \frac{1}{2}\alpha\eta + \eta - \xi\right) + \eta^\alpha(\xi-\eta) \\ &= 2(\alpha+1) - \alpha + (\alpha+1)\left(\xi\alpha - \alpha - 2 - \frac{1}{2}\xi\alpha\eta + \frac{1}{2}\alpha\eta + \eta - \xi\right) + \eta^\alpha(\xi-\eta) \\ &= (\alpha+1)\left[\alpha(\xi-1)\left(1-\frac{1}{2}\eta\right) + (\eta-\xi)\right] + \bigl[\eta^\alpha(\xi-\eta) - \alpha\bigr]. \end{aligned}$$

Obviously, by $1 < \alpha \le 2$ and $0 < \eta \le \xi < 1$, we obtain $C < 0$. The proof is complete. □

4 Example

Example 4.1 Consider the following BVP:

$$\begin{cases} {}^C D_{0+}^{\frac{2}{3}} \phi_{(t^2+2)}({}^C D_{0+}^{\frac{3}{2}} x(t)) = \frac{1}{10} + \frac{1}{30}\sin(x(t)) + \frac{1}{90}\sin(x'(t)), \quad t \in (0,1), \\ x(0) + \frac{1}{4}x'(0) = 2\int_0^{\frac{1}{2}} x(t)\,\mathrm{d}t, \\ x(1) - \frac{3}{4}x'(1) = 2\int_0^{\frac{1}{2}} x(t)\,\mathrm{d}t, \\ {}^C D_{0+}^{\alpha} x(0) = 0, \end{cases}$$

where $p(t) = t^2 + 2$, $\alpha = \frac{3}{2}$, $\beta = \frac{2}{3}$, $\theta = 2$, $f(t,x(t),x'(t)) = \frac{1}{10} + \frac{1}{30}\sin(x(t)) + \frac{1}{90}\sin(x'(t))$, $a_1 = \frac{1}{10}$, $b_1 = \frac{1}{30}$, $c_1 = \frac{1}{90}$, $b = \frac{1}{4}$, $\xi = \frac{1}{2}$, $\eta = \frac{1}{2}$, $m = \frac{3}{4}$, $\gamma = 2$, $\sigma = 2$, $P_L = 2$. It is easy to verify that (1.2) and (1.3) hold. Let B = 30, D = 90, C = $-\frac{93}{32} < 0$, if $x(t) > 30$, $x'(t) > 90$, then $f(t,x(t),x'(t)) > 0$. Clearly, (H_1) of Theorem 3.1 holds. By (3.3), we get

$$\begin{gathered} \frac{4^{\theta-1}(b_1(\alpha+1)^{\theta-1} + c_1\alpha^{\theta-1})}{\Gamma(\beta+1)(\Gamma(\alpha+1))^{\theta-1}} = \frac{2}{5\Gamma(\frac{5}{3})\Gamma(\frac{5}{2})} < \frac{1}{2}, \\ Nx(t) = \phi_{p(t)}^{-1}\bigl(I_{0+}^{\beta} f\bigl(t,x(t),x'(t)\bigr)\bigr) = \phi_{(t^2+2)}^{-1}\bigl(I_{0+}^{\frac{2}{3}} f\bigl(t,x(t),x'(t)\bigr)\bigr) > 0. \end{gathered}$$

So, (H_2) of Theorem 3.1 holds. Furthermore, by the definition of $T_2 y$, we have

$$\begin{aligned} T_2 Nu(t) = {}& \alpha\int_0^1 (1-s)^{\alpha-1} Nu(s)\,\mathrm{d}s - m\alpha(\alpha-1)\int_0^1 (1-s)^{\alpha-2} Nu(s)\,\mathrm{d}s \\ & - \sigma\int_0^\eta (\eta-s)^\alpha Nu(s)\,\mathrm{d}s \end{aligned}$$

$$= Nu(s)\left(1 - m\alpha - \frac{\sigma\eta^{\alpha+1}}{\alpha+1}\right)$$
$$= Nu(s)\left(-\frac{1}{8} - \frac{1}{5\sqrt{2}}\right) < 0.$$

Thus, (H_3) of Theorem 3.1 holds. Hence, there exists at least one solution.

5 Conclusions

The solvability for fractional Sturm-Liouville BVP with $p(t)$-Laplacian operator is discussed in the article by using Mawhin's continuation theorem, and the existence of solutions has been obtained (see Theorem 3.1). The kernel space is expanded to higher dimensions on condition that $\dim \operatorname{Ker} L = 2$, and the system is more complex in comparison with the case when $\dim \operatorname{Ker} L = 1$. Moreover, when $p(t) = p$, the $p(t)$-Laplacian operator will evolve into the famous p-Laplacian operator. Therefore, our results would develop previous findings to some extent.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally in this article. They have all read and approved the final manuscript.

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