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Blow-up phenomena for p-Laplacian parabolic problems with Neumann boundary conditions

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Abstract

In this paper, we deal with the blow-up and global solutions of the following p-Laplacian parabolic problems with Neumann boundary conditions:

$$\begin{cases} (g(u))_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u) + k(t)f(u) & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}, \end{cases}$$

where $p > 2$ and Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary $\partial\Omega$. By introducing some appropriate auxiliary functions and technically using maximum principles, we establish conditions to guarantee that the solution blows up in some finite time or remains global. In addition, the upper estimates of blow-up rate and global solution are specified. We also obtain an upper bound of blow-up time.

MSC: 35K65; 35B40

Keywords: blow-up; p-Laplacian equation; Neumann boundary condition

1 Introduction

In the past decades, many authors have researched the blow-up problems of p-Laplacian elliptic and parabolic equations (see, for instance, [1–18]). In this paper, we study the blow-up phenomena of the following p-Laplacian parabolic problems with Neumann boundary conditions:

$$\begin{cases} (g(u))_t = \nabla \cdot (|\nabla u|^{p-2}\nabla u) + k(t)f(u) & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}. \end{cases} \tag{1.1}$$

In (1.1), $p > 2$, Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary $\partial\Omega$, T is the blow-up time if blow-up occurs, otherwise $T = +\infty$, $g(s)$ is a $C^3(\overline{\mathbb{R}_+})$ function satisfying $g'(s) > 0, s \in \overline{\mathbb{R}_+}$, $f(s)$ is a positive $C^2(\overline{\mathbb{R}_+})$ function, $k(t)$ is a positive $C^2(\overline{\mathbb{R}_+})$ function, and $u_0(x)$ is a positive $C^2(\overline{\Omega})$ function which satisfies the compatibility condition

$\partial u_0(x)/\partial n = 0, x \in \partial\Omega$. With the aid of the regularity theorem [19], we know that the nonnegative classical solution of (1.1) satisfies $u \in C^3(\Omega \times (0,T)) \cap C^2(\overline{\Omega} \times [0,T))$.

There is a lot of papers dealing with the blow-up and global solutions of parabolic problems with Neumann boundary conditions. We refer readers to [20–29] and the references therein. In [25], Payne and Philippin studied the following problems:

$$\begin{cases} u_t = \Delta u + k(t)f(u) & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary. By means of the differential inequality technique, they set up conditions on data to guarantee that $u(x,t)$ blows up at some finite time. Moreover, an upper bound for blow-up time was obtained. When $\Omega \subset \mathbb{R}^3$, a lower bound of blow-up time was also derived if blow-up occurred. In [20, 24], An et al. considered the following problems:

$$\begin{cases} u_t = \nabla \cdot (a(u)\nabla u) + f(u) & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary. By using the differential inequality technique, they gave the upper and lower bounds for blow-up time when the blow-up of the solution occurred. In [28], Gao et al. researched the following problems:

$$\begin{cases} (g(u))_t = \nabla \cdot (a(u)\nabla u) + f(u) & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{\Omega}, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary. By introducing some auxiliary functions and using maximum principles, they established some conditions on nonlinearities to guarantee the existence of blow-up solution or global solution. Furthermore, the upper estimates of global solution and blow-up rate were obtained. They gave also an upper bound for blow-up time of the solution.

As far as we know, there is little information about the blow-up problem of (1.1). Inspired by the above research, in this paper, we study the blow-up phenomena of (1.1). However, it is difficult to use the differential inequality technique employed in [20, 24, 25] to study the blow-up problem of (1.1). In this paper, we technically use maximum principles to deal with problem (1.1). We note that the auxiliary functions in [28] are not suitable for (1.1). Therefore, we must introduce some new auxiliary functions to complete our research.

Our paper is constructed as follows. In Section 2, we establish conditions that ensure the solution blows up in some finite time. At the same time, an upper bound of blow-up time and an upper estimate of blow-up solution are obtained. Section 3 is dedicated to finding conditions to guarantee that the solution remains global. We also give an upper estimate of the global solution. In Section 4, two examples are presented to illustrate the abstract results of this paper.

For convenience, throughout the paper, the notation $u,_i = \frac{\partial u}{\partial x_i}, u,_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ will be used. We will also adopt the summation convention (i.e., summation over repeated indies), e.g.,

$$u,_i u,_j u,_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

2 Blow-up solution

In this section, we set up some conditions for blow-up to occur and derive an upper bound for blow-up time. In order to achieve this goal, we need to construct three auxiliary functions as follows:

$$\phi(x,t) = g'(u)u_t - k(t)f(u), \quad (x,t) \in \overline{\Omega} \times [0,T), \tag{2.1}$$

$$\psi(s) = \int_0^s k(\tau)\,\mathrm{d}\tau, \quad s \in \overline{\mathbb{R}_+}, \tag{2.2}$$

$$\eta(s) = \int_s^{+\infty} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau, \quad s \in \overline{\mathbb{R}_+}. \tag{2.3}$$

Now we have

$$\psi'(s) = k(s) > 0, \quad s \in \mathbb{R}_+,$$

which ensures the existence of the inverse function ψ^{-1} of ψ. Similarly,

$$\eta'(s) = -\frac{g'(s)}{f(s)} < 0, \quad s \in \mathbb{R}_+$$

implies that there exists the inverse function η^{-1}. By using the above three auxiliary functions, we can get the following Theorem 2.1.

Theorem 2.1 *Let u be a nonnegative classical solution of problem* (1.1). *Assume that the initial value u_0 satisfies*

$$\nabla \cdot \left(|\nabla u_0|^{p-2} \nabla u_0\right) \geq 0, \quad x \in \overline{\Omega}. \tag{2.4}$$

In addition, we assume that the functions f, g and k satisfy

$$\int_{M_0}^{\infty} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau < \int_0^{+\infty} k(\tau)\,\mathrm{d}\tau, \quad M_0 = \max_{\overline{\Omega}} u_0(x) \tag{2.5}$$

and

$$g'(s)\left(\frac{f'(s)}{g'(s)}\right)' - f(s)\left(\frac{g''(s)}{(g'(s))^2}\right)' \geq 0, \quad s \in \overline{\mathbb{R}_+}. \tag{2.6}$$

Then the solution u blows up at some finite time T and

$$T \leq \psi\left(\int_{M_0}^{+\infty} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau\right),$$

$$u(x,t) \le \eta^{-1}\left(\int_t^T k(\tau)\,\mathrm{d}\tau\right).$$

Proof Consider the auxiliary function $\phi(x,t)$ defined in (2.1). By direct calculation, we have

$$\phi_{,i} = g'' u_t u_{,i} + g' u_{t,i} - kf' u_{,i} \tag{2.7}$$

and

$$\phi_{,ij} = g''' u_t u_{,i} u_{,j} + g'' u_{t,j} u_{,i} + g'' u_{t,i} u_{,j} + g'' u_t u_{,ij} + g' u_{t,ij} - kf'' u_{,i} u_{,j} - kf' u_{,ij}. \tag{2.8}$$

It follows from (2.8) that

$$\Delta\phi = \phi_{,ii} = g'''|\nabla u|^2 u_t + 2g''(\nabla u \cdot \nabla u_t) + g'' u_t \Delta u + g' \Delta u_t - kf''|\nabla u|^2 - kf' \Delta u. \tag{2.9}$$

Now we use the first equation of (1.1) to obtain

$$\begin{aligned}
\phi_t &= \left[g'(u)u_t - k(t)f(u)\right]_t = \left[\left(g(u)\right)_t - k(t)f(u)\right]_t = \left[\nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right)\right]_t \\
&= \left[|\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4} u_{,i} u_{,j} u_{,ij}\right]_t \\
&= (p-2)|\nabla u|^{p-4}(\nabla u \cdot \nabla u_t)\Delta u + |\nabla u|^{p-2}\Delta u_t \\
&\quad + (p-2)(p-4)|\nabla u|^{p-6}(\nabla u \cdot \nabla u_t) u_{,i} u_{,j} u_{,ij} \\
&\quad + 2(p-2)|\nabla u|^{p-4} u_{t,i} u_{,j} u_{,ij} + (p-2)|\nabla u|^{p-4} u_{,i} u_{,j} u_{t,ij}.
\end{aligned} \tag{2.10}$$

Making use of (2.8), (2.9) and (2.10), we obtain

$$\begin{aligned}
&\frac{|\nabla u|^{p-2}}{g'}\Delta\phi + (p-2)\frac{|\nabla u|^{p-4}}{g'} u_{,i} u_{,j} \phi_{,ij} - \phi_t \\
&\quad = (p-1)\frac{g'''}{g'}|\nabla u|^p u_t + 2(p-1)\frac{g''}{g'}|\nabla u|^{p-2}(\nabla u \cdot \nabla u_t) \\
&\qquad + \frac{g''}{g'}|\nabla u|^{p-2} u_t \Delta u - (p-1)\frac{kf''}{g'}|\nabla u|^p \\
&\qquad - \frac{kf'}{g'}|\nabla u|^{p-2}\Delta u + (p-2)\frac{g''}{g'}|\nabla u|^{p-4} u_t u_{,i} u_{,j} u_{,ij} - (p-2)\frac{kf'}{g'}|\nabla u|^{p-4} u_{,i} u_{,j} u_{,ij} \\
&\qquad - (p-2)|\nabla u|^{p-4}(\nabla u \cdot \nabla u_t)\Delta u - (p-2)(p-4)|\nabla u|^{p-6}(\nabla u \cdot \nabla u_t) u_{,i} u_{,j} u_{,ij} \\
&\qquad - 2(p-2)|\nabla u|^{p-4} u_{t,i} u_{,j} u_{,ij}.
\end{aligned} \tag{2.11}$$

It follows from (2.7) that

$$u_{t,i} = \frac{1}{g'}\phi_{,i} - \frac{g''}{g'} u_t u_{,i} + \frac{kf'}{g'} u_{,i} \tag{2.12}$$

and

$$\nabla u_t = \frac{1}{g'}\nabla\phi - \frac{g''}{g'} u_t \nabla u + \frac{kf'}{g'}\nabla u. \tag{2.13}$$

Substituting (2.12) and (2.13) into (2.11), we arrive at

$$\begin{aligned}
&\frac{|\nabla u|^{p-2}}{g'}\Delta\phi + (p-2)\frac{|\nabla u|^{p-4}}{g'}u_{,i}u_{,j}\phi_{,ij} \\
&\quad + \frac{|\nabla u|^{p-6}}{g'}\biggl((p-2)|\nabla u|^2\Delta u + (p-2)(p-4)u_{,i}u_{,j}u_{,ij} \\
&\quad - 2(p-1)\frac{g''}{g'}|\nabla u|^4\biggr)(\nabla u\cdot\nabla\phi) \\
&\quad + 2(p-2)\frac{|\nabla u|^{p-4}}{g'}u_{,i}u_{,ij}\phi_{,i} - \phi_t \\
&= \biggl((p-1)\frac{g'''}{g'} - 2(p-1)\frac{(g'')^2}{(g')^2}\biggr)|\nabla u|^p u_t + \biggl(2(p-1)\frac{kf'g''}{(g')^2} - (p-1)\frac{kf''}{g'}\biggr)|\nabla u|^p \\
&\quad + (p-1)\frac{g''}{g'}|\nabla u|^{p-2}u_t\Delta u \\
&\quad - (p-1)\frac{kf'}{g'}|\nabla u|^{p-2}\Delta u + (p-1)(p-2)\frac{g''}{g'}|\nabla u|^{p-4}u_{,i}u_{,j}u_{,ij}u_t \\
&\quad - (p-1)(p-2)\frac{kf'}{g'}|\nabla u|^{p-4}u_{,i}u_{,j}u_{,ij}.
\end{aligned} \tag{2.14}$$

The first equation of (1.1) implies

$$|\nabla u|^{p-2}\Delta u = g'u_t - (p-2)|\nabla u|^{p-4}u_{,i}u_{,j}u_{,ij} - kf. \tag{2.15}$$

Inserting (2.15) into (2.14), we derive

$$\begin{aligned}
&\frac{|\nabla u|^{p-2}}{g'}\Delta\phi + (p-2)\frac{|\nabla u|^{p-4}}{g'}u_{,i}u_{,j}\phi_{,ij} \\
&\quad + \frac{|\nabla u|^{p-6}}{g'}\biggl((p-2)|\nabla u|^2\Delta u + (p-2)(p-4)u_{,i}u_{,j}u_{,ij} \\
&\quad - 2(p-1)\frac{g''}{g'}|\nabla u|^4\biggr)(\nabla u\cdot\nabla\phi) \\
&\quad + 2(p-2)\frac{|\nabla u|^{p-4}}{g'}u_{,i}u_{,ij}\phi_{,i} - \phi_t \\
&= \biggl((p-1)\frac{g'''}{g'} - 2(p-1)\frac{(g'')^2}{(g')^2}\biggr)|\nabla u|^p u_t + \biggl(2(p-1)\frac{kf'g''}{(g')^2} - (p-1)\frac{kf''}{g'}\biggr)|\nabla u|^p \\
&\quad + (p-1)g''(u_t)^2 - \biggl((p-1)\frac{kfg''}{g'} + (p-1)kf'\biggr)u_t + (p-1)\frac{k^2ff'}{g'}.
\end{aligned} \tag{2.16}$$

By (2.1), we have

$$u_t = \frac{1}{g'}\phi + \frac{kf}{g'}. \tag{2.17}$$

Substituting (2.17) into (2.16), we deduce

$$\begin{aligned}
&\frac{|\nabla u|^{p-2}}{g'}\Delta\phi + (p-2)\frac{|\nabla u|^{p-4}}{g'}u_{,i}u_{,j}\phi_{,ij} \\
&\quad + \frac{|\nabla u|^{p-6}}{g'}\biggl((p-2)|\nabla u|^2\Delta u + (p-2)(p-4)u_{,i}u_{,j}u_{,ij}
\end{aligned}$$

$$
\begin{aligned}
& - 2(p-1)\frac{g''}{g'}|\nabla u|^4\Big)(\nabla u\cdot\nabla\phi) \\
& + 2(p-2)\frac{|\nabla u|^{p-4}}{g'}u_{,i}u_{,ij}\phi_{,i} \\
& + (p-1)\frac{1}{g'}\left\{kf' - \frac{1}{g'}\left[g''(\phi+kf) + \left(g''' - 2\frac{(g'')^2}{g'}\right)|\nabla u|^p\right]\right\}\phi - \phi_t \\
= {}& (1-p)k\left[g'\left(\frac{f'}{g'}\right)' - f\left(\frac{g''}{(g')^2}\right)'\right]|\nabla u|^p.
\end{aligned}
\tag{2.18}
$$

Assumption (2.6) implies that the right-hand side in equality (2.18) is nonpositive. In other words, we have

$$
\begin{aligned}
& \frac{|\nabla u|^{p-2}}{g'}\Delta\phi + (p-2)\frac{|\nabla u|^{p-4}}{g'}u_{,i}u_{,j}\phi_{,ij} \\
& + \frac{|\nabla u|^{p-6}}{g'}\Big((p-2)|\nabla u|^2\Delta u + (p-2)(p-4)u_{,i}u_{,j}u_{,ij} \\
& - 2(p-1)\frac{g''}{g'}|\nabla u|^4\Big)(\nabla u\cdot\nabla\phi) \\
& + 2(p-2)\frac{|\nabla u|^{p-4}}{g'}u_{,i}u_{,ij}\phi_{,i} \\
& + (p-1)\frac{1}{g'}\left\{kf' - \frac{1}{g'}\left[g''(\phi+kf)\right.\right. \\
& \left.\left. + \left(g''' - 2\frac{(g'')^2}{g'}\right)|\nabla u|^p\right]\right\}\phi - \phi_t \le 0 \quad \text{in } \Omega\times(0,T).
\end{aligned}
\tag{2.19}
$$

Making use of the maximum principle [30], we have the following three possible cases where ϕ may take its minimum value:

(i) for $t = 0$,

(ii) at a point where $|\nabla u| = 0$,

(iii) on the boundary $\partial\Omega\times(0,T)$.

We first consider the first case. It follows from (2.4) that

$$
\begin{aligned}
\phi(x,0) &= g'(u_0)u_{0t} - k(0)f(u_0) = \big(g(u_0)\big)_t - k(0)f(u_0) \\
&= \nabla\cdot\big(|\nabla u_0|^{p-2}\nabla u_0\big) \ge 0, \quad x\in\overline{\Omega}.
\end{aligned}
\tag{2.20}
$$

Then we consider the second case. Assume that $(\bar{x},\bar{t})\in\Omega\times(0,T)$ is a point where $|\nabla u(\bar{x},\bar{t})| = 0$. Since

$$
\begin{aligned}
\big|\phi(x,t)\big| &= \big|\nabla\cdot\big(|\nabla u|^{p-2}\nabla u\big)\big| = \big||\nabla u|^{p-2}\Delta u + |\nabla u|^{p-4}u_{,i}u_{,j}u_{,ij}\big| \\
&\le |\nabla u|^{p-2}|\Delta u| + |\nabla u|^{p-4}|\nabla u||\nabla u||u_{,ij}| = |\nabla u|^{p-2}\big(|\Delta u| + |u_{,ij}|\big),
\end{aligned}
$$

the fact that $p > 2$ and $|\nabla u(\bar{x},\bar{t})| = 0$ imply

$$
\big|\phi(\bar{x},\bar{t})\big| \le \big|\nabla u(\bar{x},\bar{t})\big|^{p-2}\big(\big|\Delta u(\bar{x},\bar{t})\big| + \big|u_{,ij}(\bar{x},\bar{t})\big|\big) = 0.
\tag{2.21}
$$

Hence, we have $\phi(\bar{x},\bar{t}) = 0$. Finally, we consider the third case. Applying the boundary condition of (1.1), we get

$$\frac{\partial \phi}{\partial n} = g''(u)u_t\frac{\partial u}{\partial n} + g'\frac{\partial u_t}{\partial n} - f'(u)\frac{\partial u}{\partial n} = g'(u)\left(\frac{\partial u}{\partial n}\right)_t = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{2.22}$$

Combining (2.20)-(2.22) and the maximum principle, we now get that the minimum of ϕ in $\overline{\Omega} \times [0,T)$ is nonnegative. Thus

$$\phi \geq 0 \quad \text{in } \overline{\Omega} \times [0,T);$$

that is,

$$\frac{g'(u)}{f(u)}u_t \geq k(t). \tag{2.23}$$

At the point $x^* \in \overline{\Omega}$, where $u_0(x^*) = M_0$, we integrate (2.23) over $[0,t]$ to obtain

$$\int_0^t \frac{g'(u)}{f(u)}u_t\,\mathrm{d}t = \int_{M_0}^{u(x^*,t)} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau \geq \int_0^t k(\tau)\,\mathrm{d}\tau, \tag{2.24}$$

which ensures that u blows up in some finite time T. In fact, suppose that u remains global, then for any $t > 0$, we have

$$\int_{M_0}^{+\infty} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau \geq \int_{M_0}^{u(x^*,t)} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau \geq \int_0^t k(\tau)\,\mathrm{d}\tau. \tag{2.25}$$

Letting $t \to +\infty$ in (2.25), we deduce

$$\int_{M_0}^{+\infty} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau \geq \int_0^{+\infty} k(\tau)\,\mathrm{d}\tau,$$

which contradicts assumption (2.5). This shows that u blows up in some finite time T. Furthermore, taking the limit as $t \to T^-$ in (2.24), we arrive at

$$\lim_{t\to T^-} \int_{M_0}^{u(x^*,t)} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau \geq \lim_{t\to T^-} \int_0^t k(\tau)\,\mathrm{d}\tau$$

and

$$\int_{M_0}^{+\infty} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau \geq \int_0^T k(\tau)\,\mathrm{d}\tau = \psi(T). \tag{2.26}$$

It follows from (2.26) that

$$T \leq \psi^{-1}\left(\int_{M_0}^{+\infty} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau\right).$$

For each fixed x, we integrate inequality (2.23) over $[t,t^*]$ $(0 < t < t^* < T)$ to obtain

$$\eta\big(u(x,t)\big) \geq \eta\big(u(x,t)\big) - \eta\big(u(x,t^*)\big) = \int_{u(x,t)}^{u(x,t^*)} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau \geq \int_t^{t^*} k(\tau)\,\mathrm{d}\tau. \tag{2.27}$$

Letting $t^* \to T^-$ in (2.27), we deduce

$$\eta\big(u(x,t)\big) \geq \int_t^T k(\tau)\,\mathrm{d}\tau,$$

from which we have

$$u(x,t) \leq \eta^{-1}\left(\int_t^T k(\tau)\,\mathrm{d}\tau\right).$$

The proof of Theorem 2.1 is complete. □

Remark 2.1 When

$$\int_0^{+\infty} k(\tau)\,\mathrm{d}\tau = +\infty,$$

assumption (2.5) means

$$\int_{M_0}^{+\infty} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau < +\infty, \quad M_0 = \max_{\overline{\Omega}} u_0(x).$$

3 Global solution

In this section, we establish some conditions on functions f, g, k, and initial value u_0 to ensure that the solution of (1.1) remains global. In order to accomplish this task, in addition to using the auxiliary function ϕ defined in (2.1), we also need to construct the following auxiliary function:

$$\zeta(s) = \int_0^s \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau, \quad s \in \overline{\mathbb{R}_+}. \tag{3.1}$$

Now we have

$$\zeta'(s) = \frac{g'(s)}{f(s)} > 0, \quad s \in \mathbb{R}_+,$$

which guarantees the existence of the inverse function ζ^{-1} of function ζ. The main result of this section is the following Theorem 3.1.

Theorem 3.1 *Let u be a nonnegative classical solution of problem* (1.1). *Suppose that the initial value u_0 satisfies*

$$\nabla \cdot \big(|\nabla u_0|^{p-2}\nabla u_0\big) \leq 0, \quad x \in \overline{\Omega}, \tag{3.2}$$

and the functions f, g and k satisfy

$$\int_{M_0}^{\infty} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau \geq \int_0^{+\infty} k(\tau)\,\mathrm{d}\tau, \quad M_0 = \max_{\overline{\Omega}} u_0(x) \tag{3.3}$$

and

$$g'(s)\left(\frac{f'(s)}{g'(s)}\right)' - f(s)\left(\frac{g''(s)}{(g'(s))^2}\right)' \leq 0, \quad s \in \overline{\mathbb{R}_+}. \tag{3.4}$$

Then u exists globally and

$$u(x,t) \le \zeta^{-1}\left(\int_0^t k(\tau)\,\mathrm{d}\tau + \zeta\big(u_0(x)\big)\right).$$

Proof It follows from (2.18) and (3.4) that

$$\begin{aligned}
&\frac{|\nabla u|^{p-2}}{g'}\Delta\phi + (p-2)\frac{|\nabla u|^{p-4}}{g'}u_{,i}u_{,j}\phi_{,ij} \\
&+ \frac{|\nabla u|^{p-6}}{g'}\bigg((p-2)|\nabla u|^2\Delta u + (p-2)(p-4)u_{,i}u_{,j}u_{,ij} \\
&- 2(p-1)\frac{g''}{g'}|\nabla u|^4\bigg)(\nabla u\cdot\nabla\phi) \\
&+ 2(p-2)\frac{|\nabla u|^{p-4}}{g'}u_{,i}u_{,ij}\phi_{,i} \\
&+ (p-1)\frac{1}{g'}\bigg\{kf' - \frac{1}{g'}\bigg[g''(\phi+kf) \\
&+ \bigg(g''' - 2\frac{(g'')^2}{g'}\bigg)|\nabla u|^p\bigg]\bigg\}\phi - \phi_t \ge 0 \quad \text{in } \Omega\times(0,T).
\end{aligned}$$

The maximum principle implies that ϕ may take its maximum value in the following three possible cases:

(i) for $t = 0$,

(ii) at a point where $|\nabla u| = 0$,

(iii) on the boundary $\partial\Omega\times(0,T)$.

By using the same reasoning process as (2.20), we have

$$\phi(x,0) = \nabla\cdot\big(|\nabla u_0|^{p-2}\nabla u_0\big) \le 0, \quad x\in\overline{\Omega}, \tag{3.5}$$

where we use assumption (3.2). It follows from (3.5), (2.21)-(2.22) and the maximum principle that the maximum value of ϕ in $\overline{\Omega}\times[0,T)$ is nonpositive. In other words, we have

$$\phi(x,t) \le 0, \quad \text{in } \overline{\Omega}\times[0,T);$$

that is,

$$\frac{g'(u)}{f(u)}u_t \le k(t). \tag{3.6}$$

Now, for each fixed $x\in\overline{\Omega}$, we integrate (3.6) over $[0,t]$ to obtain

$$\int_0^t \frac{g'(u)}{f(u)}u_t\,\mathrm{d}t = \int_{u_0(x)}^{u(x,t)}\frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau \le \int_0^t k(\tau)\,\mathrm{d}\tau, \tag{3.7}$$

which implies that u remains global. In fact, suppose that $u(x,t)$ blows up in some finite time T, we have

$$\lim_{t\to T^-} u(x,t) = +\infty.$$

Letting $t \to T^-$ in (3.7), we derive

$$\int_{M_0}^{+\infty} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau \le \int_{u_0(x)}^{+\infty} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau \le \int_{u_0(x)}^{T} k(\tau)\,\mathrm{d}\tau < \int_0^{+\infty} k(\tau)\,\mathrm{d}\tau,$$

which contradicts assumption (3.3). This shows that u is a global solution. Furthermore, it follows from (3.1) and (3.7) that

$$\begin{aligned}\zeta\big(u(x,t)\big) - \zeta\big(u_0(x)\big) &= \int_0^{u(x,t)} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau - \int_0^{u_0(x)} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau = \int_{u_0(x)}^{u(x,t)} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau \\ &\le \int_0^t k(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Hence, we deduce

$$u(x,t) \le \zeta^{-1}\left(\int_0^t k(\tau)\,\mathrm{d}\tau + \zeta\big(u_0(x)\big)\right).$$

The proof of Theorem 3.1 is complete. □

Remark 3.1 When

$$\int_{M_0}^{+\infty} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau = +\infty, \qquad M_0 = \max_{\overline{\Omega}} u_0(x),$$

assumption (3.3) means

$$\int_0^{+\infty} k(\tau)\,\mathrm{d}\tau < +\infty.$$

4 Applications

In the following, we present two examples to demonstrate the applications of Theorems 2.1-3.1 obtained in this paper.

Example 4.1 Let u be a nonnegative classical solution of the following problem:

$$\begin{cases} (ue^u)_t = \nabla\cdot(|\nabla u|^{p-2}\nabla u) + \mathrm{e}^{-t}(1+u)^3\mathrm{e}^u & \text{in } \Omega\times(0,T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = 1 & \text{in } \overline{\Omega}, \end{cases}$$

where $p > 2$ and $\Omega = \{x = (x_1, x_2, x_3) | \sum_{i=1}^3 x_i^2 < 1\}$. Now we have

$$g(u) = u\mathrm{e}^u, \qquad f(u) = (1+u)^3\mathrm{e}^u, \qquad k(t) = \mathrm{e}^{-t}, \qquad u_0(x) = 1.$$

It is easy to verify that assumptions (2.4)-(2.6) hold. It follows from Theorem 2.1 that u blows up in some finite time T and

$$T \le \psi^{-1}\left(\int_{M_0}^{+\infty} \frac{g'(\tau)}{f(\tau)}\,\mathrm{d}\tau\right) = \psi^{-1}\left(\int_1^{+\infty} \frac{1}{(1+\tau)^2}\,\mathrm{d}\tau\right) = \psi^{-1}\left(\frac{1}{2}\right) = \ln 2,$$

$$u(x,t) \le \eta^{-1}\left(\int_t^T k(\tau)\,\mathrm{d}\tau\right) = \eta^{-1}\left(\int_t^T \mathrm{e}^{-\tau}\,\mathrm{d}\tau\right) = \eta^{-1}\left(\mathrm{e}^{-t} - \mathrm{e}^{-T}\right) = \frac{\mathrm{e}^T\mathrm{e}^t}{\mathrm{e}^T - \mathrm{e}^t} - 1.$$

Example 4.2 Let u be a nonnegative classical solution of the following problem:

$$\begin{cases} (\mathrm{e}^{4u})_t = \nabla\cdot(|\nabla u|^{p-2}\nabla u) + \frac{\mathrm{e}^{2u}}{(1+t)^2} & \text{in } \Omega\times(0,T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega\times(0,T), \\ u(x,0) = 1 & \text{in } \overline{\Omega}, \end{cases}$$

where $p > 2$ and $\Omega = \{x = (x_1, x_2, x_3) | \sum_{i=1}^3 x_i^2 < 1\}$. We then have

$$g(u) = \mathrm{e}^{4u}, \qquad f(u) = \mathrm{e}^{2u}, \qquad k(t) = \frac{1}{(1+t)^2}, \qquad u_0(x) = 1.$$

It can be easily seen that assumptions (3.2)-(3.4) hold. Hence, Theorem 3.1 implies that u exists globally and

$$\begin{aligned} u(x,t) &\le \zeta^{-1}\left(\int_0^t k(\tau)\,\mathrm{d}\tau + \zeta\left(u_0(x)\right)\right) = \zeta^{-1}\left(\int_0^t \frac{1}{(1+\tau)^2}\,\mathrm{d}\tau + \zeta(1)\right) \\ &= \zeta^{-1}\left(\frac{t}{1+t} + 2\left(\mathrm{e}^2 - 1\right)\right) = \frac{1}{2}\ln\left(\frac{t}{2(1+t)} + \mathrm{e}^2\right). \end{aligned}$$

5 Conclusion

In this paper, we study the blow-up and global solutions of problem (1.1). As far as we know, there is little information about the blow-up problem of (1.1). It is difficult to use a differential inequality technique employed in [20, 24, 25] to study the blow-up problem of (1.1). We technically use maximum principles to deal with problem (1.1). We note that the auxiliary functions in [28] are not suitable for (1.1). Therefore, the key to our research is to construct the appropriate auxiliary functions. We establish conditions to guarantee that the solution of (1.1) blows up in some finite time or remains global. In addition, the upper estimates of blow-up rate and global solution are specified. We also obtain an upper bound of blow-up time.

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Authors' contributions
All results belong to Juntang Ding. The author read and approved the final manuscript.

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