Wang and Wu *Boundary Value Problems* (2017) 2017:164
DOI 10.1186/s13661-017-0896-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

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Eigenvalues of stochastic Hamiltonian systems driven by Poisson process with boundary conditions

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Abstract

In this paper, we study an eigenvalue problem for stochastic Hamiltonian systems driven by a Brownian motion and Poisson process with boundary conditions. By means of dual transformation and generalized Riccati equation systems, we prove the existence of eigenvalues and construct the corresponding eigenfunctions. Moreover, a specific numerical example is considered to illustrate the phenomenon of statistic periodicity for eigenfunctions of stochastic Hamiltonian systems.

Keywords: stochastic Hamiltonian systems; Poisson process; eigenvalues and eigenfunctions; dual transformation; Riccati equations; statistic periodicity

1 Introduction

The backward stochastic differential equations driven by Poisson process (BSDEP) were first introduced and studied by Tang and Li [1]. Later, Situ Rong [2] proved the existence and uniqueness of solutions to BSDEP with non-Lipschitz coefficients. Barles et al. [3] adopted the BSDEP to provide a probabilistic interpretation for a system of parabolic integro-partial differential equations. Then, the fully coupled forward-backward stochastic differential equations driven by Poisson process (FBSDEP) were deeply investigated by Wu [4, 5], etc. Precisely, in [4], the author established the well-posedness for FBSDEP under the so-called 'monotone assumptions' via the continuation method; while in [5], the author discussed the BSDEP and FBSDEP with stopping time duration.

The stochastic Hamiltonian systems are proposed in the optimal control theory as a necessary condition for optimality, called the stochastic maximum principle. The fundamental works related to this topic include Bismut [6], Bensoussan [7], Peng [8] and so on. Due to the discontinuity of stock prices and other common 'random jump' phenomena in reality, the stochastic Hamiltonian systems driven by Poisson process, which is a special kind of FBSDEP, are very suitable for us to study the stochastic optimal control problems with random jumps. Wu and Wang [9] discussed a linear quadratic stochastic optimization problem with random jumps, and furthermore associated its Hamiltonian system with a generalized Riccati equation system to give the linear feedback optimality for this problem.

However, all literature works above only concern the uniqueness of solutions to FBSDEP as well as stochastic Hamiltonian systems. Peng [10] studied a kind of eigenvalue problem for stochastic Hamiltonian systems driven by Brownian motion with boundary conditions. In this paper, we extend that problem to stochastic Hamiltonian systems driven by Poisson process within the formulation of FBSDEP established by Wu [4]. Generally speaking, for a class of stochastic Hamiltonian systems driven by Poisson process parameterized by $\lambda \in \mathbb{R}$ which always admit the trivial solution $(x_t, y_t, z_t, k_t) \equiv (0,0,0,0)$ for all λ, our problem is to find some real numbers λ_i, $i = 1, 2, \ldots$, such that the corresponding Hamiltonian system has multi-solutions. Here, λ_i are called eigenvalues and the corresponding nontrivial solutions are called eigenfunctions of this class of stochastic Hamiltonian systems. Inspired by the method of dual transformation introduced in [11] and the relationship between stochastic Hamiltonian systems driven by Poisson process and generalized Riccati equation systems given in [9], we obtain the existence of eigenvalues and construct the eigenfunctions explicitly. We also provide some sufficient conditions for the existence of multi-solutions to FBSDEP. Moreover, it follows from the construction of eigenfunctions that they keep the 'statistic periodicity' property as that in the deterministic case and also in the stochastic case with Brownian motion. On the other hand, for any real number λ larger than 1, we establish a family of stochastic Hamiltonian systems whose eigenvalue systems contain λ and give the corresponding eigenfunctions. A numerical example is presented to illustrate the theoretical result as well as the phenomenon of statistic periodicity for eigenfunctions.

The rest of this paper is organized as follows. In Section 2, we first recall the formulation of general FBSDEP and then formulate the eigenvalue problem for stochastic Hamiltonian systems driven by Poisson process. The main results are given by two theorems in Section 3: one is the existence of eigenvalues and eigenfunctions for an arbitrarily dimensional case; the other is a more concrete conclusion for a one-dimensional case. To prove our results, we introduce the dual transformation for stochastic Hamiltonian systems and establish the relationship between stochastic Hamiltonian systems driven by Poisson process and a kind of Riccati equation systems in Section 4. Thus, the proofs of two theorems above are completed in Section 5. In Section 6, we discuss the 'statistic periodicity' property for eigenfunctions from another viewpoint and present a numerical example to show our theoretical result vividly. The last section is devoted to concluding the novelty of this paper.

2 Formulation

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$ and $\mathcal{F}_{t+} = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$, $t \geq 0$. We suppose that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by two independent processes: one is a one-dimensional standard Brownian motion $\{B_t\}_{t\geq 0}$; the other is a Poisson random measure $\{N_t\}_{t\geq 0}$ with the compensator $\hat{N}(dt) = \theta\,dt$, such that $\tilde{N}([0,t]) = (N - \hat{N})([0,t])_{t\geq 0}$ is a martingale, where $\theta > 0$ is a constant called the intensity of $\{N_t\}_{t\geq 0}$. $T > 0$ is a fixed time horizon. Denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the scalar product and the norm of an Euclidean space, respectively. We also introduce the following notations:

$$\mathcal{M}^2(\mathbb{R}^n) = \biggl\{ \{\phi_t\}_{0\leq t\leq T} \text{ is an } \mathbb{R}^n\text{-valued } \mathcal{F}_t\text{-adapted process} \\ \text{such that } \mathbb{E}\biggl[\int_0^T |\phi_t|^2\,dt\biggr] < \infty \biggr\},$$

$$F_N^2(\mathbb{R}^n) = \left\{ \{k_t\}_{0\le t\le T} \text{ is an } \mathbb{R}^n\text{-valued } \mathcal{F}_t\text{-predictable process} \right.$$
$$\left. \text{such that } \mathbb{E}\left[\int_0^T |k_t|^2 \, dt \right] < \infty \right\}.$$

First, let us recall an existence and uniqueness result of solutions to FBSDEP from Wu [4]. Consider the following FBSDEP:

$$\begin{cases} dx_t = f_2(t, x_t, y_t, z_t, k_t)\, dt + f_3(t, x_t, y_t, z_t, k_t)\, dB_t \\ \qquad + f_4(t, x_{t-}, y_{t-}, z_t, k_t)\, d\tilde{N}_t, \\ -dy_t = f_1(t, x_t, y_t, z_t, k_t)\, dt - z_t\, dB_t - k_t\, d\tilde{N}_t, \quad 0 \le t \le T, \\ x_0 = x_0, \qquad y_T = \Psi(x_T). \end{cases} \tag{1}$$

Here, $f_1, f_2, f_3, f_4 : \mathbb{R}^{4n} \times [0, T] \times \Omega \mapsto \mathbb{R}^n$ and $\Psi : \mathbb{R}^n \mapsto \mathbb{R}^n$ are all measurable functions. We denote

$$f = \begin{bmatrix} -f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} : (x, y, z, k) \in \mathbb{R}^{4n} \times [0, T] \times \Omega \mapsto \mathbb{R}^{4n}$$

and assume the following.

Assumption 2.1 The functions f and Ψ satisfy:

(i) For any $\xi = (x, y, z, k) \in \mathbb{R}^{4n}, f(\xi, \cdot) \in \mathcal{M}^2(\mathbb{R}^{4n})$;

(ii) There exists a constant $C > 0$ such that

$$\begin{aligned} &\left| f(\xi, t) - f(\xi', t) \right| \le C \left| \xi - \xi' \right|, \quad \forall \xi, \xi' \in \mathbb{R}^{4n}, \\ &\left| \Psi(x) - \Psi(x') \right| \le C \left| x - x' \right|, \quad \forall x, x' \in \mathbb{R}^n; \end{aligned}$$

(iii) There exists a constant $\alpha > 0$ such that

$$\begin{aligned} &\left\langle f(\xi, t) - f(\xi', t), \xi - \xi' \right\rangle \le -\alpha \left| \xi - \xi' \right|^2, \quad \forall \xi, \xi' \in \mathbb{R}^{4n}, \\ &\left\langle \Psi(x) - \Psi(x'), x - x' \right\rangle \ge 0, \quad \forall x, x' \in \mathbb{R}^n. \end{aligned}$$

We have the following from Theorem 3.1 of Wu [4].

Theorem 2.2 *Let Assumption* 2.1 *hold. Then FBSDEP* (1) *admits a unique solution* $(x_t, y_t, z_t, k_t) \in \mathcal{M}^2(\mathbb{R}^n) \times \mathcal{M}^2(\mathbb{R}^n) \times F_N^2(\mathbb{R}^n) \times F_N^2(\mathbb{R}^n)$.

Remark 2.3 The result above is slightly stronger than that of Wu [4]. In fact, Theorem 2.2 can be proved by the same arguments of Wu [4] since the martingale representation theorem from [1] guarantees that the component z_t belongs to $F_N^2(\mathbb{R}^n)$.

The result in Theorem 2.2 can be applied to discuss the boundary problem of stochastic Hamiltonian systems with Poisson process. Suppose that $h(x, y, z, k) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times$

$\mathbb{R}^n \mapsto \mathbb{R}$ is a $\mathcal{C}^1$ real function called the Hamiltonian function and $\Phi(x) : \mathbb{R}^n \mapsto \mathbb{R}$ is a $\mathcal{C}^1$ real function. The problem is to find a quadruple $(x_t, y_t, z_t, k_t) \in \mathcal{M}^2(\mathbb{R}^n) \times \mathcal{M}^2(\mathbb{R}^n) \times F_N^2(\mathbb{R}^n) \times F_N^2(\mathbb{R}^n)$ satisfying the following stochastic Hamiltonian system:

$$\begin{cases} dx_t = \partial_y h(x_t, y_t, z_t, k_t)\,dt + \partial_z h(x_t, y_t, z_t, k_t)\,dB_t + \partial_k h(x_{t-}, y_{t-}, z_t, k_t)\,d\tilde{N}_t, \\ -dy_t = \partial_x h(x_t, y_t, z_t, k_t)\,dt - z_t\,dB_t - k_t\,d\tilde{N}_t, \quad 0 \le t \le T, \\ x_0 = x_0, \qquad y_T = \partial_x \Phi(x_T). \end{cases} \tag{2}$$

It is obvious that (2) is a special case of (1) with

$$f = \begin{bmatrix} -\partial_x h \\ \partial_y h \\ \partial_z h \\ \partial_k h \end{bmatrix}, \quad \text{and} \quad \Psi = \partial_x \Phi.$$

From Theorem 2.2, the stochastic Hamiltonian system (2) admits a unique solution $(x_t, y_t, z_t, k_t) \in \mathcal{M}^2(\mathbb{R}^n) \times \mathcal{M}^2(\mathbb{R}^n) \times F_N^2(\mathbb{R}^n) \times F_N^2(\mathbb{R}^n)$ under Assumption 2.1.

Now, we are ready to formulate the eigenvalue problem for stochastic Hamiltonian systems with Poisson process. Suppose that $\bar{h}(x, y, z, k) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ is a $\mathcal{C}^1$ real function, and denote for each $\lambda \in \mathbb{R}$

$$h^\lambda(x, y, z, k) = h(x, y, z, k) + \lambda \bar{h}(x, y, z, k).$$

Moreover, we assume that for $(x, y, z, k) = (0, 0, 0, 0)$,

$$\partial_x h = \partial_y h = \partial_z h = \partial_k h = \partial_x \bar{h} = \partial_y \bar{h} = \partial_z \bar{h} = \partial_k \bar{h} = 0.$$

Consider the following parameterized stochastic Hamiltonian system:

$$\begin{cases} dx_t = \partial_y h^\lambda(x_t, y_t, z_t, k_t)\,dt + \partial_z h^\lambda(x_t, y_t, z_t, k_t)\,dB_t \\ \qquad + \partial_k h^\lambda(x_{t-}, y_{t-}, z_t, k_t)\,d\tilde{N}_t, \\ -dy_t = \partial_x h^\lambda(x_t, y_t, z_t, k_t)\,dt - z_t\,dB_t - k_t\,d\tilde{N}_t, \quad 0 \le t \le T, \\ x_0 = 0, \qquad y_T = 0. \end{cases} \tag{3}$$

It is obvious that $(x_t, y_t, z_t, k_t) \equiv (0, 0, 0, 0)$ is a trivial solution of (3). The eigenvalue problem is to find some real number λ such that (3) admits nontrivial solutions. Throughout this paper, we shall focus on the case that h and $\bar{h}$ are in the form of

$$h(\xi) = \frac{1}{2}\langle H\xi, \xi\rangle, \qquad \bar{h}(\xi) = \frac{1}{2}\langle \bar{H}\xi, \xi\rangle, \quad \forall \xi = (x, y, z, k) \in \mathbb{R}^{4n},$$

where H and $\bar{H}$ are both $4n \times 4n$ symmetric matrices:

$$H = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & 0 \\ H_{41} & H_{42} & 0 & H_{44} \end{bmatrix}, \qquad \bar{H} = \begin{bmatrix} \bar{H}_{11} & \bar{H}_{12} & \bar{H}_{13} & \bar{H}_{14} \\ \bar{H}_{21} & \bar{H}_{22} & \bar{H}_{23} & \bar{H}_{24} \\ \bar{H}_{31} & \bar{H}_{32} & \bar{H}_{33} & 0 \\ \bar{H}_{41} & \bar{H}_{42} & 0 & \bar{H}_{44} \end{bmatrix}.$$

Here, H_{ij} and $\bar{H}_{ij}$, $i,j = 1,2,3,4$, are all $n \times n$ matrices such that $H_{ji} = H_{ij}^T$ and $\bar{H}_{ji} = \bar{H}_{ij}^T$. We also set

$$H^{\lambda} = H - \lambda \bar{H}, \qquad H_{ij}^{\lambda} = H_{ij} - \lambda \bar{H}_{ij}, \quad i,j = 1,2,3,4.$$

Thus, (iii) of Assumption 2.1 is equivalent to

$$\begin{bmatrix} -H_{11} & -H_{12} & -H_{13} & -H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & 0 \\ H_{41} & H_{42} & 0 & H_{44} \end{bmatrix} \leq -\alpha I_{4n}. \tag{4}$$

Hence,

$$\begin{bmatrix} -H_{11} & -H_{13} & -H_{14} \\ H_{31} & H_{33} & 0 \\ H_{41} & 0 & H_{44} \end{bmatrix} \leq -\alpha I_{3n}, \quad \text{and} \quad \begin{bmatrix} H_{22} & H_{23} & H_{24} \\ H_{32} & H_{33} & 0 \\ H_{42} & 0 & H_{44} \end{bmatrix} \leq -\alpha I_{3n}. \tag{5}$$

It follows from (5) that

$$\begin{aligned} &-H_{11} + H_{13}H_{33}^{-1}H_{31} + H_{14}H_{44}^{-1}H_{41} < 0, \\ &H_{22} - H_{23}H_{33}^{-1}H_{32} - H_{24}H_{44}^{-1}H_{42} < 0. \end{aligned} \tag{6}$$

Besides, for this specific case, (3) can be written as

$$\begin{cases} dx_t = (H_{21}^{\lambda}x_t + H_{22}^{\lambda}y_t + H_{23}^{\lambda}z_t + H_{24}^{\lambda}k_t)\,dt + (H_{31}^{\lambda}x_t + H_{32}^{\lambda}y_t + H_{33}^{\lambda}z_t)\,dB_t \\ \qquad\quad + (H_{41}^{\lambda}x_{t-} + H_{42}^{\lambda}y_{t-} + H_{44}^{\lambda}k_t)\,d\tilde{N}_t, \\ -dy_t = (H_{11}^{\lambda}x_t + H_{12}^{\lambda}y_t + H_{13}^{\lambda}z_t + H_{14}^{\lambda}k_t)\,dt - z_t\,dB_t - k_t\,d\tilde{N}_t, \quad 0 \leq t \leq T, \\ x_0 = 0, \qquad y_T = 0. \end{cases} \tag{7}$$

Now, we give the definition of eigenvalues and eigenfunctions of stochastic Hamiltonian systems.

Definition 2.4 $\lambda \in \mathbb{R}$ is called an eigenvalue of stochastic Hamiltonian system (7) if (7) corresponding to λ admits nontrivial solutions $(x_t, y_t, z_t, k_t) \in \mathcal{M}^2(\mathbb{R}^n) \times \mathcal{M}^2(\mathbb{R}^n) \times F_N^2(\mathbb{R}^n) \times F_N^2(\mathbb{R}^n)$. These solutions are the eigenfunctions corresponding to λ. The linear subspace of $\mathcal{M}^2(\mathbb{R}^n) \times \mathcal{M}^2(\mathbb{R}^n) \times F_N^2(\mathbb{R}^n) \times F_N^2(\mathbb{R}^n)$ consisting all eigenfunctions corresponding to eigenvalue λ is the eigenfunction subspace corresponding to λ.

Remark 2.5 According to Theorem 2.2, if condition (4) holds, then (7) only admits a trivial solution $(x_t, y_t, z_t, k_t) \equiv (0,0,0,0)$ corresponding to $\lambda = 0$. So $\lambda = 0$ cannot be an eigenvalue of (7).

3 Main results

There are two main theoretical results in this paper. For the multi-dimensional situation, we shall study the problem in which $\bar{H}$ is taken as

$$\bar{H} = \begin{bmatrix} 0 & 0 & H_{13} & H_{14} \\ 0 & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & 0 & 0 \\ H_{41} & H_{42} & 0 & 0 \end{bmatrix}.$$

Hence, (7) can be written as

$$\begin{cases} dx_t = [H_{21}x_t + (1-\lambda)H_{22}y_t + (1-\lambda)H_{23}z_t + (1-\lambda)H_{24}k_t]\,dt \\ \qquad\quad + [(1-\lambda)H_{31}x_t + (1-\lambda)H_{32}y_t + H_{33}z_t]\,dB_t \\ \qquad\quad + [(1-\lambda)H_{41}x_{t-} + (1-\lambda)H_{42}y_{t-} + H_{44}k_t]\,d\tilde{N}_t, \\ -dy_t = [H_{11}x_t + H_{12}y_t + (1-\lambda)H_{13}z_t + (1-\lambda)H_{14}k_t]\,dt - z_t\,dB_t - k_t\,d\tilde{N}_t, \\ x_0 = 0, \qquad y_T = 0. \end{cases} \tag{8}$$

Theorem 3.1 *Let condition* (4) *hold. Then the system of all eigenvalues of stochastic Hamiltonian system* (8) *has at least one element* $\lambda > 0$, *which is the smallest eigenvalue. Moreover, the dimension of the eigenfunction subspace corresponding to* λ *is no more than* n.

As for the one-dimensional situation where $\bar{H}$ is taken as

$$\bar{H} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & H_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

(7) is in the following form:

$$\begin{cases} dx_t = [H_{21}x_t + (1-\lambda)H_{22}y_t + H_{23}z_t + H_{24}k_t]\,dt \\ \qquad\quad + [H_{31}x_t + H_{32}y_t + H_{33}z_t]\,dB_t \\ \qquad\quad + [H_{41}x_{t-} + H_{42}y_{t-} + H_{44}k_t]\,d\tilde{N}_t, \\ -dy_t = [H_{11}x_t + H_{12}y_t + H_{13}z_t + H_{14}k_t]\,dt - z_t\,dB_t - k_t\,d\tilde{N}_t, \\ x_0 = 0, \qquad y_T = 0. \end{cases} \tag{9}$$

Then we have the following theorem, a more concrete result than Theorem 3.1.

Theorem 3.2 *For* $n = 1$, *let condition* (4) *hold, and assume* $H_{23} = -H_{33}H_{13}$, $H_{24} = -H_{44}H_{14}$. *Then the system of all eigenvalues of stochastic Hamiltonian system* (9) *is a strictly increasing real number sequence* $\{\lambda_i\}_{i\geq 1}$ *with* λ_i *going to infinity as* $i \to \infty$. *Moreover, the dimension of the eigenfunction subspace corresponding to each* λ_i *is* 1.

4 Dual transformation of stochastic Hamiltonian systems with Poisson process

Inspired by the method given in Peng [11], we introduce the dual transformation of stochastic Hamiltonian systems driven by Poisson process. We shall see that this dual transformation is a powerful tool for solving the eigenvalue problem.

4.1 General case

Suppose that the stochastic Hamiltonian system (2) admits a solution (x_t, y_t, z_t, k_t). Now, we exchange the role of x_t and y_t, i.e., define $(\tilde{x}_t, \tilde{y}_t) = (y_t, x_t)$ to see whether $(\tilde{x}_t, \tilde{y}_t)$ will still satisfy some stochastic Hamiltonian system. In addition, we assume

Assumption 4.1 h and Φ are both $\mathcal{C}^2$ functions. Moreover, for all $(x,y) \in \mathbb{R}^{2n}$, $h(x,y,\cdot,\cdot)$ is concave and $\Phi(\cdot)$ is convex.

Thus, we can give the following Legendre transformation of h and Φ with respect to (z,k) and x, respectively:

$$\begin{aligned}\tilde{h}(\tilde{x},\tilde{y},\tilde{z},\tilde{k}) &= \inf_{(z,k)\in\mathbb{R}^{2n}} \bigl\{ \langle z,\tilde{z}\rangle + \langle k,\tilde{k}\rangle - h(\tilde{y},\tilde{x},z,k)\bigr\} \\ &= \bigl\langle z^*(\tilde{x},\tilde{y},\tilde{z},\tilde{k}),\tilde{z}\bigr\rangle + \bigl\langle k^*(\tilde{x},\tilde{y},\tilde{z},\tilde{k}),\tilde{k}\bigr\rangle - h\bigl(\tilde{y},\tilde{x},z^*(\tilde{x},\tilde{y},\tilde{z},\tilde{k}),k^*(\tilde{x},\tilde{y},\tilde{z},\tilde{k})\bigr), \\ \tilde{\Phi}(\tilde{x}) &= \sup_{x\in\mathbb{R}^n} \bigl\{ \langle x,\tilde{x}\rangle - \Phi(x)\bigr\} = \bigl\langle x^*(\tilde{x}),\tilde{x}\bigr\rangle - \Phi\bigl(x^*(\tilde{x})\bigr),\end{aligned}$$

where $(z^*(\tilde{x},\tilde{y},\tilde{z},\tilde{k}), k^*(\tilde{x},\tilde{y},\tilde{z},\tilde{k}))$ is the unique minimum point for each $(\tilde{x},\tilde{y},\tilde{z},\tilde{k}) \in \mathbb{R}^{4n}$, and $x^*(\tilde{x})$ is the unique maximum point for each $\tilde{x} \in \mathbb{R}^n$. $\tilde{h}$ is called the dual Hamiltonian function of (2). Inversely, we can get

$$\begin{aligned}h(x,y,z,k) &= \inf_{(\tilde{z},\tilde{k})\in\mathbb{R}^{2n}} \bigl\{ \langle z,\tilde{z}\rangle + \langle k,\tilde{k}\rangle - \tilde{h}(y,x,\tilde{z},\tilde{k})\bigr\} \\ &= \bigl\langle z,\tilde{z}^*(x,y,z,k)\bigr\rangle + \bigl\langle k,\tilde{k}^*(x,y,z,k)\bigr\rangle - \tilde{h}\bigl(y,x,\tilde{z}^*(x,y,z,k),\tilde{k}^*(x,y,z,k)\bigr), \\ \Phi(x) &= \sup_{\tilde{x}\in\mathbb{R}^n} \bigl\{ \langle x,\tilde{x}\rangle - \tilde{\Phi}(\tilde{x})\bigr\} = \bigl\langle x,\tilde{x}^*(x)\bigr\rangle - \tilde{\Phi}\bigl(\tilde{x}^*(x)\bigr).\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\tilde{z} = \partial_z h\bigl(\tilde{y},\tilde{x},z(\tilde{x},\tilde{y},\tilde{z},\tilde{k}),k(\tilde{x},\tilde{y},\tilde{z},\tilde{k})\bigr), \quad \tilde{k} = \partial_k h\bigl(\tilde{y},\tilde{x},z(\tilde{x},\tilde{y},\tilde{z},\tilde{k}),k(\tilde{x},\tilde{y},\tilde{z},\tilde{k})\bigr), \\ &\tilde{x} = \partial_x \Phi\bigl(x(\tilde{x})\bigr), \quad (\tilde{x},\tilde{y},\tilde{z},\tilde{k}) \in \mathbb{R}^{4n}, \\ &z = \partial_{\tilde{z}} \tilde{h}\bigl(y,x,\tilde{z}(x,y,z,k),\tilde{k}(x,y,z,k)\bigr), \quad k = \partial_{\tilde{k}} \tilde{h}\bigl(y,x,\tilde{z}(x,y,z,k),\tilde{k}(x,y,z,k)\bigr), \\ &x = \partial_{\tilde{x}} \tilde{\Phi}\bigl(\tilde{x}(x)\bigr), \quad (x,y,z,k) \in \mathbb{R}^{4n}.\end{aligned}$$

Then it can be easily verified that the quadruple $(\tilde{x}_t, \tilde{y}_t, \tilde{z}_t, \tilde{k}_t)$ defined by

$$\tilde{x}_t = y_t, \qquad \tilde{y}_t = x_t, \qquad \tilde{z}_t = \partial_z h(x_{t-},y_{t-},z_t,k_t), \qquad \tilde{k}_t = \partial_k h(x_{t-},y_{t-},z_t,k_t) \tag{10}$$

satisfies the stochastic Hamiltonian system driven by Poisson process:

$$\begin{cases} d\tilde{x}_t = \partial_{\tilde{y}} \tilde{h}(\tilde{x}_t,\tilde{y}_t,\tilde{z}_t,\tilde{k}_t)\,dt + \partial_{\tilde{z}} \tilde{h}(\tilde{x}_t,\tilde{y}_t,\tilde{z}_t,\tilde{k}_t)\,dB_t \\ \qquad\quad + \partial_{\tilde{k}} \tilde{h}(\tilde{x}_{t-},\tilde{y}_{t-},\tilde{z}_t,\tilde{k}_t)\,d\tilde{N}_t, \\ -d\tilde{y}_t = \partial_{\tilde{x}} \tilde{h}(\tilde{x}_t,\tilde{y}_t,\tilde{z}_t,\tilde{k}_t)\,dt - \tilde{z}_t\,dB_t - \tilde{k}_t\,d\tilde{N}_t, \quad 0 \le t \le T, \\ \tilde{x}_0 = y_0, \qquad \tilde{y}_T = \partial_{\tilde{x}} \tilde{\Phi}(\tilde{x}_T). \end{cases} \tag{11}$$

We call (11) the dual stochastic Hamiltonian system of (2). Moreover, it is obvious that the dual stochastic Hamiltonian system of (11) is our original stochastic Hamiltonian system (2).

4.2 Linear case

Now, let us consider a specific linear stochastic Hamiltonian system as follows:

$$\begin{cases} dx_t = [H_{21}x_t + H_{22}y_t + H_{23}z_t + H_{24}k_t]\,dt \\ \qquad\quad + [H_{31}x_t + H_{32}y_t + H_{33}z_t]\,dB_t \\ \qquad\quad + [H_{41}x_{t-} + H_{42}y_{t-} + H_{44}k_t]\,d\tilde{N}_t, \\ -dy_t = [H_{11}x_t + H_{12}y_t + H_{13}z_t + H_{14}k_t]\,dt \\ \qquad\quad - z_t\,dB_t - k_t\,d\tilde{N}_t, \quad 0 \le t \le T, \\ x_0 = x_0, \qquad y_T = Qx_T, \end{cases} \tag{12}$$

where Q and H_{ij} are $n \times n$ matrices such that $Q^T = Q$ and $H_{ji} = H_{ij}^T$, $i,j = 1,2,3,4$. In this situation, Assumption 4.1 can be guaranteed by

$$H_{33} < 0, \qquad H_{44} < 0, \quad \text{and} \quad Q > 0.$$

Thanks to (10), if we define

$$\begin{aligned} &\tilde{x}_t = y_t, \qquad \tilde{y}_t = x_t, \\ &\tilde{z}_t = H_{31}x_{t-} + H_{32}y_{t-} + H_{33}z_t, \\ &\tilde{k}_t = H_{41}x_{t-} + H_{42}y_{t-} + H_{44}k_t, \end{aligned}$$

then the dual stochastic Hamiltonian system of (12) is

$$\begin{cases} d\tilde{x}_t = [\tilde{H}_{21}\tilde{x}_t + \tilde{H}_{22}\tilde{y}_t + \tilde{H}_{23}\tilde{z}_t + \tilde{H}_{24}\tilde{k}_t]\,dt \\ \qquad\quad + [\tilde{H}_{31}\tilde{x}_t + \tilde{H}_{32}\tilde{y}_t + \tilde{H}_{33}\tilde{z}_t]\,dB_t \\ \qquad\quad + [\tilde{H}_{41}\tilde{x}_{t-} + \tilde{H}_{42}\tilde{y}_{t-} + \tilde{H}_{44}\tilde{k}_t]\,d\tilde{N}_t, \\ -d\tilde{y}_t = [\tilde{H}_{11}\tilde{x}_t + \tilde{H}_{12}\tilde{y}_t + \tilde{H}_{13}\tilde{z}_t + \tilde{H}_{14}\tilde{k}_t]\,dt \\ \qquad\quad - \tilde{z}_t\,dB_t - \tilde{k}_t\,d\tilde{N}_t, \quad 0 \le t \le T, \\ \tilde{x}_0 = y_0, \qquad \tilde{y}_T = Q^{-1}\tilde{x}_T, \end{cases} \tag{13}$$

where

$$\tilde{H} = \begin{bmatrix} H_{23}H_{33}^{-1}H_{32} + H_{24}H_{44}^{-1}H_{42} - H_{22} & H_{23}H_{33}^{-1}H_{31} + H_{24}H_{44}^{-1}H_{41} - H_{21} & -H_{23}H_{33}^{-1} & -H_{24}H_{44}^{-1} \\ H_{13}H_{33}^{-1}H_{32} + H_{14}H_{44}^{-1}H_{42} - H_{12} & H_{13}H_{33}^{-1}H_{31} + H_{14}H_{44}^{-1}H_{41} - H_{11} & -H_{13}H_{33}^{-1} & -H_{14}H_{44}^{-1} \\ -H_{33}^{-1}H_{32} & -H_{33}^{-1}H_{31} & H_{33}^{-1} & 0 \\ -H_{44}^{-1}H_{42} & -H_{44}^{-1}H_{41} & 0 & H_{44}^{-1} \end{bmatrix}.$$

4.3 Generalized Riccati equation systems

The Riccati equations are widely applied to investigate the linear-quadratic optimal control problems, e.g., Wonham [12], Bismut [6], Peng [8], Wu and Wang [9] and so on. Inspired by [9], we shall reformulate the Riccati equations in a general form.

Denote by $\mathcal{S}^n$ the space of $n \times n$ symmetric matrices, and denote by $\mathcal{S}^n_+$ the space of nonnegative matrices in $\mathcal{S}^n$. Now we introduce a dynamic system consisting of an $\mathcal{S}^n$-valued ODE and two algebraic equations on some interval $[T_1, T_2] \subseteq [0, T]$:

$$\begin{cases} -\dot{K}(t) = K(t)(H_{21} + H_{22}K(t) + H_{23}M(t) + H_{24}L(t)) \\ \qquad\qquad + H_{11} + H_{12}K(t) + H_{13}M(t) + H_{14}L(t), \\ M(t) = K(t)(H_{31} + H_{32}K(t) + H_{33}M(t)), \\ L(t) = K(t)(H_{41} + H_{42}K(t) + H_{44}L(t)), \\ (K(\cdot), M(\cdot), L(\cdot)) \\ \quad \in \mathcal{C}^1([T_1, T_2]; \mathcal{S}^n) \times \mathcal{L}^\infty([T_1, T_2]; \mathbb{R}^{n\times n}) \times \mathcal{L}^\infty([T_1, T_2]; \mathbb{R}^{n\times n}), \\ K(T_2) = K' \in \mathcal{S}^n. \end{cases} \tag{14}$$

This system is called a generalized Riccati equation system. Note that the two algebraic equations in (14) are equivalent to

$$\begin{aligned} &\bigl[I_n - K(t)H_{33}\bigr]M(t) = K(t)\bigl(H_{31} + H_{32}K(t)\bigr), \\ &\bigl[I_n - K(t)H_{44}\bigr]L(t) = K(t)\bigl(H_{41} + H_{42}K(t)\bigr). \end{aligned}$$

Suppose that (14) admits a unique solution $(K(\cdot), M(\cdot), L(\cdot))$, then we have

$$\det\bigl(I_n - K(t)H_{33}\bigr) \neq 0, \qquad \det\bigl(I_n - K(t)H_{44}\bigr) \neq 0, \quad \forall t \in [T_1, T_2]. \tag{15}$$

Hence, $(I_n - K(\cdot)H_{33})^{-1}$ and $(I_n - K(\cdot)H_{44})^{-1}$ exist and are both uniformly bounded because of the continuity of $\det(I_n - K(\cdot)H_{33})$ and $\det(I_n - K(\cdot)H_{44})$. In this situation, $M(\cdot)$ and $L(\cdot)$ can be represented by $K(\cdot)$:

$$\begin{aligned} M(t) &= \bigl(I_n - K(t)H_{33}\bigr)^{-1}K(t)\bigl(H_{31} + H_{32}K(t)\bigr) = F_0\bigl(K(t)\bigr)\bigl(H_{31} + H_{32}K(t)\bigr), \\ L(t) &= \bigl(I_n - K(t)H_{44}\bigr)^{-1}K(t)\bigl(H_{41} + H_{42}K(t)\bigr) = F_1\bigl(K(t)\bigr)\bigl(H_{41} + H_{42}K(t)\bigr). \end{aligned}$$

Here, $F_0(\cdot)$ and $F_1(\cdot)$ are respectively defined as

$$F_0(K) = (I_n - KH_{33})^{-1}K, \quad \text{and} \quad F_1(K) = (I_n - KH_{44})^{-1}K.$$

Thus, (14) can be rewritten as

$$\begin{aligned} -\dot{K}(t) = {} & K(t)H_{21} + H_{12}K(t) + H_{11} + H_{13}F_0\bigl(K(t)\bigr)H_{31} + H_{14}F_1\bigl(K(t)\bigr)H_{41} \\ & + \bigl[K(t)H_{23}F_0\bigl(K(t)\bigr)H_{31} + H_{13}F_0\bigl(K(t)\bigr)H_{32}K(t)\bigr] \\ & + \bigl[K(t)H_{24}F_1\bigl(K(t)\bigr)H_{41} + H_{14}F_1\bigl(K(t)\bigr)H_{42}K(t)\bigr] \\ & + K(t)\bigl[H_{22} + H_{23}F_0\bigl(K(t)\bigr)H_{32} + H_{24}F_1\bigl(K(t)\bigr)H_{42}\bigr]K(t), \end{aligned} \tag{16}$$

or equivalently,

$$\begin{aligned}-\dot{K}(t) &= K(t)H_{21} + H_{12}K(t) + H_{11} + K(t)H_{22}K(t)\\ &\quad + \bigl(H_{13} + K(t)H_{23}\bigr)F_0\bigl(K(t)\bigr)\bigl(H_{32}K(t) + H_{31}\bigr)\\ &\quad + \bigl(H_{14} + K(t)H_{24}\bigr)F_1\bigl(K(t)\bigr)\bigl(H_{42}K(t) + H_{41}\bigr).\end{aligned} \tag{17}$$

Remark 4.2 Similar to the discussion in Remark 4 and Remark 5 of [10], the following facts hold:

(i) For any given constant $\gamma < 1$ and $K \in D_F^{\gamma} = \{K \in \mathcal{S}^n : K \geq \gamma H_{33}^{-1} \vee \gamma H_{44}^{-1}\}$, $(I_n - KH_{33})^{-1}$ and $(I_n - KH_{44})^{-1}$ exist and are both uniformly bounded in D_F^{γ};

(ii) For any $K \in \mathcal{S}_+^n$, $F_0(K)$ and $F_1(K)$ are both bounded and monotone. More precisely, we have

$$0 \leq F_0(K) \leq -H_{33}^{-1}, \qquad 0 \leq F_1(K) \leq -H_{44}^{-1}, \quad \forall K \geq 0,$$

and

$$F_0(K_1) \geq F_0(K_2), \qquad F_1(K_1) \geq F_1(K_2), \quad \forall K_1 \geq K_2 \geq 0.$$

The following lemma shows the relationship between generalized Riccati equation system (14) and linear stochastic Hamiltonian system (12).

Lemma 4.3 *Suppose that* (14) *admits a solution* $(K(\cdot), M(\cdot), L(\cdot))$ *on some interval* $[T_1, T_2] \subseteq [0, T]$. *Then* (12) *with the boundary condition*

$$x_{T_1} = x_0, \qquad y_{T_2} = K' x_{T_2} \tag{18}$$

admits a solution

$$(x_t, y_t, z_t, k_t) = \bigl(x(t), K(t)x(t), M(t)x(t-), L(t)x(t-)\bigr), \quad t \in [T_1, T_2], \tag{19}$$

where $\{x(t)\}$ *satisfies*

$$\begin{cases} dx(t) = [H_{21} + H_{22}K(t) + H_{23}M(t) + H_{24}L(t)]x(t)\,dt\\ \qquad\qquad + [H_{31} + H_{32}K(t) + H_{33}M(t)]x(t)\,dB_t\\ \qquad\qquad + [H_{41} + H_{42}K(t) + H_{44}L(t)]x(t-)\,d\tilde{N}_t,\\ x(T_1) = x_0. \end{cases} \tag{20}$$

Moreover, if we assume that for $(I_n - KH_{33})$ *and* $(I_n - KH_{44})$, *condition* (15), *or the following weaker condition holds*:

$$\begin{aligned}&\bigl(I_n - K(t)H_{33}\bigr)^T\bigl(I_n - K(t)H_{33}\bigr) \geq c\bigl(H_{13} + K(t)H_{23}\bigr)^T\bigl(H_{13} + K(t)H_{23}\bigr),\\ &\bigl(I_n - K(t)H_{44}\bigr)^T\bigl(I_n - K(t)H_{44}\bigr) \geq c\bigl(H_{14} + K(t)H_{24}\bigr)^T\bigl(H_{14} + K(t)H_{24}\bigr),\end{aligned} \tag{21}$$

where c *is a positive constant, then* (12) *with boundary condition* (18) *admits a unique solution.*

Proof It can be easily verified that (19) is a solution of linear stochastic Hamiltonian system (12) with boundary condition (18) by applying Itô's formula to $K(t)x_t$.

As for the uniqueness, we first consider the case where condition (15) holds. Suppose that (x_t, y_t, z_t, k_t) is another solution of (12) with boundary condition (18) and define $(\bar{y}_t, \bar{z}_t, \bar{k}_t) = (K(t)x_t, M(t)x_{t-}, L(t)x_{t-})$. Applying Itô's formula to $K(t)x_t$, we have

$$\begin{aligned}
-d\bar{y}_t &= -\dot{K}(t)x_t\,dt - K(t)dx_t \\
&= \bigl[K(t)\bigl(H_{21} + H_{22}K(t) + H_{23}M(t) + H_{24}L(t)\bigr) \\
&\quad + H_{11} + H_{12}K(t) + H_{13}M(t) + H_{14}L(t)\bigr]x_t\,dt \\
&\quad - K(t)\bigl[(H_{21}x_t + H_{22}y_t + H_{23}z_t + H_{24}k_t)\,dt \\
&\quad + (H_{31}x_t + H_{32}y_t + H_{33}z_t)\,dB_t \\
&\quad + (H_{41}x_{t-} + H_{42}y_{t-} + H_{44}k_t)\,d\tilde{N}_t\bigr] \\
&= \bigl[K(t)(H_{22}\bar{y}_t + H_{23}\bar{z}_t + H_{24}\bar{k}_t) + H_{11}x_t + H_{12}\bar{y}_t + H_{13}\bar{z}_t + H_{14}\bar{k}_t\bigr]\,dt \\
&\quad - K(t)\bigl[(H_{22}y_t + H_{23}z_t + H_{24}k_t)\,dt + (H_{31}x_t + H_{32}y_t + H_{33}z_t)\,dB_t \\
&\quad + (H_{41}x_{t-} + H_{42}y_{t-} + H_{44}k_t)\,d\tilde{N}_t\bigr].
\end{aligned}$$

Denote $(\hat{y}_t, \hat{z}_t, \hat{k}_t) = (\bar{y}_t - y_t, \bar{z}_t - z_t, \bar{k}_t - k_t)$. Thus we get

$$\begin{aligned}
-d\bar{y}_t &= \bigl[K(t)(H_{22}\hat{y}_t + H_{23}\hat{z}_t + H_{24}\hat{k}_t) + H_{11}x_t + H_{12}\bar{y}_t + H_{13}\bar{z}_t + H_{14}\bar{k}_t\bigr]\,dt \\
&\quad - \bigl[K(t)H_{31}x_t + K(t)(H_{32}y_t + H_{33}z_t)\bigr]\,dB_t \\
&\quad - \bigl[K(t)H_{41}x_{t-} + K(t)(H_{42}y_{t-} + H_{44}k_t)\bigr]\,d\tilde{N}_t.
\end{aligned}$$

It follows from

$$\begin{aligned}
&M(t) = K(t)\bigl(H_{31} + H_{32}K(t) + H_{33}M(t)\bigr), \\
&L(t) = K(t)\bigl(H_{41} + H_{42}K(t) + H_{44}L(t)\bigr)
\end{aligned}$$

that

$$\begin{aligned}
&K(t)H_{31}x_{t-} = \bar{z}_t - K(t)(H_{32}\bar{y}_{t-} + H_{33}\bar{z}_t), \\
&K(t)H_{41}x_{t-} = \bar{k}_t - K(t)(H_{42}\bar{y}_{t-} + H_{44}\bar{k}_t).
\end{aligned}$$

So we can obtain

$$\begin{cases}
-d\hat{y}_t = [(H_{12} + K(t)H_{22})\hat{y}_t + (H_{13} + K(t)H_{23})\hat{z}_t + (H_{14} + K(t)H_{24})\hat{k}_t]\,dt \\
\qquad\quad - [-K(t)H_{32}\hat{y}_t + (I_n - K(t)H_{33})\hat{z}_t]\,dB_t \\
\qquad\quad - [-K(t)H_{42}\hat{y}_{t-} + (I_n - K(t)H_{44})\hat{k}_t]\,d\tilde{N}_t, \\
\hat{y}_T = 0.
\end{cases}$$

Since condition (15) leads to the uniform boundedness of $(I_n - K(\cdot)H_{33})^{-1}$ and $(I_n - K(\cdot)H_{44})^{-1}$, the above equation can be rewritten as

$$\begin{cases} -d\hat{y}_t = [(H_{12} + K(t)H_{22})\hat{y}_t \\ \qquad + (H_{13} + K(t)H_{23})(I_n - K(t)H_{33})^{-1}(z'_t + K(t)H_{32}\hat{y}_t) \\ \qquad + (H_{14} + K(t)H_{24})(I_n - K(t)H_{44})^{-1}(k'_t + K(t)H_{42}\hat{y}_t)]\,dt \\ \qquad - z'_t\,dB_t - k'_t\,d\tilde{N}_t, \\ \hat{y}_T = 0, \end{cases} \tag{22}$$

where we define

$$z'_t = -K(t)H_{32}\hat{y}_{t-} + \big(I_n - K(t)H_{33}\big)\hat{z}_t,$$
$$k'_t = -K(t)H_{42}\hat{y}_{t-} + \big(I_n - K(t)H_{44}\big)\hat{k}_t.$$

According to Theorem 2.1 of [4], (22) admits a unique solution $(\hat{y}_t, z'_t, k'_t) \equiv (0,0,0)$. Hence $(\hat{y}_t, \hat{z}_t, \hat{k}_t) \equiv (0,0,0)$, which implies that

$$y_t = K(t)x_t, \qquad z_t = M(t)x_{t-}, \qquad k_t = L(t)x_{t-}.$$

On the other hand, under the weaker condition (21) instead of (15), again by the similar arguments for proving Theorem 2.1 of [4], we can still show that $(\hat{y}_t, z'_t, k'_t) \equiv (0,0,0)$. Thus, it follows from the forward SDE in (12) that $\{x_t\}$ is the solution of (20). So we have $(x_t, y_t, z_t, k_t) \equiv (x(t), K(t)x(t), M(t)x(t-), L(t)x(t-))$. The proof is completed. □

Remark 4.4 Similarly, the dual linear stochastic Hamiltonian system (13) is associated with the following generalized Riccati equation system:

$$\begin{cases} -\frac{d}{dt}\tilde{K}(t) = \tilde{K}(t)(\tilde{H}_{21} + \tilde{H}_{22}\tilde{K}(t) + \tilde{H}_{23}\tilde{M}(t) + \tilde{H}_{24}\tilde{L}(t)) \\ \qquad + \tilde{H}_{11} + \tilde{H}_{12}\tilde{K}(t) + \tilde{H}_{13}\tilde{M}(t) + \tilde{H}_{14}\tilde{L}(t), \\ \tilde{M}(t) = \tilde{K}(t)(\tilde{H}_{31} + \tilde{H}_{32}\tilde{K}(t) + \tilde{H}_{33}\tilde{M}(t)), \\ \tilde{L}(t) = \tilde{K}(t)(\tilde{H}_{41} + \tilde{H}_{42}\tilde{K}(t) + \tilde{H}_{44}\tilde{L}(t)), \\ (\tilde{K}(\cdot), \tilde{M}(\cdot), \tilde{L}(\cdot)) \\ \quad \in \mathcal{C}^1([T_1,T_2];\mathcal{S}^n) \times \mathcal{L}^\infty([T_1,T_2];\mathbb{R}^{n\times n}) \times \mathcal{L}^\infty([T_1,T_2];\mathbb{R}^{n\times n}). \end{cases} \tag{23}$$

If $\det(I_n - \tilde{K}(\cdot)\tilde{H}_{33}) \neq 0$ and $\det(I_n - \tilde{K}(\cdot)\tilde{H}_{44}) \neq 0$, then it can be rewritten as

$$\begin{aligned} -\frac{d}{dt}\tilde{K}(t) &= \tilde{K}(t)\tilde{H}_{21} + \tilde{H}_{12}\tilde{K}(t) + \tilde{H}_{11} + \tilde{H}_{13}\tilde{F}_0\big(\tilde{K}(t)\big)\tilde{H}_{31} + \tilde{H}_{14}\tilde{F}_1\big(\tilde{K}(t)\big)\tilde{H}_{41} \\ &\quad + \big[\tilde{K}(t)\tilde{H}_{23}\tilde{F}_0\big(\tilde{K}(t)\big)\tilde{H}_{31} + \tilde{H}_{13}\tilde{F}_0\big(\tilde{K}(t)\big)\tilde{H}_{32}\tilde{K}(t)\big] \\ &\quad + \big[\tilde{K}(t)\tilde{H}_{24}\tilde{F}_1\big(\tilde{K}(t)\big)\tilde{H}_{41} + \tilde{H}_{14}\tilde{F}_1\big(\tilde{K}(t)\big)\tilde{H}_{42}\tilde{K}(t)\big] \\ &\quad + \tilde{K}(t)\big[\tilde{H}_{22} + \tilde{H}_{23}\tilde{F}_0\big(\tilde{K}(t)\big)\tilde{H}_{32} + \tilde{H}_{24}\tilde{F}_1\big(\tilde{K}(t)\big)\tilde{H}_{42}\big]\tilde{K}(t), \end{aligned} \tag{24}$$

where

$$\tilde{F}_0(K) = (I_n - K\tilde{H}_{33})^{-1}K, \quad \text{and} \quad \tilde{F}_1(K) = (I_n - K\tilde{H}_{44})^{-1}K.$$

At the end of this section, we present a kind of comparison theorem for Riccati equations in the form of (17), which will be used repeatedly later. Consider the following $\mathcal{S}^n$-valued ODEs: for $i = 1, 2$,

$$\begin{cases} -\dot{K}_i(t) = K_i(t)A(t) + A^T(t)K_i(t) \\ \qquad\qquad + C^T(t)K_i(t)C(t) + R_i(t) + K_i(t)N_i(t)K_i(t) \\ \qquad\qquad + [B(t) + K_i(t)D(t)]F_i(K_i(t))[B(t) + K_i(t)D(t)]^T \\ \qquad\qquad + [E(t) + K_i(t)G(t)]\bar{F}_i(K_i(t))[E(t) + K_i(t)G(t)]^T, \\ K_i(T) = Q_i, \end{cases} \tag{25}$$

where the mappings $A(\cdot), B(\cdot), C(\cdot), D(\cdot), E(\cdot), G(\cdot) : [0, T] \mapsto \mathbb{R}^{n\times n}$, $R_i(\cdot), N_i(\cdot) : [0, T] \mapsto \mathcal{S}^n$ are all continuous in $[0, T]$, and $F_i(\cdot), \bar{F}_i(\cdot) : \mathcal{S}^n \mapsto \mathcal{S}^n$ are both locally Lipschitz.

Lemma 4.5 *For* (25), *suppose that*

$$\begin{aligned} &Q_1 \geq Q_2, \qquad R_1(t) \geq R_2(t), \qquad N_1(t) \geq N_2(t), \quad \forall t \in [0, T], \\ &F_1(K) \geq F_2(K), \qquad \bar{F}_1(K) \geq \bar{F}_2(K), \quad \forall K \in \mathcal{S}^n, \\ &F_1(K) \geq F_1(K'), \qquad \bar{F}_1(K) \geq \bar{F}_1(K'), \quad K \geq K'. \end{aligned}$$

Then we have

$$K_1(t) \geq K_2(t), \quad \forall t \in [0, T].$$

The proof of Lemma 4.5 is very similar to that of Lemma 8.2 in [10]. So we just omit it.

5 Proof of main results

We shall complete the proofs of Theorems 3.1 and 3.2 in this section. It will be seen that the features of eigenvalues and corresponding eigenfunctions are dominated by the blow-up times of solutions to related Riccati equation systems of the linear stochastic Hamiltonian systems (8) and (9).

5.1 Proof of Theorem 3.1

For notational simplicity, let $\rho = 1 - \lambda$. Then, for the linear Hamiltonian system (8), the corresponding Riccati equation system is

$$\begin{cases} -\dot{K}(t) = K(t)(H_{21} + \rho H_{22}K(t) + \rho H_{23}M(t) + \rho H_{24}L(t)) \\ \qquad\qquad + H_{11} + H_{12}K(t) + \rho H_{13}M(t) + \rho H_{14}L(t), \\ M(t) = K(t)(\rho H_{31} + \rho H_{32}K(t) + H_{33}M(t)), \\ L(t) = K(t)(\rho H_{41} + \rho H_{42}K(t) + H_{44}L(t)), \\ (K(\cdot), M(\cdot), L(\cdot)) \in \mathcal{C}^1([0, T]; \mathcal{S}^n) \times \mathcal{L}^\infty([0, T]; \mathbb{R}^{n\times n}) \times \mathcal{L}^\infty([0, T]; \mathbb{R}^{n\times n}), \\ K(T) = 0. \end{cases} \tag{26}$$

If we consider the solutions of (26) among $K(\cdot) \geq \gamma H_{33}^{-1} \vee \gamma H_{44}^{-1}$ for some given $\gamma \in (0, 1)$, it follows from Remark 4.2 that $(I_n - K(\cdot)H_{33})^{-1}, (I_n - K(\cdot)H_{44})^{-1}$ and $F_0(K(\cdot)), F_1(K(\cdot))$ are all well defined. So we can rewrite (26) as

$$\begin{cases} -\dot{K}(t) = K(t)H_{21} + H_{12}K(t) + H_{11} \\ \qquad + \rho^2 H_{13}F_0(K(t))H_{31} + \rho^2 H_{14}F_1(K(t))H_{41} \\ \qquad + \rho^2[K(t)H_{23}F_0(K(t))H_{31} + H_{13}F_0(K(t))H_{32}K(t)] \\ \qquad + \rho^2[K(t)H_{24}F_1(K(t))H_{41} + H_{14}F_1(K(t))H_{42}K(t)] \\ \qquad + K(t)[\rho H_{22} + \rho^2 H_{23}F_0(K(t))H_{32} + \rho^2 H_{24}F_1(K(t))H_{42}]K(t), \\ K(T) = 0, \end{cases} \tag{27}$$

or equivalently,

$$\begin{cases} -\dot{K}(t) = K(t)H_{21} + H_{12}K(t) + H_{11} + \rho K(t)H_{22}K(t) \\ \qquad + \rho^2(H_{13} + K(t)H_{23})F_0(K(t))(H_{32}K(t) + H_{31}) \\ \qquad + \rho^2(H_{14} + K(t)H_{24})F_1(K(t))(H_{42}K(t) + H_{41}), \\ K(T) = 0. \end{cases} \tag{28}$$

Since $F_0(K)$ and $F_1(K)$ are both analytic, by the classic theory of ODEs, (28) admits a unique solution $K(t) = K(t;\rho)$ on some sufficiently small interval $(t_\rho, T]$. It follows from Lemma 4.5 that $K(t;\rho) \geq 0$, and thus $F_0(K(\cdot))$ and $F_1(K(\cdot))$ are always well defined. Here t_ρ is the so-called 'blow-up time' of Riccati equation (28), and its properties are shown in the following lemma.

Lemma 5.1 *For Riccati equation* (28), *when* $\rho \in [0,1]$, *there is no explosion occurring, i.e.*, $t_\rho = -\infty$. *When* $\rho \in (-\infty, 0)$, *the blow-up time* t_ρ *is finite*: $t_\rho \in (-\infty, T)$. *Moreover*, t_ρ *is continuous and strictly decreasing with respect to* ρ. *We also have*

$$\lim_{\rho\to-\infty} t_\rho = T, \quad \textit{and} \quad \lim_{\rho\to 0} t_\rho = -\infty.$$

Proof For $\rho \in [0,1]$, it is sufficient to verify that the quadratic term of (27) is nonpositive. That is to say,

$$K\big[\rho H_{22} + \rho^2 H_{23}F_0(K)H_{32} + \rho^2 H_{24}F_1(K)H_{42}\big]K \leq 0.$$

This can be obtained immediately from $H_{22} - H_{23}H_{33}^{-1}H_{32} + H_{24}H_{44}^{-1}H_{42} < 0$ and $F_0(K) \leq -H_{33}^{-1}$, $F_1(K) \leq -H_{44}^{-1}$ for any $K \geq 0$.

As for the case where $\rho \in (-\infty, 0)$, thanks to Lemma 4.5, we can prove all conclusions by the very similar method introduced in Lemmas 5.1 and 5.2 of [10]. So we just omit it. □

With the results above in hand, now we can give the proof of Theorem 3.1 as follows.

Proof of Theorem 3.1 According to Lemma 5.1, there exists a unique $\rho^1 < 0$ such that the blow-up time of the corresponding Riccati equation (28) is $t_{\rho^1} = 0$. Then, by the very similar arguments for the proof of Theorem 3.1 in [10], we can prove that $\lambda^1 = 1 - \rho^1$ is the smallest eigenvalue of linear stochastic Hamiltonian system (8), and the dimension of the eigenfunction subspace corresponding to λ_1 is no more than n. This completes the proof. □

5.2 Proof of Theorem 3.2

For the one-dimensional case, since $H_{23} = -H_{33}H_{13}$ and $H_{24} = -H_{44}H_{14}$, the Riccati equation corresponding to linear stochastic Hamiltonian system (9) is

$$-\dot{K}(t) = \left(2H_{21} + H_{13}^2 + H_{14}^2\right)K(t) + H_{11} + \left(\rho H_{22} - H_{33}H_{13}^2 - H_{44}H_{14}^2\right)K^2(t), \tag{29}$$

and the related dual Riccati equation is

$$-\frac{d}{dt}\tilde{K}(t) = -\left(2H_{21} + H_{13}^2 + H_{14}^2\right)\tilde{K}(t) - \left(\rho H_{22} - H_{33}H_{13}^2 - H_{44}H_{14}^2\right) - H_{11}\tilde{K}^2(t). \tag{30}$$

It can be seen from the quadratic term of (29) that the critical point for blow-up time t_ρ of (29) with the terminal condition $K(T) = 0$ is $\rho_0 = H_{22}^{-1}H_{33}H_{13}^2 + H_{22}^{-1}H_{44}H_{14}^2 > 0$. Analogously to Lemma 5.1, we can obtain the properties of t_ρ as follows.

Lemma 5.2 *For Riccati equation* (29), *when* $\rho \in (-\infty, \rho_0)$, *the blow-up time* t_ρ *is finite*: $t_\rho \in (-\infty, T)$. *Moreover*, t_ρ *is continuous and strictly decreasing with respect to* ρ. *We also have*

$$\lim_{\rho\to-\infty} t_\rho = T \quad \textit{and} \quad \lim_{\rho\to\rho_0} t_\rho = -\infty.$$

This lemma can be proved by the same arguments as Lemma 5.1. So we just omit it. Very similarly, we can get the following properties of blow-up time $\tilde{t}_\rho$ of (30) with the terminal condition $\tilde{K}(T) = 0$.

Lemma 5.3 *When* $\rho \in (-\infty, \rho_0)$, *the blow-up time* $\tilde{t}_\rho$ *is finite*: $\tilde{t}_\rho \in (-\infty, T)$. *Moreover*, $\tilde{t}_\rho$ *is continuous and strictly decreasing with respect to* ρ. *We also have*

$$\lim_{\rho\to-\infty} \tilde{t}_\rho = T \quad \textit{and} \quad \lim_{\rho\to\rho_0} \tilde{t}_\rho = -\infty.$$

Thus, just by the same arguments as the proof of Theorem 3.2 in [10], we can find a sequence of eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ and construct corresponding eigenfunctions, of which the dimension with respect to each λ_i is 1. So we omit the details for the proof of Theorem 3.2.

6 Statistic periodicity

It follows from the proof of Theorem 3.2 that the eigenfunctions of linear stochastic Hamiltonian system (9) own the 'statistic periodicity' property. Now, we observe this property for stochastic Hamiltonian systems from another viewpoint.

For any $\rho < \rho_0$, according to Lemmas 5.2 and 5.3, the Riccati equations (29) and (30) with terminal conditions $K(T) = 0$ and $\tilde{K}(T) = 0$ admit unique finite blow-up times t_ρ and $\tilde{t}_\rho$, respectively. Again, by the proof of Theorem 3.2, there exists a quadruple $(x_t, y_t, z_t, k_t) \in$

$\mathcal{M}^2(\mathbb{R}) \times \mathcal{M}^2(\mathbb{R}) \times F_N^2(\mathbb{R}) \times F_N^2(\mathbb{R})$ satisfying

$$\begin{aligned} dx_t &= (H_{21}x_t + \rho H_{22}y_t + H_{23}z_t + H_{24}k_t)\,dt \\ &\quad + (H_{31}x_t + H_{32}y_t + H_{33}z_t)\,dB_t \\ &\quad + (H_{41}x_{t-} + H_{42}y_{t-} + H_{44}k_t)\,d\tilde{N}_t, \\ -dy_t &= (H_{11}x_t + H_{12}y_t + H_{13}z_t + H_{14}k_t)\,dt - z_t\,dB_t - k_t\,d\tilde{N}_t \end{aligned} \tag{31}$$

such that

$$x_{t_\rho^{2i}} = 0, \qquad y_{t_\rho^{2i+1}} = 0, \quad i = 0,1,2,\dots.$$

Here,

$$\begin{aligned} t_\rho^{2i} &= i(T - t_\rho) + i(T - \tilde{t}_\rho), \\ t_\rho^{2i+1} &= (i+1)(T - t_\rho) + i(T - \tilde{t}_\rho). \end{aligned} \tag{32}$$

So we have the following.

Proposition 6.1 *For $n = 1$, let* (4) *hold, and assume $H_{23} = -H_{33}H_{13}$, $H_{24} = -H_{44}H_{14}$. Then, for any $\lambda > 1$, there exists a family of stochastic Hamiltonian systems whose dynamics are in the form of* (31) *with the boundary condition*

$$x_0 = 0, \qquad y_{t_\rho^{2i+1}} = 0, \quad i = 0,1,2,\dots,$$

such that they take λ as one of their eigenvalues. Moreover, the eigenfunctions corresponding to λ have the 'statistic periodicity' property with t_ρ^4-period.

In order to demonstrate Proposition 6.1 vividly, we consider a specific numerical example. Suppose $T = 1$ and

$$H = \begin{bmatrix} 5 & 2 & 1 & 0 \\ 2 & -5 & 3 & 0 \\ 1 & 3 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \qquad \bar{H} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, for $\lambda = 3$, the related Riccati equation and the dual Riccati equation are

$$\begin{cases} -\dot{K}(t) = 5K(t) + 5 + 13K^2(t), \\ K(1) = 0, \end{cases} \tag{33}$$

and

$$\begin{cases} -\frac{d}{dt}\tilde{K}(t) = -5\tilde{K}(t) - 13 - 5\tilde{K}^2(t), \\ K(1) = 0. \end{cases} \tag{34}$$

Figure 1 The solution of (33).

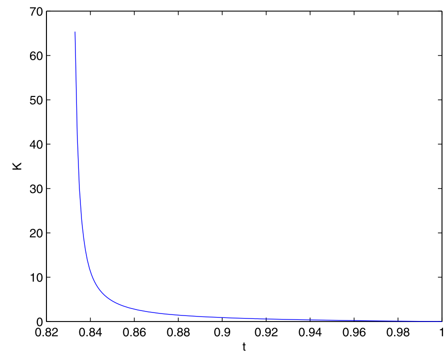
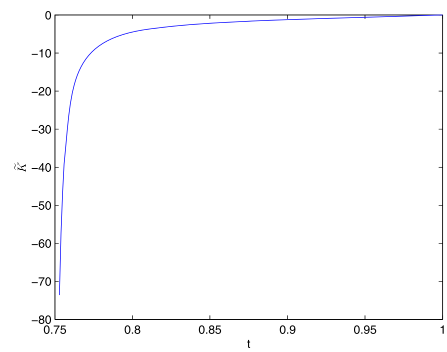


Figure 2 The solution of (34).



Since it is very difficult to obtain analytic solutions of (33) and (34), we give numerical solutions of them by Figure 1 and Figure 2. It can be seen from them that the solutions of (33) and (34) explode at blow-up times t_{-2} and $\tilde{t}_{-2}$, respectively. Moreover, according to the numerical computation, we have

$$t_{-2} \approx 0.832, \qquad \tilde{t}_{-2} \approx 0.752.$$

Thus, by Proposition 6.1, there exists a family of stochastic Hamiltonian systems whose eigenvalue systems contain $\lambda = 3$. More specifically, these stochastic Hamiltonian systems are

$$\begin{cases} dx_t = (2x_t - 5(1-\lambda)y_t + 3z_t)\,dt + (x_t + 3y_t - 3z_t)\,dB_t - 3k_t\,d\tilde{N}_t, \\ -dy_t = (5x_t + 2y_t + z_t)\,dt - z_t\,dB_t - k_t\,d\tilde{N}_t, \\ x_0 = 0, \qquad y_{T_{2i+1}} = 0, \end{cases} \tag{35}$$

where T_{2i+1} is approximately equal to $(1-0.832)(i+1) + (1-0.752)i = 0.416i + 0.168$, $i = 0, 1, \ldots$. By Lemma 4.3 and solutions to (33) and (34), we can construct the corresponding eigenfunctions explicitly on $[0, T_{2i+1}]$. For $i = 2$, Figures 3 and 4 show one approximate path of $\{x_t\}$ and $\{y_t\}$, respectively.

We can also see from Figure 3 that x_t reaches zero only at $t_{-1}^0 = 0$, $t_{-1}^2 = 0.416$, $t_{-1}^4 = 0.832$. It keeps positive on $(0, 0.416)$ and negative on $(0.416, 0.832)$. Similarly, it can be seen from Figure 4 that y_t reaches zero only at $t_{-1}^1 = 0.168$, $t_{-1}^3 = 0.584$, $t_{-1}^5 = 1$. It keeps positive on

Figure 3 $\{x_t\}$ **when** $\lambda = 3$.

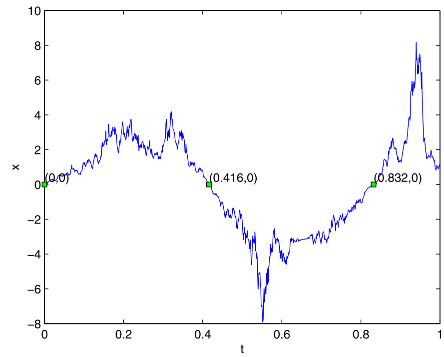


Figure 4 $\{y_t\}$ **when** $\lambda = 3$.

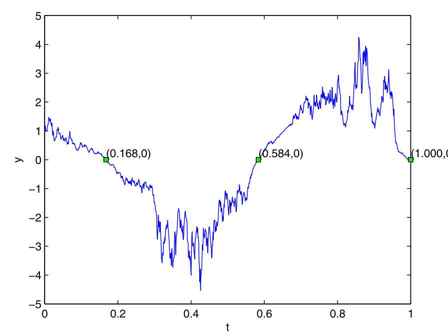
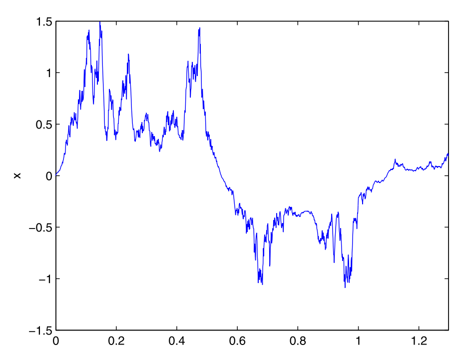


Figure 5 $\{x_t\}$ **when** $\lambda = 2$.



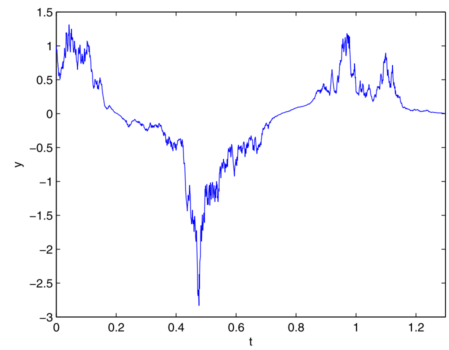
$(0, 0.168) \cup (0.584, 1)$ and negative on $(0.168, 0.584)$. Moreover, we have

$$x_{0.832} = 0, \qquad y_{0.832} > 0.$$

It means that (x_t, y_t) returns to the situation where $x_0 = 0$ and $y_0 > 0$. That is to say, a periodicity is complete on $[0, 0.832]$. This just verifies the theoretical results given in Proposition 6.1.

Let us consider another $\lambda = 2$. By the same method introduced above, we obtain the blow-up time t_{-1} and $\tilde{t}_{-1}$ of the related Riccati equation and the dual Riccati equation are 0.796 and 0.657, respectively. Thus, the stochastic Hamiltonian systems whose eigenvalue systems contain $\lambda = 2$ are in the form of (35), where T_{2i+1} is approximately equal to $(1 - 0.796)(i + 1) + (1 - 0.657)i = 0.547i + 0.204$, $i = 0, 1, \ldots$. For $i = 2$, Figures 5 and 6 show one

Figure 6 $\{y_t\}$ **when** $\lambda = 2$.



approximate path of $\{x_t\}$ and $\{y_t\}$, respectively. Moreover, (x_t, y_t) completes a periodicity on $[0, 1.094]$.

Remark 6.2

(i) Comparing Figures 5 and 6 with Figures 3 and 4, we can see that the period of eigenfunctions corresponding to $\lambda = 3$ is shorter than that corresponding to $\lambda = 2$. In fact, it follows from Lemma 4.5 that the blow-up times t_ρ and $\tilde{t}_\rho$ of (33) and (34) will rise when λ becomes larger. Thus, the period of eigenfunctions will decrease indeed as λ increases.

(ii) In the numerical example above, the eigenfunctions $\{x_t\}$ and $\{y_t\}$ are both continuous since H_{14} and H_{24} are assumed to be 0 and thus $k_t \equiv 0$ for $t \in [0, T_5]$. We demonstrate this continuous case for simplicity and convenience to see the statistic periodicity of eigenfunctions visually. The general situation can be dealt with by the same method, and the corresponding eigenfunctions are with jumps. So we omit the details.

7 Conclusions

To our best knowledge, it is the first time to consider the eigenvalue problem for stochastic Hamiltonian systems driven by Poisson process with boundary conditions. Under certain conditions, we obtain the existence of eigenvalues and corresponding eigenfunctions by means of the dual transformation and generalized Riccati equation systems. From another viewpoint, for any real number $\lambda > 1$, we can establish a family of linear stochastic Hamiltonian systems whose eigenvalue systems contain λ and give the corresponding eigenfunctions explicitly. Moreover, a specific numerical example is studied to illustrate our theoretical results above and show the 'statistic periodicity' vividly for the eigenfunctions of stochastic Hamiltonian systems. Besides, the main results of this paper can help us to construct some examples of multi-solutions for FBSDEP.

On the other hand, as is shown in [10], our problem can also be formulated as an eigenvalue problem for a bounded and self-adjoint operator in a Hilbert space, and then investigated in a standard way by the theory of functional analysis. We leave the details to interested readers.

Acknowledgements

The second author is supported by the Natural Science Foundation of China (61573217, 11125102, 11221061 and 61174092), the National High-Level personnel of special support program and the Chang Jiang Scholar Program of Chinese Education Ministry.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript, read and approved the final manuscript.

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Received: 13 June 2017 Accepted: 23 October 2017 Published online: 09 November 2017

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