Lu and Sun *Boundary Value Problems* (2017) 2017:144
DOI 10.1186/s13661-017-0874-x

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

A new technique to study the boundary behaviors of superharmonic multifunctions and their application

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Abstract

Using some recent results of the Riesz decomposition method for sharp estimates of certain boundary value problems of harmonic functions in (St. Cer. Mat. 27:323-328, 1975), the boundary behaviors of upper and lower superharmonic multifunctions are studied. Several fundamental properties of these new classes of these functions are shown. A new technique is proposed to find the exact boundary behaviors by using Levin's type boundary behaviors for harmonic functions admitting certain lower bounds in (Pacific J. Math. 15:961-970, 1965). Finally, some examples are given to illustrate the applications of our results.

Keywords: superharmonic function; superharmonic multifunction; boundary behavior

1 Introduction

In 1977, Husain [3] has initiated the concept of superharmonic-open sets, which is considered as a wider class of some known types of near-open sets. In 1983, Mashhour *et al.* [4, 5] defined the concept of S-continuity, but for a single-valued function $f:(X,\tau)\rightarrow(Y,\sigma)$. Many topological properties of the above mentioned concepts and others have been established in [6, 7]. The purpose of this paper is to present the upper (resp. lower) superharmonic-continuous multifunction as a generalization of each of upper (resp. lower) super-continuous superharmonic multifunction in the sense of Berge [7] the upper (resp. lower) sub-continuous and the upper (resp. lower) precontinuous superharmonic multifunction due to Popa [1, 8] and also upper (resp. lower) α-continuous and upper (resp. lower) β-continuous superharmonic multifunctions as given in [9, 10] recently. Moreover, these new superharmonic multifunctions are characterized and many of their properties have also been established.

2 Preliminaries

The topological space or simply space which is used here will be given by (X,τ) and (Y,σ). τ-cl(W) and τ-int(W) denote the closure and the interior of any subset W of X with respect to a topology τ. In (X,τ), the class $\tau^*\subseteq P(X)$ is called a superharmonic topology on X if $X\in\tau^*$ and τ^* is closed under arbitrary union [3], (X,τ^*) is a superharmonic-topological space or simply superharmonic space, each member of τ is superharmonic-open and its complement is superharmonic-closed [5], In (X, τ^*), the superharmonic-closure, the superharmonic-interior and superharmonic-frontier of any $A \subseteq X$ will be denoted by superharmonic-cl(A), superharmonic-int(A) and superharmonic-fr(A), respectively, which are defined in [5] and likewise the corresponding ordinary ones. Meanwhile, for any $x \in X$, we define

$$\tau^*(x) = \{W \subseteq X : W \in \tau^*, x \in W\}.$$

In (X, τ), $A \subseteq X$ is called super-open [11] if there exists $U \in \tau$ such that $U \subseteq A \subseteq \tau\text{-cl}(U)$, while A is preopen [5] if $A \subseteq \tau\text{-int}(\tau\text{-cl}(A))$. The families of all super-open and preopen sets in (X, τ) are denoted by $SO(X, \tau)$ and $PO(X, \tau)$, respectively. Moreover,

$$\tau^\alpha = SO(X, \tau) \cap PO(X, \tau)$$

and

$$\beta O(X, \tau) \supset SO(X, \tau) \cup PO(X, \tau).$$

$A \in \tau^\alpha$ and $A \in \beta O(X, \tau)$ are called a superharmonic-α-set [2] and a superharmonic-β-open set [6], respectively. A single-valued superharmonic multifunction $f : (X, \tau) \to (Y, \sigma)$ is called superharmonic-S-continuous [5], if the inverse image of each open set in (Y, σ) is τ^*-supra open in (X, τ). For a superharmonic multifunction $F : (X, \tau) \to (Y, \sigma)$, the upper and the lower inverses of any $B \subseteq Y$ are given by

$$F^+(B) = \{x \in X : F(x) \subseteq B\}$$

and

$$F^-(B) = \{x \in X : F(X) \cap B \neq \phi\},$$

respectively. Moreover, $F : (X, \tau) \to (Y, \sigma)$ is called upper (resp. lower) super-continuous [7], if for each $V \in \sigma$, $F^+(V) \in \tau$ (resp. $F^-(V) \in \tau$). If τ in super-continuity is replaced by $SO(X, \tau)$, τ^α, $PO(X, \tau)$ and $\beta O(X, \tau)$, then F is upper (resp. lower) sub-continuous [8], upper (resp. lower) superharmonic α-continuous [1], upper (resp. lower) precontinuous [9] and upper (resp. lower) superharmonic-β-continuous [10], respectively. A space (X, τ) is called superharmonic-compact [12], if every supraopen cover of X admits a finite subcover.

3 Supra-continuous superharmonic multifunctions

Definition 3.1 A superharmonic multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:

(a) upper superharmonic-continuous at a point $x \in X$ if for each open set V containing $F(x)$, there exists $W \in \tau^*(x)$ such that

$$F(W) \subseteq V;$$

(b) lower superharmonic-continuous at a point $x \in X$ if for each open set V containing $F(x)$, there exists $W \in \tau^*(x)$ such that

$$F(W) \cap V \neq \phi;$$

(c) upper (resp. lower) superharmonic-continuous if F has this property at every point of X.

Any single-valued superharmonic function $f : (X,\tau) \to (Y,\sigma)$ can be considered as a multi-valued one which assigns to any $x \in X$ the singleton $\{f(x)\}$. We apply the above definitions of both upper and lower superharmonic-continuous multifunctions to the single-valued case. It is clear that they coincide with the notion of S-continuous due to Mashhour *et al.* [5]. One characterization of the above superharmonic multifunction is established throughout the following result, of which the proof is straightforward, so it is omitted.

Remark 3.1 For a superharmonic multifunction $F : (X,\tau) \to (Y,\sigma)$, many properties of upper (resp. lower) semicontinuity [7] (resp. upper (lower)) F-continuity [9], upper (resp. lower) sub-continuity [1], upper (resp. lower) precontinuity [10] and upper (resp. lower) (G-continuity [10]) can be deduced from the upper (resp. lower) superharmonic-continuity by considering $\tau^* = \tau$ (resp. $\tau^* = \tau^\alpha$, $\tau^* = SO(X,\tau)$, $\tau^* = PO(X,\tau)$ and $\tau^* = \beta O(X,\tau)$).

Proposition 3.1 *A superharmonic multifunction* $F : (X,\tau) \to (Y,\sigma)$ *is upper* (*resp. lower*) *superharmonic-continuous at a point* $x \in X$ *if and only if for* $V \in \sigma$ *with* $F(x) \subseteq V$ (*resp.* $F(x) \cap V \neq \phi$). *Then* $x \in \text{superharmonic-int}(F^+(V))$ (*resp.* $x \in \text{superharmonic-int}(F^-(V))$.

Lemma 3.1 *For any* $A \in (X,\tau)$, *we have*

$$\tau\text{-int}(A) \subseteq \text{superharmonic-int}(A) \subseteq A \subseteq \text{superharmonic-cl}(A) \subseteq \tau\text{-cl}(A).$$

Theorem 3.1 *The following are equivalent for a superharmonic multifunction* $F : (X,\tau) \to (Y,\sigma)$:

(i) F *is upper superharmonic-continuous*;

(ii) *for each* $x \in X$ *and each* $V \in \sigma(F(x))$, *we have* $F^+(V) \in \tau^*(x)$;

(iii) *for each* $x \in X$ *and each* $V \in \sigma(F(x))$, *there exists* $W \in \tau^*$ *such that*

$$F(W) \subseteq V;$$

(iv) $F^+(V) \in \tau^*$ *for every* $V \in \sigma$;

(v) $F^-(K)$ *is superharmonic-closed for every closed set* $K \subseteq Y$;

(vi) $\text{superharmonic-cl}(F^-(B)) \subseteq F^-(\tau\text{-cl}(B))$ *for every* $B \subseteq Y$;

(vii) $F^+(\tau\text{-int}(B)) \subseteq \text{superharmonic-int}(F^+(B))$ *for every* $B \subseteq Y$;

(viii) $\text{superharmonic-fr}(F^-(B)) \subseteq F^-(\text{fr}(B))$ *for every* $B \subseteq Y$;

(ix) $F : (X,\tau^*) \to (Y,\sigma)$ *is upper superharmonic-continuous.*

Proof (i) $\Longleftrightarrow$ (ii) and (i) $\Rightarrow$ (iv): Follow from Proposition 3.1.

(ii) $\Longleftrightarrow$ (iii): This is obvious, since the arbitrary union of superharmonic-open set is superharmonic-open.

(iv) = (v): Let K be closed in Y, the result satisfies

$$F^{+}(Y\backslash K) = X\backslash F^{-}(K).$$

(v) $\Rightarrow$ (vi): By putting $K = \sigma$-cl(B) and applying Lemma 3.1.

(vi) $\Rightarrow$ (vii): Let $B \Rightarrow Y$, then σ-int$(B) \in \sigma$ and so $Y\backslash \sigma$-int(B) is super-closed in (Y, σ). Therefore by (vi) we get

$$X\backslash \text{super-int}\big(F^{+}(B)\big) = \text{super-cl}\big(X\backslash F^{+}(B)\big) \subseteq \text{sub-cl}(X\backslash F^{+}\big(\sigma\text{-int}(B)\big)$$

and

$$\text{supra-cl}(F^{-}\big(Y\,\sigma\text{-int}(B)\big) \subseteq F - \big(Y\backslash\,\sigma\text{-int}(B)\big) \subseteq X\backslash F^{+}\big(\sigma\text{-int}(B)\big).$$

This implies that

$$F^{+}\big(\sigma\text{-int}(B)\big) \subseteq \text{supra-int}\big(F^{+}(B)\big).$$

(vii) $\Rightarrow$ (ii): Let $x \in X$ be arbitrary and each $V \in \sigma(F(x))$ then

$$F^{+}(V) \subseteq \text{supra-int}\big(F^{+}(V)\big).$$

Hence $F^{+}(V) \in \tau^{*}(x)$.

(viii) $\Leftrightarrow$ (v): Clearly, a suprafrontier and frontier of any set is superharmonic-closed and closed, respectively.

(ix) $\Leftrightarrow$ (iv): Follows immediately. □

Theorem 3.2 *For a superharmonic multifunction* $F : (X, \tau) \to (Y, \sigma)$, *the following statements are equivalent*:

(i) F *is lower superharmonic-continuous*;

(ii) *for each* $X \in X$ *and each* $V \in \sigma$ *such that*

$$F(x) \cap V \neq \phi \quad \textit{and} \quad F^{-}(V) \in \tau^{*}(x);$$

(iii) *for each* $x \in X$ *and each* $V \in \sigma$ *with* $F(x) \cap V \neq \phi$, *there exists* $W \in \tau^{*}$ *such that*

$$F(W) \cap V \neq \phi;$$

(iv) $F^{-}(V) \in \tau^{*}$ *for every* $V \in \sigma$;

(v) $F^{+}(K)$ *is superharmonic-closed for every closed set* $K \subseteq Y$;

(vi) superharmonic-cl$(F^{+}(B)) \subseteq F^{+}(\sigma$ cl-$(B))$ *for any* $B \subseteq Y$;

(vii) $F^{-}(\sigma$-int$(B)) \subseteq$ superharmonic-int$(F^{-}(B))$ *for any* $B \subseteq Y$;

(viii) superharmonic-fr$(F^{+}(B)) \subseteq F^{+}(fr(B))$ *for every* $B \subseteq Y$;

(ix) $F : (X, \tau^{*}) \to (Y, \sigma)$ *is lower superharmonic-continuous.*

Proof The proof is a quite similar to that of Theorem 3.1. Recall that the net $(\chi_i)_{(i\in I)}$ is superharmonic-convergent to x_0, if for each $W \in \tau^{*}(x_O)$ there exists a $i_o \in I$ such that for each $i \geq i_o$ it implies $x_i \in W$. □

Theorem 3.3 *A superharmonic multifunction $F:(X,\tau)\to(Y,\sigma)$ is upper superharmonic-continuous if and only if for each net $(\chi_i)_{(i\in I)}$ superharmonic-convergent to x_o and for each $V\in\sigma$ with $F(x_o)\subseteq V$ there is $i_o\in I$ such that $F(X_i)\subseteq V$ for all $i\geq i_o$.*

Proof Necessity, let $V\in\sigma$ with $F(x_o)\subseteq V$. By upper superharmonic-continuity of F, there is $W\in\tau^*(X_O)$ such that $F(W)\subseteq V$. Since from the hypothesis a net $(\chi_i)_{(i\in I)}$ is superharmonic-convergent to x_o and $W\in\tau^*(x_o)$ there is one $i_o\in I$ such that $x_i\in W$ for all $i>i_o$ and then $F(X_i)\subseteq V$ for all $i>i_o$. As regards sufficiency, assume the converse, *i.e.* there is an open set V in Y with $F(x_o)\subseteq V$ such that for each $W\in\tau^*$ under inclusion we have the relation $F(W)\nsubseteq V$, *i.e.* there is $x_w\in W$ such that $F(x_w)\nsubseteq V$. Then all of x_w will form a net in X with directed set W of $\tau^*(x_o)$, clearly this net is superharmonic-convergent to x_o. But $F(x_w)\nsubseteq V$ for all $W\in\tau^*(x_o)$. This leads to a contradiction which completes the proof. □

Theorem 3.4 *A superharmonic multifunction $F:(X,\tau)\to(Y,\sigma)$ is lower superharmonic-continuous if and only if for each $y_o\in F(x_o)$ and for every net $(\chi_i)_{(i\in I)}$ superharmonic-convergent to x_o, there exists a subnet $(Z_j)_{(j\in J)}$ of the net $(\chi_i)_{(i\in I)}$ and a net $(y_i)_{(j,v)\in J}$ in Y so that $(y_i)_{(j,v)\in J}$ superharmonic-convergent to y and $y_j\in F(z_j)$.*

Proof For necessity, suppose F is lower superharmonic-continuous, $(\chi_i)_{(i\in I)}$ is a net superharmonic-convergent to x_o, $y\in F(x_o)$ and $V\in\sigma(y)$. So we have $F(x_o)\cap V\neq\phi$, by lower superharmonic-continuity of F at x_o, there is a superharmonic-open set $W\subseteq X$ containing x_o such that $W\subseteq F^-(V)$. We have superharmonic-convergence of a net $(\chi_i)_{(i\in I)}$ to x_0 and for this W, there is a $i_o\in I$ such that, for each $i>i_o$, we have $x_i\in W$ and therefore $x_i\in F^-(V)$. Hence, for each $V\in\sigma(y)$, define the sets

$$I_v=\{i_o\in I: i>i_o\Rightarrow x_i\in F^-(V)\}$$

and

$$J=\{(i,V): V\in D(y), i\in I_v\}$$

and an order $\geq$ on J given as $(i',V')\geq(i,V)$ if and only if $i'>i$ and $V'\subseteq V$. Also, define $\zeta: J\to I$ by $\zeta((j,V))=j$. Then ζ is increasing and cofinal in I, so ζ defines a subset of $(\chi_i)_{(i\in I)}$, denoted by $(z_i)_{(j,v)\in J}$. On the other hand for any $(j,V)\in J$, since $j>j_o$ implies $x_j\in F^-(V)$ we have $F(Z_j)\cap V=F(X_j)\cap V\neq\phi$. Pick $y_j\in F(Z_j)\cap V\neq\phi$. Then the net $(y_i)_{(j,v)\in J}$ is supraconvergent to y. To see this, let $V_0\in\sigma(y)$; then there is $j_0\in I$ with $j_o=\zeta(j_o,V_o)$; $(j_o,V_o)\in J$ and $y_{jo}\in V$. If $(j,V)>(j_o,V_o)$ this means that $j>j_o$ and $V\subseteq V_o$. Therefore

$$y_j\in F(z_j)\cap V\subseteq F(x_j)\cap V\subseteq F(x_j)\cap V_o.$$

So $y_j\in V_o$. Thus $(y_i)_{(j,v)\in J}$ is superharmonic-convergent to y, which shows the result.

To show the sufficiency, assume the converse, *i.e.* F is not lower superharmonic-continuous at x_o. Then there exists $V\in\sigma$ such that $F(x_o)\cap V\neq\phi$ and for any superharmonic-neighborhood $W\subseteq X$ of x_o, there exists $x_w\in W$ for which $F(x_w)\cap V=\phi$. Let us consider the net $(\chi_w)_{W\in\tau^*(\chi_0)}$, which is obviously superharmonic-convergent to

x_o. Suppose $y_o \in F(x_o) \cap V$, by hypothesis there is a superset $(z_k)_{k\in K}$ of $(\chi_w)_{W\in\tau^*(\chi_0)}$ and $y_k \in F(z_k)$ like $(y_k)_{k\in K}$ superharmonic-convergent to y_o. As $y_o \in V \in \sigma$ there is $k_0' \in K$ so that $k > k_0'$ implies $y_k \in V$. On the other hand $(z_k)_{kEK}$ is a superset of the net $(\chi^w)_{W\in\tau^*(\chi_0)}$ and so there exists a superharmonic function $\Omega : K \to \tau^*(x_o)$ such that $z_k = \chi_{\Omega(k)}$ and for each $W \in \tau^*(x_o)$ there exists $k_0'' \in K$ such that $\Omega(k_0'') \geq W$. If $k \geq k_0''$ then $\Omega(k) \geq \Omega(k_0'') \geq W$. Considering $k_0 \in K$ so that $k_o \geq k_0'$ and $k_o \geq k_0''$. Therefore $y_k \in V$ and by the meaning of the net $(\chi_W)_{W\in\tau^*(\chi_0)}$, we have

$$F(z_k) \cap V = F(\chi_{\Omega(K)}) \cap V = \phi.$$

This gives $y_k \notin V$, which contradicts the hypothesis and so the requirement holds. □

Definition 3.2 A subset W of a space (X, τ) is called superharmonic-regular, if for any $x \in W$ and any $H \in \tau^*(x)$ there exists $U \in \tau$ such that

$$x \in U \subseteq \tau\text{-cl}(U) \subseteq H.$$

Recall that $F : (X, \tau) \to (Y, \sigma)$ is punctually superharmonic-regular, if for each $X \in X$, $F(x)$ is superharmonic-regular.

Lemma 3.2 *In a superharmonic space (X, τ), if $W \subseteq X$ is superharmonic-regular and contained in a superharmonic-open set H, then there exists $U \in \tau$ such that*

$$W \subseteq U \subseteq \tau\text{-cl}(U) \subseteq H.$$

For a superharmonic multifunction $F : (X, \tau) \to (Y, \sigma)$, a superharmonic multifunction superharmonic-cl$(F) : (X, \tau) \to (Y, \sigma)$ *is defined as follows*:

$$(\text{superharmonic-cl}\,F)(x) = \text{superharmonic-cl}\big(F(x)\big)$$

for each $x \in X$.

Proposition 3.2 *For a punctually α-paracompact and punctually superharmonic-regular superharmonic multifunction $F : (X, \tau) \to (Y, \sigma)$, we have*

$$\big(\text{superharmonic-cl}(F)^+(W)\big) = F^+(W)$$

for each $W \in \sigma^*$.

Proof Let $x \in (\text{superharmonic-cl}(F))^+(W)$ for any $W \in \sigma^*$, this means

$$F(x) \subseteq \text{superharmonic-cl}\big(F(x)\big) \subseteq W,$$

which leads to $x \in F^+(W)$. Hence one inclusion holds. To show the other, let $X \in F^+(W)$ where $W \in \sigma^*(x)$. Then $F(x) \subseteq W$, by the hypothesis of F and the fact that $\sigma \subseteq \sigma^*$, applying Lemma 3.2, there exists $G \in \sigma$ such that

$$F(x) \subseteq G \in \sigma\text{-cl}(G) \subseteq W.$$

Therefore

$$\text{superharmonic-cl}\big(F(x)\big) \subseteq W.$$

This means that $x \in (\text{superharmonic-cl}\,F)^{+}(W)$. Hence the equality holds. □

Theorem 3.5 *Let $F(X,\tau) \to (Y,\sigma)$ be a punctually a-paracompact and punctually superharmonic-regular superharmonic multifunction. Then F is upper superharmonic-continuous if and only if*

$$(\text{superharmonic-cl}\,F) : (X,\tau) \to (Y,\sigma)$$

is upper superharmonic-continuous.

Proof As regards necessity, suppose $V \in \sigma$ and $x \in (\text{superharmonic-cl}\,F)^{+}(V) = F^{+}(V)$ (see Proposition 3.2). By upper superharmonic-continuity of F, there exists $H \in \tau^{*}(x)$ such that $F(H) \subseteq V$. Since $\sigma \in \sigma^{*}$, by Lemma 3.2 and the assumption of F, there exists $G \in \sigma$ such that

$$F(h) \subseteq G \subseteq \sigma\text{-cl}(G) \subseteq W$$

for each $h \in H$.

Hence

$$\text{superharmonic-cl}\big(F(h)\big) \subseteq \text{superharmonic-cl}(G) \subseteq \sigma\text{-cl}(G) \subseteq V$$

for each $h \in H$, which shows that [13]

$$(\text{superharmonic-cl}\,F)(H) \subseteq V.$$

Thus (superharmonic-cl F) is upper superharmonic-continuous. As regards sufficiency, assume $V \in \sigma$ and $X \in F^{+}(V) = (\text{superharmonic-cl}\,F)^{+}(V)$. By the hypothesis of F in this case, there is $H \in \tau^{*}(x)$ such that $(\text{superharmonic-cl}\,F)(H) \subseteq V$, which obviously gives $F(H) \subseteq V$. This completes the proof. □

Lemma 3.3 *In a space (X,τ), any $x \in X$ and $A \subseteq X, X \in$ superharmonic-cl(A) if and only if*

$$A \cap W \neq \phi$$

for each $W \in \tau^{*}(x)$.

Proposition 3.3 *For a superharmonic multifunction $F : (X,\tau) \to (Y,\sigma)$,*

$$(\text{superharmonic-cl}\,F)^{-}(W) = F^{-}(W)$$

for each $W \in \sigma^{*}$.

Proof Let $x \in (\text{superharmonic-cl}\, F)^{-}(W)$. Then

$$W \cap \text{superharmonic-cl}\big(F(x)\big) \neq \phi.$$

Since $W \in \sigma^*$, Lemma 3.3 gives $W \cap F(x) \neq \phi$ and hence $x \in F^{-}(W)$. Conversely, let $x \in F^{-}(W)$, then

$$\phi \neq F(x) \cap W \subseteq (\text{supracl}\, F)^{-}(x) \cap W$$

and so

$$x \in (\text{superharmonic-cl}\, F)^{-}(W).$$

Hence

$$x \in (\text{superharmonic-cl}\, F)^{+}(W)$$

and this completes the equality. □

Theorem 3.6 *A superharmonic multifunction $F : (X, \tau) \to (Y, \sigma)$ is lower superharmonic-continuous if and only if* (superharmonic-cl F) $: (X, \tau) \to (Y, \sigma)$ *is lower superharmonic-continuous.*

Proof This is an immediate consequence of Proposition 3.2 taking in consideration that $\tau \subseteq \tau^*$ and (iv) of Theorem 3.2. □

Theorem 3.7 *If $F : (X, \tau) \to (Y, \sigma)$ is an upper superharmonic-continuous surjection and for each $x \in X, F(x)$ is compact relative to Y. If (X, τ) is superharmonic-compact, then (Y, σ) is compact.*

Proof Let

$$\{V_i : i \in I, V_i \in \sigma\}$$

be a cover of Y; $F(x)$ is compact relative to Y, for each $x \in X$. Then there exists a finite $I_o(x)$ of I such that [14]

$$F(x) \subseteq U\big(V_i : i \in I_o(x)\big).$$

Upper superharmonic-continuity of F shows that there exists $W(x) \in \tau^*(X, x)$ such that

$$F\big(W(x)\big) \subseteq \bigcup V_i : i \in I_o(x).$$

Since (X, τ) is superharmonic-compact, there exists $x_1, x_2, \ldots, x_n$ such that

$$X = \bigcup\big(W(x_j) : 1 \leq j \leq n\big).$$

Therefore

$$Y = F(X) = \bigcup \bigl(F\bigl(W(x_j)\bigr) : 1 \leq j \leq n\bigr) \subseteq \bigcup V_i : i \in I_0(X_j) \quad 1 \leq j \leq n.$$

Hence (Y,σ) is compact. □

4 Supra-continuous superharmonic multifunctions and superharmonic-closed graphs

Definition 4.1 A superharmonic multifunction $F : (X,\tau) \to (Y,\sigma)$ is said to have a superharmonic-closed graph if there exists $W \in \tau^*(X)$ and $H \notin \sigma^*(y)$ such that

$$(W \times H) \cap G(F) = \phi$$

for each pair $(x,y) \notin G(F)$.

A superharmonic multifunction $F : (X,\tau) \to (Y,\sigma)$ is point-closed (superharmonic-closed), if for each $x \in X$, $F(x)$ is closed (superharmonic-closed) in Y.

Proposition 4.1 *A superharmonic multifunction $F : (X,\tau) \to (Y,\sigma)$ has a superharmonic-closed graph if and only if for each $x \in X$ and $y \in Y$ such that $y \notin F(x)$, there exist two superharmonic-open sets H, W containing x and y, respectively, such that*

$$F(H) \cap W = \phi.$$

Proof As regards necessity, let $x \in X$ and $y \in Y$ with $y \notin F(x)$. Then by the superharmonic-closed graph of F, there are $H \in \tau^*(x)$ and $W \in \sigma^*$ containing $F(x)$ such that $(HxW) \cap G(F) = \phi$. This implies that for every $x \in H$ and $y \in W$ where $y \notin F(x)$ we have $F(H) \cap W = \phi$.

As regards sufficiency, let $(x,y) \notin G(F)$, this means $y \notin F(x)$; then there are two disjoint superharmonic-open sets H, W containing x and y, respectively, such that $F(H) \cap W = \phi$. This implies that $(H \times W) \cap G(F) = \phi$, which completes the proof. □

Theorem 4.1 *If $F : (X,\tau) \to (Y,\sigma)$ is an upper superharmonic-continuous and point-closed superharmonic multifunction, then $G(F)$ is superharmonic-closed if (Y,σ) is regular.*

Proof Suppose that

$$(x,y) \notin G(F).$$

Then $y \notin F(x)$. Since Y is regular, there exists disjoint

$$V_i \in \sigma \quad (i = 1,2)$$

such that

$$y \in V_1$$

and

$$F(x) \subseteq V_2.$$

Since F is upper superharmonic-continuous at x, there exists

$$W \in \tau^*(x)$$

such that $F(W) \subseteq V_2$. As $V_1 \cap V_2 = \phi$, then

$$\bigcap_{i=1}^{2} \text{superharmonic-int}(V_i) \neq \phi$$

and therefore

$$x \in \text{superharmonic-int}(W) = W,$$
$$y \in \text{superharmonic-int}(V_1),$$

and

$$(x, y) \in W \times \text{superharmonic-int}(V_1) \subseteq (X \times Y) \backslash G(F).$$

Thus

$$(X \times Y) \backslash G(F) \in \tau^*(X \times Y),$$

which gives the result. □

Definition 4.2 A subset W of a space (X, τ) is called α-paracompact [12] if for every open cover ν of W in (X, τ) there exists a locally finite open cover ξ of W which refines ν.

Theorem 4.2 *Let $F : (X, \tau) \to (Y, \sigma)$ be an upper superharmonic-continuous superharmonic multifunction from (X, τ) into a Hausdorff space (Y, σ). If $F(x)$ is α-paracompact for each $x \in X$, then $G(F)$ is superharmonic-closed.*

Proof Let $(x_o, y_o) \notin G(F)$, then $y_o \notin F(x_o)$. Since (Y, σ) is Hausdorff, for each $y \in F(x_o)$ there exist $V_y \in \sigma(y)$ and $V_y^* \in \sigma(y_o)$ such that

$$V_y \cap V_y^* = \phi.$$

So the family $\{V_y : y \in F(x_0)\}$ is an open cover of $F(x_o)$. Thus, by α-paracompactness of $F(x_o)$ [15], there is a locally finite open cover $\{U_i : i \in I\}$ which refines $\{V_y : y \in F(x_o)\}$. Therefore, there exists $H_o \in \sigma(y_o)$ such that H_o intersects only finitely many members $U_{i_1}, U_{i_2}, \ldots, U_{i_n}$ of h. Choose $y_1, y_2, \ldots, y_n$ in $F(x_o)$ such that $U_{i_j} \subseteq U_{y_j}$ for each $1 < j < n$, and the set

$$H = H_o \cap \left(\bigcup_{i \in I} V_{y_i} \right).$$

Then $H \in \sigma(y_o)$ such that

$$H \cap \left(\bigcup_{i\in I} V_i\right) = \phi.$$

The upper superharmonic-continuity of F means that there exists $W \in \tau^*(xo)$ such that [16]

$$x_o \in W \subseteq F^+\left(\bigcup_{i\in I} V_i\right).$$

It follows that $(W \times H) \cap G(F) = \phi$, and hence $G(F)$ is superharmonic-closed. □

Lemma 4.1 ([14]) *The following hold for $F : (X, \tau) \to (Y, \sigma)$, $A \subseteq X$ and $B \subseteq Y$;*
(*i*)

$$G_F^+(A \times B) = A \cap F^+(B);$$

(*ii*)

$$G_F^-(A \times B) = A \cap F^-(B).$$

Theorem 4.3 *For a superharmonic multifunction $F : (X, \tau) \to (Y, \sigma)$, if GF is upper superharmonic-continuous, then F is upper superharmonic-continuous. Proof. Let $x \in X$ and $V \in \sigma(F(x))$. Since $X \times V \in \tau \times \sigma$ and*

$$G_F(x) \subseteq X \times V,$$

by Theorem 3.1, *there exists $W \in \tau^*(x)$ such that $G_F(W) \subseteq X \times V$. Therefore, by Lemma* 4.1, *we get*

$$W \subseteq G_F^-(X \times V) = X \cap G_f^+(V) = F^+(V)$$

and so $F(W) \subseteq V$. Hence Theorem 3.1 *shows also that F upper supracontinuous.*

Theorem 4.4 *If the graph G_F of a superharmonic multifunction $F : (X, \tau) \to (Y, \sigma)$ is lower superharmonic-continuous, then F is also.*

Proof Let $x \in X$ and $V \in \sigma(F(x))$ with $F(x) \cap V \neq \phi$, also since

$$X \times V \in \tau \times \sigma,$$

then

$$G_F(x) \cap (X \times V) = x \times F(x) \cap (X \times V) = x \times \big(F(x) \cap V\big) \neq \phi.$$

Theorem 3.2 shows that there exists $W \in \tau^*(x)$ such that

$$G_F(w) \subseteq (X \times V) \neq \phi$$

for each $w \in W$. Hence Lemma 4.1 obtains; we have

$$W \subseteq G^{-}(X \times V) = X \cap G^{-}(V) = F^{-}(V).$$

Therefore,

$$F(w) \cap V \neq \phi$$

for each $w \in W$ and Theorem 3.2 completes the proof. □

Acknowledgements
The authors want to thank the reviewers for much encouragement, support, productive feedback, cautious perusal and making helpful remarks, which improved the presentation and comprehensibility of the article.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally and read and approved the final manuscript.

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Publisher's Note

Received: 15 July 2017 Accepted: 26 August 2017 Published online: 02 October 2017

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