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Existence and regularity of solutions to a quasilinear elliptic problem involving variable sources

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Abstract

The authors of this paper prove the existence and regularity results for the homogeneous Dirichlet boundary value problem to the equation $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \frac{f(x)}{u^{\alpha(x)}}$ with $f \in L^m(\Omega)$ $(m \geq 1)$ and $\alpha(x) > 0$. Due to the nonlinearity of a p-Laplace operator and the anisotropic variable exponent $\alpha(x)$, some classical methods may not directly be applied to our problem. In this paper, we construct a suitable test function and apply the Leray-Schauder fixed point theorem to prove the existence of positive solutions with necessary a priori estimate and compact argument. Furthermore, we also discuss the relationship among the regularity of solutions, the summability of f and the value of $\alpha(x)$.

Keywords: quasilinear elliptic problem; nonlinear singular term; existence; regularity; variable exponent

1 Introduction

In this paper, we study the existence of solutions for the following quasilinear elliptic problem with nonlinear singular terms and variable exponent:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \frac{f(x)}{u^{\alpha(x)}}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N $(N \geq p)$ with smooth boundary $\partial\Omega$, $p > 2$, $\alpha(x)$ is a continuous function on $\overline{\Omega}$, $\alpha(x) > 0$, $\alpha^+ = \sup_{x\in\overline{\Omega}} \alpha(x)$, $\alpha^- = \inf_{x\in\overline{\Omega}} \alpha(x)$, f is a nonnegative function belonging to the Lebesgue space $L^m(\Omega)$ for some suitable $m \geq 1$.

Problem (1.1) has been widely applied in many areas such as the contexts of chemical heterogeneous catalysts, non-Newtonian fluids and also the theory of heat conduction in electrically conducting materials, see [1–4] for detailed discussion.

For constant-exponent cases, Lazer and Mckenna in [5] discussed the case when $p = 2$ and f is a positive regular function in $\overline{\Omega}$. They proved that the solution was in $H_0^1(\Omega)$ if and only if $\alpha < 3$, while it was not in $C^1(\overline{\Omega})$ if $\alpha > 1$.

Lair and Shaker in [6] improved the results of [5]. More specially speaking, they proved that this problem with $0 < \alpha < 1$ has a unique weak positive solution in $H_0^1(\Omega)$ if $f(x)$ is a nonnegative nontrivial function in $L^2(\Omega)$.

In 2004, the results of Lair and Shaker were generalized by Zhang and Cheng (see [7]) to the following problem:

$$\begin{cases} -\Delta u = f(x)g(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $g(s)$ is singular near $s = 0$. They proved the existence and uniqueness of classical solutions under the assumption that $f(x) \in C^\alpha(\Omega)$.

Recently, Boccardo and Orsina in [8] studied the existence, regularity and nonexistence of solutions for the following problem:

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = \frac{f(x)}{u^\alpha}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

They discussed how the summability of f and the value of α affected the existence and regularities of solutions to the above problems. For other results of the related problems, the interested readers may refer to [9, 10] and the references therein. Problem (1.3) was discussed and extended to the more general problem of which the right-hand side is $f(x)/u^{\alpha(x)}$ in [11]. For more related questions, refer to [12–14].

In this paper, we generalize the results in [11] to the case when the left-hand side is a p-Laplace operator. Due to the nonlinearity of a p-Laplace operator and the anisotropic variable exponent $\alpha(x)$, some classical methods may not directly be applied to our problem. We apply the method of regularization and the Schauder fixed point theorem, construct a suitable test function as well as a necessary compactness argument to overcome the difficulties arising from a variable exponent and a nonlinear differential operator and give an almost complete classification of coefficient m and variable exponent $\alpha(x)$, then we prove the existence and regularity of solutions.

2 Preliminaries

Firstly, we give the definition of weak solutions to problem (1.1).

Definition 2.1 A function $u \in W_0^{1,p}(\Omega)$ is called a weak solution of problem (1.1) if the following identity holds:

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi \,\mathrm{dx} = \int_\Omega \frac{f}{u^{\alpha(x)}}\varphi \,\mathrm{dx}, \quad \forall\varphi \in C_0^\infty(\Omega). \tag{2.1}$$

In order to prove our results, we will consider the following approximation problem:

$$\begin{cases} -\operatorname{div}(|\nabla u_n|^{p-2}\nabla u_n) = \frac{f_n}{(u_n+\frac{1}{n})^{\alpha(x)}}, & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega, \end{cases} \tag{2.2}$$

where $f_n(x) = \min\{f(x), n\}$, $n \in N$.

Lemma 2.1 *Problem* (2.2) *has a nonnegative solution* u_n *in* $W_0^{1,p}(\Omega) \bigcap L^\infty(\Omega)$.

Proof Let $n \in N$ be fixed, and ω be a function in $L^p(\Omega)$. It is not difficult to prove that the following problem has a unique solution $v \in W_0^{1,p}(\Omega) \bigcap L^\infty(\Omega)$ (see [15, 16]):

$$\begin{cases} -\operatorname{div}(|\nabla v|^{p-2}\nabla v) = \frac{f_n}{(|\omega|+\frac{1}{n})^{\alpha(x)}}, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{2.3}$$

So, for any $\omega \in L^p(\Omega)$, we define the mapping $\Gamma : L^p(\Omega) \to L^p(\Omega)$ as $\Gamma(\omega) = v$. Taking v as a test function for (2.3), we have

$$\int_\Omega |\nabla v|^p \,\mathrm{dx} = \int_\Omega \frac{f_n}{(|\omega| + \frac{1}{n})^{\alpha(x)}} v \,\mathrm{dx} \le \int_\Omega \frac{n}{(\frac{1}{n})^{\alpha^+}} v \,\mathrm{dx} \le n^{\alpha^+ + 1} \int_\Omega |v| \,\mathrm{dx}.$$

By the Poincaré inequality (on the left-hand side) and the Sobolev embedding theorem on the right-hand side ($W^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$), we get that

$$\|v\|_{W^{1,p}}^p \le Cn^{\alpha^+ + 1} \|v\|_{W^{1,p}},$$

this implies that

$$\|v\|_{W^{1,p}} \le Cn^{\frac{\alpha^+ + 1}{p-1}}.$$

Since the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, we obtain that Γ is a compact operator. Moreover, if $u = \lambda\Gamma u$ for some $0 < \lambda \le 1$, then $\Gamma u = \frac{u}{\lambda}$ and hence $\|u\|_{L^p(\Omega)} \le \|u\|_{W^{1,p}(\Omega)} \le C$ for a constant C independent of λ. Then, by Schauder's fixed point theorem, we know that there exists $u_n \in W_0^{1,p}(\Omega)$ such that $u_n = \Gamma(u_n)$, i.e., problem (2.2) has a solution. Since $\frac{f_n}{(u_n+\frac{1}{n})^{\alpha(x)}} \ge 0$, the maximum principle in [17, 18] shows that $u_n \ge 0$, $u_n \in L^\infty(\Omega)$. □

Lemma 2.2 *The sequence* $\{u_n\}$ *is increasing with respect to* n, $u_n > 0$ *in* Ω, *and for every* $\Omega' \subset\subset \Omega$, *there exists* $C_{\Omega'} > 0$ (*independent of* n) *such that*

$$u_n(x) \ge C_{\Omega'} > 0 \quad \textit{for every } x \in \Omega', \textit{for every } n \in N. \tag{2.4}$$

Proof Due to $0 \le f_n \le f_{n+1}$ and $\alpha(x) > 0$, we have that

$$\begin{aligned} -\operatorname{div}\big(|\nabla u_n|^{p-2}\nabla u_n\big) &= \frac{f_n}{(u_n + \frac{1}{n})^{\alpha(x)}} \le \frac{f_{n+1}}{(u_n + \frac{1}{n+1})^{\alpha(x)}}, \\ -\operatorname{div}\big(|\nabla u_{n+1}|^{p-2}\nabla u_{n+1}\big) &= \frac{f_{n+1}}{(u_{n+1} + \frac{1}{n+1})^{\alpha(x)}}, \end{aligned}$$

so that

$$-\operatorname{div}\big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_{n+1}|^{p-2} u_{n+1}\big) = f_{n+1} \frac{(u_{n+1} + \frac{1}{n+1})^{\alpha(x)} - (u_n + \frac{1}{n+1})^{\alpha(x)}}{(u_n + \frac{1}{n+1})^{\alpha(x)} (u_{n+1} + \frac{1}{n+1})^{\alpha(x)}}.$$

Choosing $(u_n - u_{n+1})_+ = \max\{u_n - u_{n+1}, 0\}$ as a test function and observing that

$$\big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_{n+1}|^{p-2}\nabla u_{n+1}\big)\nabla(u_n - u_{n+1})_+ \ge 0,$$

$$\left(\left(u_{n+1}+\frac{1}{n+1}\right)^{\alpha(x)}-\left(u_n+\frac{1}{n+1}\right)^{\alpha(x)}\right)(u_n-u_{n+1})_+\le 0,$$

we get that

$$0\le\int_\Omega\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_{n+1}|^{p-2}\nabla u_{n+1}\right)\nabla(u_n-u_{n+1})_+\,\mathrm{dx}\le 0,$$

which implies that $(u_n-u_{n+1})_+=0$ a.e. in Ω, that is, $u_n\le u_{n+1}$ for every $n\in N$. Since the sequence $\{u_n\}$ is increasing with respect to n, we only need to prove that (2.4) holds for u_1. Using Lemma 2.1, we know that $u_1\in L^\infty(\Omega)$, i.e., there exists a constant C (depending only on Ω and N) such that

$$\|u_1\|_{L^\infty(\Omega)}\le C\|f_1\|_{L^\infty(\Omega)}\le C,$$

then

$$-\operatorname{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right)=\frac{f_1}{(u_1+1)^{\alpha(x)}}\ge\frac{f_1}{(C+1)^{\alpha(x)}}.$$

Since $\frac{f_1}{(C+1)^{\alpha(x)}}\ge 0$, $\frac{f_1}{(C+1)^{\alpha(x)}}\not\equiv 0$, the strong maximum principle implies that $u_1>0$ in Ω and (2.4) holds for u_1. Because of the monotonicity of u_n, (2.4) holds for u_n. □

Remark 2.1 If u_n and v_n are two solutions of (2.2), following the lines of the proof of the first part in Lemma 2.2, we may show that $u_n\le v_n$. By symmetry, this implies that the solution of (2.2) is unique.

Lemma 2.3 *The solution u_1 to problem* (2.2) *with $n=1$ satisfies*

$$\int_\Omega u_1^{-r}\,\mathrm{dx}<\infty,\quad \forall r<1. \tag{2.5}$$

Proof By $\frac{\min\{f(x),1\}}{(u_1+1)^{\alpha(x)}}\le 1$ and Lemma 2.2 in [17], we know that there exists $0<\beta<1$ such that $u_1\in C^{1,\beta}(\overline{\Omega})$ and $\|u_1\|_{C^{1,\beta}}\le C$, which implies that the gradient of u_1 exists everywhere, then the Hopf lemma in [19] shows that $\frac{\partial u_1(x)}{\partial\nu}>0$ in $\overline{\Omega}$, where ν is the outward unit normal vector of $\partial\Omega$ at x. Moreover, following the lines of the proof of lemma in [5], we get that

$$\int_\Omega u_1^{r}\,\mathrm{dx}<\infty\quad\text{if and only if}\quad r>-1.$$ □

We know clearly that the estimates on u_n depend on f and $\alpha(x)$, we will discuss this in different cases.

3 The case $0<\alpha^-\le\alpha(x)\le\alpha^+<1$

In this case, we obtain a priori estimates on u_n in $H_0^1(\Omega)$ only if f is more regular than $L^1(\Omega)$. We have the following results.

Lemma 3.1 ([20]) *Let u_n be the solution of* (2.2) *with* $0<\alpha^-\le\alpha(x)\le\alpha^+<1$, *and suppose that $f\in L^m(\Omega)$ with $m=\frac{Np}{Np-N+p+(N-p)\alpha^-}=(\frac{p^*}{1-\alpha^-})'$. Then the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$.*

Once we have the boundedness of u_n, we can prove an existence result for (1.1).

Theorem 3.1 ([20]) *Suppose that* f *is a nonnegative function in* $L^m(\Omega)$ ($f \not\equiv 0$), *with* $m = \frac{Np}{Np-N+p+(N-p)\alpha^-} = (\frac{p^*}{1-\alpha^-})'$, $f \not\equiv 0$, *and let* $0 < \alpha^- \leq \alpha(x) \leq \alpha^+ < 1$. *Then problem* (1.1) *has a solution* $u \in W_0^{1,p}(\Omega)$ *satisfying* (2.1).

The summability of u depends on the summability of f, which is proved in the next lemma.

Lemma 3.2 *Suppose that* $f \in L^m(\Omega)$, $m \geq \frac{Np}{Np-N+p+(N-p)\alpha^-}$, *and let* $0 < \alpha^- \leq \alpha(x) \leq \alpha^+ < 1$. *Then the solution* u *of* (1.1) *given by Theorem* 3.1 *is such that*:

(i) *if* $m > \frac{N}{p}$, *then* $u \in L^\infty(\Omega)$;

(ii) *if* $\frac{Np}{Np-N+p+(N-p)\alpha^-} \leq m < \frac{N}{p}$, *then* $u \in L^s(\Omega)$, $s = \frac{Nm(\alpha^-+p-1)}{N-pm}$.

Proof To prove (i), let $k > 1$ and define $G_k(s) = (s-k)_+$. Taking $G_k(u_n)$ as a test function in (2.2), we obtain

$$\int_\Omega \left|\nabla G_k(u_n)\right|^p \,\mathrm{dx} \leq \int_\Omega \left(\left|\nabla G_k(u_n)\right|^{p-2} \nabla G_k(u_n)\right) \cdot \nabla G_k(u_n)\,\mathrm{dx} = \int_\Omega \frac{f_n G_k(u_n)}{(u_n + \frac{1}{n})^{\alpha(x)}}\,\mathrm{dx}.$$

Since $G_k(u_n) \neq 0$, it implies that

$$\int_\Omega \left|\nabla G_k(u_n)\right|^p \,\mathrm{dx} \leq \int_\Omega f G_k(u_n)\,\mathrm{dx}. \tag{3.1}$$

Starting from inequality (3.1), Theorem 4.2 in [21] shows that there exists a constant C (independent of n) such that

$$\|u_n\|_{L^\infty(\Omega)} \leq C\|f\|_{L^m(\Omega)},$$

which implies that u belongs to $L^\infty(\Omega)$.

To prove (ii), noting that if $m = \frac{Np}{Np-N+p+(N-p)\alpha^-}$, $s = \frac{Np}{N-p} = p^*$, since $u \in W_0^{1,p}(\Omega)$, the result when $m = \frac{Np}{Np-N+p+(N-p)\alpha^-}$ is true by the Sobolev embedding theorem. If $\frac{Np}{Np-N+p+(N-p)\alpha^-} < m < \frac{N}{p}$, letting $\delta > 1$ and choosing $u_n^{p\delta-p+1}$ as a test function in (2.2), using Hölder's inequality, we get that

$$\begin{aligned}
&(p\delta - p + 1)\int_\Omega |\nabla u_n|^p u_n^{p\delta-p}\,\mathrm{dx} \\
&\quad\leq \int_{\{x\in\Omega, u_n\geq 1\}} \frac{f u_n^{p\delta-p+1}}{u_n^{\alpha^-}}\,\mathrm{dx} + \int_{\{x\in\Omega, u_n< 1\}} \frac{f u_n^{p\delta-p+1}}{u_n^{\alpha^+}}\,\mathrm{dx} \\
&\quad= \|f\|_{L^m(\Omega)} \Bigg(\bigg(\int_\Omega u_n^{(p\delta-p+1-\alpha^-)m'}\,\mathrm{dx} \bigg)^{\frac{1}{m'}} \\
&\qquad + |\Omega|^{\frac{\alpha^+-\alpha^-}{(p\delta-p+1-\alpha^-)m'}} \bigg(\int_\Omega u_n^{(p\delta-p+1-\alpha^-)m'}\,\mathrm{dx} \bigg)^{\frac{p\delta-p+1-\alpha^+}{(p\delta-p+1-\alpha^-)m'}} \Bigg).
\end{aligned} \tag{3.2}$$

By the Sobolev inequality (on the left-hand side), we have that

$$\int_\Omega |\nabla u_n|^p u_n^{p\delta-p}\,\mathrm{dx} = \frac{1}{\delta^p}\int_\Omega \left|\nabla u_n^\delta\right|^p \,\mathrm{dx} \geq \frac{S}{\delta^p}\bigg(\int_\Omega u_n^{p^*\delta}\,\mathrm{dx} \bigg)^{\frac{p}{p^*}}, \tag{3.3}$$

where S is the constant of the Sobolev embedding theorem. Combining with (3.2) and (3.3), we have that

$$\begin{aligned}
&\frac{S(p\delta-p+1)}{\delta^p}\left(\int_\Omega u_n^{p^*\delta}\,\mathrm{dx}\right)^{\frac{p}{p^*}} \\
&\quad\le \|f\|_{L^m(\Omega)}\left(\left(\int_\Omega u_n^{(p\delta-p+1-\alpha^-)m'}\,\mathrm{dx}\right)^{\frac{1}{m'}}\right. \\
&\quad\quad\left.+|\Omega|^{\frac{\alpha^+-\alpha^-}{(p\delta-p+1-\alpha^-)m'}}\left(\int_\Omega u_n^{(p\delta-p+1-\alpha^-)m'}\,\mathrm{dx}\right)^{\frac{p\delta-p+1-\alpha^+}{(p\delta-p+1-\alpha^-)m'}}\right).
\end{aligned} \tag{3.4}$$

We choose δ in such a way that $p^*\delta=(p\delta-p+1-\alpha^-)m'$, i.e.,

$$\delta=\frac{(\alpha^-+p-1)m(N-p)}{p(N-mp)},$$

which yields that $\delta>1$ if and only if $\frac{Np}{Np-N+p+(N-p)\alpha^-}<m<\frac{N}{p}$, and that $p^*\delta=\frac{Nm(\alpha^-+p-1)}{N-pm}=s$. Therefore, (3.4) becomes

$$\begin{aligned}
\left(\int_\Omega u_n^s\,\mathrm{dx}\right)^{\frac{p}{p^*}} &\le \frac{\delta^p}{S(p\delta-p+1)}\|f\|_{L^m(\Omega)}\left(\left(\int_\Omega u_n^s\,\mathrm{dx}\right)^{\frac{1}{m'}}\right. \\
&\quad\left.+|\Omega|^{\frac{\alpha^+-\alpha^-}{(p\delta-p+1-\alpha^-)m'}}\left(\int_\Omega u_n^s\,\mathrm{dx}\right)^{\frac{p\delta-p+1-\alpha^+}{(p\delta-p+1-\alpha^-)m'}}\right),
\end{aligned}$$

which implies that

$$\left(\int_\Omega u_n^s\,\mathrm{dx}\right)^{\frac{p-1+\alpha^+}{p^*\delta}} \le \frac{\delta^p}{S(p\delta-p+1)}\|f\|_{L^m(\Omega)}\left(\left(\int_\Omega u_n^s\,\mathrm{dx}\right)^{\frac{\alpha^+-\alpha^-}{p^*\delta}}+|\Omega|^{\frac{\alpha^+-\alpha^-}{p^*\delta}}\right). \tag{3.5}$$

Using Young's inequality on the right-hand side in (3.5), we have that

$$\left(\int_\Omega u_n^s\,\mathrm{dx}\right)^{\frac{p-1+\alpha^+}{p^*\delta}} \le \frac{\delta^p}{S(p\delta-p+1)}\|f\|_{L^m(\Omega)}\left(\varepsilon\left(\int_\Omega u_n^s\,\mathrm{dx}\right)^{\frac{p-1+\alpha^+}{p^*\delta}}+\varepsilon^{-\frac{\alpha^+-\alpha^-}{p-1+\alpha^-}}+|\Omega|^{\frac{\alpha^+-\alpha^-}{p^*\delta}}\right),$$

where $\varepsilon=\frac{S(p\delta-p+1)}{2\delta^p\|f\|_{L^m(\Omega)}}$. Thus, we get that

$$\left(\int_\Omega u_n^s\,\mathrm{dx}\right)^{\frac{p-1+\alpha^+}{p^*\delta}} \le \frac{2\delta^p\|f\|_{L^m(\Omega)}}{S(p\delta-p+1)}\left(\left(\frac{2\delta^p\|f\|_{L^m(\Omega)}}{S(p\delta-p+1)}\right)^{\frac{\alpha^+-\alpha^-}{p-1+\alpha^-}}+|\Omega|^{\frac{\alpha^+-\alpha^-}{p^*\delta}}\right). \tag{3.6}$$

Therefore, we know that u_n is bounded in $L^s(\Omega)$, so is $u\in L^s(\Omega)$. □

Theorem 3.2 *Suppose that* $f\in L^m(\Omega)$, $\frac{(p-1+\alpha^+)N}{(\alpha^-+p-1)(N-p)+p(p-1+\alpha^+)}\le m<\frac{Np}{Np-N+p+(N-p)\alpha^-}$, *and* $0<\alpha^-\le\alpha(x)\le\alpha^+<1$. *Then problem* (1.1) *has a solution* u *in* $W_0^{1,q}(\Omega)$, $q=\frac{Nm(\alpha^-+p-1)}{N-m(1-\alpha^-)}$.

Proof The lines of our proof are that if we can prove that u_n is bounded in $W_0^{1,q}(\Omega)$ (with q as in the statement), the existence of a solution u in $W_0^{1,q}(\Omega)$ of (1.1) will be proved by passing to the limit in (2.2) as in the proof of Theorem 3.1. To prove that u_n is bounded in $W_0^{1,q}(\Omega)$, we begin by proving that it is bounded in $L^s(\Omega)$, with $s = \frac{Nm(\alpha^- + p - 1)}{N - pm}$. To attain this goal, we choose $u_n^{p\delta - p + 1}$ as a test function in (2.2) as in the statement of Lemma 3.2, where $\frac{p-1+\alpha^+}{p} \le \delta < 1$; however, $\nabla u_n^{p\delta-p+1}$ will be singular at $u_n = 0$. Therefore, we choose $(u_n + \varepsilon)^{p\delta-p+1} - \varepsilon^{p\delta-p+1}$ as a test function in (2.2), where $\varepsilon < \frac{1}{n}$ for n fixed. We have that

$$(p\delta - p + 1)\int_\Omega |\nabla u_n|^p (u_n + \varepsilon)^{p\delta - p}\,\mathrm{dx} \le \int_\Omega \frac{f_n(u_n + \varepsilon)^{p\delta-p+1}}{(u_n + \varepsilon)^{\alpha(x)}}\,\mathrm{dx}.$$

Since $f_n \le f$, we have that

$$\begin{aligned}(p\delta - p + 1)&\int_\Omega |\nabla u_n|^p (u_n + \varepsilon)^{p\delta - p}\,\mathrm{dx}\\ &\le \int_\Omega f(u_n + \varepsilon)^{p\delta-p+1-\alpha^-}\,\mathrm{dx} + \int_\Omega f(u_n + \varepsilon)^{p\delta-p+1-\alpha^+}\,\mathrm{dx}.\end{aligned} \tag{3.7}$$

By the Sobolev embedding theorem ($W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$) on the left-hand side, it follows that

$$\begin{aligned}\int_\Omega |\nabla u_n|^p (u_n + \varepsilon)^{p\delta - p}\,\mathrm{dx} &= \int_\Omega \frac{|\nabla((u_n + \varepsilon)^\delta - \varepsilon^\delta)|^p}{\delta^p}\,\mathrm{dx}\\ &\ge \frac{S}{\delta^p}\left(\int_\Omega \left((u_n + \varepsilon)^\delta - \varepsilon^\delta\right)^{p^*}\mathrm{dx}\right)^{\frac{p}{p^*}},\end{aligned} \tag{3.8}$$

where S is the best constant of the Sobolev embedding theorem. Combining (3.7) with (3.8), we have that

$$\begin{aligned}&\frac{S(p\delta - p + 1)}{\delta^p}\left(\int_\Omega \left((u_n + \varepsilon)^\delta - \varepsilon^\delta\right)^{p^*}\mathrm{dx}\right)^{\frac{p}{p^*}}\\ &\quad\le \int_\Omega f(u_n + \varepsilon)^{p\delta-p+1-\alpha^-}\,\mathrm{dx} + \int_\Omega f(u_n + \varepsilon)^{p\delta-p+1-\alpha^+}\,\mathrm{dx}.\end{aligned} \tag{3.9}$$

Using Hölder's inequality on the right-hand side, we get

$$\begin{aligned}&\frac{S(p\delta - p + 1)}{\delta^p}\left(\int_\Omega \left((u_n + \varepsilon)^\delta - \varepsilon^\delta\right)^{p^*}\mathrm{dx}\right)^{\frac{p}{p^*}}\\ &\quad\le \|f\|_{L^m(\Omega)}\left(\int_\Omega (u_n + \varepsilon)^{(p\delta-p+1-\alpha^-)m'}\,\mathrm{dx}\right)^{\frac{1}{m'}}\\ &\qquad+ |\Omega|^{\frac{\alpha^+ - \alpha^-}{(p\delta-p+1-\alpha^-)m'}}\|f\|_{L^m(\Omega)}\left(\int_\Omega (u_n + \varepsilon)^{(p\delta-p+1-\alpha^-)m'}\,\mathrm{dx}\right)^{\frac{p\delta-p+1-\alpha^+}{(p\delta-p+1-\alpha^-)m'}}.\end{aligned}$$

Letting $\varepsilon \to 0$, we get (3.4), i.e.,

$$\begin{aligned}\left(\int_\Omega u_n^{p^*\delta}\,\mathrm{dx}\right)^{\frac{p}{p^*}} \le \frac{\delta^p}{S(p\delta - p + 1)}\|f\|_{L^m(\Omega)}&\left(\left(\int_\Omega u_n^{(p\delta-p+1-\alpha^-)m'}\,\mathrm{dx}\right)^{\frac{1}{m'}}\right.\\ &\left.+ |\Omega|^{\frac{\alpha^+ - \alpha^-}{(p\delta-p+1-\alpha^-)m'}}\left(\int_\Omega u_n^{(p\delta-p+1-\alpha^-)m'}\,\mathrm{dx}\right)^{\frac{p\delta-p+1-\alpha^+}{(p\delta-p+1-\alpha^-)m'}}\right),\end{aligned}$$

where δ is chosen in such a way that $p^*\delta = (p\delta - p + 1 - \alpha^-)m'$, i.e.,

$$\delta = \frac{(\alpha^- + p - 1)(N - p)m}{p(N - mp)}.$$

If $m = \frac{(p-1+\alpha^+)N}{(\alpha^-+p-1)(N-p)+p(p-1+\alpha^+)}$, we choose $\delta = \frac{p-1+\alpha^+}{p}$ in (3.9), and letting $\varepsilon \to 0$, we have that

$$\left(\int_\Omega u_n^{p^*\delta}\,\mathrm{dx}\right)^{\frac{p}{p^*}} \le \frac{\delta^p}{S(p\delta - p + 1)}\left(\int_\Omega f u_n^{p\delta-p+1-\alpha^-}\,\mathrm{dx} + \int_\Omega f\,\mathrm{dx}\right).$$

Using Hölder's inequality and Young's inequality, we get that

$$\left(\int_\Omega u_n^{p^*\delta}\,\mathrm{dx}\right)^{\frac{p}{p^*}} \le \frac{\delta^p\|f\|_{L^m(\Omega)}}{S(p\delta - p + 1)}\left(\varepsilon\left(\int_\Omega u_n^{p^*\delta}\,\mathrm{dx}\right)^{\frac{p}{p^*}} + \varepsilon^{-\frac{p^*}{pm'-p^*}} + |\Omega|^{\frac{1}{m'}}\right),$$

where $\varepsilon = \frac{S(p\delta-p+1)}{2\delta^p\|f\|_{L^m(\Omega)}}$. Thus we have that

$$\left(\int_\Omega u_n^{p^*\delta}\,\mathrm{dx}\right)^{\frac{p}{p^*}} \le \frac{2\delta^p\|f\|_{L^m(\Omega)}}{S(p\delta - p + 1)}\left(\left(\frac{2\delta^p\|f\|_{L^m(\Omega)}}{S(p\delta - p + 1)}\right)^{\frac{p^*}{pm'-p^*}} + |\Omega|^{\frac{1}{m'}}\right).$$

Therefore we obtain that u_n is bounded in $L^{\frac{N(p-1+\alpha^+)}{N-p}}(\Omega)$, where $\frac{N(p-1+\alpha^+)}{N-p}$ is the value of s for $m = \frac{(p-1+\alpha^+)N}{(\alpha^-+p-1)(N-p)+p(p-1+\alpha^+)}$.

If $\frac{(p-1+\alpha^+)N}{(\alpha^-+p-1)(N-p)+p(p-1+\alpha^+)} < m < \frac{Np}{Np-N+p+(N-p)\alpha^-}$, it is clear that the inequality on m holds true if and only if $\frac{p-1+\alpha^+}{p} < \delta < 1$, starting from (3.4) and arguing as in the proof of Lemma 3.2, we also get that u_n is bounded in $L^s(\Omega)$ with $s = \frac{Nm(\alpha^-+p-1)}{N-pm}$.

The right-hand side of (3.7) is bounded with respect to n (and ε, which we take smaller than 1) by using the estimate on u_n in $L^s(\Omega)$ and the choice of δ.

Since $\delta < 1$,

$$\int_\Omega \frac{|\nabla u_n|^p}{(u_n + \varepsilon)^{p-p\delta}}\,\mathrm{dx} = \int_\Omega |\nabla u_n|^p (u_n + \varepsilon)^{p\delta - p}\,\mathrm{dx} \le C.$$

If $q = \frac{Nm(\alpha^-+p-1)}{N-m(1-\alpha^-)} < p$, by Hölder's inequality, we have that

$$\begin{aligned}\int_\Omega |\nabla u_n|^q\,\mathrm{dx} &= \int_\Omega \frac{|\nabla u_n|^q}{(u_n + \varepsilon)^{(1-\delta)q}}(u_n + \varepsilon)^{(1-\delta)q}\,\mathrm{dx} \\ &\le \left(\int_\Omega \frac{|\nabla u_n|^p}{(u_n + \varepsilon)^{p(1-\delta)}}\,\mathrm{dx}\right)^{\frac{q}{p}}\left(\int_\Omega (u_n + \varepsilon)^{\frac{pq(1-\delta)}{p-q}}\,\mathrm{dx}\right)^{1-\frac{q}{p}} \\ &\le C\left(\int_\Omega (u_n + \varepsilon)^{\frac{pq(1-\delta)}{p-q}}\,\mathrm{dx}\right)^{1-\frac{q}{p}}.\end{aligned} \tag{3.10}$$

The choice of δ and the value of q are such that $\frac{pq(1-\delta)}{p-q} = s$, so that the right-hand side of (3.10) is bounded with respect to n and ε. Hence, u_n is bounded in $W_0^{1,q}(\Omega)$. □

Theorem 3.3 *Suppose that* $f \in L^m(\Omega)$, $\frac{1}{2-p-\alpha^+ + p\delta} < m < \frac{(p-1+\alpha^+)N}{(\alpha^- + p-1)(N-p)+p(p-1+\alpha^+)}$ ($\frac{p-1+\alpha^-}{p} < \delta < \frac{p-1+\alpha^+}{p}$), *and* $0 < \alpha^- \le \alpha(x) \le \alpha^+ < 1$. *Then problem* (1.1) *has a solution* u *in* $W_0^{1,q}(\Omega)$, $q = \frac{Nm(\alpha^- + p-1)}{N-m(1-\alpha^-)}$.

Proof The lines of our proof are similar to those in the proof of Theorem 3.2. We also begin by proving that u_n is bounded in $L^s(\Omega)$, with $s = \frac{Nm(\alpha^- + p-1)}{N-pm}$. To this aim, we also choose $(u_n + \varepsilon)^{p\delta-p+1} - \varepsilon^{p\delta-p+1}$ as a test function in (2.2), where $\frac{p-1+\alpha^-}{p} < \delta < \frac{p-1+\alpha^+}{p}$, $\varepsilon < \frac{1}{n}$ for n fixed. Since $f_n \le f$, using the Sobolev embedding theorem ($W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$) on the left-hand side again, we have that

$$\begin{aligned} &\frac{S(p\delta - p + 1)}{\delta^p} \left(\int_\Omega \left((u_n + \varepsilon)^\delta - \varepsilon^\delta \right)^{p^*} \mathrm{dx} \right)^{\frac{p}{p^*}} \\ &\quad \le \int_\Omega f (u_n + \varepsilon)^{p\delta-p+1-\alpha^-} \mathrm{dx} + \int_\Omega f (u_n + \varepsilon)^{p\delta-p+1-\alpha^+} \mathrm{dx}, \end{aligned}$$

where S is the best constant of the Sobolev embedding theorem.

Using Hölder's inequality and Lemma 2.3 on the right-hand side, we get that

$$\begin{aligned} &\frac{S(p\delta - p + 1)}{\delta^p} \left(\int_\Omega \left((u_n + \varepsilon)^\delta - \varepsilon^\delta \right)^{p^*} \mathrm{dx} \right)^{\frac{p}{p^*}} \\ &\quad \le \int_\Omega f (u_n + \varepsilon)^{p\delta-p+1-\alpha^-} \mathrm{dx} + \int_\Omega \frac{f}{u_1^{p-1+\alpha^+-p\delta}} \mathrm{dx} \\ &\quad \le \|f\|_{L^m(\Omega)} \left(\int_\Omega (u_n + \varepsilon)^{(p\delta-p+1-\alpha^-)m'} \mathrm{dx} \right)^{\frac{1}{m'}} + C\|f\|_{L^m(\Omega)}. \end{aligned}$$

Letting $\varepsilon \to 0$, we have that

$$\left(\int_\Omega u_n^{p^*\delta} \mathrm{dx} \right)^{\frac{p}{p^*}} \le \frac{\delta^p}{S(p\delta - p + 1)} \|f\|_{L^m(\Omega)} \left(\left(\int_\Omega u_n^{(p\delta-p+1-\alpha^-)m'} \mathrm{dx} \right)^{\frac{1}{m'}} + C \right), \tag{3.11}$$

where δ is chosen in such a way that $p^*\delta = (p\delta - p + 1 - \alpha^-)m'$, i.e.,

$$\delta = \frac{(\alpha^- + p - 1)(N - p)m}{p(N - mp)}.$$

If $1 < m < \frac{(p-1+\alpha^+)N}{(\alpha^- + p-1)(N-p)+p(p-1+\alpha^+)}$, it is clear that the inequality on m holds true if and only if $\frac{p-1+\alpha^-}{p} < \delta < \frac{p-1+\alpha^+}{p}$, and arguing as to the case $m = \frac{(p-1+\alpha^+)N}{(\alpha^- + p-1)(N-p)+p(p-1+\alpha^+)}$ in the proof of Theorem 3.2, we also obtain that u_n is bounded in $L^s(\Omega)$, with $s = \frac{Nm(\alpha^- + p-1)}{N-pm}$.

Since $\delta < 1$,

$$\int_\Omega \frac{|\nabla u_n|^p}{(u_n + \varepsilon)^{p-p\delta}} \mathrm{dx} = \int_\Omega |\nabla u_n|^p (u_n + \varepsilon)^{p\delta-p} \mathrm{dx} \le C.$$

If $q = \frac{Nm(\alpha^- + p-1)}{N-m(1-\alpha^-)} < p$, similarly to the proof of Theorem 3.2, we have by Hölder's inequality that

$$\int_\Omega |\nabla u_n|^q \mathrm{dx} \le C \left(\int_\Omega (u_n + \varepsilon)^{\frac{pq(1-\delta)}{p-q}} \mathrm{dx} \right)^{1-\frac{q}{p}}.$$

Due to the choice of δ and the value of q, the right-hand side of the above inequality is bounded with respect to n and ε. Hence, u_n is bounded in $W_0^{1,q}(\Omega)$. □

4 The case $1 < \alpha^- \le \alpha(x) \le \alpha^+$

The case $1 < \alpha^- \le \alpha(x) \le \alpha^+$ has many analogies with the case $0 < \alpha^- < \alpha^+ < 1$. In this case, we can also prove that u_n is bounded in $W_0^{1,p}(\Omega)$ only if f is more regular than $L^1(\Omega)$ and α^+ and α^- is close to 1. Hence we obtain the existence of problem (1.1).

Lemma 4.1 *Suppose that* $f \in L^m(\Omega)$ $(m > 1)$, *let* u_n *be the solution of* (2.2) *with* $1 < \alpha^- < \alpha^+ < 2 - \frac{1}{m}$. *Then* u_n *is bounded in* $W_0^{1,p}(\Omega)$.

Proof Taking u_n as a test function in (2.2), we obtain that

$$\int_\Omega |\nabla u_n|^p \,\mathrm{dx} \le \int_\Omega \frac{f}{u_n^{\alpha(x)-1}} \,\mathrm{dx}.$$

Using Lemma 2.1 and Lemma 2.2, we know that $u_n \ge u_1$ and there exists a constant $M > 0$ s.t. $u_1 \le M$. Hence $(\frac{M}{u_1})^{\alpha(x)-1} \le (\frac{M}{u_1})^{\alpha^+-1}$, and we have that

$$\int_\Omega |\nabla u_n|^p \,\mathrm{dx} \le \int_\Omega \frac{f}{u_1^{\alpha(x)-1}} \,\mathrm{dx} \le \left(1 + M^{\alpha^+-\alpha^-}\right) \int_\Omega \frac{f}{u_1^{\alpha^+-1}} \,\mathrm{dx}.$$

Using Hölder's inequality on the right-hand side and Lemma 2.3, we obtain

$$\int_\Omega |\nabla u_n|^p \,\mathrm{dx} \le C\left(1 + M^{\alpha^+-\alpha^-}\right) \|f\|_{L^m(\Omega)}.$$

Therefore, u_n is bounded in $W_0^{1,p}(\Omega)$. □

Once we have the boundedness of u_n, we can prove the following existence theorem along the lines of Theorem 3.1.

Theorem 4.1 *Suppose that* $f \in L^m(\Omega)$ $(m > 1)$, $f \not\equiv 0$ *and* $1 < \alpha^- < \alpha^+ < 2 - \frac{1}{m}$. *Then problem* (1.1) *has a solution* u *in* $W_0^{1,p}(\Omega)$.

The summability of u can be proved along the lines of Lemma 3.2 with small changes.

Lemma 4.2 *Suppose that* $f \in L^m(\Omega)$ $(m > 1)$ *and* $1 < \alpha^- < \alpha^+ < 2 - \frac{1}{m}$. *Then the solution* u *of* (1.1) *given by Theorem* 4.1 *is such that*:

(i) *if* $m > \frac{N}{p}$, *then* $u \in L^\infty(\Omega)$;

(ii) *if* $\frac{(p-1+\alpha^+)N}{(\alpha^-+p-1)(N-p)+p(p-1+\alpha^+)} \le m < \frac{N}{p}$, *then* $u \in L^s(\Omega)$, $s = \frac{Nm(\alpha^-+p-1)}{N-pm}$.

Proof The proof of (i) is similar to the proof of Lemma 3.2(i), we omit the details here.

To prove (ii), we choose $u_n^{p\delta-p+1}$ as a test function with $\delta \ge \frac{p-1+\alpha^+}{p}$ in (2.2). Similarly to the proof of Lemma 3.2, we obtain that

$$\frac{S(p\delta - p + 1)}{\delta^p} \left(\int_\Omega u_n^{p^*\delta} \,\mathrm{dx} \right)^{\frac{p}{p^*}} \le \int_\Omega f u_n^{p\delta-p+1-\alpha^-} \,\mathrm{dx} + \int_\Omega f u_n^{p\delta-p+1-\alpha^+} \,\mathrm{dx}. \tag{4.1}$$

If $m = \frac{(p-1+\alpha^+)N}{(\alpha^-+p-1)(N-p)+p(p-1+\alpha^+)}$, choosing $\delta = \frac{p-1+\alpha^+}{p}$ in (4.1), by Hölder's inequality, we get that

$$\frac{S(p\delta - p + 1)}{\delta^p}\left(\int_\Omega u_n^{p^*\delta}\,\mathrm{dx}\right)^{\frac{p}{p^*}} \leq \|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{(p\delta-p+1-\alpha^-)m'}\,\mathrm{dx}\right)^{\frac{1}{m'}} + |\Omega|^{1-\frac{1}{m}}\|f\|_{L^m(\Omega)}.$$

We choose δ in such a way that $p^*\delta = (p\delta - p + 1 - \alpha^-)m'$, i.e., $\delta = \frac{(\alpha^-+p-1)m(N-p)}{p(N-mp)}$. Since $m = \frac{(p-1+\alpha^+)N}{(\alpha^-+p-1)(N-p)+p(p-1+\alpha^+)}$, we get that $\frac{p}{p^*} > \frac{1}{m'}$. Because $s = p^*\delta$, we have the boundedness of u_n in $L^{\frac{N(p-1+\alpha^+)}{N-p}}(\Omega)$, which is the value of s for $m = \frac{N(p-1+\alpha^+)}{(\alpha^-+p-1)(N-p)+p(p-1+\alpha^+)}$.

If $\frac{(p-1+\alpha^+)N}{(\alpha^-+p-1)(N-p)+p(p-1+\alpha^+)} < m < \frac{N}{p}$, starting from inequality (4.1), using Hölder's inequality, we get that

$$\begin{aligned}\frac{S(p\delta - p + 1)}{\delta^p}\left(\int_\Omega u_n^{p^*\delta}\,\mathrm{dx}\right)^{\frac{p}{p^*}} &\leq \|f\|_{L^m(\Omega)}\Bigg(\left(\int_\Omega u_n^{(p\delta-p+1-\alpha^-)m'}\,\mathrm{dx}\right)^{\frac{1}{m'}} \\ &\quad + |\Omega|^{\frac{\alpha^+-\alpha^-}{(p\delta-p+1-\alpha^-)m'}}\left(\int_\Omega u_n^{(p\delta-p+1-\alpha^-)m'}\,\mathrm{dx}\right)^{\frac{p\delta-p+1-\alpha^+}{(p\delta-p+1-\alpha^-)m'}}\Bigg).\end{aligned}$$

We also choose δ in such a way that $p^*\delta = (p\delta - p + 1 - \alpha^-)m'$, which yields that $\delta > \frac{p-1+\alpha^+}{p}$ if and only if $m > \frac{(p-1+\alpha^+)N}{(\alpha^-+p-1)(N-p)+p(p-1+\alpha^+)}$, and that $p^*\delta = s$. So, since $\frac{p}{p^*} > \frac{1}{m'}$ being $m < \frac{N}{p}$, we have the boundedness of u_n in $L^s(\Omega)$, so does $u \in L^s(\Omega)$. □

Moreover, we can prove that a positive power of u_n is bounded in $W_0^{1,p}(\Omega)$ only if f is more regular than $L^1(\Omega)$ and α^+ is close to α^-, and we only have the boundedness of u_n in $W_{\mathrm{loc}}^{1,p}(\Omega)$.

Lemma 4.3 *Suppose that $f \in L^m(\Omega)$ $(m > 1)$, let u_n be the solution of* (2.2) *with $1 < \alpha^- \leq \alpha(x) \leq \alpha^+$ and $\alpha^+ - \alpha^- < 1 - \frac{1}{m}$. Then $u_n^{\frac{p-1+\alpha^-}{p}}$ is bounded in $W_0^{1,p}(\Omega)$, and u_n is bounded in $W_{\mathrm{loc}}^{1,p}(\Omega)$ and in $L^s(\Omega)$, with $s = \frac{N(\alpha^-+p-1)}{N-p}$.*

Proof Taking $u_n^{\alpha^-}$ as a test function in (2.2), since $\frac{u_n^{\alpha^-}}{(u_n+\frac{1}{n})^{\alpha^-}} \leq 1$ and $f_n \leq f$, by Hölder's inequality and Lemma 2.3, we get that

$$\begin{aligned}\alpha^-\int_\Omega |\nabla u_n|^p u_n^{\alpha^--1}\,\mathrm{dx} &\leq \int_\Omega \frac{fu_n^{\alpha^-}}{(u_n+\frac{1}{n})^{\alpha^-}}\,\mathrm{dx} + \int_\Omega \frac{fu_n^{\alpha^-}}{u_n^{\alpha^+}}\,\mathrm{dx} \\ &\leq \int_\Omega f\,\mathrm{dx} + \int_\Omega \frac{f}{u_n^{\alpha^+-\alpha^-}}\,\mathrm{dx} \leq |\Omega|^{1-\frac{1}{m}}\|f\|_{L^m(\Omega)} + C\|f\|_{L^m(\Omega)}.\end{aligned}$$

Since

$$\int_\Omega |\nabla u_n|^p u_n^{\alpha^--1}\,\mathrm{dx} = \frac{p^p}{(\alpha^-+p-1)^p}\int_\Omega \left|\nabla u_n^{\frac{\alpha^-+p-1}{p}}\right|^p\mathrm{dx},$$

we have that

$$\frac{p^p\alpha^-}{(\alpha^-+p-1)^p}\int_\Omega \left|\nabla u_n^{\frac{\alpha^-+p-1}{p}}\right|^p\mathrm{dx} \leq \left(C + |\Omega|^{1-\frac{1}{m}}\right)\|f\|_{L^m(\Omega)}.$$

Thus, we have that $u_n^{\frac{\alpha^-+p-1}{p}}$ is bounded in $W_0^{1,p}(\Omega)$.

Applying the Sobolev embedding theorem to $u_n^{\frac{\alpha^-+p-1}{p}}$ $(W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega))$, we get that

$$S\left(\int_\Omega \left|u_n^{\frac{\alpha^-+p-1}{p}}\right|^{p^*} \mathrm{dx}\right)^{\frac{p}{p^*}} \leq \int_\Omega \left|\nabla u_n^{\frac{\alpha^-+p-1}{p}}\right|^p \mathrm{dx},$$

where S is the best constant of the Sobolev embedding theorem. Since the boundedness of $u_n^{\frac{\alpha^-+p-1}{p}}$ in $W_0^{1,p}(\Omega)$, we thus have the boundedness of u_n in $L^s(\Omega)$.

To prove the boundedness of u_n in $W_{\mathrm{loc}}^{1,p}(\Omega)$, we choose $u_n\varphi^p$ as a test function in (2.2), where $\varphi \in C_0^\infty(\Omega)$, $\Omega' = \{x \in \Omega, \varphi \not\equiv 0\}$. By (2.4), we have that

$$\begin{aligned}
&\int_\Omega |\nabla u_n|^p \varphi^p \,\mathrm{dx} + p\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n\right) \cdot \nabla\varphi u_n \varphi^{p-1} \,\mathrm{dx} \\
&\quad \leq \int_\Omega \frac{f_n\varphi^p}{u_n^{\alpha(x)-1}} \,\mathrm{dx} \leq \int_\Omega \frac{f_n\varphi^p}{C_{\Omega'}^{\alpha(x)-1}} \,\mathrm{dx} \leq \frac{1}{\min\{C_{\Omega'}^{\alpha^+-1}, C_{\Omega'}^{\alpha^--1}\}} \int_\Omega f_n\varphi^p \,\mathrm{dx}.
\end{aligned}$$

By Young's inequality, we have that

$$\begin{aligned}
&p\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n\right) \cdot \nabla\varphi u_n \varphi^{p-1} \,\mathrm{dx} \\
&\quad \leq \frac{p\frac{1}{2(p-1)}}{\frac{p}{p-1}} \int_\Omega \left||\nabla u_n|^{p-2}\nabla u_n\right|^{\frac{p}{p-1}} \varphi^p \,\mathrm{dx} + \frac{p(2(p-1))^{p-1}}{p} \int_\Omega |\nabla\varphi|^p u_n^p \,\mathrm{dx} \\
&\quad \leq \frac{1}{2}\int_\Omega |\nabla u_n|^p \varphi^p \,\mathrm{dx} + \left(2(p-1)\right)^{p-1} \int_\Omega |\nabla\varphi|^p u_n^p \,\mathrm{dx}.
\end{aligned}$$

Since u_n is bounded in $L^s(\Omega)$ (where $s \geq p$), by Hölder's inequality, we obtain that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla u_n|^p \varphi^p \,\mathrm{dx} \\
&\quad \leq \frac{1}{\min\{C_{\Omega'}^{\alpha^+-1}, C_{\Omega'}^{\alpha^--1}\}} \int_\Omega f_n\varphi^p \,\mathrm{dx} + \left(2(p-1)\right)^{p-1} \int_\Omega |\nabla\varphi|^p u_n^p \,\mathrm{dx} \\
&\quad \leq \frac{\|\varphi\|_{L^\infty(\Omega)}^p}{\min\{C_{\Omega'}^{\alpha^+-1}, C_{\Omega'}^{\alpha^--1}\}} \int_\Omega f \,\mathrm{dx} + \left(2(p-1)\right)^{p-1} \|\nabla\varphi\|_{L^\infty(\Omega)}^p \int_\Omega u_n^p \,\mathrm{dx} \\
&\quad \leq \frac{|\Omega|^{1-\frac{1}{m}} \|\varphi\|_{L^\infty(\Omega)}^p \|f\|_{L^m(\Omega)}}{\min\{C_{\Omega'}^{\alpha^+-1}, C_{\Omega'}^{\alpha^--1}\}} + \left(2(p-1)\right)^{p-1} |\Omega|^{1-\frac{p}{s}} \|\nabla\varphi\|_{L^\infty(\Omega)}^p \left(\int_\Omega u_n^s \,\mathrm{dx}\right)^{\frac{p}{s}},
\end{aligned}$$

and hence u_n is bounded in $W_{\mathrm{loc}}^{1,p}(\Omega)$. □

Once we have the boundedness of u_n, we can prove the following existence theorem along the lines of Theorem 3.1.

Theorem 4.2 *Suppose that f is a nonnegative function in $L^m(\Omega)$ $(m > 1)$, $(f \not\equiv 0)$, $1 < \alpha^- \leq \alpha(x) \leq \alpha^+$ and $\alpha^+ - \alpha^- < 1 - \frac{1}{m}$. Then problem* (1.1) *has a solution u in $W_{\mathrm{loc}}^{1,p}(\Omega)$. Furthermore, $u^{\frac{\alpha^-+p-1}{p}}$ belongs to $W_0^{1,p}(\Omega)$.*

The summability of u can be proved as the following lemma, the proof is similar to the proof of Lemma 4.2

Lemma 4.4 *Suppose that* $f \in L^m(\Omega)$, $1 < \alpha^- \le \alpha(x) \le \alpha^+$ *and* $\alpha^+ - \alpha^- < 1 - \frac{1}{m}$. *Then the solution* u *of* (1.1) *given by Theorem* 4.2 *is such that*:

(i) *if* $m > \frac{N}{p}$, *then* $u \in L^\infty(\Omega)$;

(ii) *if* $\frac{N(p-1+\alpha^+)}{(\alpha^-+p-1)(N-p)+p(p-1+\alpha^+)} \le m < \frac{N}{p}$, *then* $u \in L^s(\Omega)$, $s = \frac{Nm(\alpha^-+p-1)}{N-mp}$.

5 The case $0 < \alpha^- < 1 < \alpha^+$

If $0 < \alpha^- < 1 < \alpha^+$, the boundedness of u_n in $W_0^{1,p}(\Omega)$ can also be obtained only if f is more regular than $L^1(\Omega)$. Furthermore, the existence of problem (1.1) is obtained, the proof has many analogies with the case $0 < \alpha^- < \alpha^+ < 1$. We have the following results.

Lemma 5.1 *Suppose that* $f \in L^m(\Omega)$, *with* $m = \frac{Np}{Np-N+p+(N-p)\alpha^-}$, *and let* u_n *be the solution of* (2.2) *with* $0 < \alpha^- < 1 < \alpha^+ < 2 - \frac{1}{m}$. *Then the sequence* $\{u_n\}$ *is bounded in* $W_0^{1,p}(\Omega)$.

Proof We choose u_n as a test function in (2.2), by Hölder's inequality and Lemma 2.3, since $f_n \le f$, we have that

$$\begin{aligned}\int_\Omega |\nabla u_n|^p \,\mathrm{dx} &\le \int_\Omega f u_n^{1-\alpha^-} \,\mathrm{dx} + \int_\Omega \frac{f}{u_1^{\alpha^+-1}} \,\mathrm{dx} \\ &\le \|f\|_{L^m(\Omega)} \left(\int_\Omega u_n^{(1-\alpha^-)m'} \,\mathrm{dx}\right)^{\frac{1}{m'}} + C\|f\|_{L^m(\Omega)}. \end{aligned} \tag{5.1}$$

Applying the Sobolev embedding theorem on the left-hand side, we get

$$S\left(\int_\Omega u_n^{p^*} \,\mathrm{dx}\right)^{\frac{p}{p^*}} \le \int_\Omega |\nabla u_n|^p \,\mathrm{dx}. \tag{5.2}$$

Combining (5.1) with (5.2) implies that

$$S\left(\int_\Omega u_n^{p^*} \,\mathrm{dx}\right)^{\frac{p}{p^*}} \le \|f\|_{L^m(\Omega)} \left(\int_\Omega u_n^{(1-\alpha^-)m'} \,\mathrm{dx}\right)^{\frac{1}{m'}} + C\|f\|_{L^m(\Omega)}.$$

Let $p^* = (1-\alpha^-)m'$, it follows that

$$S\left(\int_\Omega u_n^{p^*} \,\mathrm{dx}\right)^{\frac{p}{p^*}} \le \|f\|_{L^m(\Omega)} \left(\int_\Omega u_n^{p^*} \,\mathrm{dx}\right)^{\frac{1}{m'}} + C\|f\|_{L^m(\Omega)}.$$

By Young's inequality, we get that

$$S\left(\int_\Omega u_n^{p^*} \,\mathrm{dx}\right)^{\frac{p}{p^*}} \le \|f\|_{L^m(\Omega)} \left(\varepsilon\left(\int_\Omega u_n^{p^*} \,\mathrm{dx}\right)^{\frac{p}{p^*}} + \varepsilon^{-\frac{p^*}{pm'-p^*}}\right) + C\|f\|_{L^m(\Omega)}.$$

Thus, we have that

$$\left(\int_\Omega u_n^{p^*} \,\mathrm{dx}\right)^{\frac{p}{p^*}} \le \frac{\|f\|_{L^m(\Omega)}}{S - \varepsilon\|f\|_{L^m(\Omega)}} \left(\varepsilon^{-\frac{p^*}{pm'-p^*}} + C\right).$$

We choose $\varepsilon = \frac{S}{2\|f\|_{L^m(\Omega)}}$ to get

$$\left(\int_\Omega u_n^{p^*}\,\mathrm{dx}\right)^{\frac{p}{p^*}} = \frac{2\|f\|_{L^m(\Omega)}}{S}\left(\left(\frac{2\|f\|_{L^m(\Omega)}}{S}\right)^{\frac{p^*}{pm'-p^*}} + C\right).$$

So the boundedness of u_n in $L^{p^*}(\Omega)$ is obtained. Using the estimate and (5.1) again, we have the estimate of u_n in $W_0^{1,p}(\Omega)$. □

Once the boundedness of u_n in $W_0^{1,p}(\Omega)$ is obtained, we can prove the following existence theorem.

Theorem 5.1 *Suppose that* $f \in L^m(\Omega)$ *with* $m = \frac{Np}{Np-N+p+(N-p)\alpha^-}$, $f \not\equiv 0$, *and* $0 < \alpha^- < 1 < \alpha^+ < 2 - \frac{1}{m}$. *Then problem* (1.1) *has a solution* u *in* $W_0^{1,p}(\Omega)$.

Lemma 5.2 *Suppose that* $f \in L^m(\Omega)$ *with* $m \geq \frac{Np}{Np-N+p+(N-p)\alpha^-}$, *and* $0 < \alpha^- < 1 < \alpha^+ < 2 - \frac{1}{m}$. *Then the solution* u *of* (1.1) *given by Theorem* 5.1 *is such that*:

(i) *if* $m > \frac{N}{p}$, *then* $u \in L^\infty(\Omega)$;

(ii) *if* $\frac{Np}{Np-N+p+(N-p)\alpha^-} \leq m < \frac{N}{p}$, *then* $u \in L^s(\Omega)$, $s = \frac{Nm(\alpha^-+p-1)}{N-mp}$.

Proof The proof of (i) is similar to that for Lemma 3.2(i), we omit the details here.

To prove (ii), if $\frac{N(p-1+\alpha^+)}{(\alpha^-+p-1)(N-p)+p(p-1+\alpha^+)} \leq m < \frac{N}{p}$, the proof is identical to that for Lemma 4.2, we also omit it here.

If $m = \frac{Np}{Np-N+p+(N-p)\alpha^-}$, we can prove the results by the Sobolev embedding theorem.

If $\frac{Np}{Np-N+p+(N-p)\alpha^-} < m < \frac{N(p-1+\alpha^+)}{(\alpha^-+p-1)(N-p)+p(p-1+\alpha^+)}$, we choose $1 < \delta < \frac{p-1+\alpha^+}{p}$, and use once again $u_n^{p\delta-p+1}$ as a test function in (2.2). Using $\delta > 1 > \frac{p-1+\alpha^-}{p}$, as well as Hölder's inequality, the Sobolev embedding theorem, Lemma 2.3, we get that

$$\begin{aligned}\frac{S(p\delta-p+1)}{\delta^p}\left(\int_\Omega u_n^{p^*\delta}\,\mathrm{dx}\right)^{\frac{p}{p^*}} &\leq \int_\Omega f u_n^{p\delta-p+1-\alpha^-}\,\mathrm{dx} + \int_\Omega \frac{f}{u_1^{p-1+\alpha^+-p\delta}}\,\mathrm{dx}\\ &\leq \|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{(p\delta-p+1-\alpha^-)m'}\,\mathrm{dx}\right)^{\frac{1}{m'}} + C\|f\|_{L^m(\Omega)}.\end{aligned}$$

The choice of δ in such a way that $p^*\delta = (p\delta - p + 1 - \alpha^-)m'$ yields that $1 < \delta < \frac{p-1+\alpha^+}{p}$ if and only if $\frac{Np}{Np-N+p+(N-p)\alpha^-} < m < \frac{N(p-1+\alpha^+)}{(\alpha^-+p-1)(N-p)+p(p-1+\alpha^+)}$, and that $p^*\delta = s$. The choice of $m < \frac{N}{p}$ implies that $\frac{p}{p^*} > \frac{1}{m'}$. Thus we have the boundedness of u_n in $L^s(\Omega)$, and so does the limit u in $L^s(\Omega)$. □

6 Conclusions

In this paper, we study the existence and regularity of solutions to the quasilinear elliptic problem with nonlinear singular terms and variable exponent. Due to the nonlinearity of a p-Laplace operator and the anisotropic variable exponent $\alpha(x)$, some classical methods may not directly be applied to our problem. We construct a suitable test function and apply the Leray-Schauder fixed point theorem to prove the existence of positive solutions with necessary a priori estimate and compact argument. Furthermore, we prove that the existence and regularity of solutions depend on the summability of f and the value of $\alpha(x)$.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors collaborated in all the steps concerning the research and achievements presented in the final manuscript. All authors read and approved the final manuscript.

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