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Lie symmetry analysis, Lie-Bäcklund symmetries, explicit solutions, and conservation laws of Drinfeld-Sokolov-Wilson system

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Abstract

The symmetry analysis method is used to study the Drinfeld-Sokolov-Wilson system. The Lie point symmetries of this system are obtained. An optimal system of one-dimensional subalgebras is derived by using Ibragimov's method. Based on the optimal system, similarity reductions and explicit solutions of the system are presented. The Lie-Bäcklund symmetry generators are also investigated. Furthermore, the method of constructing conservation laws of nonlinear partial differential equations with the aid of a new conservation theorem associated with Lie-Bäcklund symmetries is presented. Conservation laws of the Drinfeld-Sokolov-Wilson system are constructed by using this method.

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1 Introduction

The Lie symmetry method was initiated by Lie [1] in the second half of the nineteenth century. It has become one of the most powerful methods to study nonlinear partial differential equations (NLPDEs). The core idea of the Lie symmetry method is that one-parameter groups of transformations acting on the space of the independent and dependent variables leave the NLPDEs unchanged [2–5]. Most problems in science and engineering can be represented by NLPDEs [6–9]. Application of the Lie symmetry method for constructing the explicit solutions of the NLPDEs can always be regarded as one of the most important fields of mathematical physics. Many important properties of NLPDEs such as symmetry reductions, conservation laws, and explicit solutions by using symmetries can be considered successively [10–12].

Studying conservation laws is helpful in analyzing NLPDEs in the physical point of view [13]. Some studies have indicated that conservation laws play an important role in the numerical integration of PDEs [14]. Conservation laws are also helpful in solving equations by means of the double reduction theory [15, 16]. In order to construct conservation laws various methods have been developed, such as Noether's theorem [17], the partial

Noether approach [18], the multiplier approach [19], and a new conservation theorem [20, 21]. Noether’s theorem and the partial Noether approach establish a relationship between symmetries and conservation laws for NLPDEs. Nevertheless, these methods do not work on a nonlinear equation without a Lagrangian. It is notable that a new conservation theorem was proposed by Ibragimov [20]. The conservation laws of a nonlinear equation and its adjoint equation can be constructed by using formal Lagrangian.

In the present paper, we shall consider the Drinfeld-Sokolov-Wilson system (DSWS)

$$\begin{aligned} u_t + 2vv_x &= 0, \\ v_t - av_{xxx} + 3bu_xv + 3kuv_x &= 0, \end{aligned} \tag{1}$$

where $a, b,$ and k are real valued constants. System (1) is a physical model for describing nonlinear surface gravity waves propagating over a horizontal sea bed.

The DSWS has attracted the attention of many scholars. Some conservation laws of DSWS were obtained with the aid of the multiplier approach [22]. Then the double reduction analysis was employed to study the reductions of some conservation laws for DSWS [23]. This system has an infinite number of conservation laws, has a Lax representation, and is a member of KP hierarchy [24, 25], which indicates the integrality of this system. The scaling invariant Lax pairs of the DSWS were derived by Hickman and Hereman *et al.* [26]. The homotopy analysis method was applied to obtain the approximate solutions of DSWS [27]. Matjila *et al.* derived the exact solutions of system (1) by using the (G'/G) -expansion function method. They also constructed conservation laws using Noether’s approach [28]. We gave some symmetry reductions and conservation laws of this system [29].

This paper is arranged as follows. In Section 2, we derive the Lie point symmetries of the DSWS using Lie group analysis and find the transformed solutions. In Section 3, a new optimal system of subalgebras of system (1) is constructed by using a concise method. The new optimal system contains five operators. Then in Section 4, based on the optimal system, the similarity reduced equations and the explicit solutions of system (1) are investigated systematically. In Section 5, the method of constructing conservation laws of NLPDEs with the aid of the new conservation theorem associated with Lie-Bäcklund symmetries is presented. In Section 6, the conservation laws of the DSWS are constructed. Finally, the conclusions are given in the last section.

2 Lie point symmetries

The infinitesimal generators

$$X = \xi^1(t, x, u, v) \frac{\partial}{\partial t} + \xi^2(t, x, u, v) \frac{\partial}{\partial x} + \eta^1(t, x, u, v) \frac{\partial}{\partial u} + \eta^2(t, x, u, v) \frac{\partial}{\partial v} \tag{2}$$

will give rise to the Lie group of symmetries [30]. Then X should satisfy the following invariant surface conditions:

$$pr^{(3)}X(\Delta_1)|_{\Delta_1=0} = 0, \tag{3}$$

$$pr^{(3)}X(\Delta_2)|_{\Delta_2=0} = 0, \tag{4}$$

where $\Delta_1 = u_t + 2vv_x$ and $\Delta_2 = v_t - av_{xxx} + 3bu_xv + 3kuv_x$. The invariant surface conditions give the overdetermined system of PDEs

$$\begin{aligned} \xi_u^1 &= 0, & \xi_v^1 &= 0, & \xi_x^1 &= 0, & \xi_{tt}^1 &= 0, & \xi_t^2 &= 0, \\ \xi_u^2 &= 0, & \xi_v^2 &= 0, & \xi_x^2 &= \frac{1}{3}\xi_t^1, & \eta^1 &= -\frac{2}{3}u\xi_t^1, & \eta^2 &= -\frac{2}{3}v\xi_t^1. \end{aligned} \tag{5}$$

By solving (5), we get

$$\xi^1 = 3c_1t + c_2, \quad \xi^2 = c_1x + c_3, \quad \eta^1 = -2c_1u, \quad \eta^2 = -2c_1v, \tag{6}$$

where c_1, c_2 , and c_3 are arbitrary constants. Therefore we obtain the three infinitesimal symmetry generators

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}. \end{aligned} \tag{7}$$

The above three infinitesimal symmetry generators form a three-dimensional Lie algebra L_3 by commutation.

One can obtain the group transformation generated by the Lie point symmetry operator X_i ($i = 1, \dots, 3$) by solving the ordinary differential equations with initial conditions

$$\begin{aligned} \frac{dt^*}{d\varepsilon} &= \xi^1(t^*, x^*, u^*, v^*), & t^*|_{\varepsilon=0} &= t, \\ \frac{dx^*}{d\varepsilon} &= \xi^2(t^*, x^*, u^*, v^*), & x^*|_{\varepsilon=0} &= x, \\ \frac{du^*}{d\varepsilon} &= \eta^1(t^*, x^*, u^*, v^*), & u^*|_{\varepsilon=0} &= u, \\ \frac{dv^*}{d\varepsilon} &= \eta^2(t^*, x^*, u^*, v^*), & v^*|_{\varepsilon=0} &= v. \end{aligned} \tag{8}$$

Then the one-parameter transformation groups G_i of the DSWS are given as follows:

$$\begin{aligned} G_1 &: (t, x, u, v) \rightarrow (t + \varepsilon, x, u, v), \\ G_2 &: (t, x, u, v) \rightarrow (t, x + \varepsilon, u, v), \\ G_3 &: (t, x, u, v) \rightarrow (te^{3\varepsilon}, xe^\varepsilon, ue^{-2\varepsilon}, ve^{-2\varepsilon}). \end{aligned} \tag{9}$$

Therefore, the following theorem is established.

Theorem 1 *If $\{u(x, t), v(x, t)\}$ is a solution of the DSWS, so are the transformed solutions*

$$\begin{aligned} G_1(\varepsilon) \cdot u(x, t) &= u(x, t - \varepsilon), & G_1(\varepsilon) \cdot v(x, t) &= v(x, t - \varepsilon), \\ G_2(\varepsilon) \cdot u(x, t) &= u(x - \varepsilon, t), & G_2(\varepsilon) \cdot v(x, t) &= v(x - \varepsilon, t), \\ G_3(\varepsilon) \cdot u(x, t) &= e^{-2\varepsilon}u(e^{-\varepsilon}x, e^{-3\varepsilon}t), & G_3(\varepsilon) \cdot v(x, t) &= e^{-2\varepsilon}v(e^{-\varepsilon}x, e^{-3\varepsilon}t). \end{aligned} \tag{10}$$

3 Optimal system of subalgebras

Over the past few decades, the problem of classifying the subgroup was studied by many researchers. In order to demonstrate the classification of the group invariant solutions, Ovsianikov put forward the concept of optimal systems of subalgebras of a Lie algebra [2]. Then this method extended to many examples of optimal systems of subgroups for the Lie group of mathematical physics models by Winternitz *et al.* [31, 32]. Notably, Olver [4] used a different technique for constructing a one-dimensional optimal system, which was based on a commutator table and adjoint representation. Furthermore, Ibragimov presented a concise method to get the optimal system in their paper [33]. Zhao and Han extended this method to the Heisenberg equation [11], the AKNS system [12], and the Broer-Kaup system [34]. This method only relies on a commutator table. In this section, we shall use the Ibragimov method to construct an optimal system of one-dimensional subalgebra of the Lie algebra L_3 for the DSWS.

An arbitrary infinitesimal operator X of Lie algebra L_3 can be written

$$X = l^1 X_1 + l^2 X_2 + l^3 X_3. \tag{11}$$

The transformations of the symmetry group with the Lie algebra L_3 provide the three-parameter group of linear transformations. The following generators

$$E_i = c_{ij}^\lambda l^j \frac{\partial}{\partial l^\lambda}, \quad i = 1, 2, 3, \tag{12}$$

are useful in finding these transformations, where c_{ij} is determined by $[X_i, X_j] = c_{ij}^\lambda X_\lambda$. The commutator table of generators $X_1, X_2,$ and X_3 is given in Table 1. Using equation (12) and commutator Table 1, $E_1, E_2,$ and E_3 can be written

$$\begin{aligned} E_1 &= 3l^3 \frac{\partial}{\partial l^1}, \\ E_2 &= l^3 \frac{\partial}{\partial l^2}, \\ E_3 &= -3l^1 \frac{\partial}{\partial l^1} - l^2 \frac{\partial}{\partial l^2}. \end{aligned} \tag{13}$$

For the generator E_1 , the Lie equations with the initial condition $\tilde{l}|_{a_1=0} = l$ are written

$$\frac{d\tilde{l}^1}{da_1} = 3\tilde{l}^3, \quad \frac{d\tilde{l}^2}{da_1} = 0, \quad \frac{d\tilde{l}^3}{da_1} = 0, \tag{14}$$

where a_1 is the parameter. The solutions of the above initial value problem give the transformation

$$\tilde{l}^1 = l^1 + 3a_1 l^3, \quad \tilde{l}^2 = l^2, \quad \tilde{l}^3 = l^3. \tag{15}$$

Table 1 Commutator table

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$3X_1$
X_2	0	0	X_2
X_3	$-3X_1$	$-X_2$	0

Other Lie equations for E_2 and E_3 are

$$\frac{d\tilde{l}^1}{da_2} = 0, \quad \frac{d\tilde{l}^2}{da_2} = \tilde{l}^3, \quad \frac{d\tilde{l}^3}{da_2} = 0, \tag{16}$$

$$\frac{d\tilde{l}^1}{da_3} = -3\tilde{l}^1, \quad \frac{d\tilde{l}^2}{da_3} = -\tilde{l}^2, \quad \frac{d\tilde{l}^3}{da_3} = 0, \tag{17}$$

where a_2 and a_3 are the parameters. In a similar way, we obtain the following transformations:

$$\tilde{l}^1 = l^1, \quad \tilde{l}^2 = l^2 + a_2 l^3, \quad \tilde{l}^3 = l^3, \tag{18}$$

$$\tilde{l}^1 = a_3^{-3} l^1, \quad \tilde{l}^2 = a_3^{-1} l^2, \quad \tilde{l}^3 = l^3, \quad a_3 > 0. \tag{19}$$

Theorem 2 *The following five operators constitute an optimal system of one-dimensional subalgebras of the three-dimensional Lie algebra L_3 :*

$$X_1, \quad X_2, \quad X_3, \quad X_2 - X_1, \quad X_2 + X_1. \tag{20}$$

Proof The process of constructing optimal systems is equivalent to the classification of the vector

$$l = (l^1, l^2, l^3), \tag{21}$$

by means of the transformations (15), (18), and (19). We focus on finding the simplest representatives of each class of similar vectors (21).

Case 1. $l^3 \neq 0$. First of all, we take $a_2 = -\frac{l^2}{l^3}$ in (18) and reduce the vector (21) to the form

$$(l^1, 0, l^3). \tag{22}$$

By taking $a_1 = -\frac{l^1}{3l^3}$ in the transformations (15), we make $\tilde{l}^1 = 0$. Hence the vector (22) is reduced to the form

$$(0, 0, l^3). \tag{23}$$

Thus this case provides the operator X_3 .

Case 2. $l^3 = 0$.

(I) $l^2 \neq 0$. We assume $l^2 = 1$ and make $l^1 = \pm 1$ by the transformations (19). In addition, taking into account the possibility $l^1 = 0$, we obtain the following representatives for the optimal system

$$X_2, \quad X_2 - X_1, \quad X_2 + X_1. \tag{24}$$

(II) Let $l_2 = 0$. If $l^1 \neq 0$ we can set $l^1 = 1$ and obtain the vector $(1, 0, 0)$. This gives rise to the operator X_1 . □

4 Symmetry reductions and explicit solutions of the DSWS

In Section 3, an optimal system of one-dimensional subalgebras of the DSWS has been obtained. We now make use of the symmetries of the optimal system (20) to reduce system (1) to a system of nonlinear ordinary differential equations in one variable. All the symmetry reductions are presented in Table 2. By solving the reduced equations, the group invariant solutions to the DSWS can be obtained.

4.1 Explicit solutions of system (1) using the simplest equation method

The method of the simplest equation was first proposed by Kudryashov [35, 36] and systematically developed by Vitanov [37–39]. We have summarized the main steps of the simplest equation method and have obtained the traveling wave solutions of the (3 + 1)-dimensional KP equation and the generalized Fisher equation [40]. In this section, we will employ the simplest equation method and obtain explicit solutions of reduced equations of case (5) in Table 2. The equation of Bernoulli is used as the simplest equation. Consider a solution of (B) of case (5) in Table 2 of the form

$$\begin{aligned}
 F(\xi) &= \sum_{i=0}^m a_i (G(\xi))^i, \\
 H(\xi) &= \sum_{i=0}^n b_i (G(\xi))^i,
 \end{aligned}
 \tag{25}$$

where m and n are positive integers that can be determined by a homogeneous balance procedure and $G(\xi)$ satisfies the Bernoulli equation

$$G'(\xi) = cG(\xi) + dG^2(\xi).
 \tag{26}$$

The Bernoulli equation (26) has the solution

$$G(\xi) = c \left\{ \frac{\cosh[c(\xi + C)] + \sinh[c(\xi + C)]}{1 - d \cosh[c(\xi + C)] - d \sinh[c(\xi + C)]} \right\}.
 \tag{27}$$

Table 2 Reduction of the DSWS

Case	Similarity variables	Reduced equations
(1) X_1	$\xi = x,$ $u(x, t) = F(\xi),$ $v(x, t) = H(\xi).$	$H' = 0, 3kFH' + 3bF'H - aH''' = 0.$
(2) X_2	$\xi = t,$ $u(x, t) = F(\xi),$ $v(x, t) = H(\xi).$	$F' = 0, H' = 0.$
(3) X_3	$\xi = xt^{-\frac{1}{3}},$ $u(x, t) = t^{-\frac{2}{3}}F(\xi),$ $v(x, t) = t^{-\frac{2}{3}}H(\xi).$	$\xi F' - 6HH' + 2F = 0, -9kFH' - 9bF'H + \xi H' + 3aH''' + 2H = 0. (A)$
(4) $X_2 - X_1$	$\xi = x + t,$ $u(x, t) = F(\xi),$ $v(x, t) = H(\xi).$	$2HH' + F' = 0, 3bF'H + 3kFH' - aH''' + H' = 0.$
(5) $X_2 + X_1$	$\xi = x - t,$ $u(x, t) = F(\xi),$ $v(x, t) = H(\xi).$	$2HH' - F' = 0, 3bF'H + 3kFH' - aH''' - H' = 0. (B)$

We get $m = 2, n = 1$ by balancing system (B). Hence the solution of (B) can be written

$$\begin{aligned} F(\xi) &= a_0 + a_1G + a_2G^2, \\ H(\xi) &= b_0 + b_1G. \end{aligned} \tag{28}$$

Substituting system (28) along with equation (26) into (B) and collecting the coefficient of the same power G^i leads to a set of constraining equations for a_i and b_i . Solving these constraining equations, we obtain one solution

$$\begin{aligned} a_0 &= \frac{-abc^2 + ac^2k + 2b + k}{3k(2b + k)}, & a_1 &= \frac{2acd}{2b + k}, & a_2 &= \frac{2ad^2}{2b + k}, \\ b_0 &= \pm \frac{\sqrt{c}\sqrt{a_1}}{2\sqrt{d}}, & b_1 &= \frac{2db_0}{c}. \end{aligned} \tag{29}$$

Therefore, the solutions of system (1) are

$$\begin{aligned} u(x, t) &= a_0 + a_1c \left\{ \frac{\cosh[c(\xi + C)] + \sinh[c(\xi + C)]}{1 - d \cosh[c(\xi + C)] - d \sinh[c(\xi + C)]} \right\} \\ &\quad + a_2c^2 \left\{ \frac{\cosh[c(\xi + C)] + \sinh[c(\xi + C)]}{1 - d \cosh[c(\xi + C)] - d \sinh[c(\xi + C)]} \right\}^2, \\ v(x, t) &= b_0 + b_1c \left\{ \frac{\cosh[c(\xi + C)] + \sinh[c(\xi + C)]}{1 - d \cosh[c(\xi + C)] - d \sinh[c(\xi + C)]} \right\}, \end{aligned} \tag{30}$$

where $\xi = x - t$ and C is a constant of integration (see Figure 1).

4.2 Explicit solutions of system (1) using generalized tanh method

The generalized tanh method was proposed by Fan [41], who constructed a series of traveling wave solutions for some special types of equations [42–44]. We take the solution of reduced equations (B) of case (5) of Table 2 in the form

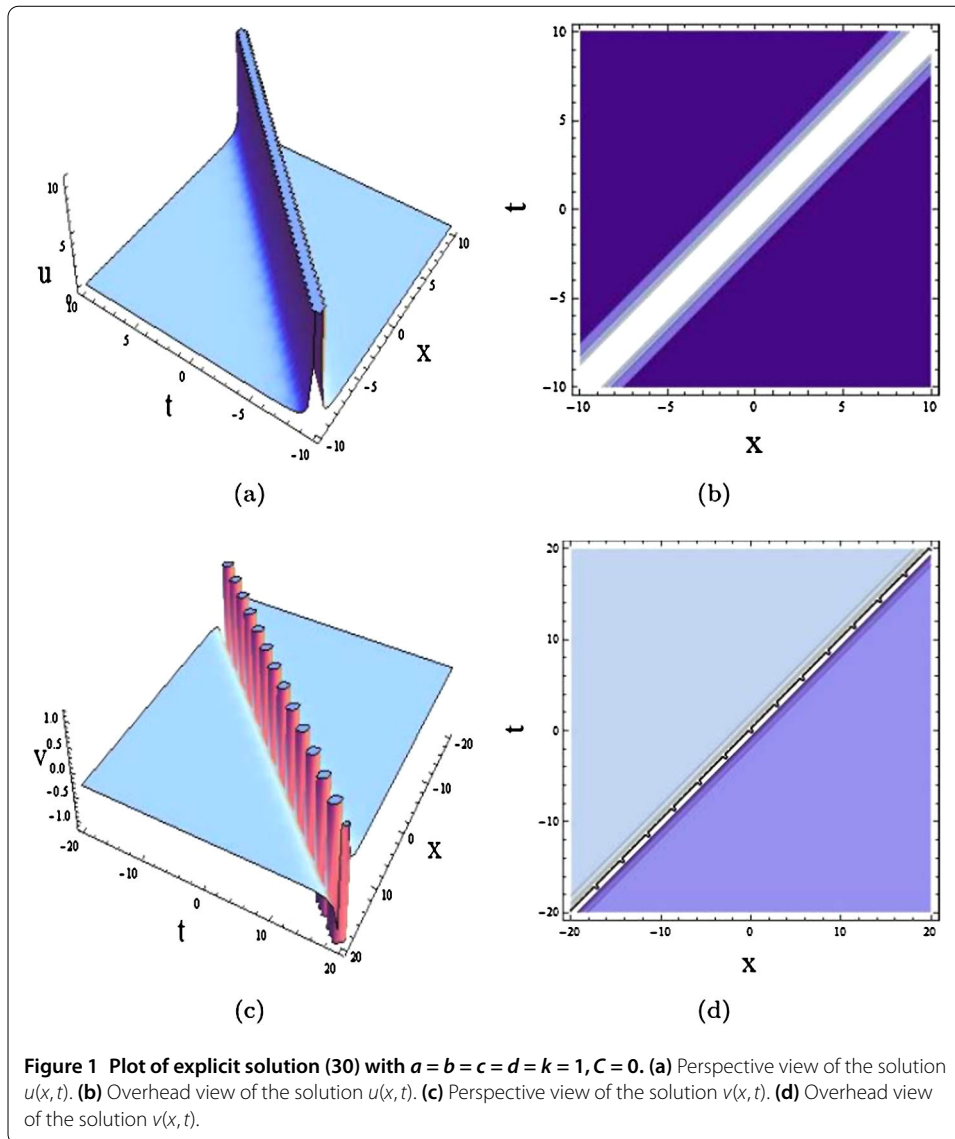
$$F(\xi) = \sum_{i=1}^m a_i \varphi^i, \quad H(\xi) = \sum_{i=1}^m b_i \varphi^i, \tag{31}$$

where $\varphi = \varphi(\xi)$ is a solution of the Riccati equation

$$\varphi' = q + \varphi^2. \tag{32}$$

The Riccati equation has a series of solutions

$$\begin{aligned} \varphi &= \begin{cases} -\sqrt{-q} \coth(\sqrt{-q}\xi), \\ -\sqrt{-q} \tanh(\sqrt{-q}\xi), \end{cases} & q < 0, \\ \varphi &= -\frac{1}{\xi}, & q = 0, \\ \varphi &= \begin{cases} -\sqrt{q} \cot(\sqrt{q}\xi), \\ -\sqrt{q} \tan(\sqrt{q}\xi), \end{cases} & q > 0. \end{aligned} \tag{33}$$



The solutions of (B) are of the form

$$\begin{aligned}
 F(\xi) &= a_0 + a_1\varphi + a_2\varphi^2, \\
 H(\xi) &= b_0 + b_1\varphi.
 \end{aligned}
 \tag{34}$$

Substituting system (34) along with the equation (32) into (B) and collecting the coefficient of the same power φ^i , we can obtain a set of constraining equations for a_i and b_i . Solving these constraining equations, we obtain one solution

$$\begin{aligned}
 a_0 &= \frac{2aq + 1}{3k}, & a_1 &= 0, & a_2 &= \frac{2a}{2b + k}, \\
 b_0 &= 0, & b_1 &= \pm\sqrt{a_2}.
 \end{aligned}
 \tag{35}$$

Therefore we obtain five solutions, namely

$$\begin{cases}
 u_1(\xi) = \frac{2aq+1}{3k} + \frac{2a}{2b+k} [-\sqrt{-q} \tanh(\sqrt{-q}\xi)]^2, \\
 v_1(\xi) = \mp \sqrt{\frac{2a}{2b+k}} \sqrt{-q} \tanh(\sqrt{-q}\xi), \quad q < 0, \\
 \\
 u_2(\xi) = \frac{2aq+1}{3k} + \frac{2a}{2b+k} \frac{1}{\xi^2}, \\
 v_2(\xi) = \mp \sqrt{\frac{2a}{2b+k}} \frac{1}{\xi}, \quad q = 0, \\
 \\
 u_3(\xi) = \frac{2aq+1}{3k} + \frac{2a}{2b+k} [\sqrt{q} \tan(\sqrt{q}\xi)]^2, \\
 v_3(\xi) = \mp \sqrt{\frac{2a}{2b+k}} \sqrt{q} \tan(\sqrt{q}\xi), \quad q > 0, \\
 \\
 u_4(\xi) = \frac{2aq+1}{3k} + \frac{2a}{2b+k} [-\sqrt{-q} \coth(\sqrt{-q}\xi)]^2, \\
 v_4(\xi) = \mp \sqrt{\frac{2a}{2b+k}} \sqrt{-q} \coth(\sqrt{-q}\xi), \quad q < 0, \\
 \\
 u_5(\xi) = \frac{2aq+1}{3k} + \frac{2a}{2b+k} [\sqrt{q} \cot(\sqrt{q}\xi)]^2, \\
 v_5(\xi) = \mp \sqrt{\frac{2a}{2b+k}} \sqrt{q} \cot(\sqrt{q}\xi), \quad q > 0,
 \end{cases} \tag{36}$$

where $\xi = x - t$ (see Figure 2 and Figure 3).

4.3 Explicit power series solutions of system (1)

We shall seek a solution to reduced equations (A) of case 3 in Table 2 in a power series of the form

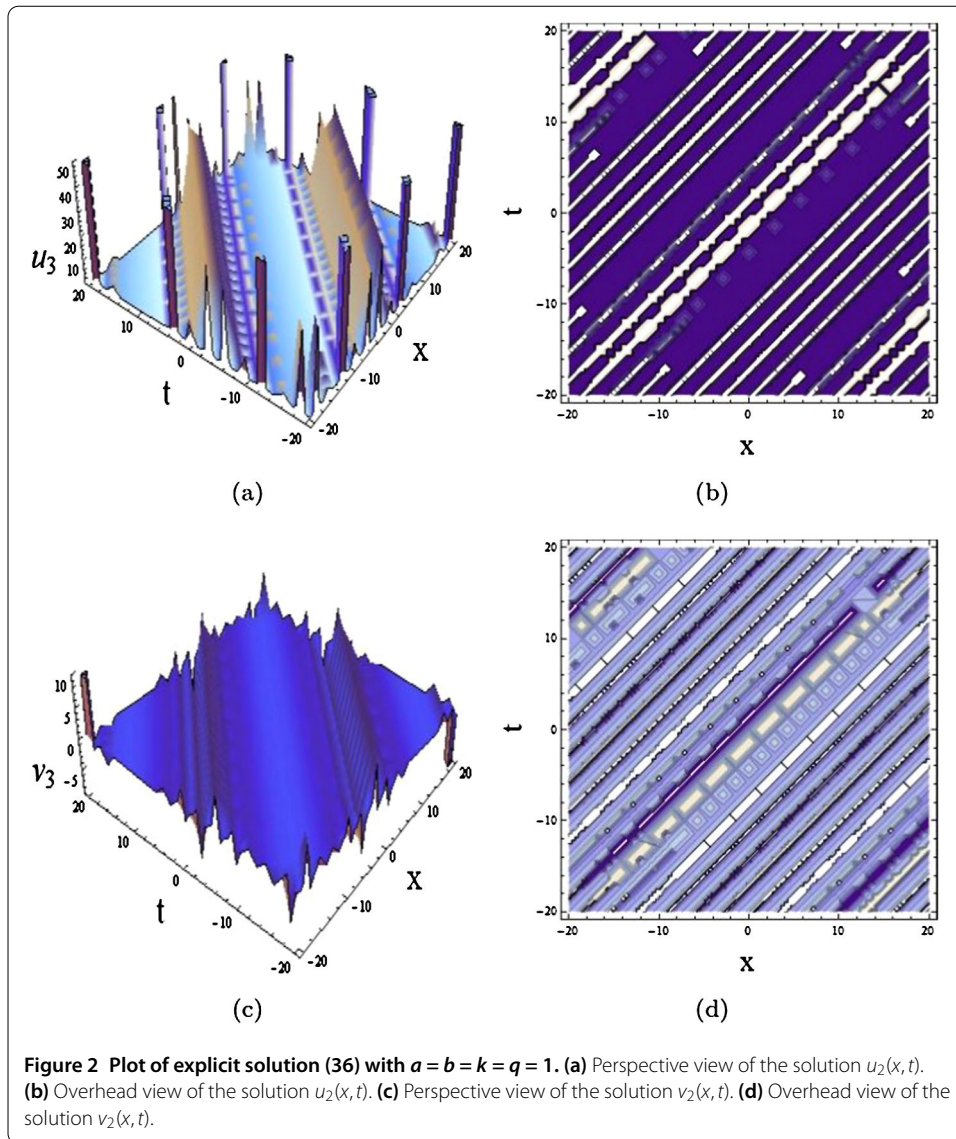
$$F(\xi) = \sum_{n=0}^{\infty} r_n \xi^n, \quad H(\xi) = \sum_{n=0}^{\infty} s_n \xi^n. \tag{37}$$

Substituting (37) into (A) and comparing coefficients, we obtain

$$\begin{aligned}
 r_0 &= 3s_0s_1, \\
 s_0 &= \frac{9kr_0s_1 + 9br_1s_0 - 18as_3}{2}.
 \end{aligned} \tag{38}$$

For the case $n = 0, 1, 2, \dots$, we have

$$\begin{aligned}
 n &\geq 0, \\
 r_n - 6 \sum_{l=0}^{n+1} (n+2-l) s_l s_{n+2-l} + 2r_{n+1} &= 0, \\
 -9k \sum_{l=1}^{n+1} (n+2-l) r_l s_{n+2-l} - 9b \sum_{l=1}^{n+1} (n+2-l) s_l r_{n+1-l} + s_n \\
 + 3a(n+1)(n+2)(n+3)s_{n+3} + 2s_{n+1} &= 0.
 \end{aligned} \tag{39}$$



Therefore, we get the following relationship between the coefficients r_i and s_i :

$$\begin{aligned}
 r_{n+1} &= -\frac{1}{2}r_n + 3 \sum_{l=0}^{n+1} (n+2-l)s_l s_{n+2-l}, \\
 s_{n+1} &= \frac{9}{2}k \sum_{l=1}^{n+1} (n+2-l)r_l s_{n+2-l} + \frac{9}{2}b \sum_{l=1}^{n+1} (n+2-l)s_l r_{n+1-l} - \frac{1}{2}s_n \\
 &\quad - \frac{3}{2}a(n+1)(n+2)(n+3)s_{n+3}.
 \end{aligned}
 \tag{40}$$

The exact power series solution to system (A) can be written as follows:

$$\begin{aligned}
 F(\xi) &= r_0 + r_1\xi + r_2\xi^2 + r_3\xi^3 + r_4\xi^4 + \dots, \\
 H(\xi) &= s_0 + s_1\xi + s_2\xi^2 + s_3\xi^3 + s_4\xi^4 + \dots.
 \end{aligned}
 \tag{41}$$

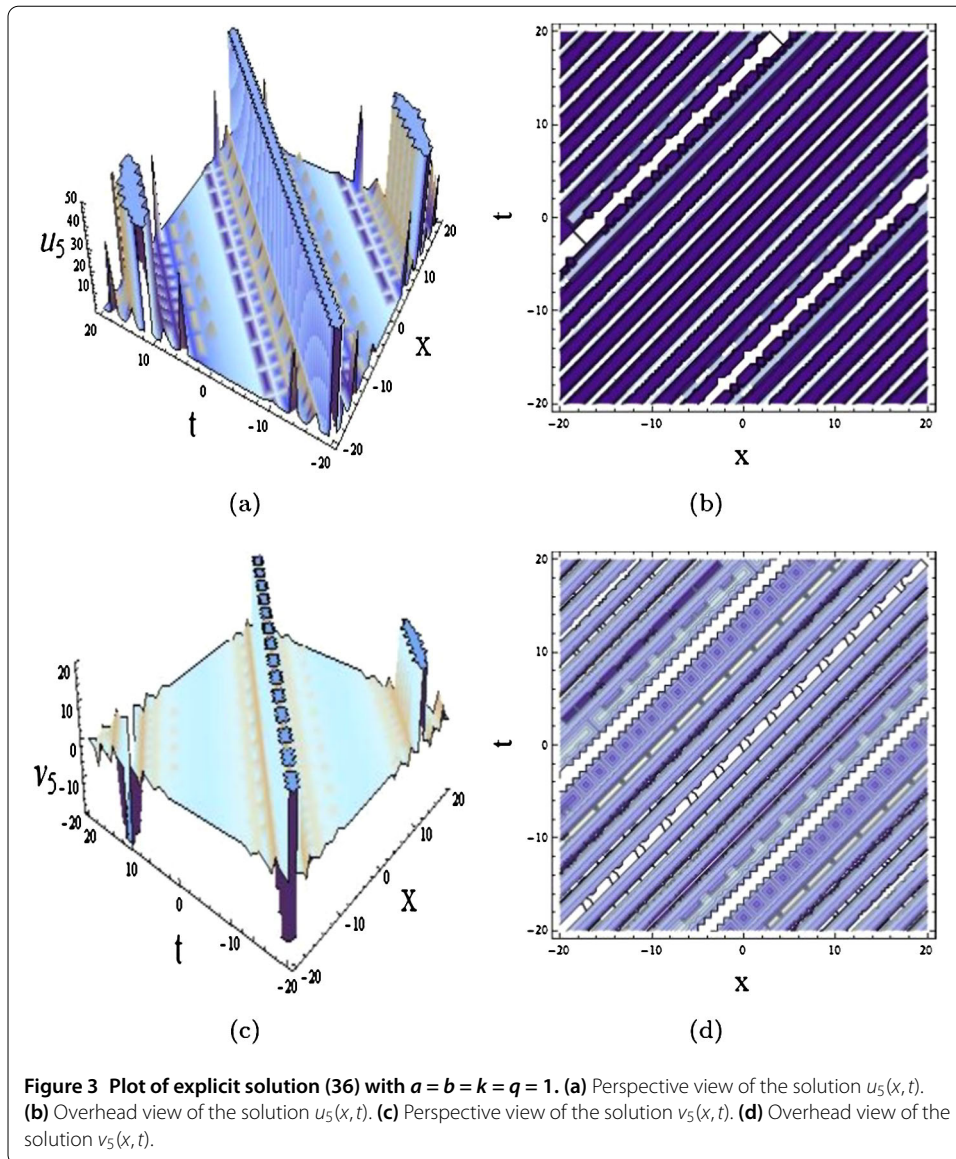


Figure 3 Plot of explicit solution (36) with $a = b = k = q = 1$. (a) Perspective view of the solution $u_5(x, t)$. (b) Overhead view of the solution $u_5(x, t)$. (c) Perspective view of the solution $v_5(x, t)$. (d) Overhead view of the solution $v_5(x, t)$.

Thus, we obtain the exact power series solution of system (1) as follows:

$$\begin{aligned}
 u(x, t) &= t^{-\frac{2}{3}} \left[r_0 + r_1 x t^{-\frac{1}{3}} + r_2 (x t^{-\frac{1}{3}})^2 + r_3 (x t^{-\frac{1}{3}})^3 + r_4 (x t^{-\frac{1}{3}})^4 + \dots \right], \\
 v(x, t) &= t^{-\frac{2}{3}} \left[s_0 + s_1 x t^{-\frac{1}{3}} + s_2 (x t^{-\frac{1}{3}})^2 + s_3 (x t^{-\frac{1}{3}})^3 + s_4 (x t^{-\frac{1}{3}})^4 + \dots \right].
 \end{aligned}
 \tag{42}$$

Remark 1 The power series method is a useful approach to solve higher order variable coefficient ordinary differential equations. A large number of solutions for ordinary differential equations are constructed by utilizing the method by Liu *et al.* [45, 46]. Moreover, we can show the convergence of the power series solution (42) just as the method from the paper [46]. If we regard the series solution (42) to a particular section, we get the polynomial solution. In addition, we may obtain the approximate solutions of system (1) by using Newton’s interpolating series [47].

5 Construction of conservation laws using Lie-Bäcklund symmetries

In this section, we briefly present the notations and theorem which are useful for constructing conservation laws with the aid of Lie-Bäcklund symmetries. For more detailed information, the reader is referred to the literature [20, 48–51].

Consider a system of PDEs

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(k)}) = 0, \quad \alpha = 1, 2, \dots, m, \tag{43}$$

where $u_{(k)}$ are the k th order derivatives, $x = (x^1, x^2, \dots, x^n)$, and $u = (u^1, u^2, \dots, u^m)$. u_i^α denotes the derivatives of u^α with respect to x^i , which is explicitly expressed by $u_i^\alpha = D_i(u^\alpha)$, where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, 2, \dots, n. \tag{44}$$

D_i is the total derivative operator. A Lie-Bäcklund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{j=1}^\infty \zeta_{i_1 i_2 \dots i_j}^\alpha \frac{\partial}{\partial u_{i_1 \dots i_j}^\alpha}, \tag{45}$$

where $\zeta_{i_1 i_2 \dots i_j}^\alpha$ are determined by the following relations:

$$\begin{aligned} \zeta_i^\alpha &= D_i(\eta^\alpha) - u_s^\alpha D_i(\xi^s), \\ \zeta_{i_1 i_2 \dots i_j}^\alpha &= D_{i_j}(\zeta_{i_1 i_2 \dots i_{j-1}}^\alpha) - u_{s i_1 i_2 \dots i_{j-1}}^\alpha D_{i_j}(\xi^s), \quad j > 1. \end{aligned} \tag{46}$$

The Lie-Bäcklund operators (45) which have characteristic functions are

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{j=1}^\infty D_{i_1} \dots D_{i_j}(W^\alpha) \frac{\partial}{\partial u_{i_1 \dots i_j}^\alpha}, \tag{47}$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$, $\alpha = 1, 2, \dots, m$ are the Lie characteristic functions. The Euler-Lagrange operator is represented by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{j=1}^\infty (-1)^j D_{i_1} \dots D_{i_j} \frac{\partial}{\partial u_{i_1 \dots i_j}^\alpha}, \quad \alpha = 1, 2, \dots, m. \tag{48}$$

A Noether operator associated with a Lie-Bäcklund operator X is defined as

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{j=1}^\infty D_{i_1} \dots D_{i_j}(W^\alpha) \frac{\delta}{\delta u_{i i_1 \dots i_j}^\alpha}, \quad i = 1, 2, \dots, n. \tag{49}$$

For example, the first order derivative of the Euler-Lagrange operator is given by

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{j=1}^\infty (-1)^j D_{i_1} \dots D_{i_j} \frac{\partial}{\partial u_{i i_1 \dots i_j}^\alpha}, \quad i = 1, 2, \dots, n, \alpha = 1, 2, \dots, m. \tag{50}$$

A vector $T = (T^1, T^2, \dots, T^n)$ is a conserved vector of system (43) if T^i satisfies

$$D_i T^i|_{(43)} = 0. \tag{51}$$

The adjoint equations for the system of PDEs (43) are defined by

$$E_\alpha^*(x, u, v, \dots, u_{(k)}, v_{(k)}) = 0, \quad \alpha = 1, \dots, m, \tag{52}$$

where

$$E_\alpha^*(x, u, v, \dots, u_{(k)}, v_{(k)}) = \frac{\delta(v^\beta E_\beta)}{\delta u^\alpha}, \quad \alpha, \beta = 1, \dots, m, v = v(x). \tag{53}$$

The relationship between Lie-Bäcklund symmetries and conservation laws can be established by the following theorem.

Theorem 3 Every Lie-Bäcklund symmetry generator [52, 53]

$$U = \eta^\alpha(x, u, u_{(1)}, u_{(2)}, \dots) \frac{\partial}{\partial u^\alpha}, \quad \alpha = 1, 2, \dots, m, \tag{54}$$

of PDEs (43) can give rise to a conservation law for the system consisting of (43) and the adjoint system (52). The components T^i of the conserved vector $T = (T^1, \dots, T^n)$ are determined by the formula

$$\begin{aligned} T^i = & W^\alpha \left[\frac{\partial L}{\partial u_i^\alpha} - D_j \left(\frac{\partial L}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial L}{\partial u_{ijk}^\alpha} \right) - \dots \right] \\ & + D_j (W^\alpha) \left[\left(\frac{\partial L}{\partial u_{ij}^\alpha} \right) - D_k \left(\frac{\partial L}{\partial u_{ijk}^\alpha} \right) + \dots \right] \\ & + D_j D_k (W^\alpha) \left(\frac{\partial L}{\partial u_{ijk}^\alpha} - \dots \right) + \dots, \end{aligned} \tag{55}$$

where the formal Lagrangian is

$$L = v^\alpha E_\alpha(x, u, \dots, u_{(k)}). \tag{56}$$

Remark 2 For the formal Lagrangian L which has a third order derivative, the conservation laws equation (55) can be written

$$\begin{aligned} T^i = & W^\alpha \left[\frac{\partial L}{\partial u_i^\alpha} - D_j \left(\frac{\partial L}{\partial u_{ij}^\alpha} \right) + D_j D_k \left(\frac{\partial L}{\partial u_{ijk}^\alpha} \right) \right] + D_j (W^\alpha) \left[\left(\frac{\partial L}{\partial u_{ij}^\alpha} \right) - D_k \left(\frac{\partial L}{\partial u_{ijk}^\alpha} \right) \right] \\ & + D_j D_k (W^\alpha) \left(\frac{\partial L}{\partial u_{ijk}^\alpha} \right). \end{aligned} \tag{57}$$

6 Conservation laws of the DSWS

The Lie-Bäcklund transformation group [52, 53] can be regarded as a tangent transformation group. It is the extension of one-parameter continuous symmetry group transformations. Lie-Bäcklund symmetry group of (1) will be generated by the vector field of the

form

$$U = \eta^u(x, t, u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx}) \frac{\partial}{\partial u} + \eta^v(x, t, u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx}) \frac{\partial}{\partial v},$$

where U satisfies $U^{(3)} \Delta_i|_{\Delta_i} = 0, i = 1, 2$. By applying the third prolongation $U^{(3)}$ to system (1), we obtain an overdetermined system of PDEs. The general solutions of the overdetermined system are

$$\begin{aligned} \eta^u &= 6\vartheta_3tvv_x + 2\vartheta_2vv_x - \vartheta_3u_{xx} + \vartheta_1u_x - 2\vartheta_3u, \\ \eta^v &= \vartheta_1v_x + 3\vartheta_2bu_xv + 3\vartheta_2kuv_x - a\vartheta_2v_{xxx} + 9\vartheta_3btu_xv - 2\vartheta_3v - \vartheta_3v_{xx} + 9\vartheta_3ktuv_x \\ &\quad - 3\vartheta_3atv_{xxx}, \end{aligned}$$

where $\vartheta_1, \vartheta_2, \vartheta_3$ are arbitrary constants. Thus, the third order Lie-Bäcklund symmetries of (1) are given by

$$\begin{aligned} U_1 &= u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v}, \\ U_2 &= 2vv_x \frac{\partial}{\partial u} + (3bu_xv + 3kuv_x - av_{xxx}) \frac{\partial}{\partial v}, \\ U_3 &= (6tvv_x - u_{xx} - 2u) \frac{\partial}{\partial u} + (9btu_xv + 9ktuv_x - 3atv_{xxx} - v_{xx} - 2v) \frac{\partial}{\partial v}. \end{aligned}$$

For system (1), the formal Lagrangian is

$$L = \phi(x, t)(u_t + 2vv_x) + \tau(x, t)(v_t - av_{xxx} + 3bu_xv + 3kuv_x). \tag{58}$$

The adjoint equations of system (1) are given by

$$\begin{aligned} -\phi_t - 3b(\tau_xv + \tau v_x) + 3k\tau v_x &= 0, \\ -2\phi_xv - \tau_t + a\tau_{xxx} + 3bu_x\tau - 3k(\tau_xu + \tau u_x) &= 0, \end{aligned} \tag{59}$$

where $\phi(x, t)$ and $\tau(x, t)$ are the new dependent variables. On the basis of equation (57), one can derive components of the conservation vector for system (1) as follows:

$$\begin{aligned} T^t &= W^1 \frac{\partial L}{\partial u_t} + W^2 \frac{\partial L}{\partial v_t}, \\ T^x &= W^1 \frac{\partial L}{\partial u_x} + W^2 \left[\frac{\partial L}{\partial v_x} + D_x^2 \left(\frac{\partial L}{\partial v_{xxx}} \right) \right] \\ &\quad + D_x(W^2) \left[-D_x \left(\frac{\partial L}{\partial v_{xxx}} \right) \right] + D_x^2(W^2) \frac{\partial L}{\partial v_{xxx}}, \end{aligned} \tag{60}$$

where $W^1 = \eta^u$ and $W^2 = \eta^v$.

- (1) For the Lie-Bäcklund symmetry generator $U_1 = u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v}$, the Lie characteristic functions are $W^1 = u_x$ and $W^2 = v_x$. Using equation (60), we obtain the following components of the conserved vector:

$$\begin{aligned} T_1^t &= \phi u_x + \tau v_x, \\ T_1^x &= 3b\tau u_xv + 2\phi vv_x + 3k\tau uv_x - a\tau_{xx}v_x + a\tau_xv_{xx} - a\tau v_{xxx}. \end{aligned} \tag{61}$$

(II) The Lie-Bäcklund symmetry generator $U_2 = 2vv_x \frac{\partial}{\partial u} + (3bu_x v + 3kuv_x - av_{xxx}) \frac{\partial}{\partial v}$ has the Lie characteristic functions $W^1 = 2vv_x$ and $W^2 = 3bu_x v + 3kuv_x - av_{xxx}$. Using equation (60), we obtain the following components of the conserved vector:

$$\begin{aligned}
 T_2^t &= 2\phi vv_x + (3bu_x v + 3kuv_x - av_{xxx})\tau, \\
 T_2^x &= 6b\tau v^2 v_x + (3bu_x v + 3kuv_x - av_{xxx})(3k\tau u + 2\phi v - a\tau_{xx}) \\
 &\quad + a\tau_x(3bu_{xx} v + 3bu_x v_x + 3ku_x v_x + 3kuv_{xx} - av_{xxx}) \\
 &\quad - a\tau(3bu_{xxx} v + 6bu_{xx} v_x + 3bu_x v_{xx} + 3ku_{xx} v_x + 6ku_x v_{xx} \\
 &\quad + 3kuv_{xxx} - av_{xxxx}).
 \end{aligned} \tag{62}$$

(III) Finally, for the Lie-Bäcklund symmetry generator

$$U_3 = (6tvv_x - u_{xx} - 2u) \frac{\partial}{\partial u} + (9btu_x v + 9ktuv_x - 3atv_{xxx} - v_{xx} - 2v) \frac{\partial}{\partial v},$$

the Lie characteristic functions are $W^1 = 6tvv_x - u_{xx} - 2u$ and $W^2 = 9btu_x v + 9ktuv_x - 3atv_{xxx} - v_{xx} - 2v$. Using equation (60), we obtain the following components of the conserved vector:

$$\begin{aligned}
 T_3^t &= \phi(6tvv_x - u_{xx} - 2u) + (9btu_x v + 9ktuv_x - 3atv_{xxx} - v_{xx} - 2v)\tau, \\
 T_3^x &= 3b\tau v(6tvv_x - u_{xx} - 2u) \\
 &\quad + (9btu_x v + 9ktuv_x - 3atv_{xxx} - v_{xx} - 2v)(3k\tau u + 2\phi v - a\tau_{xx}) \\
 &\quad + a\tau_x(9btu_{xx} v + 9btu_x v_x + 9ktu_x v_x + 9ktuv_{xx} - 3atv_{xxx} - v_{xxx} - 2v_x) \\
 &\quad - a\tau(9btu_{xxx} v + 18btu_{xx} v_x + 9btu_x v_{xx} + 9ktu_{xx} v_x + 18ktu_x v_{xx} \\
 &\quad + 9ktuv_{xxx} - 3atv_{xxxx} - v_{xxxx} - 2v_{xx}).
 \end{aligned} \tag{63}$$

7 Conclusions

Lie symmetry analysis has been employed to investigate Lie point symmetries of the Drinfeld-Sokolov-Wilson system. The symmetries X_1 , X_2 , and X_3 form a three-dimensional Lie algebra L_3 . By using Ibragimov’s method, we have derived an optimal system of one-dimensional subalgebra. It is proved that the optimal system has five operators. Based on the optimal system, we have considered the symmetry reductions and group invariant solutions of the DSWS. To the best of our knowledge, very little work has been devoted to constructing conservation laws of NLPDEs by using Lie-Bäcklund symmetries. Lie-Bäcklund symmetries of the DSWS have been derived. The method of constructing conservation laws of NLPDEs with the aid of a new conservation theorem associated with Lie-Bäcklund symmetries has been presented. Conservation laws of the DSWS have been constructed by using this method.

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