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Pullback attractors for a class of non-autonomous reaction-diffusion equations in $\mathbb{R}^n$

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Abstract
The aim of this paper is to consider the dynamical behaviour for a class of non-autonomous reaction-diffusion equations in $\mathbb{R}^n$, where the external force $g(x,t)$ satisfies only a certain integrability condition. The existence of $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$-$\boldsymbol{\mathcal{D}}$-pullback attractors and $(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$-$\boldsymbol{\mathcal{D}}$-pullback attractors is obtained for this evolution equation.

MSC: Primary 35B41; secondary 35B45; 35K15

Keywords: pullback attractors; dynamical behaviour; non-autonomous equations

1 Introduction

In this paper, we consider the asymptotic behaviour of solutions for the following non-autonomous reaction-diffusion equations defined in the whole space:

$$\begin{cases} u_t - \nu\Delta u + \lambda u + f_1(u) + a(x)f_2(u) = g(x,t), & \text{in } \mathbb{R}^n \times [\tau,\infty), \\ u(x,\tau) = u_\tau, & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where ν and λ are positive constants. Assume that nonlinear terms $f_1(u), f_2(u) \in C^1(\mathbb{R};\mathbb{R})$ satisfy the following conditions:

$$\alpha_1|u|^p - \beta_1|u|^2 \leq f_1(u)u \leq \alpha_2|u|^p + \beta_2|u|^2 \quad \text{and} \quad f_1'(u) \geq -l_1 \tag{1.2}$$

with $p > 2$ and $\lambda > \beta_1$,

$$\alpha_3|u|^p - \beta_3 \leq f_2(u)u \leq \alpha_4|u|^p + \beta_4 \quad \text{and} \quad f_2'(u) \geq -l_2 \tag{1.3}$$

with $p > 2$, where α_i, β_i, $i = 1,2,3,4$, and l_i, $i = 1,2$ are positive constants. Furthermore, $a(x)$ is a function in $\mathbb{R}^n$ and the external force $g(x,t) \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^n))$ satisfies the following conditions:

$$a(x) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad \text{and} \quad a(x) > 0, \tag{1.4}$$

$$\int_{-\infty}^t e^{\sigma s}\|g(x,s)\|^2_{L^2(\mathbb{R}^n)}\,ds < \infty, \quad \text{for all } t \in \mathbb{R}, \sigma \in (0, \lambda - \beta_1). \tag{1.5}$$

In the last decade, the autonomous and non-autonomous infinite dimensional dynamical systems have been studied extensively by many authors (see, e.g. [1–8] and the references therein). The concept of pullback attractors was proposed in [9] when the authors considered the asymptotic behaviour of random dynamical systems. Such attractors is a parameterised family $\{A(t)\}_{t\in\mathbb{R}}$ of invariant compact sets, which attract the trajectories of the systems when the initial instant of time goes to $-\infty$ and the final time remains fixed. Later on, the pullback attractors were extended to non-autonomous dynamical systems. In the last two decades, the theory of pullback attractors has been developed for non-autonomous dynamical systems and random dynamical systems (see, e.g. [10–12] and the references therein). In [10], the authors introduced the notion of $\mathcal{D}$-pullback attractors, which requires that the process $U(t,\tau)$ associated with the systems be $\mathcal{D}$-pullback asymptotically compact.

It is well known that the Sobolev embeddings are no longer compact in unbounded domain, and so it is difficult to verify the process $U(t,\tau)$ associated with the systems to be pullback asymptotically compact. To overcome this drawback, in [13], using the idea of Wang [5], the authors proved the existence of pullback attractors in $L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ for non-autonomous reaction-diffusion equations defined on $\mathbb{R}^n$. Recently, motivated by [3], the authors of [14] gave a new method to prove the existence of $\mathcal{D}$-pullback attractors by using the technique of non-compactness measure, and this method only needs the process $U(t,\tau)$ associated with the systems to be norm-to-weak continuous (see Definition 2.1) in the phase space.

As we know, the solutions may be unbounded for many non-autonomous systems when time tends to infinity, and we cannot obtain the existence of a uniform attractor for these systems. So we prove the existence of a pullback attractor to overcome this drawback. In this paper, we use a different approach from the article [13] to prove the existence of pullback attractors, and we improve the model equation as Eq. (1.1), which amounts to putting a weight function partially on the nonlinearity. We can also replace the conditions for the nonlinearity $f(u)$ as given in [13] that $f(u)$ satisfies only a Sobolev growth rate with some weak assumptions. For Eq. (1.1), the $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$-global attractor, $(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$-global attractor and $(L^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$-global attractor were proved in [15, 16]. Using the new method in [14], we prove the existence of $\mathcal{D}$-pullback attractors in $L^2(\mathbb{R}^n)$ for Eq. (1.1) and, motivated by the idea in [17, 18], we obtain the existence of $\mathcal{D}$-pullback attractors in $L^p(\mathbb{R}^n)$ for Eq. (1.1). This new method has been used successfully in many papers (see, e.g. [14, 17, 19, 20] and the references therein).

For convenience, the letter C denotes a constant which may be different from line to line and even in the same line. We use $\|\cdot\|$ and $(\cdot,\cdot)$ for the usual norm and the inner product of $L^2(\mathbb{R}^n)$, respectively. We denote by $\|\cdot\|_p$ the norm of $L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$ and by $\|\cdot\|_{H^1}$ the norm of $H^1(\mathbb{R}^n)$. In general, $m(e)$ is the Lebesgue measure of $e \subset \mathbb{R}^n$. $\|\cdot\|_E$ denotes the norm of any Banach space E and $B(E)$ is the set of all bounded subsets of E. Let $X, Y \subset E$, denote by dist $(X,Y) = \sup_{x\in X}\inf_{y\in Y} d(x,y)$ the semidistance between X and Y.

2 Preliminaries

In this section, we first recall the basic definitions and theorems.

Definition 2.1 ([14]) Let X be a complete metric space and $\{U(t,\tau)\} = \{U(t,\tau): t \ge \tau, \tau \in \mathbb{R}\}$ be a two-parameter family of mappings acting on X: $U(t,\tau): X \to X$, $t \ge \tau$, $\tau \in \mathbb{R}$.

We say that $\{U(t,\tau)\}_{\tau\leq t}$ is a continuous process (or norm-to-weak continuous process) in X if

(1) $U(t,s)U(s,\tau) = U(t,\tau)$, $\forall t \geq s \geq \tau$,

(2) $U(\tau,\tau) = Id$ is the identity operator, $\tau \in \mathbb{R}$,

(3) $x \to U(t,\tau)x$ is continuous in X

(or $U(t,\tau)x_n \rightharpoonup U(t,\tau)x$ if $x_n \to x$, $\forall t \geq \tau$, $\tau \in \mathbb{R}$).

Suppose that $\mathcal{D}$ is a nonempty class of parameterised sets $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset B(E)$.

Definition 2.2 ([14]) The process $\{U(t,\tau)\}_{\tau\leq t}$ is said to be $\mathcal{D}$-pullback asymptotically compact if, for any $t \in \mathbb{R}$ and any $\hat{\mathcal{D}} \in \mathcal{D}$, and any sequence $\tau_n \to -\infty$, any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}$ is precompact in X.

Definition 2.3 ([14]) It is said that $\hat{\mathcal{B}} \in \mathcal{D}$ is $\mathcal{D}$-pullback absorbing for the process $\{U(t,\tau)\}_{\tau\leq t}$ if, for any $t \in \mathbb{R}$ and any $\hat{\mathcal{D}} \in \mathcal{D}$, there exists $\tau_0(t,\hat{\mathcal{D}}) \leq t$ such that $U(t,\tau) \times D(\tau) \subset B(t)$ for all $\tau \leq \tau_0(t,\hat{\mathcal{D}})$.

Definition 2.4 ([14]) The family $\hat{A} = \{A(t) : t \in \mathbb{R}\} \subset B(E)$ is said to be a $\mathcal{D}$-pullback attractor for $U(t,\tau)$ if

(1) $A(t)$ is compact for all $t \in \mathbb{R}$,

(2) $\hat{A}$ is invariant, i.e.

$$U(t,\tau)A(\tau) = A(\tau) \quad \text{for all } t \geq \tau,$$

(3) $\hat{A}$ is $\mathcal{D}$-pullback attracting, i.e.

$$\lim_{\tau\to-\infty} \operatorname{dist}\bigl(U(t,\tau)D(\tau), A(t)\bigr) = 0 \quad \text{for all } \hat{\mathcal{D}} \in \mathcal{D} \text{ and all } t \in \mathbb{R},$$

(4) if $\{C(t)\}_{t\in\mathbb{R}}$ is another family of closed attracting sets, then $A(t) \subset C(t)$ for all $t \in \mathbb{R}$.

Definition 2.5 ([21]) Let M be a metric space and A be a bounded subset of M. The Kuratowski measure of non-compactness $\alpha(A)$ is defined by

$$\alpha(A) = \inf\{\delta > 0 \mid A \text{ admits a finite cover by sets of diameter} \leq \delta\}.$$

It has the following properties.

Lemma 2.1 ([21]) *Let $B, B_1, B_2 \in B(E)$. Then*

(1) $\alpha(B) = 0 \Leftrightarrow \alpha(N(B,\varepsilon)) \leq 2\varepsilon \Leftrightarrow \bar{B}$ *is compact*;

(2) $\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$;

(3) $\alpha(B_1) \leq \alpha(B_2)$ *whenever* $B_1 \subset B_2$;

(4) $\alpha(B_1 \cup B_2) \leq \max\{\alpha(B_1), \alpha(B_2)\}$;

(5) $\alpha(B) = \alpha(\bar{B})$;

(6) *if B is a ball of radius ε, then* $\alpha(B) \leq 2\varepsilon$.

Definition 2.6 ([14]) A process $\{U(t,\tau)\}_{\tau\leq t}$ is called $\mathcal{D}$-pullback ω-limit compact if for any $\varepsilon > 0$ and $\hat{\mathcal{D}} \in \mathcal{D}$, there exists $\tau_0(t,\hat{\mathcal{D}}) \leq t$ such that $\alpha(\bigcup_{\tau\leq\tau_0} U(t,\tau)D(\tau)) \leq \varepsilon$.

Theorem 2.1 ([14]) *Let $\{U(t,\tau)\}_{\tau\le t}$ be a process on X. Then $\{U(t,\tau)\}_{\tau\le t}$ is $\mathcal{D}$-pullback asymptotically compact if and only if $\{U(t,\tau)\}_{\tau\le t}$ is $\mathcal{D}$-pullback ω-limit compact.*

Theorem 2.2 ([14]) *Let $\{U(t,\tau)\}_{\tau\le t}$ be a norm-to-weak continuous process such that $\{U(t,\tau)\}_{\tau\le t}$ is $\mathcal{D}$-pullback ω-limit compact. If there exists a family of $\mathcal{D}$-pullback absorbing sets $\{B(t):t\in\mathbb{R}\}\in\mathcal{D}$, i.e. for any $t\in\mathbb{R}$ and $\hat{\mathcal{D}}\in\mathcal{D}$, there exists $\tau_0(t,\hat{\mathcal{D}})\le t$ such that $U(t,\tau)D(\tau)\subset B(t)$ for all $\tau\le\tau_0$, then there exists a $\mathcal{D}$-pullback attractor $\mathcal{A}=\{A(t):t\in\mathbb{R}\}$ and*

$$A(t)=\omega(\hat{B},t)=\bigcap_{s\le t}\overline{\bigcup_{\tau\le s}U(t,\tau)B(\tau)}.$$

Remark Obviously, a continuous process and a weak continuous process are both norm-to-weak continuous processes.

Theorem 2.3 ([17]) *Let Ω be a domain in $\mathbb{R}^n$, $\{U(t,\tau)\}_{\tau\le t}$ be a process on $L^p(\Omega)$ and $L^q(\Omega)$ $(p>q\ge 1)$ and $\{U(t,\tau)\}_{\tau\le t}$ satisfy the following two assumptions:*

(1) *$\{U(t,\tau)\}_{\tau\le t}$ is $\mathcal{D}$-pullback ω-limit compact in $L^q(\Omega)$;*
(2) *for any $\varepsilon>0$, $\hat{\mathcal{B}}\in\mathcal{D}$, there exist $M(\varepsilon,\hat{\mathcal{B}})$ and $\tau_1=\tau_1(\varepsilon,\hat{\mathcal{B}})\le t$ such that*

$$\left(\int_{\Omega(|U(t,\tau)|\ge M)}\left|U(t,\tau)u_\tau\right|^p\,dx\right)^{\frac{1}{p}}<2^{-\frac{2p+2}{p}}\varepsilon\quad \textit{for any } u_\tau\in B(\tau) \textit{ and } \tau\ge\tau_1.$$

Then $\{U(t,\tau)\}_{\tau\le t}$ is $\mathcal{D}$-pullback ω-limit compact in $L^p(\Omega)$.

Theorem 2.4 ([13]) *Let X, Y be two Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let $\{U(t,\tau)\}_{\tau\le t}$ be a continuous process on X and a process on Y. Assume that the family $\hat{\mathcal{B}}_0=\{B_0(t):t\in\mathbb{R}\}$ is (X,X)-$\mathcal{D}$-pullback absorbing for $U(t,\tau)$, and for any $t\in\mathbb{R}$ and any sequence $\tau_n\to-\infty$, any sequence $x_n\in B_0(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}$ is precompact in X. Then the family of sets $\mathcal{A}=\{A(t):t\in\mathbb{R}\}$, where*

$$A(t)=\bigcap_{s\le t}\overline{\bigcup_{\tau\le s}U(t,\tau)B(\tau)}^X$$

is a (X,X)-$\mathcal{D}$-pullback attractor for $\{U(t,\tau)\}_{\tau\le t}$, where $\overline{A}^X$ denotes the closure of A with respect to the norm topology in X.

Furthermore, if the family $\hat{\mathcal{B}}_1=\{B_1(t):t\in\mathbb{R}\}$ is (X,Y)-$\mathcal{D}$-pullback absorbing for $\{U(t,\tau)\}_{\tau\le t}$, and it satisfies that, for any $t\in\mathbb{R}$ and any sequence $\tau_n\to-\infty$, any sequence $x_n\in B_1(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}$ is precompact in Y. Then the family of sets $\mathcal{A}'=\{A'(t):t\in\mathbb{R}\}$, where

$$A'(t)=\bigcap_{s\le t}\overline{\bigcup_{\tau\le s}U(t,\tau)\bigl(B_0(\tau)\cap B_1(\tau)\bigr)}^X=\bigcap_{s\le t}\overline{\bigcup_{\tau\le s}U(t,\tau)\bigl(B_0(\tau)\cap B_1(\tau)\bigr)}^Y$$

is a (X,Y)-$\mathcal{D}$-pullback attractors for $\{U(t,\tau)\}_{\tau\le t}$.

Remark When $\{U(t,\tau)\}_{\tau\le t}$ is only a process on Y, we also prove $\mathcal{A}'=\{A'(t):t\in\mathbb{R}\}$ is a (X,Y)-$\mathcal{D}$-pullback attractor for $\{U(t,\tau)\}_{\tau\le t}$.

Lemma 2.2 *Let $\{U(t,\tau)\}_{\tau\le t}$ be a process on $L^p(\mathbb{R}^n)$ $(p\ge 1)$, $\hat{\mathcal{B}}_1=\{B_1(t):t\in\mathbb{R}\}$ is (X,Y)-$\mathcal{D}$-pullback absorbing for $\{U(t,\tau)\}_{\tau\le t}$. Then, for any $\varepsilon>0$, $t\in\mathbb{R}$ and $\hat{\mathcal{D}}\in\mathcal{D}\subset B(L^p(\mathbb{R}^n))$, there exist $M(t,\varepsilon)$ and $\tau_0=\tau_0(t,\varepsilon)$ such that*

$$m\bigl(\mathbb{R}^n\bigl(\bigl|U(t,\tau)\bigr|\ge M(t,\varepsilon)\bigr)\bigr)<\varepsilon \quad \textit{for all } u_\tau\in D(\tau) \textit{ and } \tau\le\tau_0.$$

The proof of the above lemma is identical to the proof of Lemma 5.2 in [18].

Using the standard Faedo-Galerkin method (see [6, 7]), it is easy to prove the following lemma.

Lemma 2.3 *Assume that* (1.2)-(1.5) *hold and $g\in L^2_{\mathrm{loc}}(\mathbb{R},L^2(\mathbb{R}^n))$. Then, for any $T>0$, $u_\tau\in L^2(\mathbb{R}^n)$, $\tau\in\mathbb{R}$ and $T\ge\tau$, there exists a unique weak solution $u(x,t)$ for Eq.* (1.1) *satisfying*

$$u\in C\bigl([\tau,T];L^2\bigl(\mathbb{R}^n\bigr)\bigr)\cap L^p\bigl(\tau,T;L^p\bigl(\mathbb{R}^n\bigr)\bigr)\cap L^2\bigl(\tau,T;H^1\bigl(\mathbb{R}^n\bigr)\bigr).$$

Furthermore, $u_\tau\mapsto u(t,\tau;u_\tau)$ is continuous in $L^2(\mathbb{R}^n)$.

Based on Lemma 2.3, we can define a continuous process $\{U(t,\tau)\}_{\tau\le t}$ in $L^2(\mathbb{R}^n)$ by

$$U(t,\tau)u_\tau=u(t)\quad \text{for all } t\ge\tau, \tag{2.1}$$

where $u(t)$ is the solution of Eq. (1.1) with the initial value $u(x,\tau)=u_\tau\in L^2(\mathbb{R}^n)$. Moreover, we also know that $\{U(t,\tau)\}_{\tau\le t}$ is a process in $L^p(\mathbb{R}^n)$.

3 Main results

3.1 $(L^2(\mathbb{R}^n),L^2(\mathbb{R}^n))$-$\mathcal{D}$-pullback attractors

Firstly, the following lemma ensures a $\mathcal{D}$-pullback absorbing set in $L^2(\mathbb{R}^n)$.

Lemma 3.1 *Assume that* (1.2)-(1.4) *hold and the external force $g\in L^2_{\mathrm{loc}}(\mathbb{R},L^2(\mathbb{R}^n))$ satisfies* (1.5). *Then, for any $\hat{\mathcal{D}}\in\mathcal{D}\subset B(L^2(\mathbb{R}^n))$ and any $t\in\mathbb{R}$, there exists $\tau_0(t,\hat{\mathcal{D}})\le t$ such that*

$$\bigl\|U(t,\tau)u_\tau\bigr\|\le R_0(t)\quad \textit{for all } \tau\le\tau_0(t,\hat{\mathcal{D}}) \textit{ and all } u_\tau\in D(\tau), \tag{3.1}$$

where $R_0(t)=(\frac{2\beta_3\|a(x)\|_1}{\sigma}+\frac{2e^{-\sigma t}}{\lambda-\beta_1}\int_{-\infty}^t e^{\sigma r}\|g(x,r)\|^2\,dr)^{\frac{1}{2}}$.

Proof Taking the inner product of (1.1) with u in $L^2(\mathbb{R}^n)$, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2+\nu\|\nabla u\|^2+\lambda\|u\|^2+\bigl(f_1(u),u\bigr)+\bigl(a(x)f_2(u),u\bigr)=\bigl(g(x,t),u\bigr).$$

Due to (1.2)-(1.4) and Young's inequality, we get

$$\frac{d}{dt}\|u\|^2+(\lambda-\beta_1)\|u\|^2\le 2\beta_3\bigl\|a(x)\bigr\|_1+\frac{\|g(x,t)\|^2}{\lambda-\beta_1}, \tag{3.2}$$

$$\begin{aligned}&\frac{d}{dt}\|u\|^2+(\lambda-\beta_1)\|u\|^2+2\nu\|\nabla u\|^2+2\alpha_1\|u\|_p^p+2\alpha_3\int_{\mathbb{R}^n}a(x)|u|^p\,dx\\&\quad\le 2\beta_3\bigl\|a(x)\bigr\|_1+\frac{\|g(x,t)\|^2}{\lambda-\beta_1}.\end{aligned} \tag{3.3}$$

By (3.2), we obtain

$$\frac{d}{dr}\left(e^{\sigma r}\|u\|^2\right) + (\lambda - \beta_1 - \sigma)e^{\sigma r}\|u\|^2 \leq 2\beta_3\left\|a(x)\right\|_1 e^{\sigma r} + \frac{\|g(x,r)\|^2}{\lambda - \beta_1}e^{\sigma r}.$$

Integrating over the interval $[\tau, t]$ and noting that $\sigma \in (0, \lambda - \beta_1)$, we have

$$\begin{aligned} e^{\sigma t}\left\|u(t)\right\|^2 &\leq \frac{2\beta_3\|a(x)\|_1}{\sigma}e^{\sigma t} + \frac{1}{\lambda - \beta_1}\int_\tau^t e^{\sigma r}\left\|g(x,r)\right\|^2 dr + e^{\sigma\tau}\|u_\tau\|^2 \\ &\leq \frac{2\beta_3\|a(x)\|_1}{\sigma}e^{\sigma t} + \frac{1}{\lambda - \beta_1}\int_{-\infty}^t e^{\sigma r}\left\|g(x,r)\right\|^2 dr + e^{\sigma\tau}\|u_\tau\|^2. \end{aligned} \tag{3.4}$$

Thus, we get

$$\left\|u(t)\right\|^2 \leq \frac{2\beta_3\|a(x)\|_1}{\sigma} + e^{-\sigma t}e^{\sigma\tau}\|u_\tau\|^2 + \frac{e^{-\sigma t}}{\lambda - \beta_1}\int_{-\infty}^t e^{\sigma r}\left\|g(x,r)\right\|^2 dr,$$

and this implies (3.1). □

Let $\hat{\mathcal{B}}_0 = \{B_0(t) : t \in \mathbb{R}\}$, where

$$B_0(t) = \left\{u \in L^2\left(\mathbb{R}^n\right) : \|u\| \leq R_0(t)\right\}. \tag{3.5}$$

By Lemma 3.1, it is easy to know that the family $\hat{\mathcal{B}}_0$ is $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$-$\mathcal{D}$-pullback absorbing for the process $\{U(t,\tau)\}_{\tau\leq t}$ defined by (2.1) and

$$e^{\sigma t}\left(R_0(t)\right)^2 \to 0 \quad \text{as } t \to -\infty. \tag{3.6}$$

Let $F_1(u) = \int_0^u f_1(s)\,ds$ and $F_2(u) = \int_0^u f_2(s)\,ds$. By (1.2)-(1.3), there exist positive constants $\tilde{\alpha}_i$, $\tilde{\beta}_i$, $i = 1, 2, 3, 4$, such that

$$\tilde{\alpha}_1|u|^p - \tilde{\beta}_1|u|^2 \leq F_1(u) \leq \tilde{\alpha}_2|u|^p + \tilde{\beta}_2|u|^2, \qquad \lambda > 2\tilde{\beta}_1, \tag{3.7}$$

$$\tilde{\alpha}_3|u|^p - \tilde{\beta}_3 \leq F_2(u) \leq \tilde{\alpha}_4|u|^p + \tilde{\beta}_4. \tag{3.8}$$

Lemma 3.2 *Assume that* (1.2)-(1.4) *hold and the external force* $g \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{R}^n))$ *satisfies* (1.5). *Then, for any* $\hat{\mathfrak{D}} \in \mathcal{D} \subset B(L^2(\mathbb{R}^n))$ *and any* $t \in \mathbb{R}$, *there exists* $\tau_1(t, \hat{\mathfrak{D}}) \leq t$ *such that*

$$\left\|u(t)\right\|^2 + \left\|\nabla u(t)\right\|^2 + \left\|u(t)\right\|_p^p \leq \left(R_1(t)\right)^2 \quad \textit{for all } \tau \leq \tau_1(t, \hat{\mathfrak{D}}) \textit{ and all } u_\tau \in D(\tau), \tag{3.9}$$

where $R_1(t) = C(\frac{\beta_3\|a(x)\|_1}{\sigma} + \frac{e^{-\sigma t}}{\lambda-\beta_1}\int_{-\infty}^t e^{\sigma r}\|g(x,r)\|^2\,dr)^{\frac{1}{2}}$ *and the positive constant* C *is independent of* t *and* $\hat{\mathfrak{D}}$.

Proof Multiplying (3.3) by $e^{\sigma t}$, we have

$$\begin{aligned} &\frac{d}{dt}\left(e^{\sigma t}\left\|u(t)\right\|^2\right) + (\lambda - \beta_1 - \sigma)e^{\sigma t}\left\|u(t)\right\|^2 + 2\nu e^{\sigma t}\left\|\nabla u(t)\right\|^2 \\ &\quad + 2\alpha_1 e^{\sigma t}\|u\|_p^p + 2\alpha_3 e^{\sigma t}\int_{\mathbb{R}^n} a(x)|u|^p\,dx \\ &\leq 2\beta_3 e^{\sigma t}\left\|a(x)\right\|_1 + e^{\sigma t}\frac{\|g(x,t)\|^2}{\lambda - \beta_1}. \end{aligned}$$

Let $\tau < t-1$ and $r \in [\tau, t-1]$, integrating over the interval $[r, r+1]$, we get

$$\begin{aligned}
&e^{\sigma(r+1)}\|u(r+1)\|^2 + (\lambda-\beta_1-\sigma)\int_r^{r+1} e^{\sigma s}\|u(s)\|^2\,ds + 2\nu\int_r^{r+1} e^{\sigma s}\|\nabla u(s)\|^2\,ds \\
&\quad + 2\alpha_1\int_r^{r+1} e^{\sigma s}\|u(s)\|_p^p\,ds + 2\alpha_3\int_r^{r+1} e^{\sigma s}\int_{\mathbb{R}^n} a(x)|u(s)|^p\,dx\,ds \\
&\quad \le 2\beta_3\|a(x)\|_1\int_r^{r+1} e^{\sigma s}\,ds + \int_r^{r+1} e^{\sigma s}\frac{\|g(x,s)\|^2}{\lambda-\beta_1}\,ds + e^{\sigma r}\|u(r)\|^2.
\end{aligned}$$

By (3.4), we find

$$\begin{aligned}
&\int_r^{r+1} e^{\sigma s}\left(\|u(s)\|^2 + \|\nabla u(s)\|^2 + \|u(s)\|_p^p + \int_{\mathbb{R}^n} a(x)|u(s)|^p\,dx\right)ds \\
&\quad \le C\left(\frac{2\beta_3\|a(x)\|_1}{\sigma}e^{\sigma(r+1)} + \frac{1}{\lambda-\beta_1}\int_\tau^{r+1} e^{\sigma s}\|g(x,s)\|^2\,ds + e^{\sigma\tau}\|u_\tau\|^2\right) \\
&\quad \le C\left(\frac{2\beta_3\|a(x)\|_1}{\sigma}e^{\sigma t} + e^{\sigma\tau}\|u_\tau\|^2 + \frac{1}{\lambda-\beta_1}\int_{-\infty}^t e^{\sigma s}\|g(x,s)\|^2\,ds\right).
\end{aligned}$$

Thus, by (3.7) and (3.8), we can obtain

$$\begin{aligned}
&\int_r^{r+1} e^{\sigma s}\left(\frac{\nu}{2}\|\nabla u\|^2 + \frac{\lambda}{2}\|u\|^2 + \int_{\mathbb{R}^n} F_1(u)\,dx + \int_{\mathbb{R}^n} a(x)F_2(u)\,dx\right)ds \\
&\quad \le C\left(\frac{2\beta_3\|a(x)\|_1}{\sigma}e^{\sigma t} + e^{\sigma\tau}\|u_\tau\|^2 + \frac{1}{\lambda-\beta_1}\int_{-\infty}^t e^{\sigma s}\|g(x,s)\|^2\,ds\right).
\end{aligned} \tag{3.10}$$

Multiplying (1.1) by u_t and integrating on $\mathbb{R}^n$, we have

$$\begin{aligned}
&\|u_t\|^2 + \frac{d}{dt}\left(\frac{\nu}{2}\|\nabla u\|^2 + \frac{\lambda}{2}\|u\|^2 + \int_{\mathbb{R}^n} F_1(u)\,dx + \int_{\mathbb{R}^n} a(x)F_2(u)\,dx\right) \\
&\quad \le \frac{1}{2}\left(\|g(x,t)\|^2 + \|u_t\|^2\right).
\end{aligned}$$

And then

$$\frac{d}{dt}\left(\frac{\nu}{2}\|\nabla u\|^2 + \frac{\lambda}{2}\|u\|^2 + \int_{\mathbb{R}^n} F_1(u)\,dx + \int_{\mathbb{R}^n} a(x)F_2(u)\,dx\right) \le \frac{1}{2}\|g(x,t)\|^2. \tag{3.11}$$

It follows from (3.11) that

$$\begin{aligned}
&\frac{d}{dr}e^{\sigma r}\left(\frac{\nu}{2}\|\nabla u\|^2 + \frac{\lambda}{2}\|u\|^2 + \int_{\mathbb{R}^n} F_1(u)\,dx + \int_{\mathbb{R}^n} a(x)F_2(u)\,dx\right) \\
&\quad \le \sigma e^{\sigma r}\left(\frac{\nu}{2}\|\nabla u\|^2 + \frac{\lambda}{2}\|u\|^2 + \int_{\mathbb{R}^n} F_1(u)\,dx + \int_{\mathbb{R}^n} a(x)F_2(u)\,dx\right) + \frac{e^{\sigma r}}{2}\|g(x,t)\|^2.
\end{aligned}$$

By (1.5), (3.7), (3.8), (3.10) and the uniform Gronwall inequality, we obtain

$$\begin{aligned}
&\frac{\nu}{2}\|\nabla u\|^2 + \frac{\lambda}{2}\|u\|^2 + \int_{\mathbb{R}^n} F_1(u)\,dx + \int_{\mathbb{R}^n} a(x)F_2(u)\,dx \\
&\quad \le C\left(\frac{2\beta_3\|a(x)\|_1}{\sigma} + e^{-\sigma(t-\tau)}\|u_\tau\|^2 + \frac{e^{-\sigma t}}{\lambda-\beta_1}\int_{-\infty}^t e^{\sigma s}\|g(x,s)\|^2\,ds\right).
\end{aligned} \tag{3.12}$$

It follows from (3.7) and (3.8) that

$$\begin{aligned}&\|u(t)\|^2+\|\nabla u(t)\|^2+\|u(t)\|_p^p\\&\le C\biggl(\frac{2\beta_3\|a(x)\|_1}{\sigma}+e^{-\sigma(t-\tau)}\|u_\tau\|^2+\frac{e^{-\sigma t}}{\lambda-\beta_1}\int_{-\infty}^t e^{\sigma s}\|g(x,s)\|^2\,ds\biggr),\end{aligned}$$

and this implies (3.9). □

Lemma 3.3 *Assume that* (1.2)-(1.4) *hold and the external force* $g\in L^2_{\mathrm{loc}}(\mathbb{R},L^2(\mathbb{R}^n))$ *satisfies* (1.5). *Let the family* $\hat{\mathcal{B}}_0=\{B_0(t):t\in\mathbb{R}\}$ *be defined by* (3.5). *Then, for any* $\varepsilon\ge0$ *and any* $t\in\mathbb{R}$, *there exist* $\tilde{k}=\tilde{k}(t,\varepsilon)>0$ *and* $\tau'(t,\varepsilon)$ *such that*

$$\int_{|x|\ge k}\bigl|U(t,\tau)u_\tau\bigr|^2\,dx\le\varepsilon\quad\textit{for all }k\ge\tilde{k},\tau\le\tau'(t,\varepsilon)\textit{ and }u_\tau\in B_0(\tau).\tag{3.13}$$

Proof Choose a smooth function θ such that $0\le\theta(s)\le1$ for $s\in\mathbb{R}^+$,

$$\theta(s)=\begin{cases}0,& 0\le s\le1,\\1,& s\ge2,\end{cases}$$

and there exists a constant c such that $|\theta'(s)|\le c$.

Multiplying (1.1) by $\theta^2(\frac{|x|^2}{k^2})u$ and integrating on $\mathbb{R}^n$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)|u|^2\,dx-\nu\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)u\Delta u\,dx+\lambda\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)|u|^2\,dx\\&\quad=-\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)f_1(u)u\,dx-\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)a(x)f_2(u)u\,dx+\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)g(x,t)u\,dx\\&\quad\le\beta_1\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)|u|^2\,dx-\alpha_1\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)|u|^p\,dx+\beta_3\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)a(x)\,dx\\&\qquad-\alpha_3\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)a(x)|u|^p\,dx+\frac{\lambda-\beta_1}{2}\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)|u|^2\,dx\\&\qquad+\frac{1}{2(\lambda-\beta_1)}\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)\bigl|g(x,t)\bigr|^2\,dx.\end{aligned}$$

And so

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)|u|^2\,dx-2\nu\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)u\Delta u\,dx+(\lambda-\beta_1)\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)|u|^2\,dx\\&\quad\le2\beta_3\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)a(x)\,dx+\frac{1}{(\lambda-\beta_1)}\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)\bigl|g(x,t)\bigr|^2\,dx.\end{aligned}\tag{3.14}$$

For the second term on the left-hand side of (3.14), we know

$$\begin{aligned}&-\nu\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)u\Delta u\,dx\\&\quad=\nu\int_{\mathbb{R}^n}\theta^2\biggl(\frac{|x|^2}{k^2}\biggr)|\nabla u|^2\,dx+\nu\int_{\mathbb{R}^n}\theta'\biggl(\frac{|x|^2}{k^2}\biggr)\theta\biggl(\frac{|x|^2}{k^2}\biggr)u\frac{4x}{k^2}\cdot\nabla u\,dx\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \theta'\left(\frac{|x|^2}{k^2}\right)\theta\left(\frac{|x|^2}{k^2}\right) u \frac{4x}{k^2}\cdot\nabla u\,dx\right| \\
&\quad\le \frac{4\sqrt{2}c}{k}\int_{k\le|x|\le\sqrt{2}k}|u||\nabla u|\,dx \le \frac{C}{k}\|u\|\|\nabla u\|. \qquad (3.16)
\end{aligned}$$

It follows from (3.15) and (3.16) that

$$\begin{aligned}
&\frac{d}{dr}\left(e^{\sigma r}\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)|u|^2\,dx\right) \\
&\quad\le 2\beta_3 e^{\sigma r}\int_{|x|\ge k} a(x)\,dx + \frac{e^{\sigma r}}{(\lambda-\beta_1)}\int_{|x|\ge k}\left|g(x,t)\right|^2\,dx + \frac{C}{k}e^{\sigma r}\|u\|\|\nabla u\|.
\end{aligned}$$

Integrating over the interval $[\tau,t]$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)\left|u(t)\right|^2\,dx \\
&\quad\le 2\beta_3 e^{-\sigma t}\int_\tau^t e^{\sigma r}\int_{|x|\ge k} a(x)\,dx\,dr + \frac{e^{-\sigma t}}{(\lambda-\beta_1)}\int_\tau^t e^{\sigma r}\int_{|x|\ge k}\left|g(x,r)\right|^2\,dx\,dr \\
&\qquad + \frac{C}{k}e^{-\sigma t}\int_\tau^t e^{\sigma r}\|u\|\|\nabla u\|\,dr + e^{-\sigma t}e^{\sigma\tau}\|u_\tau\|^2, \qquad (3.17)
\end{aligned}$$

where $\tau\le\tau_0(t,\hat{\mathcal{D}})$. We now treat each term on the right-hand side of (3.17). For the first term,

$$2\beta_3 e^{-\sigma t}\int_\tau^t e^{\sigma r}\int_{|x|\ge k} a(x)\,dx\,dr \le 2\beta_3 e^{-\sigma t}e^{\sigma t}\int_{|x|\ge k} a(x)\,dx \le 2\beta_3\int_{|x|\ge k} a(x)\,dx,$$

by (1.4), for any $\varepsilon>0$, there exists $k_1(\varepsilon,t)$ such that

$$2\beta_3\int_{|x|\ge k} a(x)\,dx < \frac{\varepsilon}{4}\quad\text{for all } k\ge k_1. \qquad (3.18)$$

For the second term, by (1.5), for any $\varepsilon>0$, there exists $k_2(\varepsilon,t)$ such that

$$\begin{aligned}
&\frac{1}{(\lambda-\beta_1)}\int_\tau^t e^{\sigma r}\int_{|x|\ge k}\left|g(x,r)\right|^2\,dx\,dr \\
&\quad\le \frac{1}{(\lambda-\beta_1)}\int_{-\infty}^t e^{\sigma r}\int_{|x|\ge k}\left|g(x,r)\right|^2\,dx\,dr < \frac{\varepsilon}{4}\quad\text{for all } k\ge k_2. \qquad (3.19)
\end{aligned}$$

For the forth term, since $u_\tau\in B_0(\tau)$, by (3.6), for any $t\in\mathbb{R}$, we get

$$e^{-\sigma t}e^{\sigma\tau}\|u_\tau\|^2\to 0\quad\text{as }\tau\to-\infty. \qquad (3.20)$$

We now handle the third term on the right-hand side of (3.17). By Young's inequality, we know

$$\frac{C}{k}e^{-\sigma t}\int_\tau^t e^{\sigma r}\|u\|\|\nabla u\|\,dr \le \frac{C}{2k}e^{-\sigma t}\int_\tau^t e^{\sigma r}\|u\|^2\,dr + \frac{C}{2k}e^{-\sigma t}\int_\tau^t e^{\sigma r}\|\nabla u\|^2\,dr.$$

We can find $\delta_0 > 0$ such that

$$\int_\tau^t e^{\sigma r}\|u\|^2\,dr \le \int_\tau^t e^{(\sigma+\delta_0)r}\|u\|^2\,dr,$$

and by (3.4), we have

$$\begin{aligned}
&\int_\tau^t e^{(\sigma+\delta_0)r}\|u(r)\|^2\,dr\\
&\quad\le \int_\tau^t e^{(\sigma+\delta_0)r}\left(\frac{2\beta_3\|a(x)\|_1}{\sigma}+e^{-\sigma r}e^{\sigma\tau}\|u_\tau\|^2+\frac{e^{-\sigma r}}{\lambda-\beta_1}\int_{-\infty}^r e^{\sigma s}\|g(x,s)\|^2\,ds\right)dr\\
&\quad\le \frac{2\beta_3\|a(x)\|_1}{\sigma}e^{(\sigma+\delta_0)t}+e^{\sigma\tau}\|u_\tau\|^2\int_\tau^t e^{\delta_0 r}\,dr+\frac{1}{\lambda-\beta_1}\int_\tau^t e^{\delta_0 s}\,ds\int_{-\infty}^t e^{\sigma s}\|g(x,s)\|^2\,ds\\
&\quad\le \frac{2\beta_3\|a(x)\|_1}{\sigma}e^{(\sigma+\delta_0)t}+\frac{1}{\delta_0}e^{\delta_0 t}e^{\sigma\tau}\|u_\tau\|^2+\frac{1}{\delta_0(\lambda-\beta_1)}e^{\delta_0 t}\int_{-\infty}^t e^{\sigma s}\|g(x,s)\|^2\,ds\\
&\quad<\infty.
\end{aligned}$$

Analogously, we can obtain

$$\int_\tau^t e^{\sigma r}\|\nabla u\|^2\,dr < \infty.$$

Thus, for any $\varepsilon > 0$, there exists $k_3(\varepsilon, t)$ such that

$$\frac{C}{k}e^{-\sigma t}\int_\tau^t e^{\sigma r}\|u\|\|\nabla u\|\,dr < \frac{\varepsilon}{4}\quad\text{for all } k \ge k_2. \tag{3.21}$$

It follows from (3.18)-(3.21) that

$$\int_{|x|\ge 2k}\left|U(t,\tau)u_\tau\right|^2 dx \le \int_{\mathbb{R}^n}\theta^2\left(\frac{|x|^2}{k^2}\right)\left|u(t)\right|^2 dx < \varepsilon.$$

So, the proof is complete. □

Next, we utilise Definition 2.6 to prove that the process $\{U(t,\tau)\}_{\tau\le t}$ associated with the initial value problem (1.1) is $\mathcal{D}$-pullback ω-limit compact.

Lemma 3.4 *Assume that* (1.2)-(1.4) *hold and the external force* $g \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{R}^n))$ *satisfies* (1.5). *Then the process* $\{U(t,\tau)\}_{\tau\le t}$ *associated with the initial value problem* (1.1) *is* $\mathcal{D}$-*pullback* ω-*limit compact in* $L^2(\mathbb{R}^n)$.

Proof Denote $B_r = B(0,r)\cap\mathbb{R}^n$, we can split $u(t)$ as

$$u(t) = \chi(x)u(t) + \big(1-\chi(x)\big)u(t),$$

where $\chi(x)$ is a smooth function satisfying $0 \le \chi(x) \le 1$, $|\chi'(x)| \le c_0$, and it is defined by

$$\chi(x) = \begin{cases} 0, & x \in B_r,\\ 1, & x \notin B_{r+1}.\end{cases}$$

And so, we have

$$u_1(t) = \begin{cases} u(t), & x \in B_r, \\ 0, & x \notin B_{r+1}, \\ \chi(x)u(t), & \text{others}, \end{cases} \qquad u_2(t) = \begin{cases} 0, & x \in B_r, \\ u(t), & x \notin B_{r+1}, \\ (1-\chi(x))u(t), & \text{others}. \end{cases}$$

For any $\hat{\mathcal{D}} \in \mathcal{D} \subset B(L^2(\mathbb{R}^n))$, $\{U(t,\tau)D(\tau)\} = \{U(t,\tau)u_\tau \mid u_\tau \in D(\tau)\}$ can be split as

$$U(t,\tau)D(\tau) = \chi(x)U(t,\tau)D(\tau) + \bigl(1-\chi(x)\bigr)U(t,\tau)D(\tau).$$

By Lemma 2.1, we have

$$\alpha\bigl(U(t,\tau)D(\tau)\bigr) \le \alpha\bigl(\chi(x)U(t,\tau)D(\tau)\bigr) + \alpha\bigl(\bigl(1-\chi(x)\bigr)U(t,\tau)D(\tau)\bigr). \tag{3.22}$$

By Lemma 3.1, we get $u_1(t) \in L^2(B_r)$ as $\tau \le \tau_0(t,\hat{\mathcal{D}})$ and

$$\chi(x)U(t,\tau)D(\tau) = \bigl\{\chi(x)U(t,\tau)u_\tau = u_1(t) \mid u_\tau \in D(\tau)\bigr\}.$$

By Lemma 3.2, we have

$$\bigl\|u_1(t)\bigr\|_{H_0^1(B_{r+1})} = \bigl\|\nabla u_1(t)\bigr\|_{L^2(B_{r+1})} \le R_1(t) \quad \text{for all } \tau \le \tau_1(t,\hat{\mathcal{D}}).$$

Since $H_0^1(B_{r+1}) \hookrightarrow L^2(B_{r+1})$ is compact, $\chi(x)U(t,\tau)D(\tau)$ is compact in $L^2(B_{r+1})$. By Lemma 2.1, we obtain

$$\alpha\bigl(\chi(x)U(t,\tau)D(\tau)\bigr) = 0. \tag{3.23}$$

By Lemma 3.3, for any $\varepsilon > 0$, we can choose r large enough such that

$$\int_{|x|\ge r} |u|^2 \le \varepsilon.$$

And then

$$\|u_2\| \le \varepsilon \quad \text{for all } \tau \le \tau'(t,\varepsilon). \tag{3.24}$$

We know

$$\bigl(1-\chi(x)\bigr)U(t,\tau)D(\tau) = \bigl\{\bigl(1-\chi(x)\bigr)U(t,\tau)u_\tau = u_2(t) \mid u_\tau \in D(\tau)\bigr\}.$$

By (3.24), we obtain

$$\alpha\bigl(\bigl(1-\chi(x)\bigr)U(t,\tau)D(\tau)\bigr) \le \varepsilon \quad \text{for all } \tau \le \tau'(t,\varepsilon). \tag{3.25}$$

It follows from (3.22), (3.23) and (3.25) that

$$\alpha\bigl(U(t,\tau)D(\tau)\bigr) \le \varepsilon \quad \text{for all } \tau \le \min\bigl\{\tau_0(t,\hat{\mathcal{D}}), \tau_1(t,\hat{\mathcal{D}}), \tau'(t,\varepsilon)\bigr\}.$$

By Definition 2.6, we obtain $\{U(t,\tau)\}_{\tau\le t}$ is $\mathcal{D}$-pullback ω-limit compact in $L^2(\mathbb{R}^n)$. $\square$

Using Theorem 2.2 or Theorem 2.4, it is easy to prove the following theorem by Lemma 3.1 and Lemma 3.4.

Theorem 3.1 *Assume that* (1.2)-(1.4) *hold and the external force* $g \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{R}^n))$ *satisfies* (1.5). *Then the process* $\{U(t,\tau)\}_{\tau\le t}$ *associated with the initial value problem* (1.1) *has a* $\mathcal{D}$*-pullback attractor* $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$ *in* $L^2(\mathbb{R}^n)$.

3.2 $(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$-$\mathcal{D}$-pullback attractors

In this subsection, we prove the existence of $\mathcal{D}$-pullback attractors in $L^p(\mathbb{R}^n)$. We set $\hat{\mathcal{B}}_1 = \{B_1(t) : t \in \mathbb{R}\}$, where

$$B_1(t) = \{u \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) : \|u\| + \|u\|_p \le R_1(t)\} \quad \text{for all } t \in \mathbb{R}, \tag{3.26}$$

and $R_1(t)$ is defined in Lemma 3.2. So by Lemma 3.2, we obtain the family $\hat{\mathcal{B}}_1 = \{B_1(t) : t \in \mathbb{R}\}$ is $(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$-$\mathcal{D}$-pullback absorbing for the process $\{U(t,\tau)\}_{\tau\le t}$, i.e. for any $\hat{\mathcal{D}} \in \mathcal{D} \subset B(L^2(\mathbb{R}^n))$, there exists $\tau_1(t, \hat{\mathcal{D}}) \le t$ such that $U(t,\tau)D(\tau) \subset B_1(t)$ for all $\tau \le \tau_1(t, \hat{\mathcal{D}})$. We also know

$$e^{\sigma t}(R_1(t))^2 \to 0 \quad \text{as } t \to -\infty. \tag{3.27}$$

Based on Theorem 2.4, we only prove that the process $\{U(t,\tau)\}_{\tau\le t}$ associated with the initial value problem (1.1) is $\mathcal{D}$-pullback ω-limit compact in $L^p(\mathbb{R}^n)$. Firstly, we prove the following lemma.

Lemma 3.5 *Assume that* (1.2)-(1.4) *hold and the external force* $g \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{R}^n))$ *satisfies* (1.5). *Let the family* $\hat{\mathcal{B}}_1 = \{B_1(t) : t \in \mathbb{R}\}$ *be defined by* (3.26). *Then, for any* $\varepsilon \ge 0$, *any* $\hat{\mathcal{D}} \in \mathcal{D} \subset B(L^2(\mathbb{R}^n))$ *and any* $t \in \mathbb{R}$, *there exist* $M = M(t,\varepsilon) > 0$ *and* $\tau''(t,\varepsilon)$ *such that*

$$\begin{aligned} &\int_{\mathbb{R}^n(|U(t,\tau)u_\tau|\ge M)} |U(t,\tau)u_\tau|^p \, dx \le \varepsilon \\ &\quad \textit{for all } u_\tau \in D(\tau), \tau \le \tau''(t,\varepsilon) \textit{ and } M \ge 2M_0. \end{aligned} \tag{3.28}$$

Proof For any $\varepsilon > 0$ be given, by (1.5), there exists $\delta_1 > 0$ such that

$$\int_{-\infty}^t e^{\sigma s} \int_{e_1} |g(x,s)|^2 \, dx \, ds < \varepsilon, \tag{3.29}$$

where $e_1 \subset \mathbb{R}^n$ and $m(e_1) \le \delta_1$. By Lemma 2.2 and Lemma 3.2, we know that there exist $M_1 = M_1(t,\varepsilon)$ and $\tau_2 = \tau_2(t,\varepsilon)$ such that

$$m(\mathbb{R}^n(|U(t,\tau)u_\tau| \ge M_1)) \le \delta_1 \quad \text{for all } u_\tau \in D(\tau) \text{ and } \tau \le \tau_2. \tag{3.30}$$

By (1.2) and (1.3), we can choose M_2 large enough such that

$$\alpha_1|u|^{p-1} - \beta_1|u| \le f_1(u) \le \alpha_2|u|^{p-1} + \beta_2|u| \quad \text{in } \mathbb{R}^n\ (U(t,\tau)u_\tau \ge M_2), \tag{3.31}$$

$$\alpha_3|u|^{p-1} \le f_2(u) \le \alpha_4|u|^{p-1} \quad \text{in } \mathbb{R}^n\ (U(t,\tau)u_\tau \ge M_2). \tag{3.32}$$

Let $M_0 = \max\{M_1, M_2\}$ and $\tau \le \tau_2$. Multiplying Eq. (1.1) by $(u - M_0)_+^{p-1}$ and integrating on $\mathbb{R}^n$, we have

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^n}\big|(u-M_0)_+\big|^p\,dx - \nu\int_{\mathbb{R}^n}\Delta u(u-M_0)_+^{p-1}\,dx + \lambda\int_{\mathbb{R}^n}u(u-M_0)_+^{p-1}\,dx\\&\quad+\int_{\mathbb{R}^n}f_1(u)(u-M_0)_+^{p-1}\,dx+\int_{\mathbb{R}^n}a(x)f_2(u)(u-M_0)_+^{p-1}\,dx=\int_{\mathbb{R}^n}g(x,t)(u-M_0)_+^{p-1}\,dx,\end{aligned}$$

where $(u - M_0)_+$ denotes the positive part of $u - M_0$, that is

$$(u-M_0)_+ = \begin{cases} u - M_0, & u \ge M_0,\\ 0, & u < M_0.\end{cases}$$

Let $\Omega_1 = \mathbb{R}^n(U(t,\tau)u_\tau \ge M_0)$, we get

$$(u-M_0)_+^{2p-2} \le |u|^{p-1}(u-M_0)_+^{p-1} \quad \text{and} \quad (u-M_0)_+^p \le u(u-M_0)_+^{p-1} \quad \text{in } \Omega_1.$$

It follows from (3.31), (3.32), Young's inequality and Hölder's inequality that

$$-\nu\int_{\mathbb{R}^n}\Delta u(u-M_0)_+^{p-1}\,dx = \nu(p-1)\int_{\Omega_1}|\nabla u|^2(u-M_0)_+^{p-2}\,dx \ge 0, \tag{3.33}$$

$$\begin{aligned}\int_{\mathbb{R}^n}f_1(u)(u-M_0)_+^{p-1}\,dx &\ge \int_{\Omega_1}\alpha_1|u|^{p-1}(u-M_0)_+^{p-1}\,dx\\&\quad-\int_{\Omega_1}\beta_1|u|(u-M_0)_+^{p-1}\,dx,\end{aligned}\tag{3.34}$$

$$\int_{\mathbb{R}^n}a(x)f_2(u)(u-M_0)_+^{p-1}\,dx \ge \alpha_3\int_{\Omega_1}a(x)|u|^{p-1}(u-M_0)_+^{p-1}\,dx \ge 0, \tag{3.35}$$

$$\begin{aligned}\int_{\mathbb{R}^n}g(x,t)(u-M_0)_+^{p-1}\,dx &\le \frac{1}{2\alpha_1}\int_{\Omega_1}\big|g(x,t)\big|^2\,dx + \frac{\alpha_1}{2}\int_{\Omega_1}(u-M_0)_+^{2p-2}\,dx\\&\le \frac{1}{2\alpha_1}\int_{\Omega_1}\big|g(x,t)\big|^2\,dx + \frac{\alpha_1}{2}\int_{\Omega_1}|u|^{p-1}(u-M_0)_+^{p-1}\,dx.\end{aligned}\tag{3.36}$$

By (3.33)-(3.36), we get

$$\frac{1}{p}\frac{d}{dt}\int_{\Omega_1}\big|(u-M_0)_+\big|^p\,dx + (\lambda-\beta_1)\int_{\Omega_1}\big|(u-M_0)_+\big|^p\,dx \le \frac{1}{2\alpha_1}\int_{\Omega_1}\big|g(x,t)\big|^2\,dx,$$

which implies that

$$\begin{aligned}&\frac{d}{dt}(t-\tau)e^{\sigma t}\int_{\Omega_1}\big|(u-M_0)_+\big|^p\,dx + c_0e^{\sigma t}\int_{\Omega_1}\big|(u-M_0)_+\big|^p\,dx\\&\quad\le\frac{p(t-\tau)}{2\alpha_1}e^{\sigma t}\int_{\Omega_1}\big|g(x,t)\big|^2\,dx,\end{aligned}\tag{3.37}$$

where $u > 0$ in Ω_1 and $c_0 = (p(\lambda - \beta_1) - \sigma)(t - \tau) - 1$. Since $\sigma \in (0, \lambda - \beta_1)$ and $p > 2$, there exists $\tau_3 = \tau_3(t, \varepsilon) < 0$ such that

$$\big(p(\lambda-\beta_1)-\sigma\big)(t-\tau) \ge 1 \quad \text{for all } \tau \le \tau_3.$$

So integrating (3.37) over the interval $[\tau, t]$, we have

$$\int_{\Omega_1} \left|(u - M_0)_+\right|^p dx \le \frac{p}{2\alpha_1} e^{-\sigma t} \int_{-\infty}^{t} e^{\sigma s} \int_{\Omega_1} \left|g(x,s)\right|^2 dx\, ds.$$

By (3.29), we can obtain

$$\int_{\Omega_1} \left|(u - M_0)_+\right|^p dx \le C\varepsilon \quad \text{for all } \tau \le \tau_3 \text{ and } u_\tau \in D(\tau), \tag{3.38}$$

where $C > 0$ is a constant independent of M_0. Set $\Omega_2 = \mathbb{R}^n(U(t,\tau)u_\tau \le -M_0)$. Likewise, replacing $(u - M_0)_+$ with $(u + M_0)_-$, we can also obtain that there exists $\tau_4 = \tau_4(t,\varepsilon)$ such that

$$\int_{\Omega_2} \left|(u + M_0)_-\right|^p dx \le C\varepsilon \quad \text{for all } \tau \le \tau_4 \text{ and } u_\tau \in D(\tau), \tag{3.39}$$

where $(u + M_0)_-$ is the negative part of $u + M_0$, that is

$$(u + M_0)_- = \begin{cases} u + M_0, & u \le -M_0, \\ 0, & u > -M_0. \end{cases}$$

Then it follows from (3.38) and (3.39) that

$$\int_{\mathbb{R}^n(|U(t,\tau)u_\tau| \ge M_0)} \left|\left(|u| - M_0\right)\right|^p dx \le \varepsilon \quad \text{for all } \tau \le \tau''(t,\varepsilon) \text{ and } u_\tau \in D(\tau),$$

where $\tau''(t,\varepsilon) = \min\{\tau_3, \tau_4\}$. Hence, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n(|U(t,\tau)u_\tau| \ge 2M_0)} \left|U(t,\tau)u_\tau\right|^p dx \\
&\quad = \int_{\mathbb{R}^n(|U(t,\tau)u_\tau| \ge 2M_0)} \left(|u| - M_0 + M_0\right)^p dx \\
&\quad \le 2^{p-1} \left(\int_{\mathbb{R}^n(|U(t,\tau)u_\tau| \ge 2M_0)} \left(|u| - M_0\right)^p dx + \int_{\mathbb{R}^n(|U(t,\tau)u_\tau| \ge 2M_0)} (M_0)^p dx \right) \\
&\quad \le 2^{p-1} \left(\int_{\mathbb{R}^n(|U(t,\tau)u_\tau| \ge M_0)} \left(|u| - M_0\right)^p dx + \int_{\mathbb{R}^n(|U(t,\tau)u_\tau| \ge M_0)} \left(|u| - M_0\right)^p dx \right) \le 2^p \varepsilon.
\end{aligned}$$

Finally, we obtain (3.28) and the proof is complete. □

By Theorem 2.3, Lemma 3.4 and Lemma 3.5, we can obtain that the process $\{U(t,\tau)\}_{\tau \le t}$ associated with the initial value problem (1.1) is $\mathcal{D}$-pullback ω-limit compact in $L^p(\mathbb{R}^n)$. So it is easy to prove the following theorem.

Theorem 3.2 *Assume that* (1.2)-(1.4) *hold and the external force* $g \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{R}^n))$ *satisfies* (1.5). *Then the family of sets* $\mathcal{A}' = \{A'(t) : t \in \mathbb{R}\}$ *is* $(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$*-*$\mathcal{D}$*-pullback attractors for* $\{U(t,\tau)\}_{\tau \le t}$.

Proof We know that the family $\hat{\mathcal{B}}_1 = \{B_1(t) : t \in \mathbb{R}\}$ is $(L^2(\mathbb{R}^n), L^p(\mathbb{R}^n))$-$\mathcal{D}$-pullback absorbing for the process $\{U(t,\tau)\}_{\tau\leq t}$, where $B_1(t)$ is defined by (3.26). Thus, by Theorem 2.4, we can deduce that the theorem is true. □

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References

1. Chepyzhov, VV, Vishik, MI: Attractors of non-autonomous dynamical systems and their dimension. J. Math. Pures Appl. **73**, 279-333 (1994)
2. Song, HT, Zhong, CK: Attractors of non-autonomous reaction-diffusion equations in L^p. Nonlinear Anal. **68**, 1890-1897 (2008)
3. Ma, QF, Wang, SH, Zhong, CK: Necessary and sufficient conditions for the existence of global attractors for semigroups and applications. Indiana Univ. Math. J. **51**, 1541-1559 (2002)
4. Zhong, CK, Yang, MH, Sun, CY: The existence of global attractors for the norm-to-weak continuous semigroup and application to the nonlinear reaction-diffusion equations. J. Differ. Equ. **223**, 367-399 (2006)
5. Wang, BX: Attractors for reaction-diffusion equations in unbounded domains. Physica D **128**, 41-52 (1999)
6. Robinson, JC: Infinite-Dimensional Dynamical Systems. Cambridge University Press, Cambridge (2001)
7. Temam, R: Infinite-Dimensional Dynamical Systems in Mechanics and Physics. Springer, New York (1997)
8. Chepyzhov, VV, Vishik, MI: Attractors for Equations of Mathematical Physics. American Mathematical Society Colloquium Publications, vol. 49. Am. Math. Soc., Providence (2002)
9. Crauel, H, Flandoli, F: Attractors for random dynamical systems. Probab. Theory Relat. Fields **100**, 365-393 (1994)
10. Caraballo, T, Lukaszewicz, G, Real, J: Pullback attractors for asymptotically compact nonautonomous dynamical systems. Nonlinear Anal. **64**, 484-498 (2006)
11. Crauel, H, Debussche, A, Flandoli, F: Random attractors. J. Dyn. Differ. Equ. **9**, 307-341 (1995)
12. Kloeden, PE, Schmalfuss, B: Asymptotic behaviour of nonautonomous difference inclusions. Syst. Control Lett. **33**, 275-280 (1998)
13. Wang, YH, Wang, LZ, Zhao, WJ: Pullback attractors for nonautonomous reaction-diffusion equations in unbounded domains. J. Math. Anal. Appl. **336**, 330-347 (2007)
14. Li, YJ, Zhong, CK: Pullback attractors for the norm-to-weak continuous process and application to the non-autonomous reaction-diffusion equations. Appl. Math. Comput. **190**, 1020-1029 (2007)
15. Zhang, YH, Zhong, CK, Wang, SY: Attractors in $L^2(\mathbb{R}^n)$ for a class of reaction-diffusion. Nonlinear Anal. **71**, 1901-1908 (2009)
16. Zhang, YH, Zhong, CK, Wang, SY: Attractors in $L^p(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ for a class of reaction-diffusion. Nonlinear Anal. **72**, 2228-2237 (2010)
17. Li, YJ, Wang, SY, Zhong, CK: Pullback attractors for non-autonomous reaction-diffusion equations in L^p. Appl. Math. Comput. **207**, 373-379 (2009)
18. Yan, XJ, Zhong, CK: L^p-Uniform attractor for nonautonomous reaction-diffusion equations in unbounded domains. J. Math. Phys. **49**, 102705 (2008)
19. Park, SH, Park, JY: Pullback attractor for a non-autonomous modified Swift-Hohenberg equation. Comput. Math. Appl. **67**, 542-548 (2014)
20. Yang, L, Yang, MH, Kloeden, PE: Pullback attractors for non-autonomous quasi-linear parabolic equations with dynamical boundary conditions. Discrete Contin. Dyn. Syst. **17**, 2635-2651 (2012)
21. Deimling, K: Nonlinear Functional Analysis. Springer, New York (1995)