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# Existence theorems and Hyers-Ulam stability for a coupled system of fractional differential equations with $p$-Laplacian operator

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**Abstract**

In this paper, we study the existence and uniqueness of solution (EUS) as well as Hyers-Ulam stability for a coupled system of FDEs in Caputo's sense with nonlinear $p$-Laplacian operator. For this purpose, the suggested coupled system is transferred to an integral system with the help of four Green functions $\mathcal{G}^{\alpha_1}(t,s)$, $\mathcal{G}^{\beta_1}(t,s)$, $\mathcal{G}^{\alpha_2}(t,s)$, $\mathcal{G}^{\beta_2}(t,s)$. Then using topological degree theory and Leray-Schauder's-type fixed point theorem, existence and uniqueness results are proved. An illustrative and expressive example is given as an application of the results.

**Keywords:** Caputo's fractional derivative; coupled system of FDEs; topological degree theory; existence and uniqueness; Hyers-Ulam stability

## 1 Introduction

Due to the applications of FDEs, fractional calculus has got the attention of scientists in the fields like fractals theory, electromagnetic theory, metallurgy, plasma physics, signal and image processing, control theory ecology, economics, biology. For instance, see the applications of FDEs in different scientific fields in [1–11] and the references therein.

There exist a large number of nonlinear mathematical models in the scientific fields for the study of dynamical systems. One of the most important nonlinear operators frequently used is the classical $p$-Laplacian operator, which satisfies

$$\frac{1}{p}+\frac{1}{q}=1,\qquad \phi_p(s)=|s|^{p-2}s,\qquad p>1\quad\text{and}\quad \phi_q(\theta)=\phi_p^{-1}(\theta).$$

For the details and applications as regards the nonlinear $p$-Laplacian operator, the reader is referred to [12–20] and the references therein.

Here we highlight some recent contributions of the researchers which are related to our work. Lu *et al.* [21] discussed a Sturm-Liouville boundary value problem (BVP) of FDEs with $p$-Laplacian for the existence of two or three positive solutions by the Leggett-

Williams fixed point theorem. Their problem is given by

$$\begin{cases} \mathcal{D}_{0_+}^{\beta}(\phi_p(\mathcal{D}_{0_+}^{\alpha}\mu(x))) + f(x,\mu(x)) = 0, \quad 0 < x < 1, \\ a\mu(0) - \eta\mu'(0) = 0, \qquad \gamma\mu(1) + \delta\mu'(1) = 0, \qquad \mathcal{D}_{0_+}^{\alpha}\mu(0) = 0, \end{cases}$$

where $\mathcal{D}_{0_+}^{\alpha}$, $\mathcal{D}_{0_+}^{\beta}$ denote the standard Caputo fractional derivatives, $1 < \alpha \le 2$, $0 < \beta \le 1$, $\rho = a\gamma + a\delta + \eta\gamma > 0$, $a, \eta, \delta, \gamma \ge 0$, and $f$ is a continuous function.

Hu *et al.* [22] investigated the following nonlinear FDEs with $p$-Laplacian operator addressing the existence of a solution:

$$\begin{cases} \mathcal{D}_{0_+}^{\beta}(\phi_p(\mathcal{D}_{0_+}^{\alpha}\mu(x))) + f(x,\mu(x),\mathcal{D}_{0_+}^{\alpha}\mu(x)) = 0, \quad x \in (0,1), \\ \mathcal{D}_{0_+}^{\alpha}\mu(0) = 0 = \mathcal{D}_{0_+}^{\alpha}\mu(1), \end{cases}$$

where $0 < \alpha, \beta < 1$, $1 < \alpha + \beta < 2$, $\mathcal{D}_{0_+}^{\alpha}$, $\mathcal{D}_{0_+}^{\beta}$ represent the standard Caputo fractional derivatives, and $f$ is continuous.

Hu and Zhang [23] recently studied a coupled system of FDEs with $p$-Laplacian operator with infinite boundary conditions,

$$\begin{cases} \mathcal{D}_{0_+}^{\beta_1}\phi_p(\mathcal{D}_{0_+}^{\alpha_1}x(t)) = h(t, y(t), \mathcal{D}_{0_+}^{\alpha_2-1}, \mathcal{D}_{0_+}^{\alpha_2-1}y(t), \ldots, \mathcal{D}_{0_+}^{\alpha_2-(n-1)}y(t)), \quad t \in (0,1), \\ \mathcal{D}_{0_+}^{\beta_2}\phi_p(\mathcal{D}_{0_+}^{\alpha_2}y(t)) = g(t, x(t), \mathcal{D}_{0_+}^{\alpha_1-1}, \mathcal{D}_{0_+}^{\alpha_1-1}x(t), \ldots, \mathcal{D}_{0_+}^{\alpha_1-(n-1)}x(t)), \quad t \in (0,1), \\ x'(0) = \cdots = x^{(n-1)}(0) = \mathcal{D}_{0_+}^{\alpha_1}x(0) = 0, \qquad x(0) = \sum_{i=1}^{\infty} a_i x(\mu_i), \\ y'(0) = \cdots = y^{(n-1)}(0) = \mathcal{D}_{0_+}^{\alpha_2}y(0) = 0, \qquad y(0) = \sum_{i=1}^{\infty} b_i y(\nu_i), \end{cases}$$

where $0 < \beta_1, \beta_2 < 1$, $n-1 < \alpha_1, \alpha_2 < n$, $0 < \mu_1 < \mu_2 < \cdots < \mu_i < \cdots < 1$, $0 < \nu_1 < \nu_2 < \cdots < \nu_i < \cdots < 1$, $\sum_{i=1}^{\infty}|a_i| < \infty$, $\sum_{i=1}^{\infty}|b_i| < \infty$, $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} a_i = 1$, and $\mathcal{D}_{0_+}^{\beta_i}$, $\mathcal{D}_{0_+}^{\alpha_i}$, for $i = 1,2$, are Caputo fractional derivatives, and $h, g$ are real valued continuous functions.

Zhi *et al.* [15] have investigated the existence of positive solutions for a nonlocal BVP of FDEs with $p$-Laplacian operator and illustrated the problem with an illustrative example. The corresponding problem is

$$\begin{cases} (\phi_p(\mathcal{D}_{0_+}^{\alpha}\mu(x)))'' = \mathfrak{F}(x,\mu(x),\mathcal{D}_{0_+}^{\beta}\mu(x)), \quad x \in (0,1), \\ \mu(x)|_{x=0} = \mu''(x)|_{x=0} = 0, \qquad \mu(1) = \int_0^1 g(\theta)\mu(\theta)\,d\theta, \\ (\phi_p(\mathcal{D}_{0_+}^{\alpha})\mu(0)))' = \xi_1(\phi_p(\mathcal{D}_{0_+}^{\alpha})\mu(a_1))', \\ \phi_p(\mathcal{D}_{0_+}^{\alpha})\mu(1) = \xi_2(\phi_p(\mathcal{D}_{0_+}^{\alpha})\mu(b_2)), \end{cases}$$

where $\phi_p$ is a $p$-Laplacian operator and $2 < \alpha \le 3$, $1 < \beta < \alpha - 1 < 2$, $0 < a_1 \le b_2 < 1$, $0 \le \xi_1, \xi_2 < 1$, and $\mathcal{D}_{0_+}^{\alpha}$ expresses the Caputo derivative of order $\alpha$.

Ahmad *et al.* [24] studied a nonlinear FDE with nonlocal Erdélyi-Kober and generalized Riemann-Liouville-type fractional integral IBCs for the EUS by a different approach. They considered the following problem:

$$\begin{cases} {}^cD^{\alpha}u(t) = f(t,u(t)), \quad t \in [0,T], \\ u(0) = \theta\mathcal{I}_{\eta}^{\gamma,\delta}u(\zeta), \qquad u(T) = \beta^{\rho}\mathcal{I}^{p}u(\xi), \quad 0 < \xi, \zeta < T, \end{cases}$$

where ${}^cD^\alpha$ is a fractional order Caputo differential operator. $\mathcal{I}_\eta^{\gamma,\delta}$ is a fractional integral in the Erdélyi-Kober sense, ${}^\rho\mathcal{I}^p$ is a fractional order Riemann-Liouville integral, $\delta > 0$, $\eta > 0$, $\rho > 0, p > 0, \theta, \beta, \gamma \in \mathbb{R}, 1 < \alpha \leq 2, f : [0, T] \times \mathbb{R} \to \mathbb{R}$.

Stability analysis plays a significant role in the optimization and numerical analysis of the FDEs. Recently, the study of different sorts of stabilities of FDEs has attracted the attention of researchers. For example, exponential, Mittag-Leffler, and Lyapunov stabilities have been considered by some researchers [25, 26]. Stability was importantly given by Ulam [27], which was formally introduced by Hyers [28]. Urs [29] investigated the Hyers-Ulam stability for the following coupled periodic BVPs:

$$\begin{cases} \mu''(x) - \mathfrak{F}_1(x, \mu(x)) = \mathfrak{F}_2(x, \nu(x)), & x \in [0, T], \\ \nu''(x) - \mathfrak{F}_1(x, \nu(x)) = \mathfrak{F}_2(x, \mu(x)), \\ \mu(x)|_{x=0} = \mu(x)|_{x=T}, \qquad \nu(x)|_{x=0} = \nu(x)|_{x=T}. \end{cases}$$

Recently, Ali *et al.* [30] studied the following coupled system of FDEs with fractional order integral boundary conditions for the EUS and Hyers-Ulam stability:

$$\begin{cases} {}^c\mathcal{D}_{0_+}^\alpha u(t) = f(t, \nu(t)), & t \in [0,1], \\ {}^c\mathcal{D}_{0_+}^\beta \nu(t) = f(t, u(t)), & t \in [0,1], \\ u(0) = 0, \qquad u(t)|_{t=1} = \frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} p(u(s))\, ds, \\ \nu(0) = 0, \qquad \nu(t)|_{t=1} = \frac{1}{\Gamma(\delta)} \int_0^T (T-s)^{\delta-1} q(\nu(s))\, ds, \end{cases}$$

where $\alpha, \beta, \gamma, \delta \in (1,2]$, ${}^c\mathcal{D}_{0_+}^\alpha$, ${}^c\mathcal{D}_{0_+}^\beta$ are Caputo fractional derivatives, and $p, q \in L[0,1]$.

Using classical fixed point theory one needs strong conditions to establish the case of FDEs and therefore restrict the applicability to certain classes of FDEs and their systems. To relax the criteria degree theory plays an excellent role for the existence of solutions to FDEs and their systems. Various degree theories including Brouwer and Leray-Schauder degree theories have been established to deal with the existence theory of differential equations. A version of degree theory known as topological degree theory was importantly introduced by Mawhin [31] and later on extended by Isaia [32]; it has been used to establish existence theory of nonlinear differential and integral equations. The mentioned method is called a prior estimate method which does not require compactness of the operator and relaxing the conditions for existence and uniqueness of solutions of differential and integral equations. Recently, the aforesaid degree theory has been applied to investigate certain classes of FDEs with boundary conditions, in the references [33–35].

Inspired by the aforementioned research, we use the topological degree method to investigate EUS and Hyers-Ulam stability of a coupled system with IBCs and nonlinear $p$-Laplacian operator given by

$$\begin{cases} \mathcal{D}_{0_+}^{\beta_1}(\phi_p(\mathcal{D}_{0_+}^{\alpha_1} u(t))) + \psi_1(t, \nu(t)) = 0, \qquad \mathcal{D}_{0_+}^{\beta_2}(\phi_p(\mathcal{D}_{0_+}^{\alpha_2} \nu(t))) + \psi_2(t, u(t)) = 0, \\ (\phi_p(\mathcal{D}_{0_+}^{\alpha_1} u(t)))|_{t=1} = \mathcal{I}_{0_+}^{\beta_1 - 1}(\psi_1(t, \nu(t)))|_{t=1}, \\ (\phi_p(\mathcal{D}_{0_+}^{\alpha_1} u(t)))'|_{t=1} = 0 = (\phi_p(\mathcal{D}_{0_+}^{\alpha_1} u(t)))''|_{t=0}, \\ (\phi_p(\mathcal{D}_{0_+}^{\alpha_2} \nu(t)))|_{t=1} = \mathcal{I}_{0_+}^{\beta_2 - 1}(\psi_2(t, u(t)))|_{t=1}, \\ (\phi_p(\mathcal{D}_{0_+}^{\alpha_2} \nu(t)))'|_{t=1} = 0 = (\phi_p(\mathcal{D}_{0_+}^{\alpha_2} \nu(t)))''|_{t=0}, \\ u(0) = 0 = u''(0), \qquad u(1) = 0, \qquad \nu(0) = 0 = \nu''(0), \qquad \nu(1) = 0, \end{cases} \tag{1.1}$$

where $2 < \alpha_i, \beta_i < 3$, $\psi_1, \psi_2 \in L[0,1]$, and $\mathcal{D}_{0_+}^{\alpha_i}$, $\mathcal{D}_{0_+}^{\beta_i}$ for $i = 1,2$ stand for the Caputo fractional derivatives, $\phi_p(\kappa) = |\kappa|^{p-2}\kappa$ is the $p$-Laplacian operator where $1/p + 1/q = 1$, and $\phi_q$ denotes the inverse of the $p$-Laplacian operator. Here we remark that the application of the degree method to deal with the existence and uniqueness and to find conditions for Hyers-Ulam stability to a coupled system of FDEs with $p$-Laplacian operator (1.1) has not been investigated to the best of our knowledge. We prove sufficient conditions for EUS and Hyers-Ulam stability for the coupled system (1.1). The sufficient conditions for the EUS are obtained with the help of coincidence degree theory and nonlinear functional analysis as suggested by Deimling [36]. Our problem is more general and complicated than the work in [37]. Some new and related results obtained via the topological degree method can be found in [38–42]. For the application of the results, an illustrative example is also presented.

## 2 Auxiliary results

Here we recall some definitions, theorems, and Hyers-Ulam stability results from the literature [2–4, 43], which have an important role in the results of the paper.

**Definition 2.1** The Riemann-Liouville-type fractional integral of order $\alpha$ of a function $f(t)$ is defined as

$$\mathcal{I}_{0_+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\theta)^{\alpha-1} f(\theta)\, d\theta, \tag{2.1}$$

provided that the integral on the right converges pointwise on $(0,\infty)$.

**Definition 2.2** The Caputo fractional derivative of order $\alpha > 0$ of a function $f(t)$ is

$$\mathcal{D}_{0_+}^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\theta)^{m-\alpha-1} f^{(m)}(\theta)\, d\theta, \tag{2.2}$$

where $m = [\alpha] + 1$, $[\alpha]$ is the integer part of $\alpha$, provided that the integral on the right hand side converges pointwise on the interval $(0,\infty)$.

**Lemma 2.3** *Let $\alpha > 0$ and $\lambda \in C(0,1) \cap L^1(0,1)$. Then the general solution of the FDE*

$$\mathcal{D}_{0_+}^{\alpha} \lambda(t) = y(t)$$

*is given by*

$$\lambda(t) = y(t) + b_0 + b_1 t + b_2 t^2 + \cdots + b_{m-1} t^{m-1},$$

*where, for some $b_i \in \mathbb{R}$, $i = 0,1,2,\dots,m-1$, $m$ is the smallest integer such that $m \geq \alpha$.*

**Lemma 2.4** ([4]) *Let $\alpha \in (n-1,n]$, $\psi \in AC^{n-1}$. Then*

$$I_{0_+}^{\alpha} \mathfrak{D}_{0_+}^{\alpha} \psi(t) = \psi(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

*for $c_i \in \mathbb{R}$ for $i = 0,1,2,\dots,n-1$.*

Let $\mathcal{L} = C([0,1],\mathbb{R})$ be the Banach space of continuous functions with a topology of uniform convergence and endowed with a norm $\|u(t)\| = \sup\{|u(t)| : t \in [0,1]\}$. The product space denoted by $\omega^* = \mathcal{L} \times \mathcal{L}$ under the norms $\|(u,v)(t)\| = \|u(t)\| + \|v(t)\|$ is also a Banach space. We recall some basic definitions and results related to the coincidence degree theory and nonlinear functional analysis, from the available literature; see [32, 36, 44–52] and the references therein.

**Definition 2.5** Let the class of all bounded sets of $P(\mathcal{L})$ be denoted by $\aleph$. Then the mapping $\mathcal{F} : \aleph \to (0,\infty)$ for the Kuratowski measure of noncompactness is defined as

$$\mathcal{F}(z) = \inf\{d > 0 : z \text{ is the finite cover for sets of diameter} \leq d\},$$

where $z \in \aleph$.

**Proposition 2.6** *The following are the characteristics of the measure* $\mathcal{F}$:

(1) *for relative compact* $A$, *the Kuratowski measure* $\mathcal{F}(A) = 0$;
(2) *semi-norm* $\mathcal{F}$, *that is* $\mathcal{F}(\kappa A) = |\kappa|\mathcal{F}(A)$, $\kappa \in \mathbb{R}$, *and* $\mathcal{F}(A_1 + A_2) \leq \mathcal{F}(A_1) + \mathcal{F}(A_2)$;
(3) $A_1 \subset A_2$ *yields* $\mathcal{F}(A_1)$; $\mathcal{F}(A_1 \cup A_2) = \sup\{\mathcal{F}(A_1), \mathcal{F}(A_2)\}$;
(4) $\mathcal{F}(\operatorname{conv} A) = \mathcal{F}(A)$;
(5) $\mathcal{F}(\bar{A}) = \mathcal{F}(A)$.

**Definition 2.7** Assume that $\varphi : \vartheta \to \mathcal{L}$ is bounded and a continuous mapping such that $\vartheta \subset \mathcal{L}$. Then $\varphi$ is an $\mathcal{F}$-Lipschitz, where $\zeta \geq 0$ such that

$$\mathcal{F}\bigl(\varphi(A)\bigr) \leq \zeta \jmath(A) \quad \text{for all bounded } A \subset \vartheta.$$

Then $\varphi$ is called a strict $\mathcal{F}$-contraction under the condition $\zeta < 1$.

**Definition 2.8** The function $\varphi$ is $\mathcal{F}$-condensing if

$$\mathcal{F}\bigl(\varphi(A)\bigr) < \mathcal{F}(A), \quad \text{for all bounded } A \subset \vartheta \quad \text{such that} \quad \mathcal{F}(A) > 0.$$

Therefore $\mathcal{F}(\varphi(A)) \geq \jmath(A)$ yields $\mathcal{F}(A) = 0$.

Further we have $\varphi : \vartheta \to \mathcal{L}$ is Lipschitz for $\zeta > 0$, such that

$$\bigl\|\varphi(\upsilon) - \varphi(\bar{\upsilon})\bigr\| \leq \zeta \|\upsilon - \bar{\upsilon}\|, \quad \text{for all } \upsilon, \bar{\upsilon} \in \vartheta.$$

The condition $\zeta < 1$ causes $\varphi$ to be a strict contraction.

**Proposition 2.9** *The mapping* $\varphi$ *is* $\mathcal{F}$*-Lipschitz with constant* $\zeta = 0$ *if and only if* $\varphi : \vartheta \to \mathcal{L}$ *is said to be compact.*

**Proposition 2.10** *The operator* $\varphi$ *is* $\mathcal{F}$*-Lipschitz for some constant* $\zeta$ *if and only if* $\varphi : \vartheta \to \mathcal{L}$ *is Lipschitz with constant* $\zeta$.

**Theorem 2.11** *Let $\varphi : \mathcal{L} \to \mathcal{L}$ be a $\mathcal{F}$-contraction and*

$$\mathcal{G} = \{z \in \mathcal{L} : \text{there exist } 0 \leq \lambda \leq 1 \text{ such that } z = \lambda\varphi(z)\}.$$

*If $\mathcal{G}$ is bounded in $\mathcal{L}$, there exists $r > 0$ and $\mathcal{G} \subset z_r(0)$, with the degree*

$$\deg(I - \lambda\varphi, z_r(0), 0) = 1, \quad \text{for every } \lambda \in [0,1].$$

*Consequently, $\varphi$ has at least one fixed point and the set of fixed points of $\varphi$ lies in $z_r(0)$.*

**Lemma 2.12** ([16]) *Let $\phi_p$ be a p-Laplacian operator. Then we have*
(i) *if $1 < p \leq 2$, $\kappa_1\kappa_2 > 0$, and $|\kappa_1|, |\kappa_2| \geq \rho > 0$, then*

$$\left|\phi_p(\kappa_1) - \phi_p(\kappa_2)\right| \leq (p-1)\rho^{p-2}|\kappa_1 - \kappa_2|;$$

(ii) *if $p > 2$ and $|\kappa_1|, |\kappa_2| \leq \rho$, then*

$$\left|\phi_p(\kappa_1) - \phi_p(\kappa_2)\right| \leq (p-1)\rho^{p-2}|\kappa_1 - \kappa_2|.$$

## 3 Main results

**Theorem 3.1** *Let $\psi_1 \in C[0,1]$ be an integrable function satisfying* (1.1). *Then the solution of*

$$\begin{cases} \mathcal{D}_{0_+}^{\beta_1}(\phi_p(\mathcal{D}_{0_+}^{\alpha_1}u(t))) + \psi_1(t,\nu(t)) = 0, \\ (\phi_p(\mathcal{D}_{0_+}^{\alpha_1}u(t)))|_{t=1} = \mathcal{I}_{0_+}^{\beta_1-1}(\psi_1(t,\nu(t)))|_{t=1}, \\ (\phi_p(\mathcal{D}_{0_+}^{\alpha_1}u(t)))'|_{t=1} = 0 = (\phi_p(\mathcal{D}_{0_+}^{\alpha_1}u(t)))''|_{t=0}, \\ u(0) = 0 = u''(0), \qquad u(1) = 0, \end{cases} \tag{3.1}$$

*is given by the integral equation*

$$u(x) = \int_0^1 \mathcal{G}^{\alpha_1}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(s,\theta)\psi_1\big(\theta,\nu(\theta)\big)\,d\theta\right)ds, \tag{3.2}$$

*where $\mathcal{G}^{\alpha_1}(t,s)$, $\mathcal{G}^{\beta_1}(s,\theta)$ are Green functions defined by*

$$\mathcal{G}^{\alpha_1}(t,s) = \begin{cases} \frac{(t-s)^{\alpha_1-1}-t(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}, & 0 \leq s \leq t \leq 1, \\ \frac{-t(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}, & 0 \leq s \leq t \leq 1, \end{cases} \tag{3.3}$$

$$\mathcal{G}^{\beta_1}(t,s) = \begin{cases} \frac{-(t-s)^{\beta_1-1}+(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} + \frac{t(1-s)^{\beta_1-2}}{\Gamma(\beta_1-2)}, & 0 \leq s \leq t \leq 1, \\ \frac{(1-s)^{\beta_1-1}}{\Gamma(\beta_1)} + \frac{t(1-s)^{\beta_1-2}}{\Gamma(\beta_1-2)}, & 0 \leq s \leq t \leq 1. \end{cases} \tag{3.4}$$

*Proof* Applying operator $\mathcal{I}_{0_+}^{\beta_1}$ to (3.1) and using Lemma 2.3, we get the following equivalent integral form of (3.1):

$$\phi_p\big(\mathcal{D}_{0_+}^{\alpha_1}u(t)\big) = -\mathcal{I}_{0_+}^{\beta_1}\psi_1\big(t,\nu(t)\big) + c_1 + c_2 t + c_3 t^2. \tag{3.5}$$

The condition $(\phi_p(\mathcal{D}_{0_+}^{\alpha_1}u(t)))''|_{t=0} = 0$ results in $c_3 = 0$. The idea that $(\phi_p(\mathcal{D}_{0_+}^{\alpha_1}u(t)))'|_{t=1} = 0$ implies

$$c_2 = \mathcal{I}_{0_+}^{\beta_1-1}\psi_1\big(t,\nu(t)\big)\big|_{t=1} = \frac{1}{\Gamma(\beta_1-1)}\int_0^1 (1-s)^{\beta_1-2}\psi_1\big(s,\nu(s)\big)\,ds. \tag{3.6}$$

With condition $(\phi_p(\mathcal{D}_{0_+}^{\alpha_1}u(t)))|_{t=1} = \mathcal{I}_{0_+}^{\beta_1-1}\psi_1(t,\nu(t))|_{t=1}$, we have

$$c_1 = \mathcal{I}_{0_+}^{\beta_1}\psi_1\big(t,\nu(t)\big)\big|_{t=1} = \frac{1}{\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-1}\psi_1\big(s,\nu(s)\big)\,ds. \tag{3.7}$$

From the values of $c_i$ for $i = 1,2,3$ and (3.5), we have

$$\begin{aligned}\phi_p\big(\mathcal{D}_{0_+}^{\alpha_1}u(t)\big) &= -\mathcal{I}_{0_+}^{\beta_1}\psi_1\big(t,\nu(t)\big) + \mathcal{I}_{0_+}^{\beta_1}\psi_1\big(t,\nu(t)\big)\big|_{t=1} + t\mathcal{I}^{\beta_1-1}\psi_1\big(t,\nu(t)\big)\\ &= \frac{-1}{\Gamma(\beta_1)}\int_0^t (t-s)^{\beta_1-1}\psi_1\big(s,\nu(s)\big)\,ds + \frac{1}{\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-1}\psi_1\big(s,\nu(s)\big)\,ds\\ &\quad + \frac{t}{\Gamma(\beta_1)}\int_0^1 (1-s)^{\beta_1-1}\psi_1\big(s,\nu(s)\big)\,ds\\ &= \int_0^1 \mathcal{G}^{\beta_1}(t,s)\psi_1\big(s,\nu(s)\big)\,ds,\end{aligned} \tag{3.8}$$

where $\mathcal{G}^{\beta_1}(t,s)$ is a Green function given in (3.4). From (3.8), we further have

$$\mathcal{D}_{0_+}^{\alpha_1}u(t) = \phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(t,s)\psi_1\big(s,\nu(s)\big)\,ds\right). \tag{3.9}$$

Applying the fractional integral operator $\mathcal{I}_{0_+}^{\alpha_1}$ on (3.9) and using Lemma 2.3 again, we have

$$u(t) = \mathcal{I}_{0_+}^{\alpha_1}\left(\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(t,s)\psi_1\big(s,\nu(s)\big)\,ds\right)\right) + k_1 + k_2 t + k_3 t^2. \tag{3.10}$$

Using the condition $u(0) = 0 = u''(0)$ in (3.10), we obtain $k_1 = 0 = k_3$. From the condition $u(1) = 0$, we have $k_2 = -\mathcal{I}_{0_+}^{\alpha_1}(\phi_q(\int_0^1 \mathcal{G}^{\beta_1}(t,s)\psi_1(s,\nu(s))\,ds))|_{t=1}$. Putting the values of $k_i$ for $i = 1,2,3$ in (3.10), we get the solution $u(t)$ in the following integral form:

$$\begin{aligned}u(t) &= \mathcal{I}_{0_+}^{\alpha_1}\left(\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(t,s)\psi_1\big(s,\nu(s)\big)\,ds\right)\right)\\ &\quad - t\mathcal{I}^{\alpha_1}\left(\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(t,s)\psi_1\big(s,\nu(s)\big)\,ds\right)\right)\bigg|_{t=1}\\ &= \left(\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} - t\int_0^1 \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}\right)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(s,\theta)\psi_1\big(\theta,\nu(\theta)\big)\,d\theta\right)ds\\ &= \int_0^1 \mathcal{G}^{\alpha_1}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(s,\theta)\psi_1\big(\theta,\nu(\theta)\big)\,d\theta\right)ds,\end{aligned} \tag{3.11}$$

where $\mathcal{G}^{\alpha_1}(t,s)$, $\mathcal{G}^{\beta_1}(s,\theta)$ are Green functions defined by (3.3), (3.4), respectively. □

Theorem 3.1 implies that our problem (1.1) is equivalent to the following coupled system of Hammerstein-type integral equations:

$$u(t) = \int_0^1 \mathcal{G}^{\alpha_1}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(s,\theta)\psi_1\bigl(\theta,v(\theta)\bigr)\,d\theta\right)ds, \tag{3.12}$$

$$v(t) = \int_0^1 \mathcal{G}^{\alpha_2}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_2}(s,\theta)\psi_2\bigl(\theta,v(\theta)\bigr)\,d\theta\right)ds, \tag{3.13}$$

where $\mathcal{G}^{\alpha_2}(t,s)$, $\mathcal{G}^{\beta_2}(t,s)$ are the following Green functions:

$$\mathcal{G}^{\alpha_2}(t,s) = \begin{cases} \frac{(t-s)^{\alpha_2-1}-t(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}, & 0 \le s \le t \le 1, \\ \frac{-t(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}, & 0 \le s \le t \le 1, \end{cases} \tag{3.14}$$

$$\mathcal{G}^{\beta_2}(t,s) = \begin{cases} \frac{-(t-s)^{\beta_2-1}+(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} + \frac{t(1-s)^{\beta_2-2}}{\Gamma(\beta_2-2)}, & 0 \le s \le t \le 1, \\ \frac{(1-s)^{\beta_2-1}}{\Gamma(\beta_2)} + \frac{t(1-s)^{\beta_2-2}}{\Gamma(\beta_2-2)}, & 0 \le s \le t \le 1. \end{cases} \tag{3.15}$$

Define $\mathcal{T}_i^* : \mathcal{L} \to \mathcal{L}$ for $(i = 1, 2)$ by

$$\mathcal{T}_1^* u(t) = \int_0^1 \mathcal{G}^{\alpha_1}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(s,\theta)\psi_1\bigl(\theta,v(\theta)\bigr)\,d\theta\right)ds, \tag{3.16}$$

$$\mathcal{T}_2^* v(t) = \int_0^1 \mathcal{G}^{\alpha_2}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_2}(s,\theta)\psi_2\bigl(\theta,u(\theta)\bigr)\,d\theta\right)ds. \tag{3.17}$$

By Theorem 3.1, the solution of the coupled system of the Hammerstein-type integral equations (3.12), (3.13) is equivalent to the fixed point, say $(u, v)$, of the operator equation

$$(u,v) = \mathcal{T}^*(u,v) = \bigl(\mathcal{T}_1^*(u), \mathcal{T}_2^*(v)\bigr)(t), \tag{3.18}$$

for $\mathcal{T}^* = (\mathcal{T}_1^*, \mathcal{T}_2^*)$.

To proceed, we introduce the following assumptions:

$(\mathcal{Q}_1)$ The functions $\psi_1$, $\psi_2$ satisfy the following growth conditions for the constants $a$, $b$, $\mathbb{M}^*_{\psi_1}$, $\mathbb{M}^*_{\psi_2}$:

$$\bigl|\psi_1(x,u)\bigr| \le a|u| + \mathbb{M}^*_{\psi_1},$$
$$\bigl|\psi_2(x,v)\bigr| \le b|v| + \mathbb{M}^*_{\psi_2}.$$

$(\mathcal{Q}_2)$ There exist real valued constants $\lambda_{\psi_1}$, $\lambda_{\psi_2}$ such that, for all $u, v, x, y \in \mathcal{L}$,

$$\bigl|\psi_1(t,\upsilon) - \psi_1(t,x)\bigr| \le \lambda_{\psi_1}|\upsilon - x|,$$
$$\bigl|\psi_2(t,\mu) - \psi_2(t,y)\bigr| \le \lambda_{\psi_2}|\mu - y|.$$

**Theorem 3.2** *With assumption* $(\mathcal{Q}_1)$, *the operator* $\mathcal{T}^* : \omega^* \to \omega^*$ *is continuous and satisfies the following growth condition*:

$$\bigl\|\mathcal{T}^*(u,v)\bigr\| \le \mathcal{B}\bigl\|(u,v)\bigr\| + \mathbb{K}, \tag{3.19}$$

*where* $\mathcal{B} = \Omega(a + b)$, $\mathbb{K} = \Omega(\mathcal{M}_1^* + \mathcal{M}_2^*)$, *and*

$$\Omega = \max\left\{ \frac{2(q-1)\rho_1^{q-2}}{\Gamma(\alpha_1+1)}\left(\frac{1}{\Gamma(\beta_1)} + \frac{2}{\Gamma(\beta_1+1)}\right), \frac{2(q-1)\rho_2^{q-2}}{\Gamma(\alpha_2+1)}\left(\frac{1}{\Gamma(\beta_2)} + \frac{2}{\Gamma(\beta_2+1)}\right)\right\}, \tag{3.20}$$

*for each* $(u,v) \in \wp_r \subset \omega^*$.

*Proof* Consider the bounded set $\wp_r = \{(u,v) \in \omega : \|(u,v)\| \leq r\}$ with sequence $\{(u_n, v_n)\}$ converging to $(u,v)$ in $\wp_r$. To show that $\|\mathcal{T}^*(u_n,v_n) - \mathcal{T}^*(u,v)\| \to 0$ as $n \to \infty$, let us consider

$$\begin{aligned} \left|\mathcal{T}_1^* u_n(t) - \mathcal{T}_1^* u(t)\right| &= \left| \int_0^1 \mathcal{G}^{\alpha_1}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(s,\theta)\psi_1\big(\theta, v_n(\theta)\big)\,d\theta\right) ds \right. \\ &\quad \left. - \int_0^1 \mathcal{G}^{\alpha_1}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(s,\theta)\psi_1\big(\theta, v(\theta)\big)\,d\theta\right) ds \right| \\ &\leq (q-1)\rho_1^{q-2}\left(\int_0^1 \left|\mathcal{G}^{\alpha_1}(t,s)\right| \int_0^1 \left|\mathcal{G}^{\beta_1}(s,\theta)\right| \right. \\ &\quad \left. \times \left|\psi_1\big(\theta, v_n(\theta)\big) - \psi_1\big(\theta, v(\theta)\big)\right| d\theta\, ds\right) \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} \left|\mathcal{T}_2^* v_n(t) - \mathcal{T}_2^* v(t)\right| &= \left| \int_0^1 \mathcal{G}^{\alpha_2}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_2}(s,\theta)\psi_2\big(\theta, u_n(\theta)\big)\,d\theta\right) ds \right. \\ &\quad \left. - \int_0^1 \mathcal{G}^{\alpha_2}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_2}(s,\theta)\psi_2\big(\theta, u(\theta)\big)\,d\theta\right) ds \right| \\ &\leq (q-1)\rho_2^{q-2}\left(\int_0^1 \left|\mathcal{G}^{\alpha_2}(t,s)\right| \int_0^1 \left|\mathcal{G}^{\beta_2}(s,\theta)\right| \right. \\ &\quad \left. \times \left|\psi_2\big(\theta, u_n(\theta)\big) - \psi_1\big(\theta, u(\theta)\big)\right| d\theta\, ds\right). \end{aligned} \tag{3.22}$$

From (3.21) and (3.22), we have

$$\begin{aligned} \left|\mathcal{T}^*(u_n,v_n)(t) - \mathcal{T}^*(u,v)(t)\right| &\leq (q-1)\rho_1^{q-2}\left(\int_0^1 \left|\mathcal{G}^{\alpha_1}(t,s)\right| \int_0^1 \left|\mathcal{G}^{\beta_1}(s,\theta)\right| \right. \\ &\quad \left. \times \left|\psi_1\big(\theta, v_n(\theta)\big) - \psi_1\big(\theta, v(\theta)\big)\right| d\theta\, ds\right) \\ &\quad + (q-1)\rho_2^{q-2}\left(\int_0^1 \left|\mathcal{G}^{\alpha_2}(t,s)\right| \int_0^1 \left|\mathcal{G}^{\beta_1}(s,\theta)\right| \right. \\ &\quad \left. \times \left|\psi_2\big(\theta, u_n(\theta)\big) - \psi_2\big(\theta, u(\theta)\big)\right| d\theta\, ds\right). \end{aligned} \tag{3.23}$$

From the continuity of $\psi_1$, $\psi_2$ and (3.23), we have $|\mathcal{T}^*(u_n,v_n)(t) - \mathcal{T}^*(u,v)(t)| \to 0$, as $n \to \infty$. Thus the operator $\mathcal{T}^*$ is a continuous operator. Further, with the help of (3.16) and

(3.17), we proceed as follows:

$$\begin{aligned}
\left|\mathcal{T}_1^* u(t)\right| &= \left|\int_0^1 \mathcal{G}^{\alpha_1}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(s,\theta)\psi_1\big(\theta,v(\theta)\big)\,d\theta\,ds\right)\right| \\
&\leq (q-1)\rho_1^{q-2}\left|\int_0^1 \mathcal{G}^{\alpha_1}(t,s)\int_0^1 \mathcal{G}^{\beta_1}(s,\theta)\psi_1\big(\theta,v(\theta)\big)\,d\theta\,ds\right| \\
&\leq (q-1)\rho_1^{q-2}\left|\left(\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} - t\int_0^1 \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}\left(\frac{-1}{\Gamma(\beta_1)}\int_0^s (s-\theta)^{\beta_1-1}\right.\right.\right. \\
&\quad \left.\left. + \frac{1}{\Gamma(\beta_1)}\int_0^1 (1-\theta)^{\beta_1-1} + \frac{s}{\Gamma(\beta_1)}\int_0^1 (1-\theta)^{\beta_1-1}\right)d\theta\,ds\right)\Big|\big(a|v| + \mathbb{M}_{\psi_1}^*\big) \\
&\leq \frac{2(q-1)\rho_1^{q-2}}{\Gamma(\alpha_1+1)}\left(\frac{1}{\Gamma(\beta_1)} + \frac{2}{\Gamma(\beta_1+1)}\right)\big(a|v| + \mathbb{M}_{\psi_1}^*\big), \qquad (3.24) \\
\left|\mathcal{T}_2^* v(t)\right| &= \left|\int_0^1 \mathcal{G}^{\alpha_2}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_2}(s,\theta)\psi_1\big(\theta,u(\theta)\big)\,d\theta\right)ds\right| \\
&\leq (q-1)\rho_2^{q-2}\left|\int_0^1 \mathcal{G}^{\alpha_2}(t,s)\left(\int_0^1 |\mathcal{G}^{\beta_2}(s,\theta)\psi_1\big(\theta,u(\theta)\big)\,d\theta\right)ds\right| \\
&\leq (q-1)\rho_2^{q-2}\left|\left(\int_0^t \frac{(t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)} - t\int_0^1 \frac{(1-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}\right)\left(\frac{-1}{\Gamma(\beta_2)}\int_0^s (s-\theta)^{\beta_2-1}\right.\right. \\
&\quad \left. + \frac{1}{\Gamma(\beta_2)}\int_0^1 (1-\theta)^{\beta_2-1} + \frac{s}{\Gamma(\beta_2)}\int_0^1 (1-\theta)^{\beta_2-1}\right)d\theta\,ds\Big|\big(b|u| + \mathbb{M}_{\psi_2}^*\big) \\
&\leq \frac{2(q-1)\rho_2^{q-2}}{\Gamma(\alpha_2+1)}\left(\frac{1}{\Gamma(\beta_2)} + \frac{2}{\Gamma(\beta_2+1)}\right)\big(b|u| + \mathbb{M}_{\psi_2}^*\big). \qquad (3.25)
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\left|\mathcal{T}^*(u,v)(t)\right| &\leq \Omega\big(a|v| + \mathbb{M}_{\psi_1}^*\big) + \Omega\big(b|u| + \mathbb{M}_{\psi_2}^*\big) \\
&\leq \Omega(a+b)\big(|v| + |u|\big) + \Omega\big(\mathbb{M}_{\psi_1}^* + \mathbb{M}_{\psi_2}^*\big) = \mathcal{B}\big\|(u,v)\big\| + \mathbb{K}. \qquad (3.26)
\end{aligned}$$

This completes the proof. □

**Theorem 3.3** *Let assumption* $(\mathcal{Q}_1)$ *hold. Then the operator* $\mathcal{T}^*: \omega^* \to \omega^*$ *is compact and* $\xi$*-Lipschitz with constant zero.*

*Proof* With the help of Theorem 3.2, we deduce that the operator $\mathcal{T}^*: \omega \to \omega$ is bounded. Next, using assumption $(\mathcal{Q}_1)$, Lemma 3.1, and equations (3.12), (3.13), for any $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}
\left|\mathcal{T}_1^* u(t_1) - \mathcal{T}_1^* u(t_2)\right| &= \left|\int_0^1 \mathcal{G}^{\alpha_1}(t_1,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(s,\theta)\psi_1\big(\theta,v(\theta)\big)\,d\theta\right)ds\right. \\
&\quad \left. - \int_0^1 \mathcal{G}^{\alpha_1}(t_2,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(s,\theta)\psi_1\big(\theta,v(\theta)\big)\,d\theta\right)ds\right| \\
&\leq (q-1)\rho_1^{q-2}\frac{|t_1^{\alpha_1} - t_2^{\alpha_1}| + |t_1 - t_2|}{\Gamma(\alpha_1+1)}\left(\frac{1}{\Gamma(\beta_1)} + \frac{2}{\Gamma(\beta_1+1)}\right) \\
&\quad \times \big(a|v| + \mathbb{M}_{\psi_1}^*\big), \qquad (3.27)
\end{aligned}$$

$$\begin{aligned}\left|\mathcal{T}_2^* v(t_1) - \mathcal{T}_2^* v(t_2)\right| &= \left|\int_0^1 \mathcal{G}^{\alpha_2}(t_1,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_2}(s,\theta)\psi_1\big(\theta,u(\theta)\big)\,d\theta\right)ds\right. \\ &\quad \left. - \int_0^1 \mathcal{G}^{\alpha_2}(t_2,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_2}(s,\theta)\psi_1\big(\theta,u(\theta)\big)\,d\theta\right)\right| \\ &\le (q-1)\rho_2^{q-2}\frac{|t_1^{\alpha_2} - t_2^{\alpha_2}| + |t_1 - t_2|}{\Gamma(\alpha_2+1)}\left(\frac{1}{\Gamma(\beta_2)} + \frac{2}{\Gamma(\beta_2+1)}\right) \\ &\quad \times \big(a|u| + \mathbb{M}_{\psi_2}^*\big).\end{aligned} \tag{3.28}$$

From (3.27), (3.28), we have

$$\begin{aligned}&\left|\mathcal{T}^*(u,v)(t_1) - \mathcal{T}^*(u,v)(t_2)\right| \\ &\quad \le (q-1)\rho_1^{q-2}\frac{|t_1^{\alpha_1} - t_2^{\alpha_1}| + |t_1 - t_2|}{\Gamma(\alpha_1+1)}\left(\frac{1}{\Gamma(\beta_1)} + \frac{2}{\Gamma(\beta_1+1)}\right) \\ &\qquad \times \big(a|v| + \mathbb{M}_{\psi_1}^*\big) \\ &\qquad + (q-1)\rho_2^{q-2}\frac{|t_1^{\alpha_2} - t_2^{\alpha_2}| + |t_1 - t_2|}{\Gamma(\alpha_2+1)}\left(\frac{1}{\Gamma(\beta_2)} + \frac{2}{\Gamma(\beta_2+1)}\right) \\ &\qquad \times \big(a|u| + \mathbb{M}_{\psi_2}^*\big).\end{aligned} \tag{3.29}$$

As $t_1 \to t_2$, the right hand side of (3.29) approaches zero. Thus $\mathcal{T}^* = (\mathcal{T}_1^*, \mathcal{T}_2^*)$ is an equicontinuous operator on $D$. By Arzela-Ascoli's theorem, the operator $\mathcal{T}^*(D)$ is compact. Hence $D$ is $\xi$-Lipschitz with constant zero. □

**Theorem 3.4** *Let assumptions* $(\mathcal{Q}_1)$, $(\mathcal{Q}_2)$ *hold. Then the coupled system* (1.1) *has a unique solution provided that* $\Omega^* < 1$, *where*

$$\begin{aligned}\Omega^* &= \frac{2(p-1)\rho_1^{p-2}\lambda_{\psi_1}}{\Gamma(\alpha_1+1)}\left(\frac{1}{\Gamma(\beta_1)} + \frac{2}{\Gamma(\beta_1+1)}\right) \\ &\quad + \frac{2(q-1)\rho_2^{q-2}\lambda_{\psi_2}}{\Gamma(\alpha_2+1)}\left(\frac{1}{\Gamma(\beta_2)} + \frac{2}{\Gamma(\beta_2+1)}\right).\end{aligned} \tag{3.30}$$

*Proof* From (3.16), (3.17), and assumptions $(\mathcal{Q}_1)$ and $(\mathcal{Q}_2)$, we have

$$\begin{aligned}\left|\mathcal{T}_1^* u(t) - \mathcal{T}_1^* \bar{u}(t)\right| &= \left|\int_0^1 \mathcal{G}^{\alpha_1}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(s,\theta)\psi_1\big(\theta,v(\theta)\big)\,d\theta\right)ds\right. \\ &\quad \left. - \int_0^1 \mathcal{G}^{\alpha_1}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(s,\theta)\psi_1\big(\theta,\bar{v}(\theta)\big)\,d\theta\right)ds\right| \\ &\le \frac{2(q-1)\rho_1^{q-2}\lambda_{\psi_1}}{\Gamma(\alpha_1+1)}\left(\frac{1}{\Gamma(\beta_1)} + \frac{2}{\Gamma(\beta_1+1)}\right)\big|v(t) - \bar{v}(t)\big|\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}\left|\mathcal{T}_2^* v(t) - \mathcal{T}_2^* \bar{v}(t)\right| &= \left|\int_0^1 \mathcal{G}^{\alpha_2}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_2}(s,\theta)\psi_1\big(\theta,u(\theta)\big)\,d\theta\right)ds\right. \\ &\quad \left. - \int_0^1 \mathcal{G}^{\alpha_2}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_2}(s,\theta)\psi_1\big(\theta,\bar{u}(\theta)\big)\,d\theta\right)ds\right| \\ &\le \frac{2(q-1)\rho_1^{q-2}\lambda_{\psi_2}}{\Gamma(\alpha_2+1)}\left(\frac{1}{\Gamma(\beta_2)} + \frac{2}{\Gamma(\beta_2+1)}\right)\big|u(t) - \bar{u}(t)\big|.\end{aligned} \tag{3.32}$$

From (3.31), (3.32) we have

$$\begin{aligned}
\left|\mathcal{T}^*(u,v)(t)-\mathcal{T}^*(\bar{u},\bar{v})(t)\right| &\leq \frac{2(q-1)\rho_1^{q-2}\lambda_{\psi_1}}{\Gamma(\alpha_1+1)}\left(\frac{1}{\Gamma(\beta_1)}+\frac{2}{\Gamma(\beta_1+1)}\right)\left(\left|v(t)-\bar{v}(t)\right|\right) \\
&\quad+\frac{2(q-1)\rho_1^{q-2}\lambda_{\psi_2}}{\Gamma(\alpha_2+1)}\left(\frac{1}{\Gamma(\beta_2)}+\frac{2}{\Gamma(\beta_2+1)}\right)\left(\left|u(t)-\bar{u}(t)\right|\right) \\
&\leq \left[\frac{2(q-1)\rho_1^{q-2}\lambda_{\psi_1}}{\Gamma(\alpha_1+1)}\left(\frac{1}{\Gamma(\beta_1)}+\frac{2}{\Gamma(\beta_1+1)}\right)\right. \\
&\quad\left.+\frac{2(q-1)\rho_1^{q-2}\lambda_{\psi_2}}{\Gamma(\alpha_2+1)}\left(\frac{1}{\Gamma(\beta_2)}+\frac{2}{\Gamma(\beta_2+1)}\right)\right] \\
&\quad\times\left(\left\|(u,v)(t)-(\bar{u},\bar{v})(t)\right\|\right) \\
&= \Omega^*\left(\left\|(u,v)(t)-(\bar{u},\bar{v})(t)\right\|\right).
\end{aligned} \tag{3.33}$$

With the help of Banach's FPT and our assumption $\Omega^* < 1$, the contraction $\mathcal{T}^*$ has a unique fixed point. Thus, the coupled system of FDEs with $p$-Laplacian operator (1.1) has a unique solution. □

## 4 Hyers-Ulam stability

Here we study Hyers-Ulam stability for the coupled system of FDEs with fractional differential and integral IBCs and $p$-Laplacian operator (1.1). In view of the definitions of Hyers-Ulam stability given in [38–42], we present the following definition.

**Definition 4.1** The coupled system of Hammerstein-type integral equations (3.12), (3.13) is Hyers-Ulam stable if there exist positive constants $\mathcal{D}_1^*$, $\mathcal{D}_2^*$ satisfying the following conditions:

For every $\lambda_1, \lambda_2 > 0$, if

$$\begin{aligned}
&\left|u(t)-\int_0^1 \mathcal{G}_{\alpha_1}(t,s)\phi_q\left(\int_0^1 \mathcal{G}_{\beta_1}\psi_1\bigl(\tau,v(\tau)\bigr)\right)ds\right| \leq \lambda_1, \\
&\left|v(t)-\int_0^1 \mathcal{G}_{\alpha_2}(t,s)\phi_q\left(\int_0^1 \mathcal{G}_{\beta_2}\psi_2\bigl(\tau,u(\tau)\bigr)\right)ds\right| \leq \lambda_2,
\end{aligned} \tag{4.1}$$

there exists a pair, say $(u^*(t), v^*(t))$, satisfying

$$\begin{aligned}
u^*(t) &= \int_0^1 \mathcal{G}_{\alpha_1}(t,s)\phi_q\left(\int_0^1 \mathcal{G}_{\beta_1}(s,\theta)\psi_1\bigl(\theta,v^*(\theta)\bigr)\,d\theta\right)ds, \\
v^*(t) &= \int_0^1 \mathcal{G}_{\alpha_2}(t,s)\phi_q\left(\int_0^1 \mathcal{G}_{\beta_2}(s,\theta)\psi_2\bigl(\theta,u^*(\theta)\bigr)\,d\theta\right)ds,
\end{aligned} \tag{4.2}$$

such that

$$\begin{aligned}
&\left|u(t)-u^*(t)\right| \leq \mathcal{D}_1^*\lambda_1, \\
&\left|v(t)-v^*(t)\right| \leq \mathcal{D}_2^*\lambda_2.
\end{aligned} \tag{4.3}$$

**Theorem 4.2** *Under the assumptions* $(\mathcal{Q}_1)$ *and* $(\mathcal{Q}_2)$, *the solution of the coupled system of FDEs with nonlinear p-Laplacian operator* (1.1) *is Hyers-Ulam stable.*

*Proof* With the help of Theorem 3.4 and definition 4.1, let $(\upsilon(t), z(t))$ be a solution of the coupled system of Hammerstein-type integral equations (3.12), (3.13). Let $(u^*(t), v^*(t))$ be any other approximation satisfying (4.2). Then we have

$$\begin{aligned}
&\left|u(t) - u^*(t)\right| \\
&\quad= \left|\int_0^1 \mathcal{G}^{\alpha_1}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(s,\theta)\psi_1\big(\theta, v(\theta)\big)\,d\theta\right)ds \right. \\
&\qquad \left. - \int_0^1 \mathcal{G}^{\alpha_1}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_1}(s,\theta)\psi_1\big(\theta, v^*(\theta)\big)\,d\theta\,ds\right)\right| \\
&\quad\leq (q-1)\rho_1^{q-2}\left(\int_0^1 \left|\mathcal{G}^{\alpha_1}(t,s)\right| \int_0^1 \left|\mathcal{G}^{\beta_1}(s,\theta)\right|\left|\psi_1\big(\theta, v(\theta)\big) - \psi_1\big(\theta, v^*(\theta)\big)\right|\,d\theta\,ds\right) \\
&\quad\leq \frac{2(q-1)\rho_1^{q-2}\lambda_{\psi_1}}{\Gamma(\alpha_1+1)}\left(\frac{1}{\Gamma(\beta_1)} + \frac{2}{\Gamma(\beta_1+1)}\right)\left|v(t) - v^*(t)\right| = \mathcal{D}_1^*\lambda_1
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
&\left|v(t) - v^*(t)\right| \\
&\quad= \left|\int_0^1 \mathcal{G}^{\alpha_2}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_2}(s,\theta)\psi_2\big(\theta, u(\theta)\big)\,d\theta\right)ds \right. \\
&\qquad \left. - \int_0^1 \mathcal{G}^{\alpha_2}(t,s)\phi_q\left(\int_0^1 \mathcal{G}^{\beta_2}(s,\theta)\psi_2\big(\theta, u^*(\theta)\big)\,d\theta\right)ds\right| \\
&\quad\leq (q-1)\rho_2^{q-2}\left(\int_0^1 \left|\mathcal{G}^{\alpha_2}(t,s)\right| \int_0^1 \left|\mathcal{G}^{\beta_2}(s,\theta)\right|\left|\psi_2\big(\theta, u(\theta)\big) - \psi_1\big(\theta, u^*(\theta)\big)\right|\,d\theta\,ds\right) \\
&\quad\leq \frac{2(q-1)\rho_2^{q-2}\lambda_{\psi_2}}{\Gamma(\alpha_2+1)}\left(\frac{1}{\Gamma(\beta_2)} + \frac{2}{\Gamma(\beta_2+1)}\right)\left|x(t) - u^*(t)\right| = \mathcal{D}_2^*\lambda_2,
\end{aligned} \tag{4.5}$$

where $\mathcal{D}_1^* = \frac{2(p-1)\rho_1^{p-2}\lambda_{\psi_1}}{\Gamma(\alpha_1+1)}(\frac{1}{\Gamma(\beta_1)} + \frac{2}{\Gamma(\beta_1+1)})$, $\mathcal{D}_2^* = \frac{2(q-1)\rho_2^{q-2}\lambda_{\psi_2}}{\Gamma(\alpha_2+1)}(\frac{1}{\Gamma(\beta_2)} + \frac{2}{\Gamma(\beta_2+1)})$. Hence, with the help of (4.4) and (4.5), the coupled system of the Hammerstein-type integral equations (3.12) and (3.13) is Hyers-Ulam stable. Consequently, the coupled system with the $p$-Laplacian operator (1.1) is Hyers-Ulam stable. □

## 5 Illustrative example

Here we give an application of the results proved in Sections 2 and 3.

**Example 5.1** Consider the coupled system of FDEs with $p$-Laplacian operator with fractional order differential and integral IBCs of the following type:

$$\begin{aligned}
&\mathcal{D}_{0_+}^{\frac{8}{3}}\left(\phi_5\big(\mathcal{D}_{0_+}^{\frac{7}{3}}u(t)\big)\right) + \psi_1\big(t, v(t)\big) = 0, \qquad \mathcal{D}_{0_+}^{\frac{8}{3}}\left(\phi_5\big(\mathcal{D}_{0_+}^{\frac{7}{3}}v(t)\big)\right) + \psi_2\big(t, u(t)\big) = 0, \\
&\left(\phi_5\big(\mathcal{D}_{0_+}^{\frac{7}{3}}u(t)\big)\right)\Big|_{t=1} = \mathcal{I}_{0_+}^{\frac{8}{3}}\left(\psi_1\big(t, v(t)\big)\right)\Big|_{t=1}, \\
&\left(\phi_5\big(\mathcal{D}_{0_+}^{\frac{7}{3}}u(t)\big)\right)'\Big|_{t=1} = 0 = \left(\phi_5\big(\mathcal{D}_{0_+}^{\frac{7}{3}}u(t)\big)\right)''\Big|_{t=0}, \\
&\left(\phi_5\big(\mathcal{D}_{0_+}^{\frac{7}{3}}v(t)\big)\right)\Big|_{t=1} = \mathcal{I}_{0_+}^{\frac{8}{3}}\left(\psi_2\big(t, u(t)\big)\right)\Big|_{t=1}, \\
&\left(\phi_5\big(\mathcal{D}_{0_+}^{\frac{7}{3}}v(t)\big)\right)'\Big|_{t=1} = 0 = \left(\phi_5\big(\mathcal{D}_{0_+}^{\frac{7}{3}}v(t)\big)\right)''\Big|_{t=0}, \\
&u(0) = 0 = u''(0), \qquad u(1) = 0, \qquad v(0) = 0 = v''(0), \qquad v(1) = 0,
\end{aligned} \tag{5.1}$$

where $t \in [0,1]$, $p = 5$, $\alpha_i = 7/3$, $\beta_i = 8/3$, for $i = 1,2$ $\psi_1(t,u(t)) = \frac{-25}{17} + \frac{1}{15}\sin(v)$, $\psi_2(t,v(t)) = \frac{30}{18} + \frac{1}{15}\cos(u)$, which implies $\mathbb{M}^*_{\psi_1} = \mathbb{M}^*_{\psi_2} = 3$, $L_{\psi_1} = L_{\psi_2} = 1/15$. By simple calculations, we obtain

$$\begin{aligned}\Omega^* &= \frac{2(p-1)\rho_1^{p-2}\lambda_{\psi_1}}{\Gamma(\alpha_1+1)}\left(\frac{1}{\Gamma(\beta_1)} + \frac{2}{\Gamma(\beta_1+1)}\right) \\ &\quad + \frac{2(p-1)\rho_2^{p-2}\lambda_{\psi_2}}{\Gamma(\alpha_2+1)}\left(\frac{1}{\Gamma(\beta_2)} + \frac{2}{\Gamma(\beta_2+1)}\right) \\ &= 0.0122447 < 1. \end{aligned} \tag{5.2}$$

With the help of Theorem 3.4 and equation (5.2), we conclude that (5.1) has a unique solution. Similarly, the conditions of Theorem 4.2 can be checked easily. Thus the coupled system (5.1) is Hyers-Ulam stable.

## 6 Conclusion

In this paper, we applied the topological degree method to deal with EUS to a coupled system of FDEs with $p$-Laplacian operator (1.1). We have also given the notion of Hyers-Ulam stability for our problem and have given sufficient conditions for EUS and Hyers-Ulam stability. This work provides a base to the study of EUS and different sorts of stabilities for the FDEs with fractional order integral and differential IBCs and $p$-Laplacian operator. For future work, we suggest the reader the consider the problem for multiple solutions. The problem may also be studied for the EUS using different definitions of the fractional order derivative.

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**Competing interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Authors' contributions**

All the authors have made equally contributions to the publication of this article. All authors read and approved the final manuscript.

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