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The method of lower and upper solutions for fourth order equations with the Navier condition

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Abstract

The aim of this paper is to explore the method of lower and upper solutions in order to give some existence results for equations of the form

$$y^{(4)}(x)+(k_1+k_2)y''(x)+k_1k_2y(x)=f\bigl(x,y(x)\bigr),\quad x\in(0,1),$$

with the Navier condition

$$y(0)=y(1)=y''(0)=y''(1)=0$$

under the condition $k_1<0<k_2<\pi^2$. The main tool is the Schauder fixed point theorem.

MSC: 34B10; 34B18

Keywords: elastic beam; fourth order equations; lower and upper solutions; Green function

1 Introduction

The aim of this paper is to explore the method of lower and upper solutions in order to give some existence of solutions for equations of the form

$$y^{(4)}(x)+(k_1+k_2)y''(x)+k_1k_2y(x)=f\bigl(x,y(x)\bigr),\quad x\in(0,1),\tag{1.1}$$

with the Navier condition

$$y(0)=y(1)=y''(0)=y''(1)=0.\tag{1.2}$$

Such boundary value problems appear, as it is well known [1–3], in the theory of hinged beams.

Recently, Vrabel [4] studied problem (1.1), (1.2) under the assumption

(H1) k_1 and k_2 are two constants with

$$k_2<k_1<0.\tag{1.3}$$

He constructed the Green function for the linear problem

$$\begin{aligned} &L(y)(x) \equiv y^{(4)}(x) + (k_1 + k_2)y''(x) + k_1k_2y(x) = 0, \quad x \in (0,1), \\ &y(0) = y(1) = y''(0) = y''(1) = 0, \end{aligned} \tag{1.4}$$

and proved its non-negativity and established the method of lower and upper solutions for (1.1), (1.2).

Definition 1.1 ([4]) The function $\alpha \in C^4[0,1]$ is said to be a *lower solution* for (1.1), (1.2) if

$$L\bigl(\alpha(x)\bigr) \le f\bigl(x, \alpha(x)\bigr) \quad \text{for } x \in (0,1), \tag{1.5}$$

and

$$\alpha(0) \le 0, \qquad \alpha(1) \le 0, \qquad \alpha''(0) \ge 0, \qquad \alpha''(1) \ge 0. \tag{1.6}$$

An *upper solution* $\beta \in C^4[0,1]$ is defined analogously by reversing the inequalities in (1.5), (1.6).

Theorem A ([4, Theorem 7]) *Let* (H1) *hold. Suppose that for problem* (1.1), (1.2) *there exist a lower solution* α *and an upper solution* β *such that*

$$\alpha(x) \le \beta(x) \quad \textit{for } x \in [0,1]. \tag{1.7}$$

If $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous and satisfies*

$$f(x, u_1) \le f(x, u_2) \quad \textit{for } \alpha(x) \le u_1 \le u_2 \le \beta(x) \textit{ and } x \in [0,1], \tag{1.8}$$

then there exists a solution $y(x)$ *for* (1.1), (1.2) *satisfying* $\alpha(x) \le y(x) \le \beta(x)$ *for* $0 \le x \le 1$.

Of course the natural question is what would happen if (H1) is replaced with the condition

(H2) $k_1 < 0 < k_2$.

Roughly speaking, for some kind of second order boundary value problems, it is well known that the existence of a lower solution α and an upper solution β, which are well ordered, that is, $\alpha \le \beta$, implies the existence of a solution between them (see [5]). However, the use of lower and upper solutions in boundary value problems of the fourth order, even for the simple boundary conditions (1.2), is heavily dependent on the positiveness properties for the corresponding linear operators, see the counterexample in [5, Remark 3.1].

It is the purpose of this paper to establish the method of lower and upper solutions for fourth order problem (1.1), (1.2) under condition (H2). To do that, we study the positiveness properties of the solutions of the nonhomogeneous linear problems

$$\begin{aligned} &Ly(x) = 0, \quad x \in (0,1), \\ &y(0) = 1, \qquad y''(0) = y(1) = y''(1) = 0, \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} &Ly(x) = 0, \quad x \in (0,1),\\ &y''(0) = 1, \qquad y(0) = y(1) = y''(1) = 0. \end{aligned} \tag{1.10}$$

Since the general solution of $Ly = 0$ under (H2) is different from that under (H1), we determine the sign of solution of (1.10) via its equivalent second order systems.

In [5], Cabada et al. have extensively studied the positiveness properties of the operator

$$\mathcal{L}y = y^{(4)} - My$$

with the homogeneous boundary value conditions (1.2) as well as the more general nonhomogeneous boundary value conditions, and then applied the positiveness properties in a systematic way to obtain existence theorems in the presence of lower and upper solutions allowing the case where they are not ordered. Obviously, Cabada et al. [5] only dealt with the case that

$$k_1 + k_2 = 0 \tag{1.11}$$

in (1.4) and (1.1).

For the related results on the existence and multiplicity of positive solutions or sign-changing solutions for fourth order problems, see Bai and Wang [6], Chu and O'Regan [7], Cid et al. [8], Drábek and Holubová [9, 10], Hernandez and Manasevich [11], Korman [12], Liu and Li [13], Ma et al. [14–18], Rynne [19, 20], Schröder [3], Webb et al. [21], Yang [22] and Yao [23] and the references therein.

The rest of the paper is arranged as follows. In Section 2, we show that the Green function of (1.4) possesses the positiveness properties under the condition $k_1 < 0 < k_2 < \pi^2$. Finally, in Section 3, we develop the method of lower and upper solutions for (1.1), (1.2) under some monotonic condition on the nonlinearity f, and give some applications of our main results.

2 Green function in the case $k_1 < 0 < k_2$

Let $E = C[0,1]$ be the Banach space of continuous functions defined on $[0,1]$ with its usual normal $\|\cdot\|$. Denote

$$k_1 = -r^2, \qquad k_2 = m^2$$

with some $r > 0$ and $m > 0$. Let us consider

$$\begin{aligned} &y''''(x) + \bigl(m^2 - r^2\bigr)y''(x) - r^2m^2y(x) = 0, \quad x \in (0,1),\\ &y(0) = y(1) = y''(0) = y''(1) = 0. \end{aligned} \tag{2.1}$$

Define a linear operator $L : D(L) \to E$

$$Ly := y'''' + \bigl(m^2 - r^2\bigr)y'' - r^2m^2y, \quad y \in D(L),$$

with the domain

$$D(L) := \{y \in C^4[0,1] : y(0) = y(1) = y''(0) = y''(1) = 0\}.$$

Firstly, we construct the Green function $G(x,s)$ for $Ly = 0$.
Define a linear operator

$$L_1 y := y'' - r^2 y, \qquad D(L_1) := \{y \in C^2[0,1] : y(0) = y(1) = 0\}.$$

The Green function of $L_1 y = 0$ is

$$G_1(t,s) = \begin{cases} \frac{\sinh(rt)\sinh(r(1-s))}{r\sinh r}, & 0 \le t \le s \le 1, \\ \frac{\sinh(rs)\sinh(r(1-t))}{r\sinh r}, & 0 \le s \le t \le 1. \end{cases}$$

Define a linear operator

$$L_2 y := y'' + m^2 y, \qquad D(L_2) := \{y \in C^2[0,1] : y(0) = y(1) = 0\}.$$

The Green function of $L_2 y = 0$ is

$$G_2(t,s) = \begin{cases} \frac{\sin(mt)\sin(m(1-s))}{m\sin m}, & 0 \le t \le s \le 1, \\ \frac{\sin(ms)\sin(m(1-t))}{m\sin m}, & 0 \le s \le t \le 1. \end{cases}$$

Obviously,

$$Ly = L_2 \circ L_1 y,$$

and the Green function of $Ly = 0$ is

$$G(x,s) := \int_0^1 G_2(x,t)G_1(t,s)\,dt, \quad (x,s) \in [0,1] \times [0,1],$$

which can be explicitly given by

$$G(x,s) = \begin{cases} \frac{1}{m^2+r^2}\left[\frac{\sin(mx)\sin(m(s-1))}{m\sin m} + \frac{\sinh(rx)\sinh(r(1-s))}{r\sinh r}\right], & 0 \le x \le s \le 1, \\ \frac{1}{m^2+r^2}\left[\frac{\sin(ms)\sin(m(x-1))}{m\sin m} + \frac{\sinh(rs)\sinh(r(1-x))}{r\sinh r}\right], & 0 \le s \le x \le 1. \end{cases} \tag{2.2}$$

Theorem 2.1 *Let $m \in (0,\pi)$ and $r \in (0,\infty)$. Then*

$$G(x,s) \ge 0, \quad (x,s) \in [0,1] \times [0,1].$$

Proof It is an immediate consequence of the facts that for $m \in (0,\pi)$,

$$G_2(t,s) \ge 0, \quad (t,s) \in [0,1] \times [0,1],$$

and for $r \in (0,\infty)$,

$$G_1(t,s) \ge 0, \quad (t,s) \in [0,1] \times [0,1].$$

□

3 Method of lower and upper solutions

In this section, we will establish the method of lower and upper solutions for (1.1), (1.2) in the case $k_1 < 0 < k_2$.

Denote

$$g_\alpha(x) = L(\alpha(x)) - f(x,\alpha(x)), \qquad g_\beta(x) = L(\beta(x)) - f(x,\beta(x)), \quad x \in [0,1]. \tag{3.1}$$

Then

$$g_\alpha(x) \le 0, \qquad g_\beta(x) \ge 0, \quad x \in [0,1]. \tag{3.2}$$

Now let $v_\alpha(x)$ be the solution of

$$\begin{aligned} &Lv_\alpha(x) = 0, \quad x \in (0,1), \\ &v_\alpha(0) = \alpha(0), \qquad v_\alpha(1) = \alpha(1), \qquad v_\alpha''(0) = \alpha''(0), \qquad v_\alpha''(1) = \alpha''(1). \end{aligned} \tag{3.3}$$

Then $v_\alpha(x)$ is uniquely determined as

$$v_\alpha(x) = \alpha(0)w(x) + \alpha(1)w(1-x) + \alpha''(0)\chi(x) + \alpha''(1)\chi(1-x), \tag{3.4}$$

where $w(x)$ is the unique solution of the nonhomogeneous problem

$$L(y) = 0, \qquad y(0) = 1, \qquad y''(0) = y(1) = y''(1) = 0, \tag{3.5}$$

and it can be explicitly given by

$$w(x) = \frac{m^2}{r^2+m^2}\frac{\sinh[r(1-x)]}{\sinh r} + \frac{r^2}{r^2+m^2}\frac{\sin[m(1-x)]}{\sin m}, \tag{3.6}$$

$\chi(x)$ is the unique solution of the nonhomogeneous problem

$$L(y) = 0, \qquad y(0) = 0, \qquad y''(0) = 1, \qquad y(1) = y''(1) = 0, \tag{3.7}$$

and it can be explicitly given by

$$\chi(x) = \frac{1}{(r^2+m^2)}\frac{\sinh[r(1-x)]}{\sinh r} - \frac{1}{(r^2+m^2)}\frac{\sin[m(1-x)]}{\sin m}.$$

Let $v_\beta(x)$ be the solution of

$$\begin{aligned} &Lv_\beta(x) = 0, \quad x \in (0,1), \\ &v_\beta(0) = \beta(0), \qquad v_\beta(1) = \beta(1), \qquad v_\beta''(0) = \beta''(0), \qquad v_\beta''(1) = \beta''(1). \end{aligned}$$

Then $v_\beta(x)$ is uniquely determined as

$$v_\beta(x) = \beta(0)w(x) + \beta(1)w(1-x) + \beta''(0)\chi(x) + \beta''(1)\chi(1-x).$$

Lemma 3.1

(1) *Let* $0 < r < \infty$ *and* $0 < m < \pi$. *Then* $w(x) > 0$ *for* $x \in (0,1)$.

(2) *Let* $0 < r < \infty$ *and* $0 < m < \pi$. *Then* $\chi(x) < 0$ *for* $x \in (0,1)$.

Proof (1) Since $r(1-x) > 0$ and $-\infty < m(1-x) < \pi$ for $x \in (0,1)$, it follows from (3.6) that $w(x) > 0$ for $x \in (0,1)$.

(2) Obviously, (3.7) is equivalent to the system

$$L_2\chi = Z, \quad \chi(0) = 0, \chi(1) = 0, \tag{3.8}$$

$$L_1 Z = 0, \quad Z(0) = 1, Z(1) = 0. \tag{3.9}$$

It is easy to see from (3.9) and the fact $G_1(t,s) > 0$ for $(t,s) \in (0,1) \times (0,1)$ that

$$Z(x) > 0 \quad \text{for } x \in [0,1).$$

Combining this with (3.8) and using the fact $G_2(t,s) > 0$ for $(t,s) \in (0,1) \times (0,1)$, we deduce that $\chi(x) < 0$ in $(0,1)$. □

From Lemma 3.1 and the definitions of v_α and v_β, it follows that

$$v_\alpha(x) \le 0, \qquad v_\beta(x) \ge 0, \quad x \in [0,1]. \tag{3.10}$$

Now, for a lower solution α of (1.1), (1.2), we have the following implications:

$$\begin{aligned} &L(\alpha(x)) = f\bigl(x,\alpha(x)\bigr) + g_\alpha(x) \\ &\quad\Rightarrow \quad \alpha(x) = v_\alpha(x) + \int_0^1 G(x,s) f\bigl(s,\alpha(s)\bigr)\,ds + \int_0^1 G(x,s) g_\alpha(s)\,ds \\ &\quad\Rightarrow \quad \alpha(x) \le T\alpha(x) \quad \text{on } [0,1], \end{aligned}$$

and, by a similar way, we obtain $\beta(x) \ge T\beta(x)$ on $[0,1]$, where $T : C[0,1] \to C^4[0,1]$ is the operator defined by

$$T\phi(x) = \int_0^1 G(x,s) f\bigl(s,\phi(s)\bigr)\,ds, \quad 0 \le x \le 1, \tag{3.11}$$

where the Green function G is as in (2.2). It is easy to check that (1.1), (1.2) is equivalent to the operator equation

$$y = Ty. \tag{3.12}$$

As a direct consequence of the Schauder fixed point theorem [4, Theorem 5], we have the following lemma.

Lemma 3.2 *Let there exist a constant M such that*

$$\bigl|f(x,y)\bigr| \le M$$

for $(x,y) \in [0,1] \times \mathbb{R}$. *Then* (1.1), (1.2) *has a solution*.

Theorem 3.1 *Let $k_1 < 0 < k_2 < \pi^2$. Suppose that for problem* (1.1), (1.2) *there exist a lower solution α and an upper solution β such that*

$$\alpha(x) \le \beta(x) \quad \textit{for } x \in [0,1].$$

If $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$f(x,u_1) \le f(x,u_2) \quad \textit{for } \alpha(x) \le u_1 \le u_2 \le \beta(x), \textit{ and } x \in [0,1], \tag{3.13}$$

then there exists a solution $y(x)$ for (1.1), (1.2) *satisfying*

$$\alpha(x) \le y(x) \le \beta(x) \quad \textit{for } 0 \le x \le 1. \tag{3.14}$$

Proof Define the function F on $[0,1] \times \mathbb{R}$ by setting

$$F(x,y) = \begin{cases} f(x,\beta(x)), & y > \beta(x), \\ f(x,y), & \alpha(x) \le y \le \beta(x), \\ f(x,\alpha(x)), & y < \alpha(x). \end{cases}$$

Since F is continuous and bounded on $[0,1] \times \mathbb{R}$, by Lemma 3.2, there exists a solution y of the problem

$$L(y) = F(x,y),$$
$$y(0) = y(1) = y''(0) = y''(1) = 0.$$

We now show that inequality (3.14) is true. We have

$$L\bigl(y(x) - \beta(x)\bigr) = L\bigl(y(x)\bigr) - L\bigl(\beta(x)\bigr) \le F\bigl(x,y(x)\bigr) - f\bigl(x,\beta(x)\bigr) \le 0.$$

Thus $L(y(x) - \beta(x)) = h_1(x) \le 0$ for $x \in [0,1]$, that is, from Theorem 2.1 and (3.10)

$$y(x) - \beta(x) = -v_\beta(x) + \int_0^1 G(t,s)h_1(s)\,ds \le 0 \quad \text{for } x \in [0,1].$$

By a similar way,

$$L\bigl(y(x) - \alpha(x)\bigr) = L\bigl(y(x)\bigr) - L\bigl(\alpha(x)\bigr) \ge F\bigl(x,y(x)\bigr) - f\bigl(x,\alpha(x)\bigr) \ge 0.$$

Thus $L(y(x) - \alpha(x)) = h_2(x) \ge 0$ for $x \in [0,1]$, that is, from Theorem 2.1 and (3.10)

$$y(x) - \alpha(x) = -v_\alpha(x) + \int_0^1 G(t,s)h_2(s)\,ds \ge 0 \quad \text{for } x \in [0,1].$$

Therefore, $\alpha(x) \le y(x) \le \beta(x)$ for $x \in [0,1]$, and accordingly, y is a solution of (1.1), (1.2). □

Remark 3.1 It is worth remarking that if (3.13) is not valid, then the existence of a lower solution α and an upper solution β with $\alpha(x) \le \beta(x)$ in $[0,1]$ cannot guarantee the existence of solutions in the order interval $[\alpha(x),\beta(x)]$. Let us see the counterexample in Cabada et al. [5, Remark 3.1].

Remark 3.2 In the case $|k_1| > k_2$, the assertions of Theorem 3.1 can be deduced from Habets and Sanchez [24, Theorem 4.1].

Remark 3.3 Let us consider the problem

$$\begin{aligned} &u^{(4)}(x) - 4u''(x) + 3u(x) = u^3 + \sin x, \quad x \in (0,1), \\ &u(0) = u(1) = u''(0) = u''(1) = 0. \end{aligned} \tag{3.15}$$

It is easy to verify that $f(x,u) = u^3 + \sin x$, $k_1 = -1$ and $k_2 = 4$, and

$$\alpha(x) \equiv -1, \qquad \beta(x) \equiv 1$$

satisfy all of the conditions in Theorem 3.1. Therefore, (3.15) has a solution u satisfying

$$-1 \le u(x) \le 1, \quad x \in [0,1].$$

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Authors' contributions
RM and JW completed the main study together. RM wrote the manuscript, DY checked the proofs process and verified the calculation. Moreover, all the authors read and approved the last version of the manuscript.

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References

1. Gupta, CP: Existence and uniqueness theorems for the bending of an elastic beam equation. Appl. Anal. **26**(4), 289-304 (1988)
2. Lazer, AC, McKenna, PJ: Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis. SIAM Rev. **32**, 537-578 (1990)
3. Schröder, J: Operator Inequalities. Mathematics in Science and Engineering, vol. 147. Academic Press, New York (1980)
4. Vrabel, R: On the lower and upper solutions method for the problem of elastic beam with hinged ends. J. Math. Anal. Appl. **421**(2), 1455-1468 (2015)
5. Cabada, A, Cid, J, Sanchez, L: Positivity and lower and upper solutions for fourth order boundary value problems. Nonlinear Anal. **67**(5), 1599-1612 (2007)
6. Bai, Z, Wang, H: On positive solutions of some nonlinear fourth order beam equations. J. Math. Anal. Appl. **270**, 357-368 (2002)
7. Chu, J, O'Regan, D: Positive solutions for regular and singular fourth-order boundary value problems. Commun. Appl. Anal. **10**, 185-199 (2006)
8. Cid, JA, Franco, D, Minhós, F: Positive fixed points and fourth-order equations. Bull. Lond. Math. Soc. **41**, 72-78 (2009)
9. Drábek, P, Holubová, G: Positive and negative solutions of one-dimensional beam equation. Appl. Math. Lett. **51**, 1-7 (2016)
10. Drábek, P, Holubová, G: On the maximum and antimaximum principles for the beam equation. Appl. Math. Lett. **56**, 29-33 (2016)

11. Hernandez, GE, Manasevich, R: Existence and multiplicity of solutions of a fourth order equation. Appl. Anal. **54**, 237-250 (1994)
12. Korman, P: Uniqueness and exact multiplicity of solutions for a class of fourth-order semilinear problems. Proc. R. Soc. Edinb., Sect. A **134**(1), 179-190 (2004)
13. Liu, X, Li, W: Existence and multiplicity of solutions for fourth-order boundary value problems with parameters. J. Math. Anal. Appl. **327**, 362-375 (2007)
14. Ma, R: Existence of positive solutions of a fourth-order boundary value problem. Appl. Math. Comput. **168**, 1219-1231 (2005)
15. Ma, R, Wang, H: On the existence of positive solutions of fourth-order ordinary differential equations. Appl. Anal. **59**, 225-231 (1995)
16. Ma, R: Nodal solutions for a fourth-order two-point boundary value problem. J. Math. Anal. Appl. **314**(1), 254-265 (2006)
17. Ma, R: Nodal solutions of boundary value problems of fourth-order ordinary differential equations. J. Math. Anal. Appl. **319**(2), 424-434 (2006)
18. Ma, R, Xu, J: Bifurcation from interval and positive solutions of a nonlinear fourth-order boundary value problem. Nonlinear Anal. TMA **72**(1), 113-122 (2010)
19. Rynne, BP: Infinitely many solutions of superlinear fourth order boundary value problems. Topol. Methods Nonlinear Anal. **19**(2), 303-312 (2002)
20. Rynne, BP: Global bifurcation for 2*m*th-order boundary value problems and infinitely many solutions of superlinear problems. J. Differ. Equ. **188**, 461-472 (2003)
21. Webb, JRL, Infante, G, Franco, D: Positive solutions of nonlinear fourth-order boundary value problems with local and non-local boundary conditions. Proc. R. Soc. Edinb., Sect. A **138**(2), 427-446 (2008)
22. Yang, Z: Existence and uniqueness of positive solutions for higher order boundary value problem. Comput. Math. Appl. **54**(2), 220-228 (2007)
23. Yao, Q: Positive solutions for eigenvalue problems of fourth-order elastic beam equations. Appl. Math. Lett. **17**(2), 237-243 (2004)
24. Habets, P, Sanchez, L: A monotone method for fourth order boundary value problems involving a factorizable linear operator. Port. Math. **64**(3), 255-279 (2007)