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Convergence of a linearly extrapolated BDF2 finite element scheme for viscoelastic fluid flow

Yunzhang Zhang[1*], Chao Xu[2] and Jiaquan Zhou[2]

*Correspondence:
z2004228@126.com
[1]School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang, 471023, China
Full list of author information is available at the end of the article

Abstract
The stability and convergence of a linearly extrapolated second order backward difference (BDF2-LE) time-stepping scheme for solving viscoelastic fluid flow in $\mathbb{R}^d$, $d = 2, 3$, are presented in this paper. The time discretization is based on the implicit scheme for the linear term and the two-step linearly extrapolated scheme for the nonlinear term. Mixed finite element (MFE) method is applied for the spatial discretization. The approximations of stress tensor $\boldsymbol{\sigma}$, velocity vector $\mathbf{u}$ and pressure p are P_m-discontinuous, P_k-continuous and P_q-continuous elements, respectively. Upwinding needed for convection of $\boldsymbol{\sigma}$ is made by a discontinuous Galerkin (DG) FE method. For the time step Δt small enough, the existence of an approximate solution is proven. If $m, k \ge \frac{d}{2}$, $q + 1 \ge \frac{d}{2}$, and $\Delta t \le C_0 h^{\frac{d}{4}}$, then the discrete H^1 and L^2 errors for the velocity and stress, and L^2 error for the pressure, are bounded by $C(\Delta t^2 + h^{\min\{m,k,q+1\}})$, where h denotes the mesh size. The derived theoretical results are supported by numerical tests.

MSC: 65N30; 65N12; 76A10

Keywords: viscoelastic fluid flow; linearly extrapolated BDF2; mixed finite element; discontinuous Galerkin; stability analysis; error estimate

1 Introduction

In this paper, we consider the time-dependent incompressible viscoelastic fluid flow problem

$$Re(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \nabla \cdot \sigma - 2(1-\alpha)\nabla \cdot D(\mathbf{u}) + \nabla p = \mathbf{f}, \tag{1.1a}$$

$$\lambda(\partial_t \sigma + \mathbf{u} \cdot \nabla \sigma) + \sigma + \lambda g_a(\sigma, \nabla \mathbf{u}) - 2\alpha D(\mathbf{u}) = 0, \tag{1.1b}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.1c}$$

for $\mathbf{x} \in \Omega$ and $t \in (0, T]$, where $\Omega \subset \mathbb{R}^d$ (d=2, 3) is a connected, bounded polygonal domain with the Lipschitz continuous boundary $\partial\Omega$. $p(\mathbf{x}, t)$ represents the pressure, $\mathbf{u} = (u_1(\mathbf{x}, t), \ldots, u_d(\mathbf{x}, t))$ the velocity vector, and $\sigma(\mathbf{x}, t)$ the stress tensor. σ is the viscoelastic part of the total stress tensor $\sigma_{\text{tot}} = \sigma + 2(1-\alpha)D(\mathbf{u}) - p\mathbf{I}$. λ is the Weissenberg number, Re the Reynolds number, $\mathbf{f}(\mathbf{x}, t)$ the body forces acting on the fluid and $0 < \alpha < 1$

may be considered as the fraction of viscoelastic viscosity. The gradient of $\mathbf{u}$ is defined as $(\nabla\mathbf{u})_{i,j} = \partial u_i/\partial x_j$. $D(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the rate of the strain tensor. For all $a \in [-1,1]$, $g_a(\sigma,\nabla\mathbf{u})$ is defined by

$$g_a(\sigma,\nabla\mathbf{u}) = \frac{1-a}{2}\big(\sigma\nabla\mathbf{u} + (\nabla\mathbf{u})^T\sigma\big) - \frac{1+a}{2}\big((\nabla\mathbf{u})\sigma + \sigma(\nabla\mathbf{u})^T\big). \tag{1.2}$$

The boundary and initial conditions are given by

$$\mathbf{u}(\mathbf{x},t) = \mathbf{0}, \quad \text{on } \partial\Omega \times (0,T], \tag{1.3}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \qquad \sigma(\mathbf{x},0) = \sigma_0(\mathbf{x}), \quad \forall\mathbf{x} \in \Omega. \tag{1.4}$$

Time-dependent calculations of viscoelastic fluid flows are important to the understanding of many problems in non-Newtonian fluid mechanics, particularity those related to flow instabilities [1–3]. The existence and uniqueness of solutions to viscoelastic fluid flow (1.1a)-(1.4) were discussed in [4, 5].

Numerical methods for solving the time-dependent incompressible viscoelastic fluid flow have been investigated extensively [6–18]. For the analysis of the time-dependent problem, Baranger and Wardi [11] studied a DG approximation to inertialess flow in $\mathbb{R}^2$. Assuming the Hood-Taylor FE pair approximation for the velocity and pressure, and a discontinuous linear FE approximation for the stress, and Euler implicit method in time, under the assumption $\triangle t \leq Ch^{3/2}$, they obtained that the discrete H^1 and L^2 errors for the velocity and stress, respectively, are bounded by $C(\triangle t + h^{3/2})$. Ervin and Heuer [14] analyzed a fully discrete approximation for the time-dependent viscoelasticity equations with an Oldroyd B constitutive equation in $\mathbb{R}^d$, $d = 2,3$. They used a Crank-Nicolson discretization for the time derivatives. At each time level a linear system of equations is solved. To resolve the nonlinear terms, they used a three-step extrapolation for the prediction of the velocity and stress at the new time level. The approximation is stabilized by using a discontinuous Galerkin approximation for the constitutive equation. Assume that $\triangle t$ is sufficiently small and satisfying $\triangle t \leq Ch^{d/4}$, the existence of an approximate solution is proven. A priori error estimate for the approximation in terms of $\triangle t$ and h is also derived. In [12], Ervin and Miles analyzed the finite element spacial semi-discrete and Euler semi-implicit fully discrete schemes, which were stabilized by using a streamline upwind Petrov-Galerkin (SUPG) for the constitutive equation. Bensaada and Esselaoui in [15] presented error analysis of a modified Euler-SUPG approximation for the time-dependent viscoelastic flow problem. In [18], based on a splitting of the error into two parts: the error from the time discretization of the PDEs and the error from the finite element approximation of corresponding iterated time-discrete PDEs, the authors carried on unconditional error estimates for time-dependent viscoelastic fluid flow.

In this work, we consider the convergence of BDF2-LE in time and MFE in space for the viscoelastic fluid flow. The backward difference formula (BDF) class of multi-step schemes has been widely used as time integration method for both ordinary and partial differential equations, see [19–27]. The BDF2 is one of the most popular BDF schemes due to its stability and damping properties [28]. Girault and Raviart introduced and analyzed a first-order and second-order BDF temporal semi-discrete schemes for Navier-Stokes equations in [21]. An unconditionally stable decoupled BDF2 time-stepping scheme was analyzed for

Boussinesq type Navier-Stokes equation in [24]. To the best of our knowledge, there is no rigorous convergence analysis available yet for the viscoelastic fluid flow by using BDF2-LE in time. We will propose and analyze a coupled scheme which belongs to this class.

This article is organized as follows. In the next section, we introduce some notations and preliminaries related to a continuum and discrete problem. In Section 3, we propose the extrapolated time-stepping scheme, prove the existence of the numerical solution and establish the stability analysis. The error analysis for the general scheme is presented in Section 4. We also present numerical tests to confirm the theoretical results in Section 5. Finally, some conclusions are drawn.

2 Notation and preliminaries

We denote the $L^2(\Omega)$ norms and corresponding inner products by $\|\cdot\|$ and $(\cdot,\cdot)$. Likewise, the $L^2(\Omega)$ norms and the Sobolev $W^{k,p}(\Omega)$ norms [29] are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{k,p}}$, respectively. $H^k(\Omega)$ is used to represent the Sobolev space $W^{k,2}(\Omega)$ and $\|\cdot\|_k$ denotes the norm in $H^k(\Omega)$. The space $H^{-k}(\Omega)$ denotes the dual spaces of $H_0^k(\Omega)$. All other norms will be clearly labeled with subscripts.

The velocity and pressure spaces are $X = H_0^1(\Omega)^d$, $Q = L_0^2(\Omega)$, respectively. The stress space S and divergence-free functions space V are given by

$$\begin{aligned} S &= \{\tau = (\tau_{ij}); \tau_{ij} = \tau_{ji}; \tau_{ij} \in L^2(\Omega); 1 \le i,j \le d\} \\ &\quad \cap \{\tau = (\tau_{ij}); \mathbf{v}\cdot\nabla\tau \in L^2(\Omega)^{d\times d}, \forall \mathbf{v} \in X\}, \\ V &= \{\mathbf{v} \in X; (q, \nabla\cdot\mathbf{v}) = 0, \forall q \in Q\}. \end{aligned}$$

A weak formulation of (1.1a)-(1.1c) is as follows: Find $(\sigma,\mathbf{u},p):[0,T]\to(S,X,Q)$ for a.e. $t\in(0,T]$ satisfying

$$Re(\partial_t\mathbf{u},\mathbf{v}) + Rec(\mathbf{u},\mathbf{u},\mathbf{v}) + (\sigma, D(\mathbf{v})) + 2(1-\alpha)(D(\mathbf{u}),D(\mathbf{v})) - (p,\nabla\cdot\mathbf{v}) = (\mathbf{f},\mathbf{v}), \tag{2.1a}$$

$$(q,\nabla\cdot\mathbf{u}) = 0, \tag{2.1b}$$

$$\lambda(\partial_t\sigma + \mathbf{u}\cdot\nabla\sigma,\tau) + \lambda(g_a(\sigma,\nabla\mathbf{u}),\tau) + (\sigma,\tau) - 2\alpha(D(\mathbf{u}),\tau) = \mathbf{0} \tag{2.1c}$$

for all $(\tau,\mathbf{v},q)\in(S,X,Q)$ with the initial condition (1.4) a.e. in Ω, where the trilinear operator c on $X\times X\times X$ is

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) = (\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}).$$

By virtue of the divergence-free space V, the weak formulation of (1.1a)-(1.1c) can be written as follows: Find $(\sigma,\mathbf{u})\in(S,V)$ such that, for all $(\tau,\mathbf{v})\in(S,V)$,

$$Re(\partial_t\mathbf{u},\mathbf{v}) + Rec(\mathbf{u},\mathbf{u},\mathbf{v}) + (\sigma, D(\mathbf{v})) + 2(1-\alpha)(D(\mathbf{u}),D(\mathbf{v})) = (\mathbf{f},\mathbf{v}), \tag{2.2a}$$

$$\lambda(\partial_t\sigma + \mathbf{u}\cdot\nabla\sigma,\tau) + (\sigma,\tau) + \lambda(g_a(\sigma,\nabla\mathbf{u}),\tau) - 2\alpha(D(\mathbf{u}),\tau) = \mathbf{0}. \tag{2.2b}$$

Here we assume that the initial-boundary value problem (1.1a)-(1.4) has a unique solution satisfying the regularity conditions

$$\begin{aligned}
&\mathbf{u}\in L^2\big(0,T;H^{k+1}(\Omega)^d\big),\qquad \partial_t\mathbf{u}\in L^2\big(0,T;H^{k+1}(\Omega)^d\big),\qquad \partial_t^3\mathbf{u}\in L^2\big(0,T;L^2(\Omega)^d\big),\\
&p\in L^2\big(0,T;H^{q+1}(\Omega)\big),\qquad \sigma\in L^2\big(0,T;H^{m+1}(\Omega)^{d\times d}\big),\\
&\partial_t\sigma\in L^2\big(0,T;H^{m+1}(\Omega)^{d\times d}\big),\qquad \partial_t^3\sigma\in L^2\big(0,T;L^2(\Omega)^{d\times d}\big),\\
&\|\mathbf{u}\|_\infty,\|\sigma\|_\infty,\|\nabla\mathbf{u}\|_\infty,\|\nabla\sigma\|_\infty\le M\quad\text{for all } t\in[0,T].
\end{aligned}\tag{2.3}$$

By using $\nabla\cdot\mathbf{u}=0$ and $\mathbf{u}=\mathbf{0}$ on $\partial\Omega$, it is easy to see that $2(D(\mathbf{u}),D(\mathbf{v}))=(\nabla\mathbf{u},\nabla\mathbf{v})$, and $\|D(\mathbf{u})\|\le\|\nabla\mathbf{u}\|$.

In order to keep the exposition simple, we restrict our attention to convex polyhedral domains. Suppose that T^h is a uniformly regular triangulation of Ω such that $\Omega=\{\bigcup K: K\in T^h\}$ and assume that there exist positive constants ν_1, ν_2 such that $\nu_1 h\le h_K\le\nu_2\rho_K$, where h_K is the diameter of K, ρ_K is the diameter of the greatest ball included in K, and $h=\max_{K\in T^h}h_K$. The corresponding FE spaces are

$$\begin{aligned}
&X_h=\big\{\mathbf{v}\in X\cap C^0(\overline{\Omega})^d;\mathbf{v}_{|K}\in P_k(K)^d,\forall K\in T^h\big\},\\
&S_h=\big\{\tau\in S;\tau_{|K}\in P_m(K)^{d\times d};\forall K\in T^h\big\},\\
&Q_h=\big\{q\in Q\cap C^0(\overline{\Omega});q_{|K}\in P_q(K);\forall K\in T^h\big\},
\end{aligned}$$

where $P_m(K)$ denotes the space of polynomials of degree $\le m$ on $K\in T^h$.

We make the following assumptions on the finite dimensional subspaces.

Assumption A1 For $(\mathbf{u},p)\in H^{k+1}(\Omega)^d\times H^{q+1}(\Omega)$, there exists $(\Pi_{\mathbf{u}}(\mathbf{u}),\Pi_p(p))\in V_h\times Q_h$ such that [21, 30–33]

$$\big\|\mathbf{u}-\Pi_{\mathbf{u}}(\mathbf{u})\big\|+h\big\|\nabla\big(\mathbf{u}-\Pi_{\mathbf{u}}(\mathbf{u})\big)\big\|\le C_{ip}h^{k+1}\|\mathbf{u}\|_{k+1},\tag{2.4}$$

$$\big\|p-\Pi_p(p)\big\|\le C_{ip}h^{q+1}\|p\|_{q+1}.\tag{2.5}$$

Let $\Pi_\sigma(\sigma)\in S_h$ be a P_m continuous interpolant of σ, and if $\sigma\in H^{m+1}(\Omega)^{d\times d}$, we have that

$$\big\|\sigma-\Pi_\sigma(\sigma)\big\|+h\big\|\nabla\big(\sigma-\Pi_\sigma(\sigma)\big)\big\|\le C_{ip}h^{m+1}\|\sigma\|_{m+1}.\tag{2.6}$$

Assumption A2 (Discrete inf-sup condition) For each $q_h\in Q_h$, there exists a nonzero function $\mathbf{v}_h\in X_h$ such that

$$\inf_{q_h\in Q_h}\sup_{\mathbf{v}_h\in X_h}\frac{(q_h,\nabla\cdot\mathbf{v}_h)}{\|q_h\|\|\nabla\mathbf{v}_h\|}\ge\beta>0,\tag{2.7}$$

where β is a positive constant independent of the mesh size h.

Assumption A3 For each $\omega_h \in X_h$, one has the inverse inequality, the Poincare inequality and the second Korn's inequality

$$\begin{aligned} &\|\nabla\omega_h\| \le C_i h^{-1}\|\omega_h\|, \qquad \|\omega_h\|_\infty \le C_i h^{-\frac{d}{2}}\|\omega_h\|, \\ &\|\omega_h\| \le C_p\|\nabla\omega_h\|, \\ &\|\nabla\omega_h\| \le C_k\|D(\omega_h)\|, \end{aligned} \tag{2.8}$$

where C_i, C_p and C_k are the positive constants, which only depend on Ω.

There are many finite element spaces satisfying Assumptions A1-A3, such as the MINI (P_1b, P_1) elements, or the Hood-Taylor (P_2, P_1) elements for the velocity $\mathbf{u}$ and pressure p, and P_1 (or P_2) discontinuous element for stress tensor σ.

The discretely divergence-free velocity space is denoted by

$$V_h = \{\mathbf{v} \in X_h; (q, \nabla\cdot\mathbf{v}) = 0, \text{ for all } q \in Q_h\}.$$

Remark 2.1 The divergence-free space V_h is introduced only for theoretical analysis. The practical computation should be based on the finite element space pair (X_h, Q_h) for velocity and pressure. We refer the readers to Heywood and Rannacher [34, 35] for the details on the construction of (X_h, Q_h).

Here we present a result which will be used in the stability analysis and error estimate for pressure. Since the divergence-free space $V_h \subset X_h$, we can define the norms of the dual spaces X_h, V_h by

$$\|\omega\|_{X_h'} = \sup_{\mathbf{v}_h\in X_h}\frac{(\omega, \mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|}, \qquad \|\omega\|_{V_h'} = \sup_{\mathbf{v}_h\in V_h}\frac{(\omega, \mathbf{v}_h)}{\|\nabla\mathbf{v}_h\|}.$$

Lemma 2.2 ([36–38]) *For* $\forall\mathbf{v} \in V_h$, *the norms* $\|\mathbf{v}\|_{X_h'}$ *and* $\|\mathbf{v}\|_{V_h'}$ *are equivalent.*

In order to describe the approximation of the constitutive equation by the method of discontinuous finite elements, following [6], we define

$$\partial K^-(\mathbf{u}) = \{\mathbf{x} \in \partial K; \mathbf{u}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) < 0\},$$

where ∂K is the boundary of $K \in T^h$, and $\mathbf{n}$ is the outward unit normal to ∂K, and

$$\partial\Omega^h = \left\{\bigcup \partial K : K \in T^h\right\} \setminus \partial\Omega, \qquad \tau^\pm(\mathbf{u})(\mathbf{x}) = \lim_{\varepsilon\to 0^\pm}\tau(\mathbf{x} + \varepsilon\mathbf{u}(\mathbf{x})).$$

Also, for all functions in $\prod_{K\in T_h}[H^1(K)]^{d\times d}$, we define

$$\begin{aligned} &(\sigma, \tau)_h = \sum_{K\in T_h}(\sigma, \tau)_K, \\ &\langle\sigma^\pm, \tau^\pm\rangle_{h,u} = \sum_{K\in T_h}\int_{\partial K^-(u)}(\sigma^\pm(\mathbf{u}), \tau^\pm(\mathbf{u}))|\mathbf{n}\cdot\mathbf{u}|\,ds, \\ &\langle\!\langle\sigma^\pm\rangle\!\rangle_{h,u}^2 = \langle\sigma^\pm, \sigma^\pm\rangle_{h,u}, \qquad \|\tau\|_{0,\Gamma^h} = \left(\sum_{K\in T_h}|\tau|_{0,\partial K}^2\right)^{1/2}. \end{aligned}$$

The convection term $((\mathbf{u}\cdot\nabla)\sigma,\tau)$ is approximated by means of an operator B on $X_h \times S_h \times S_h$, defined by

$$\begin{aligned} &B(\mathbf{u},\sigma,\tau) \\ &\quad = \big((\mathbf{u}\cdot\nabla)\sigma,\tau\big)_h + \frac{1}{2}(\nabla\cdot\mathbf{u}\sigma,\tau) + \langle\sigma^+ - \sigma^-,\tau^+\rangle_{h,\mathbf{u}} \\ &\quad = -\big((\mathbf{u}\cdot\nabla)\tau,\sigma\big)_h - \frac{1}{2}(\nabla\cdot\mathbf{u}\tau,\sigma) + \langle\sigma^-,\tau^- - \tau^+\rangle_{h,\mathbf{u}}, \end{aligned} \tag{2.9}$$

which implies some 'coercivity' of B [39]:

$$B(\mathbf{u},\sigma,\sigma) = \frac{1}{2}\langle\!\langle\sigma^+ - \sigma^-\rangle\!\rangle^2_{h,\mathbf{u}}. \tag{2.10}$$

Let $\{t_n | t_n = n\triangle t; 0 \le n \le N\}$ be a uniform partition of [0,T] with the time step $\triangle t = T/N$. We denote $\omega^m = \omega(\mathbf{x}, t_m)$. For a sequence of functions $\{\omega^n\}_{n=0}^N$, we define the BDF2 operator $\daleth(\omega^{n+1})$ and the linearly extrapolated operator $\digamma(\omega^{n+1})$

$$\daleth\big(\omega^{n+1}\big) = \frac{3\omega^{n+1} - 4\omega^n + \omega^{n-1}}{2\triangle t}, \qquad \digamma\big(\omega^{n+1}\big) = 2\omega^n - \omega^{n-1}.$$

It follows from Taylor's formula with integral remainder that [26]

$$\begin{aligned} &\daleth\big(\omega(t_n)\big) = \partial_t\omega(t_n) + \frac{1}{2\triangle t}\int_{t_{n-2}}^{t_n}\left\{2(t-t_{n-1})_+^2 - \frac{1}{2}(t-t_{n-2})^2\right\}\partial_t^3\omega\,dt, \\ &\digamma\big(\omega(t_n)\big) = \omega(t_n) + \int_{t_{n-2}}^{t_n}\big\{2(t-t_{n-1})_+ - (t-t_{n-2})\big\}\partial_t^2\omega\,dt, \end{aligned}$$

where $(t-t_{n-1})_+ = \max((t-t_{n-1}),0)$. By the Cauchy-Schwarz inequality, we have the truncation error

$$\big\|\daleth\big(\omega(t_n)\big) - \partial_t\omega(t_n)\big\| \le C_T(\triangle t)^{3/2}\big\|\partial_t^3\omega(t)\big\|_{L^2(t_{n-2},t_n;L^2(\Omega))}, \tag{2.11}$$

$$\big\|\digamma\big(\omega(t_n)\big) - \omega(t_n)\big\| \le C_T(\triangle t)^{3/2}\big\|\partial_t^2\omega(t)\big\|_{L^2(t_{n-2},t_n;L^2(\Omega))}, \tag{2.12}$$

where the constant C_T is derived from Taylor's formula.

The BDF2 operator $\daleth(\omega(t_{n+1}))$ satisfies the relation [26]

$$\begin{aligned} &\big(\daleth\big(\omega^{n+1}\big),\omega^{n+1}\big) \\ &\quad = \frac{3}{4\triangle t}\big\|\omega^{n+1}\big\|^2 + \frac{1}{4\triangle t}\big\|\omega^{n+1} - 2\omega^n + \omega^{n-1}\big\|^2 - \frac{1}{\triangle t}\big\|\omega^n\big\|^2 \\ &\qquad + \frac{1}{2\triangle t}\big[\big\|\omega^{n+1} - \omega^n\big\|^2 - \big\|\omega^n - \omega^{n-1}\big\|^2\big] + \frac{1}{4\triangle t}\big\|\omega^{n-1}\big\|^2. \end{aligned} \tag{2.13}$$

The discrete Gronwall's lemma [35] plays an important role in the following analysis.

Lemma 2.3 *Let $\triangle t$, H, and a_n, b_n, c_n, γ_n (for integers $n \ge 0$) be nonnegative numbers such that*

$$a_l + \triangle t\sum_{n=0}^{l} b_n \le \triangle t\sum_{n=0}^{l}\gamma_n a_n + \triangle t\sum_{n=0}^{l} c_n + H \quad \text{for } l \ge 0. \tag{2.14}$$

Suppose that $\triangle t\gamma_n < 1$ *for all n, and set* $\zeta_n = (1 - \triangle t\gamma_n)^{-1}$. *Then*

$$a_l + \triangle t \sum_{n=0}^{l} b_n \le \exp\left(\triangle t \sum_{n=0}^{l} \gamma_n \zeta_n\right)\left(\triangle t \sum_{n=0}^{l} c_n + H\right) \quad \text{for } l \ge 0. \tag{2.15}$$

Remark 2.4 If the first sum on the right in (2.14) extends only up to $l - 1$, then estimate (2.15) holds for all $n > 0$ with $\zeta_n = 1$.

Throughout the paper, the constants $C_1, C_2, \ldots$ denote different constants which are independent of h and $\triangle t$.

3 Numerical scheme and its stability

In this section, we first present the linearly extrapolated BDF2 scheme, then study the existence of numerical solutions, and finally establish stability of the numerical scheme.

3.1 Numerical scheme

Scheme 3.1 (BDF2-LE Galerkin FEM) Given $\mathbf{u}_h^{-1} = \mathbf{u}_h^0 = \Pi_{\mathbf{u}}(\mathbf{u}_0) \in V_h$, $\sigma_h^{-1} = \sigma_h^0 = \Pi_\sigma(\sigma_0) \in S_h$, find $\mathbf{u}_h^{n+1} \in X_h$, $p_h^{n+1} \in Q_h$, $\sigma_h^{n+1} \in S_h$ for $n = 0, 1, 2, \ldots, N - 1$ satisfying

$$\begin{aligned} &Re\big(\sqsupset(\mathbf{u}_h^{n+1}), \mathbf{v}_h\big) + Rec\big(\digamma(\mathbf{u}_h^{n+1}), \mathbf{u}_h^{n+1}, \mathbf{v}_h\big) + \big(\sigma_h^{n+1}, D(\mathbf{v}_h)\big) \\ &\quad + 2(1-\alpha)\big(D(\mathbf{u}_h^{n+1}), D(\mathbf{v}_h)\big) - \big(p_h^{n+1}, \nabla \cdot \mathbf{v}_h\big) = \big(\mathbf{f}^{n+1}, \mathbf{v}_h\big), \end{aligned} \tag{3.1a}$$

$$\big(q_h, \nabla \cdot \mathbf{u}_h^{n+1}\big) = 0, \tag{3.1b}$$

$$\begin{aligned} &\lambda\big(\sqsupset(\sigma_h^{n+1}), \tau_h\big) + \big(\sigma_h^{n+1}, \tau_h\big) + \lambda B\big(\digamma(\mathbf{u}_h^{n+1}), \sigma_h^{n+1}, \tau_h\big) \\ &\quad - 2\alpha\big(D(\mathbf{u}_h^{n+1}), \tau_h\big) + \lambda\big(g_a\big(\digamma(\sigma_h^{n+1}), \nabla \mathbf{u}_h^{n+1}\big), \tau_h\big) = 0 \end{aligned} \tag{3.1c}$$

for all $\mathbf{v}_h \in X_h$, $q_h \in Q_h$ and $\tau_h \in S_h$.

By virtue of the divergence-free subspace V_h, Scheme 3.1 can be written as another form (Scheme 3.2) which is used in stability analysis in this section and error analysis in Section 4.

Scheme 3.2 (BDF2-LE Galerkin FEM) Given $\mathbf{u}_h^{-1} = \mathbf{u}_h^0 = \Pi_{\mathbf{u}}(\mathbf{u}_0) \in V_h$, $\sigma_h^{-1} = \sigma_h^0 = \Pi_\sigma(\sigma_0) \in S_h$, find $\mathbf{u}_h^{n+1} \in V_h$, $\sigma_h^{n+1} \in S_h$ for $n = 0, 1, 2, \ldots, N - 1$, satisfying

$$\begin{aligned} &Re\big(\sqsupset(\mathbf{u}_h^{n+1}), \mathbf{v}_h\big) + Rec\big(\digamma(\mathbf{u}_h^{n+1}), \mathbf{u}_h^{n+1}, \mathbf{v}_h\big) + \big(\sigma_h^{n+1}, D(\mathbf{v}_h)\big) \\ &\quad + 2(1-\alpha)\big(D(\mathbf{u}_h^{n+1}), D(\mathbf{v}_h)\big) = \big(\mathbf{f}^{n+1}, \mathbf{v}_h\big), \end{aligned} \tag{3.2a}$$

$$\begin{aligned} &\lambda\big(\sqsupset(\sigma_h^{n+1}), \tau_h\big) + \big(\sigma_h^{n+1}, \tau_h\big) + \lambda B\big(\digamma(\mathbf{u}_h^{n+1}), \sigma_h^{n+1}, \tau_h\big) \\ &\quad - 2\alpha\big(D(\mathbf{u}_h^{n+1}), \tau_h\big) + \lambda\big(g_a\big(\digamma(\sigma_h^{n+1}), \nabla \mathbf{u}_h^{n+1}\big), \tau_h\big) = 0 \end{aligned} \tag{3.2b}$$

for all $\mathbf{v}_h \in V_h$ and $\tau_h \in S_h$.

We see that Scheme 3.1 (or Scheme 3.2) is a linear extrapolation (semi-implicit) scheme, which is preferred over a fully implicit scheme (see Scheme 5.1 in Section 5) as it requires only solving the linear system in each time level.

Remark 3.3 Since Scheme 3.1 is a two-step scheme, it requires starting values $(\mathbf{u}_h^0, \sigma_h^0)$ and $(\mathbf{u}_h^1, \sigma_h^1)$ and both with second order accuracy. For simplicity, here we take $(\mathbf{u}_h^{-1}, \sigma_h^{-1}) = (\mathbf{u}_h^0, \sigma_h^0) = (\Pi_{\mathbf{u}}(\mathbf{u}_0), \Pi_\sigma(\sigma_0))$, it ensures that $(\mathbf{u}_h^1, \sigma_h^1)$ is second order accuracy. We can also use the way as [20, 24] to get the value $(\mathbf{u}_h^0, \sigma_h^0) = (\Pi_{\mathbf{u}}(\mathbf{u}_0), \Pi_\sigma(\sigma_0))$ and $(\mathbf{u}_h^1, \sigma_h^1) = (\frac{\mathbf{u}_h^{2/3}+\mathbf{u}_h^{4/3}}{2}, \frac{\sigma_h^{2/3}+\sigma_h^{4/3}}{2})$, where $(\mathbf{u}_h^{2/3}, \sigma_h^{2/3})$ and $(\mathbf{u}_h^{4/3}, \sigma_h^{4/3})$ are solutions of the first order backward Euler scheme with time step $\frac{2}{3}\triangle t$ at $t = \frac{2}{3}\triangle t$ and $t = \frac{4}{3}\triangle t$, respectively. For details, please see [20, 24]. Of course, we can follow the Crank-Nicolson/Adams-Bashforth scheme [40] for Navier-Stokes equations to obtain $(\mathbf{u}_h^1, \sigma_h^1)$.

3.2 The existence and uniqueness of the numerical solution

To ensure the computability of Scheme 3.2, we begin by showing that it is uniquely solvable for $\mathbf{u}_h$ and σ_h at each time level.

Before proving the existence of solutions, we need to introduce the following induction hypothesis:

$$\|\mathbf{u}_h^n\|_\infty, \|\sigma_h^n\|_\infty \le K. \quad \textbf{IH1} \tag{3.3}$$

In Section 4, we will prove that the induction hypothesis **IH1** is right for any $n = 0, 1, \dots, N$.

Lemma 3.4 *Under the condition of hypothesis* **IH1**, *for* $\triangle t \le \min\{\frac{1-\alpha}{3RedC_k^2K^2}, \frac{3\lambda\alpha(1-\alpha)}{72dC_k^2K^2-2\alpha(1-\alpha)}\}$, *there exists a unique solution* $(\mathbf{u}_h^{n+1}, \sigma_h^{n+1}, p_h^{n+1}) \in X_h \times S_h \times Q_h$ *satisfying* (3.1a)-(3.1c).

Proof Taking $\mathbf{v}_h = 2\alpha\mathbf{u}_h^{n+1}$ in (3.1a), $q_h = 2\alpha p_h^{n+1}$ in (3.1b) and $\tau_h = \sigma_h^{n+1}$ in (3.1c), adding together the three equations thus obtained, we deduce that

$$\begin{aligned} A(\mathbf{u}_h^{n+1}, \sigma_h^{n+1}; \mathbf{u}_h^{n+1}, \sigma_h^{n+1}) &= \frac{2\alpha Re}{2\triangle t}(4\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^{n+1}) + \frac{\lambda}{2\triangle t}(4\sigma_h^n, \sigma_h^{n+1}) \\ &\quad - \frac{\lambda}{2\triangle t}(\sigma_h^{n-1}, \sigma_h^{n+1}) + 2\alpha(\mathbf{f}^{n+1}, \mathbf{u}_h^{n+1}), \end{aligned} \tag{3.4}$$

where the bilinear form $A(\mathbf{u}_h^{n+1}, \sigma_h^{n+1}; \mathbf{v}_h, \tau_h)$ is defined by

$$\begin{aligned} A(\mathbf{u}_h^{n+1}, \sigma_h^{n+1}; \mathbf{v}_h, \tau_h) &= \frac{2\alpha Re}{2\triangle t}(3\mathbf{u}_h^{n+1}, \mathbf{v}_h) + \frac{\lambda}{2\triangle t}(3\sigma_h^{n+1}, \tau_h) + (\sigma_h^{n+1}, \tau_h) \\ &\quad + 4\alpha(1-\alpha)(D(\mathbf{u}_h^{n+1}), D(\mathbf{v}_h)) + 2\alpha Rec(F(\mathbf{u}_h^{n+1}), \mathbf{u}_h^{n+1}, \mathbf{v}_h) \\ &\quad + \lambda(g_a(F(\sigma_h^{n+1}), \nabla\mathbf{u}_h^{n+1}), \tau_h) + \lambda B(F(\mathbf{u}_h^{n+1}), \sigma_h^{n+1}, \tau_h). \end{aligned} \tag{3.5}$$

We now estimate the nonlinear terms on the right-hand sides (RHS) of $A(\mathbf{u}_h^{n+1}, \sigma_h^{n+1}; \mathbf{v}_h, \tau_h)$ in (3.5). In view of (2.8) and the Holder inequality, we deduce that

$$\begin{aligned} 2\alpha Re|c(F(\mathbf{u}_h^{n+1}), \mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1})| &= 2\alpha Re|(F(\mathbf{u}_h^{n+1}) \cdot \nabla\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1})| \\ &\le 2\alpha Re\sqrt{d}\|F(\mathbf{u}_h^{n+1})\|_\infty \|\nabla\mathbf{u}_h^{n+1}\| \|\mathbf{u}_h^{n+1}\| \\ &\le 6\alpha Re\sqrt{d}K\|\nabla\mathbf{u}_h^{n+1}\| \|\mathbf{u}_h^{n+1}\| \\ &\le \epsilon_0\|D(\mathbf{u}_h^{n+1})\|^2 + \frac{9d\alpha^2Re^2K^2C_k^2}{\epsilon_0}\|\mathbf{u}_h^{n+1}\|^2, \end{aligned}$$

$$
\begin{aligned}
\lambda\left|\left(g_a\left(\digamma\left(\sigma_h^{n+1}\right),\nabla\mathbf{u}_h^{n+1}\right),\sigma_h^{n+1}\right)\right| &\le 4\left\|\digamma\left(\sigma_h^{n+1}\right)\nabla\mathbf{u}_h^{n+1}\right\|\left\|\sigma_h^{n+1}\right\| \\
&\le 4\sqrt{d}\left\|\digamma\left(\sigma_h^{n+1}\right)\right\|_\infty\left\|\nabla\mathbf{u}_h^{n+1}\right\|\left\|\sigma_h^{n+1}\right\| \\
&\le 4\sqrt{d}\left(2\left\|\sigma_h^{n}\right\|_\infty+\left\|\sigma_h^{n-1}\right\|_\infty\right)C_k\left\|D\left(\mathbf{u}_h^{n+1}\right)\right\|\left\|\sigma_h^{n+1}\right\| \\
&\le 12\sqrt{d}C_kK\left\|D\left(\mathbf{u}_h^{n+1}\right)\right\|\left\|\sigma_h^{n+1}\right\| \\
&\le \bar{\epsilon_0}\left\|D\left(\mathbf{u}_h^{n+1}\right)\right\|^2+\frac{36dC_k^2K^2}{\bar{\epsilon_0}}\left\|\sigma_h^{n+1}\right\|^2.
\end{aligned}
$$

Note that $\lambda B(\digamma(\mathbf{u}_h^{n+1}),\sigma_h^{n+1},\sigma_h^{n+1})=\frac{\lambda}{2}\langle\!\langle\sigma_h^{n+1,+}-\sigma_h^{n+1,-}\rangle\!\rangle_{h,\digamma(\mathbf{u}_h^{n+1})}^2$ due to the 'coercivity' (2.10) of $B(\cdot,\cdot,\cdot)$.

Combining the above inequalities with (3.5) yields

$$
\begin{aligned}
A\left(\mathbf{u}_h^{n+1},\sigma_h^{n+1};\mathbf{u}_h^{n+1},\sigma_h^{n+1}\right) \ge{}& \left(\frac{6\alpha Re}{2\triangle t}-\frac{9d\alpha^2Re^2K^2C_k^2}{\epsilon_0}\right)\left\|\mathbf{u}_h^{n+1}\right\|^2 \\
&+\left(\frac{3\lambda}{2\triangle t}-\frac{36dC_k^2K^2}{\bar{\epsilon_0}}+1\right)\left\|\sigma_h^{n+1}\right\|^2 \\
&+\left(4\alpha(1-\alpha)-\epsilon_0-\bar{\epsilon_0}\right)\left\|D\left(\mathbf{u}_h^{n+1}\right)\right\|^2 \\
&+\frac{\lambda}{2}\langle\!\langle\sigma_h^{n+1,+}-\sigma_h^{n+1,-}\rangle\!\rangle_{h,\digamma(\mathbf{u}_h^{n+1})}^2. \qquad (3.6)
\end{aligned}
$$

Choose $\epsilon_0=\bar{\epsilon_0}=\alpha(1-\alpha)$ and

$$
\triangle t\le\min\left\{\frac{1-\alpha}{3RedC_k^2K^2},\frac{3\lambda\alpha(1-\alpha)}{72dC_k^2K^2-2\alpha(1-\alpha)}\right\},
$$

thus the bilinear form $A(\mathbf{u}_h^{n+1},\sigma_h^{n+1};\mathbf{v}_h,\tau_h)$ is positive. Since system (3.4) is a finite dimensional linear system, then the existence and uniqueness of solutions $(\mathbf{u}_h^{n+1},\sigma_h^{n+1},p_h^{n+1})$ to Scheme 3.1 follow from the Lax-Milgram theorem and inf-sup condition (2.7). □

3.3 Numerical stability of Scheme 3.1

Theorem 3.5 *Suppose that* $\mathbf{f}\in L^2(0,T;H^{-1}(\Omega)^d)$, *the initial value* $\mathbf{u}_0\in L^2(\Omega)^d$ *and* $\sigma_0\in L^2(\Omega)^{d\times d}$. *For time step* $\triangle t$ *small enough, Scheme* 3.1 *is stable and satisfying*

$$
\begin{aligned}
&\alpha Re\left\|\mathbf{u}_h^l\right\|^2+\frac{\lambda}{2}\left\|\sigma_h^l\right\|^2+\triangle t\sum_{n=0}^{l-1}\left[2\alpha(1-\alpha)\left\|D\left(\mathbf{u}_h^{n+1}\right)\right\|^2+2\left\|\sigma_h^{n+1}\right\|^2\right] \\
&\quad\le\exp(2T\gamma_n)\left[\frac{2C_k^2\alpha}{(1-\alpha)}\triangle t\sum_{n=0}^{l-1}\left\|\mathbf{f}^{n+1}\right\|_{-1}^2+2\alpha Re\|\mathbf{u}_0\|^2+\lambda\|\sigma_0\|^2\right], \qquad (3.7)
\end{aligned}
$$

$$
\beta^2\triangle t\sum_{n=0}^{l-1}\left\|p_h^{n+1}\right\|^2\le C(Re,\alpha,d,\lambda,T,\Omega,K,\mathbf{f},\mathbf{u}_0,\sigma_0). \qquad (3.8)
$$

Proof Choosing $\mathbf{v}_h=2\triangle t\mathbf{u}_h^{n+1}$ in (3.1a), $q_h=2\triangle tp_h^{n+1}$ in (3.1b) and $\tau_h=2\triangle t\sigma_h^{n+1}$ in (3.1c), we get

$$
\begin{aligned}
&2Re\triangle t\left(\daleth\left(\mathbf{u}_h^{n+1}\right),\mathbf{u}_h^{n+1}\right)+2Re\triangle tc\left(\digamma\left(\mathbf{u}_h^{n+1}\right),\mathbf{u}_h^{n+1},\mathbf{u}_h^{n+1}\right) \\
&\quad+2\triangle t\left(\sigma_h^{n+1},D\left(\mathbf{u}_h^{n+1}\right)\right)+4(1-\alpha)\triangle t\left\|D\left(\mathbf{u}_h^{n+1}\right)\right\|^2=2\triangle t\left(\mathbf{f}^{n+1},\mathbf{u}_h^{n+1}\right), \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
&2\lambda\triangle t\big(\sqsupset(\sigma_h^{n+1}),\sigma_h^{n+1}\big)+2\triangle t\|\sigma_h^{n+1}\|^2+2\lambda\triangle t\big(g_a\big(\mathcal{F}(\sigma_h^{n+1}),\nabla\mathbf{u}_h^{n+1}\big),\sigma_h^{n+1}\big)\\
&\quad+\triangle t\lambda\langle\!\langle\sigma_h^{n+1,+}-\sigma_h^{n+1,-}\rangle\!\rangle^2_{h,\mathcal{F}(\mathbf{u}_h^{n+1})}-4\alpha\triangle t\big(D(\mathbf{u}_h^{n+1}),\sigma_h^{n+1}\big)=0.
\end{aligned}\tag{3.10}
$$

Multiplying (3.9) by 2α and adding to (3.10) yield the single equation

$$
\begin{aligned}
&4\alpha Re\triangle t\big(\sqsupset(\mathbf{u}_h^{n+1}),\mathbf{u}_h^{n+1}\big)+4Re\alpha\triangle tc\big(\mathcal{F}(\mathbf{u}_h^{n+1}),\mathbf{u}_h^{n+1},\mathbf{u}_h^{n+1}\big)\\
&\qquad+2\lambda\triangle t\big(\sqsupset(\sigma_h^{n+1}),\sigma_h^{n+1}\big)+8\alpha(1-\alpha)\triangle t\|D(\mathbf{u}_h^{n+1})\|^2+2\triangle t\|\sigma_h^{n+1}\|^2\\
&\qquad+2\lambda\triangle t\big(g_a\big(\mathcal{F}(\sigma_h^{n+1}),\nabla\mathbf{u}_h^{n+1}\big),\sigma_h^{n+1}\big)+\triangle t\lambda\langle\!\langle\sigma_h^{n+1,+}-\sigma_h^{n+1,-}\rangle\!\rangle^2_{h,\mathcal{F}(u_h^{n+1})}\\
&\quad=4\alpha\triangle t\big(\mathbf{f}^{n+1},\mathbf{u}_h^{n+1}\big).
\end{aligned}\tag{3.11}
$$

Furthermore, we have

$$
\begin{aligned}
&4Re\alpha\triangle t\big|c\big(\mathcal{F}(\mathbf{u}_h^{n+1}),\mathbf{u}_h^{n+1},\mathbf{u}_h^{n+1}\big)\big|\\
&\quad=4Re\alpha\triangle t\big|\big((\mathcal{F}(\mathbf{u}_h^{n+1})\cdot\nabla\mathbf{u}_h^{n+1},\mathbf{u}_h^{n+1}\big)\big|\\
&\quad\le 4Re\alpha\triangle t\|\mathcal{F}(\mathbf{u}_h^{n+1})\cdot\nabla\mathbf{u}_h^{n+1}\|\|\mathbf{u}_h^{n+1}\|\\
&\quad\le 4Re\alpha\sqrt{d}C_k\triangle t\|\mathcal{F}(\mathbf{u}_h^{n+1})\|_\infty\|D(\mathbf{u}_h^{n+1})\|\|\mathbf{u}_h^{n+1}\|\\
&\quad\le 12Re\alpha\sqrt{d}KC_k\triangle t\|D(\mathbf{u}_h^{n+1})\|\|\mathbf{u}_h^{n+1}\|\\
&\quad\le \epsilon_1\triangle t\|D(\mathbf{u}_h^{n+1})\|^2+\frac{36Re^2\alpha^2dC_k^2K^2}{\epsilon_1}\triangle t\|\mathbf{u}_h^{n+1}\|^2,
\end{aligned}\tag{3.12}
$$

$$
\begin{aligned}
&2\lambda\triangle t\big|\big(g_a\big(\mathcal{F}(\sigma_h^{n+1}),\nabla\mathbf{u}_h^{n+1}\big),\sigma_h^{n+1}\big)\big|\\
&\quad\le 8\lambda\triangle t\sqrt{d}\|\mathcal{F}(\sigma_h^{n+1})\|_\infty\|\nabla\mathbf{u}_h^{n+1}\|\|\sigma_h^{n+1}\|\\
&\quad\le 24\lambda\triangle t\sqrt{d}C_kK\|D(\mathbf{u}_h^{n+1})\|\|\sigma_h^{n+1}\|\\
&\quad\le \triangle t\bar{\epsilon}_1\|D(\mathbf{u}_h^{n+1})\|^2+\triangle t\frac{144\lambda^2dC_k^2K^2}{\bar{\epsilon}_1}\|\sigma_h^{n+1}\|^2
\end{aligned}\tag{3.13}
$$

and

$$
\begin{aligned}
\alpha\triangle t\big|\big(\mathbf{f}^{n+1},\mathbf{u}_h^{n+1}\big)\big|&\le 4\alpha\triangle t\|\mathbf{f}^{n+1}\|_{-1}\|\nabla\mathbf{u}_h^{n+1}\|\\
&\le 4\alpha C_k\triangle t\|\mathbf{f}^{n+1}\|_{-1}\|D(\mathbf{u}_h^{n+1})\|\\
&\le \triangle t\bar{\bar{\epsilon}}_1\|D(\mathbf{u}_h^{n+1})\|^2+\triangle t\frac{4C_k^2\alpha^2}{\bar{\bar{\epsilon}}_1}\|\mathbf{f}^{n+1}\|_{-1}^2.
\end{aligned}\tag{3.14}
$$

Plugging (3.12)-(3.14) into (3.11) and setting $\epsilon_1=\bar{\epsilon}_1=\bar{\bar{\epsilon}}_1=2\alpha(1-\alpha)$ yield

$$
\begin{aligned}
&4\alpha Re\triangle t\big(\sqsupset(\mathbf{u}_h^{n+1}),\mathbf{u}_h^{n+1}\big)+2\lambda\triangle t\big(\sqsupset(\sigma_h^{n+1}),\sigma_h^{n+1}\big)\\
&\qquad+2\alpha(1-\alpha)\triangle t\|D(\mathbf{u}_h^{n+1})\|^2+2\triangle t\|\sigma_h^{n+1}\|^2\\
&\quad\le\frac{72\lambda^2dC_k^2K^2}{\alpha(1-\alpha)}\triangle t\|\sigma_h^{n+1}\|^2+\frac{18Re^2dC_k^2K^2}{\alpha(1-\alpha)}\triangle t\|\mathbf{u}_h^{n+1}\|^2\\
&\qquad+\triangle t\frac{2C_k^2\alpha}{(1-\alpha)}\|\mathbf{f}^{n+1}\|_{-1}^2.
\end{aligned}\tag{3.15}
$$

Using identity (2.13) to (3.15), we get

$$\begin{aligned}
&\alpha Re\bigl[3\|\mathbf{u}_h^{n+1}\|^2 - 4\|\mathbf{u}_h^n\|^2 + \|\mathbf{u}_h^{n-1}\|^2\bigr] + 2\alpha Re\bigl[\|\mathbf{u}_h^{n+1}-\mathbf{u}_h^n\|^2 - \|\mathbf{u}_h^n-\mathbf{u}_h^{n-1}\|^2\bigr] \\
&\quad + \alpha Re\|\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}\|^2 + \frac{\lambda}{2}\bigl[3\|\sigma_h^{n+1}\|^2 - 4\|\sigma_h^n\|^2 + \|\sigma_h^{n-1}\|^2\bigr] \\
&\quad + \lambda\bigl[\|\sigma_h^{n+1}-\sigma_h^n\|^2 - \|\sigma_h^n - \sigma_h^{n-1}\|^2\bigr] + 2\triangle t\|\sigma_h^{n+1}\|^2 \\
&\quad + \frac{\lambda}{2}\|\sigma_h^{n+1} - 2\sigma_h^n + \sigma_h^{n-1}\|^2 + 2\alpha(1-\alpha)\triangle t\|D(\mathbf{u}_h^{n+1})\|^2 \\
&\le \frac{72\lambda^2 dC_k^2K^2}{\alpha(1-\alpha)}\triangle t\|\sigma_h^{n+1}\|^2 + \frac{18Re^2dC_k^2K^2}{\alpha(1-\alpha)}\triangle t\|\mathbf{u}_h^{n+1}\|^2 \\
&\quad + \triangle t\frac{2C_k^2\alpha}{(1-\alpha)}\|\mathbf{f}^{n+1}\|_{-1}^2. \qquad (3.16)
\end{aligned}$$

Summing (3.16) from $n = 0$ to $l - 1$ and using the identity

$$\frac{3}{2}a^2 - \frac{1}{2}b^2 + (a-b)^2 = \frac{1}{2}a^2 + \left(\sqrt{2}a - \frac{b}{\sqrt{2}}\right)^2 \qquad (3.17)$$

to (3.16) yield

$$\begin{aligned}
&\alpha Re\bigl[\|\mathbf{u}_h^l\|^2 + \|2\mathbf{u}_h^l - \mathbf{u}_h^{l-1}\|^2\bigr] + \frac{\lambda}{2}\bigl[\|\sigma_h^l\|^2 + \|2\sigma_h^l - \sigma_h^{l-1}\|^2\bigr] \\
&\quad + 2\alpha(1-\alpha)\triangle t\sum_{n=0}^{l-1}\|D(\mathbf{u}_h^{n+1})\|^2 + 2\triangle t\sum_{n=0}^{l-1}\|\sigma_h^{n+1}\|^2 \\
&\le \frac{18Re^2dC_k^2K^2}{\alpha(1-\alpha)}\triangle t\sum_{n=0}^{l-1}\|\mathbf{u}_h^{n+1}\|^2 + \frac{72\lambda^2dC_k^2K^2}{\alpha(1-\alpha)}\triangle t\sum_{n=0}^{l-1}\|\sigma_h^{n+1}\|^2 \\
&\quad + \triangle t\frac{2C_k^2\alpha}{(1-\alpha)}\sum_{n=0}^{l-1}\|\mathbf{f}^{n+1}\|_{-1}^2 + 2\alpha Re\|\mathbf{u}_0\|^2 + \lambda\|\sigma_0\|^2. \qquad (3.18)
\end{aligned}$$

In order to use the discrete Gronwall Lemma 2.3, here we set

$$\begin{aligned}
&a_l = \alpha Re\|\mathbf{u}_h^l\|^2 + \frac{\lambda}{2}\|\sigma_h^l\|^2, \qquad b_n = 2\alpha(1-\alpha)\|D(\mathbf{u}_h^{n+1})\|^2 + 2\|\sigma_h^{n+1}\|^2, \\
&a_n = \alpha Re\|\mathbf{u}_h^{n+1}\|^2 + \frac{\lambda}{2}\|\sigma_h^{n+1}\|^2, \\
&\gamma_n = \max\left\{\frac{18RedC_k^2K^2}{\alpha^2(1-\alpha)}, \frac{144\lambda dC_k^2K^2}{\alpha(1-\alpha)}\right\}, \qquad \zeta_n = \frac{1}{1-\triangle t\gamma_n}, \\
&c_n = \frac{2C_k^2\alpha}{(1-\alpha)}\|\mathbf{f}^{n+1}\|_{-1}^2, \qquad H = 2\alpha Re\|\mathbf{u}_0\|^2 + \lambda\|\sigma_0\|^2.
\end{aligned}$$

For time step $\triangle t$ such that $\gamma_n\triangle t \le \frac{1}{2}$, thus using the discrete Gronwall lemma to (3.18) yields the result (3.7).

Now we bound the pressure. As $V_h \subset X_h$, for all $\mathbf{v}_h \in V_h$, we have from (3.2a)

$$\begin{aligned}
&Re(\beth(\mathbf{u}_h^{n+1}), \mathbf{v}_h) + Rec(\digamma(\mathbf{u}_h^{n+1}), \mathbf{u}_h^{n+1}, \mathbf{v}_h) + (\sigma_h^{n+1}, D(\mathbf{v}_h)) \\
&\quad + (1-\alpha)(\nabla\mathbf{u}_h^{n+1}, \nabla\mathbf{v}_h) = (\mathbf{f}^{n+1}, \mathbf{v}_h).
\end{aligned}$$

Dividing by $\|\nabla \mathbf{v}_h\|$ and using the Cauchy-Schwarz inequality, we get

$$Re\frac{(\beth(\mathbf{u}_h^{n+1}),\mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|} \le Re\sqrt{d}C_p\left\|F\left(\mathbf{u}_h^{n+1}\right)\right\|_\infty \left\|\nabla \mathbf{u}_h^{n+1}\right\| + \left\|\sigma_h^{n+1}\right\| + (1-\alpha)\left\|\nabla \mathbf{u}_h^{n+1}\right\| + \left\|\mathbf{f}^{n+1}\right\|_{-1}.$$

Taking the supremum over $\mathbf{v}_h \in V_h$ yields

$$Re\left\|\beth\left(\mathbf{u}_h^{n+1}\right)\right\|_{V_h'} \le (3Re\sqrt{d}C_pK + 1 - \alpha)\left\|\nabla \mathbf{u}_h^{n+1}\right\| + \left\|\sigma_h^{n+1}\right\| + \left\|\mathbf{f}^{n+1}\right\|_{-1}.$$

The bound along with Lemma 2.2 provides the following estimate:

$$Re\left\|\beth\left(\mathbf{u}_h^{n+1}\right)\right\|_{X_h'} \le (3Re\sqrt{d}C_pK + 1 - \alpha)\left\|\nabla \mathbf{u}_h^{n+1}\right\| + \left\|\sigma_h^{n+1}\right\| + \left\|\mathbf{f}^{n+1}\right\|_{-1}.$$

From (3.1a) we have

$$\left(\nabla\cdot \mathbf{v}_h, p_h^{n+1}\right) = Re\left(\beth\left(\mathbf{u}_h^{n+1}\right),\mathbf{v}_h\right) + Rec\left(F\left(\mathbf{u}_h^{n+1}\right),\mathbf{u}_h^{n+1},\mathbf{v}_h\right) + \left(\sigma_h^{n+1}, D(\mathbf{v}_h)\right) + (1-\alpha)\left(\nabla \mathbf{u}_h^{n+1},\nabla \mathbf{v}_h\right) - \left(\mathbf{f}^{n+1},\mathbf{v}_h\right).$$

Dividing by $\|\nabla \mathbf{v}_h\|$ and using the Cauchy-Schwarz inequality yield

$$\frac{(\nabla\cdot \mathbf{v}_h, p_h^{n+1})}{\|\nabla \mathbf{v}_h\|} \le Re\frac{(\beth(\mathbf{u}_h^{n+1}),\mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|} + (3Re\sqrt{d}C_pK + 1 - \alpha)\left\|\nabla \mathbf{u}_h^{n+1}\right\| + \left\|\sigma_h^{n+1}\right\| + \left\|\mathbf{f}^{n+1}\right\|_{-1}.$$

Taking the supremum over $\mathbf{v}_h \in X_h$ and using the inf-sup conditions (2.7), we have

$$\begin{aligned}\beta\left\|p_h^{n+1}\right\| &\le Re\left\|\beth\left(\mathbf{u}_h^{n+1}\right)\right\|_{X_h'} + (3Re\sqrt{d}C_pK + 1 - \alpha)\left\|\nabla \mathbf{u}_h^{n+1}\right\| + \left\|\sigma_h^{n+1}\right\| + \left\|\mathbf{f}^{n+1}\right\|_{-1} \\ &\le (6Re\sqrt{d}C_pK + 2 - 2\alpha)\left\|\nabla \mathbf{u}_h^{n+1}\right\| + 2\left\|\sigma_h^{n+1}\right\| + 2\left\|\mathbf{f}^{n+1}\right\|_{-1}. \end{aligned} \tag{3.19}$$

Applying $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ to (3.19) yields

$$\beta^2\left\|p_h^{n+1}\right\|^2 \le 3(6Re\sqrt{d}C_pK + 2 - 2\alpha)^2\left\|\nabla \mathbf{u}_h^{n+1}\right\|^2 + 12\left\|\sigma_h^{n+1}\right\|^2 + 12\left\|\mathbf{f}^{n+1}\right\|_{-1}^2.$$

Now multiplying by $\triangle t$, summing over n from 0 to $l-1$ and using the bound (3.7), we get the required result (3.8). □

4 Error analysis of BDF2-LE scheme

We proceed to give an a priori error estimate for the BDF2-LE Galerkin FEM. In order to simplify the descriptions, we denote

$$e_{\mathbf{u}}^n = \mathbf{u}^n - \mathbf{u}_h^n, \qquad e_p^n = p^n - p_h^n, \qquad e_\sigma^n = \sigma^n - \sigma_h^n,$$

where $(\mathbf{u}^n, p^n, \sigma^n)$ and $(\mathbf{u}_h^n, p_h^n, \sigma_h^n)$ are the solutions of problems (2.1a)-(2.1c) and (3.1a)-(3.1c), respectively. We construct the error equations for velocity $e_{\mathbf{u}}^n$, pressure e_p^n and stress tensor e_σ^n. Decompose

$$\begin{aligned} e_{\mathbf{u}}^n &= (\mathbf{u}^n - \Pi_{\mathbf{u}}(\mathbf{u}^n)) + (\Pi_{\mathbf{u}}(\mathbf{u}^n) - \mathbf{u}_h^n) = \eta_{\mathbf{u}}^n + \varphi_{\mathbf{u}}^n, \\ e_p^n &= (p^n - \Pi_p(p^n)) + (\Pi_p(p^n) - p_h^n) = \eta_p^n + \varphi_p^n, \\ e_\sigma^n &= (\sigma^n - \Pi_\sigma(\sigma^n)) + (\Pi_\sigma(\sigma^n) - \sigma_h^n) = \eta_\sigma^n + \varphi_\sigma^n, \end{aligned} \tag{4.1}$$

where $(\Pi_{\mathbf{u}}(\mathbf{u}), \Pi_p(p), \Pi_\sigma(\sigma))$ denote the elements in $X_h \times Q_h \times S_h$ and satisfy the approximation properties (2.4)-(2.6). To establish the error estimate, we introduce the following discrete norms:

$$|||\omega|||_{\infty,k} = \max_{0\le n\le N-1} \|\omega^{n+1}\|_k, \qquad |||\omega|||_{0,k} = \left[\triangle t \sum_{n=0}^{N-1} \|\omega^{n+1}\|_k^2\right]^{\frac{1}{2}}. \tag{4.2}$$

Theorem 4.1 *Suppose that* $(\mathbf{u}, p, \sigma)$ *is a weak solution to* (2.1a)-(2.1c) *with additional regularities* (2.3). $(\mathbf{u}_h^l, p_h^l, \sigma_h^l)$ *is given by* (3.1a)-(3.1c) *for* $l \in \{0, 1, \dots, N-1\}$. *For hypothesis* **IH1** *and* $\triangle t\gamma_n \le \frac{1}{2}$, *we have*

$$\begin{aligned} &\alpha Re\|\varphi_{\mathbf{u}}^{l+1}\|^2 + \alpha Re\|2\varphi_{\mathbf{u}}^{l+1} - \varphi_{\mathbf{u}}^l\|^2 + \frac{\lambda}{2}\|\varphi_\sigma^{l+1}\|^2 + \frac{\lambda}{2}\|2\varphi_\sigma^{l+1} - \varphi_\sigma^l\|^2 \\ &\quad + 4\alpha(1-\alpha)\triangle t\sum_{n=0}^{l}\|D(\varphi_{\mathbf{u}}^{n+1})\|^2 + \triangle t\sum_{n=0}^{l}\|\varphi_\sigma^{n+1}\|^2 \le W(\triangle t, h), \end{aligned} \tag{4.3}$$

where

$$\gamma_n = \max\left\{1, \frac{d^2M^2}{\alpha}\left(20 + \frac{13\lambda^2 d}{Re}\right), \frac{d^2C_k^2\lambda}{\alpha(1-\alpha)}\left(\frac{15dM^2}{2} + 432K^2\right), 416\lambda d^2M^2\right\},$$

and

$$\begin{aligned} W(\triangle t, h) = \exp(2T\gamma_n)\big[&(\triangle t)^4\big(\|\partial_t^3\mathbf{u}\|_{L^2(0,T;L^2(\Omega)^d)}^2 + \|\partial_t^3\sigma\|_{L^2(0,T;L^2(\Omega)^{d\times d})}^2 \\ &+ \|\partial_t^2\mathbf{u}\|_{L^2(0,T;L^2(\Omega)^d)}^2 + \|\partial_t^2\sigma\|_{L^2(0,T;L^2(\Omega)^{d\times d})}^2\big) \\ &+ \big(h^{2q+2}|||p|||_{0,q+1}^2 + h^{2m}|||\sigma|||_{0,m+1}^2 + h^{2k}|||\mathbf{u}|||_{0,k+1}^2 \\ &+ h^{2k}\|\partial_t\mathbf{u}\|_{L^2(0,T;H^{k+1}(\Omega)^d)}^2 + h^{2m}\|\partial_t\sigma\|_{L^2(0,T;H^{m+1}(\Omega)^{d\times d})}^2\big)\big]. \end{aligned} \tag{4.4}$$

Proof At time $t_{n+1} = (n+1)\triangle t$, the true solution $(\mathbf{u}, p, \sigma)$ of (2.2a)-(2.2b) satisfies

$$\begin{aligned} &Re(\daleth(\mathbf{u}^{n+1}), \mathbf{v}_h) + Rec(\digamma(\mathbf{u}^{n+1}), \mathbf{u}^{n+1}, \mathbf{v}_h) + (\sigma^{n+1}, D(\mathbf{v}_h)) \\ &\quad + 2(1-\alpha)(D(\mathbf{u}^{n+1}), D(\mathbf{v}_h)) \\ &\quad = Re(\daleth(\mathbf{u}^{n+1}) - \partial_t\mathbf{u}^{n+1}, \mathbf{v}_h) \\ &\qquad + (\mathbf{f}^{n+1}, \mathbf{v}_h) + (p^{n+1}, \nabla\cdot\mathbf{v}_h) + Rec(\digamma(\mathbf{u}^{n+1}) - \mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{v}_h), \end{aligned} \tag{4.5a}$$

$$\begin{aligned}
&\lambda\big(\sqsupset(\sigma^{n+1}),\tau_h\big)+\big(\sigma^{n+1},\tau_h\big)+\lambda B\big(\digamma(\mathbf{u}^{n+1}),\sigma^{n+1},\tau_h\big)\\
&\quad-2\alpha\big(D(\mathbf{u}^{n+1}),\tau_h\big)+\lambda\big(g_a\big(\digamma(\sigma^{n+1}),\nabla\mathbf{u}^{n+1}\big),\tau_h\big)\\
&\quad=\lambda\big(\sqsupset(\sigma^{n+1})-\partial_t\sigma^{n+1},\tau_h\big)+\lambda B\big(\digamma(\mathbf{u}^{n+1})-\mathbf{u}^{n+1},\sigma^{n+1},\tau_h\big)\\
&\quad\quad+\lambda\big(g_a\big(\digamma(\sigma^{n+1})-\sigma^{n+1},\nabla\mathbf{u}^{n+1}\big),\tau_h\big)
\end{aligned}\tag{4.5b}$$

for all $(\mathbf{v}_h,\tau_h)\in V_h\times S_h$. Subtract (4.5a)-(4.5b) from (3.2a)-(3.2b) to yield the following error equations for $e_{\mathbf{u}}$ and e_σ:

$$\begin{aligned}
&Re\big(\sqsupset(e_{\mathbf{u}}^{n+1}),\mathbf{v}_h\big)+\big(e_\sigma^{n+1},D(\mathbf{v}_h)\big)+2(1-\alpha)\big(D(e_{\mathbf{u}}^{n+1}),D(\mathbf{v}_h)\big)\\
&\quad+Rec\big(\digamma(\mathbf{u}^{n+1}),\mathbf{u}^{n+1},\mathbf{v}_h\big)-Rec\big(\digamma(\mathbf{u}_h^{n+1}),\mathbf{u}_h^{n+1},\mathbf{v}_h\big)\\
&\quad=\big(p^{n+1},\nabla\cdot\mathbf{v}_h\big)+Re\big(\sqsupset(\mathbf{u}^{n+1})-\partial_t\mathbf{u}^{n+1},\mathbf{v}_h\big)\\
&\quad\quad+Rec\big(\digamma(\mathbf{u}^{n+1})-\mathbf{u}^{n+1},\mathbf{u}^{n+1},\mathbf{v}_h\big),
\end{aligned}\tag{4.6a}$$

$$\begin{aligned}
&\lambda\big(\sqsupset(e_\sigma^{n+1}),\tau_h\big)+\big(e_\sigma^{n+1},\tau_h\big)-2\alpha\big(D(e_{\mathbf{u}}^{n+1}),\tau_h\big)\\
&\quad+\lambda B\big(\digamma(\mathbf{u}^{n+1}),\sigma^{n+1},\tau_h\big)-\lambda B\big(\digamma(\mathbf{u}_h^{n+1}),\sigma_h^{n+1},\tau_h\big)\\
&\quad+\lambda\big(g_a\big(\digamma(\sigma^{n+1}),\nabla\mathbf{u}^{n+1}\big),\tau_h\big)-\lambda\big(g_a\big(\digamma(\sigma_h^{n+1}),\nabla\mathbf{u}_h^{n+1}\big),\tau_h\big)\\
&\quad=\lambda\big(\sqsupset(\sigma^{n+1})-\partial_t\sigma^{n+1},\tau_h\big)+\lambda B\big(\digamma(\mathbf{u}^{n+1})-\mathbf{u}^{n+1},\sigma^{n+1},\tau_h\big)\\
&\quad\quad+\lambda\big(g_a\big(\digamma(\sigma^{n+1})-\sigma^{n+1},\nabla\mathbf{u}^{n+1}\big),\tau_h\big).
\end{aligned}\tag{4.6b}$$

Taking $\mathbf{v}_h=\varphi_{\mathbf{u}}^{n+1}$ in (4.6a) and $\tau_h=\varphi_\sigma^{n+1}$ in (4.6b) yields

$$\begin{aligned}
&Re\big(\sqsupset(\varphi_{\mathbf{u}}^{n+1}),\varphi_{\mathbf{u}}^{n+1}\big)+2(1-\alpha)\big(D(\varphi_{\mathbf{u}}^{n+1}),D(\varphi_{\mathbf{u}}^{n+1})\big)\\
&\quad+\big(\varphi_\sigma^{n+1},D(\varphi_{\mathbf{u}}^{n+1})\big)+Rec\big(\digamma(\mathbf{u}_h^{n+1}),\varphi_{\mathbf{u}}^{n+1},\varphi_{\mathbf{u}}^{n+1}\big)=F_1\big(\varphi_{\mathbf{u}}^{n+1}\big),
\end{aligned}\tag{4.7a}$$

$$\begin{aligned}
&\lambda\big(\sqsupset(\varphi_\sigma^{n+1}),\varphi_\sigma^{n+1}\big)+\big(\varphi_\sigma^{n+1},\varphi_\sigma^{n+1}\big)-2\alpha\big(D(\varphi_{\mathbf{u}}^{n+1}),\varphi_\sigma^{n+1}\big)\\
&\quad+\lambda B\big(\digamma(\mathbf{u}_h^{n+1}),\varphi_\sigma^{n+1},\varphi_\sigma^{n+1}\big)=F_2\big(\varphi_\sigma^{n+1}\big),
\end{aligned}\tag{4.7b}$$

where

$$\begin{aligned}
F_1\big(\varphi_{\mathbf{u}}^{n+1}\big)&=Re\big(\sqsupset\big(\Pi_{\mathbf{u}}(\mathbf{u}^{n+1})\big)-\partial_t\mathbf{u}^{n+1},\varphi_{\mathbf{u}}^{n+1}\big)+\big(p^{n+1},\nabla\cdot\varphi_{\mathbf{u}}^{n+1}\big)\\
&\quad-\big(\eta_\sigma^{n+1},D(\varphi_{\mathbf{u}}^{n+1})\big)-2(1-\alpha)\big(D(\eta_{\mathbf{u}}^{n+1}),D(\varphi_{\mathbf{u}}^{n+1})\big)\\
&\quad+Rec\big(\digamma(\mathbf{u}^{n+1})-\mathbf{u}^{n+1},\mathbf{u}^{n+1},\varphi_{\mathbf{u}}^{n+1}\big)\\
&\quad-Rec\big(\digamma(\varphi_{\mathbf{u}}^{n+1}),\mathbf{u}^{n+1},\varphi_{\mathbf{u}}^{n+1}\big)-Rec\big(\digamma(\eta_{\mathbf{u}}^{n+1}),\mathbf{u}^{n+1},\varphi_{\mathbf{u}}^{n+1}\big)\\
&\quad-Rec\big(\digamma(\mathbf{u}_h^{n+1}),\eta_{\mathbf{u}}^{n+1},\varphi_{\mathbf{u}}^{n+1}\big),
\end{aligned}\tag{4.8}$$

and

$$\begin{aligned}
F_2\big(\varphi_\sigma^{n+1}\big)&=\lambda\big(\sqsupset\big(\Pi_\sigma(\sigma^{n+1})\big)-\partial_t\sigma^{n+1},\varphi_\sigma^{n+1}\big)+2\alpha\big(D(\eta_{\mathbf{u}}^{n+1}),\varphi_\sigma^{n+1}\big)\\
&\quad-\big(\eta_\sigma^{n+1},\varphi_\sigma^{n+1}\big)-\lambda B\big(\digamma(\varphi_{\mathbf{u}}^{n+1}),\sigma^{n+1},\varphi_\sigma^{n+1}\big)-\lambda B\big(\digamma(\eta_{\mathbf{u}}^{n+1}),\sigma^{n+1},\varphi_\sigma^{n+1}\big)\\
&\quad-\lambda B\big(\digamma(\mathbf{u}_h^{n+1}),\eta_\sigma^{n+1},\varphi_\sigma^{n+1}\big)+\lambda B\big(\digamma(\mathbf{u}^{n+1})-\mathbf{u}^{n+1},\sigma^{n+1},\varphi_\sigma^{n+1}\big)
\end{aligned}$$

$$
\begin{aligned}
&-\lambda\bigl(g_a\bigl(F(\varphi_\sigma^{n+1}),\nabla\mathbf{u}^{n+1}\bigr),\varphi_\sigma^{n+1}\bigr)-\lambda\bigl(g_a\bigl(F(\eta_\sigma^{n+1}),\nabla\mathbf{u}^{n+1}\bigr),\varphi_\sigma^{n+1}\bigr)\\
&-\lambda\bigl(g_a\bigl(F(\sigma_h^{n+1}),\nabla\varphi_{\mathbf{u}}^{n+1}\bigr),\varphi_\sigma^{n+1}\bigr)-\lambda\bigl(g_a\bigl(F(\sigma_h^{n+1}),\nabla\eta_{\mathbf{u}}^{n+1}\bigr),\varphi_\sigma^{n+1}\bigr)\\
&+\lambda\bigl(g_a\bigl(F(\sigma^{n+1})-\sigma^{n+1},\nabla\mathbf{u}^{n+1}\bigr),\varphi_\sigma^{n+1}\bigr). \qquad (4.9)
\end{aligned}
$$

Multiplying (4.7a) by 2α and adding to (4.7b), using the 'coercivity' (2.10) of $B(\cdot,\cdot,\cdot)$ yield the single equation

$$
\begin{aligned}
&2\alpha Re\bigl(\sqsupset(\varphi_{\mathbf{u}}^{n+1}),\varphi_{\mathbf{u}}^{n+1}\bigr)+4\alpha(1-\alpha)\bigl\|D(\varphi_{\mathbf{u}}^{n+1})\bigr\|^2+\lambda\bigl(\sqsupset(\varphi_\sigma^{n+1}),\varphi_\sigma^{n+1}\bigr)\\
&\quad+\bigl\|\varphi_\sigma^{n+1}\bigr\|^2+\frac{\lambda}{2}\langle\!\langle\varphi_\sigma^{n+1,+}-\varphi_\sigma^{n+1,-}\rangle\!\rangle^2_{h,F(\mathbf{u}_h^{n+1})}\\
&=2\alpha F_1(\varphi_{\mathbf{u}}^{n+1})+F_2(\varphi_\sigma^{n+1}). \qquad (4.10)
\end{aligned}
$$

Applying identity (2.13) to (4.10) yields

$$
\begin{aligned}
&\frac{2\alpha Re}{4\triangle t}\bigl[3\bigl\|\varphi_{\mathbf{u}}^{n+1}\bigr\|^2-4\bigl\|\varphi_{\mathbf{u}}^{n}\bigr\|^2+\bigl\|\varphi_{\mathbf{u}}^{n-1}\bigr\|^2\bigr]+\frac{2\alpha Re}{2\triangle t}\bigl[\bigl\|\varphi_{\mathbf{u}}^{n+1}-\varphi_{\mathbf{u}}^{n}\bigr\|^2\\
&\quad-\bigl\|\varphi_{\mathbf{u}}^{n}-\varphi_{\mathbf{u}}^{n-1}\bigr\|^2\bigr]+\frac{2\alpha Re}{4\triangle t}\bigl[\bigl\|\varphi_{\mathbf{u}}^{n+1}-2\varphi_{\mathbf{u}}^{n}+\varphi_{\mathbf{u}}^{n-1}\bigr\|^2\bigr]+\frac{\lambda}{4\triangle t}\bigl[3\bigl\|\varphi_\sigma^{n+1}\bigr\|^2\\
&\quad-4\bigl\|\varphi_\sigma^{n}\bigr\|^2+\bigl\|\varphi_\sigma^{n-1}\bigr\|^2\bigr]+\frac{\lambda}{2\triangle t}\bigl[\bigl\|\varphi_\sigma^{n+1}-\varphi_\sigma^{n}\bigr\|^2-\bigl\|\varphi_\sigma^{n}-\varphi_\sigma^{n-1}\bigr\|^2\bigr]\\
&\quad+\frac{\lambda}{4\triangle t}\bigl[\bigl\|\varphi_\sigma^{n+1}-2\varphi_\sigma^{n}+\varphi_\sigma^{n-1}\bigr\|^2\bigr]+\bigl\|\varphi_\sigma^{n+1}\bigr\|^2+4\alpha(1-\alpha)\bigl\|D(\varphi_{\mathbf{u}}^{n+1})\bigr\|^2\\
&\quad+\frac{\lambda}{2}\langle\!\langle\varphi_\sigma^{n+1,+}-\varphi_\sigma^{n+1,-}\rangle\!\rangle^2_{h,F(\mathbf{u}_h^{n+1})}\\
&=2\alpha F_1(\varphi_{\mathbf{u}}^{n+1})+F_2(\varphi_\sigma^{n+1}). \qquad (4.11)
\end{aligned}
$$

Multiplying both sides of (4.11) by $2\triangle t$, summing (4.11) with respect to n from 0 to l and using identity (3.17) give

$$
\begin{aligned}
&\alpha Re\bigl\|\varphi_{\mathbf{u}}^{l+1}\bigr\|^2+\alpha Re\bigl\|2\varphi_{\mathbf{u}}^{l+1}-\varphi_{\mathbf{u}}^{l}\bigr\|^2+\frac{\lambda}{2}\bigl\|\varphi_\sigma^{l+1}\bigr\|^2+\frac{\lambda}{2}\bigl\|2\varphi_\sigma^{l+1}-\varphi_\sigma^{l}\bigr\|^2\\
&\quad+2\triangle t\sum_{n=0}^{l}\bigl\|\varphi_\sigma^{n+1}\bigr\|^2+8\alpha(1-\alpha)\triangle t\sum_{n=0}^{l}\bigl\|D(\varphi_{\mathbf{u}}^{n+1})\bigr\|^2\\
&\quad+\lambda\triangle t\sum_{n=0}^{l}\langle\!\langle\varphi_\sigma^{n+1,+}-\varphi_\sigma^{n+1,-}\rangle\!\rangle^2_{h,F(\mathbf{u}_h^{n+1})}\\
&\le 2\alpha Re\bigl\|\varphi_{\mathbf{u}}^{0}\bigr\|^2+\lambda\bigl\|\varphi_\sigma^{0}\bigr\|^2+2\triangle t\sum_{n=0}^{l}\bigl[2\alpha F_1(\varphi_{\mathbf{u}}^{n+1})+F_2(\varphi_\sigma^{n+1})\bigr]. \qquad (4.12)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\bigl\|F(\sigma_h^{n+1})\bigr\|^2&=\bigl\|2\sigma_h^{n}-\sigma_h^{n-1}\bigr\|^2\le 4\bigl\|\sigma_h^{n}\bigr\|^2+\bigl\|\sigma_h^{n-1}\bigr\|^2+4\bigl\|\sigma_h^{n}\bigr\|\bigl\|\sigma_h^{n-1}\bigr\|\\
&\le 4\bigl\|\sigma_h^{n}\bigr\|^2+\bigl\|\sigma_h^{n-1}\bigr\|^2+4\biggl[\frac{1}{4}\bigl\|\sigma_h^{n}\bigr\|^2+\bigl\|\sigma_h^{n-1}\bigr\|^2\biggr]\\
&=5\bigl\|\sigma_h^{n}\bigr\|^2+5\bigl\|\sigma_h^{n-1}\bigr\|^2 \qquad (4.13)
\end{aligned}
$$

and

$$\left\| F\left(\mathbf{u}_h^{n+1}\right)\right\| = \left\|2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\right\| \le 2\left\|\mathbf{u}_h^n\right\| + \left\|\mathbf{u}_h^{n-1}\right\|. \tag{4.14}$$

We proceed to bound each term on the RHS of (4.12), absorb like-terms into the left-hand side.

We first estimate the linear terms of $F_1(\varphi_{\mathbf{u}}^{n+1})$ in (4.12). For the pressure term, using the Cauchy-Schwarz, Korn's and Young's inequalities, we have

$$\begin{aligned}
\left|\left(p^{n+1}, \nabla\cdot\varphi_{\mathbf{u}}^{n+1}\right)\right| &= \left|\left(p^{n+1} - \Pi_p\left(p^{n+1}\right), \nabla\cdot\varphi_{\mathbf{u}}^{n+1}\right)\right| \\
&\le \sqrt{d}\left\|p^{n+1} - \Pi_p\left(p^{n+1}\right)\right\| \left\|\nabla\varphi_{\mathbf{u}}^{n+1}\right\| \\
&\le \sqrt{d}C_k\left\|p^{n+1} - \Pi_p\left(p^{n+1}\right)\right\| \left\|D\left(\varphi_{\mathbf{u}}^{n+1}\right)\right\| \\
&\le \epsilon_2\left\|D\left(\varphi_{\mathbf{u}}^{n+1}\right)\right\|^2 + \frac{dC_k^2}{4\epsilon_2}\left\|p^{n+1} - \Pi_p\left(p^{n+1}\right)\right\|^2.
\end{aligned} \tag{4.15}$$

Similarly, we see that

$$\left(\eta_\sigma^{n+1}, D\left(\varphi_{\mathbf{u}}^{n+1}\right)\right) \le \epsilon_3\left\|D\left(\varphi_{\mathbf{u}}^{n+1}\right)\right\|^2 + \frac{1}{4\epsilon_3}\left\|\eta_\sigma^{n+1}\right\|^2, \tag{4.16}$$

$$2(1-\alpha)\left(D\left(\eta_{\mathbf{u}}^{n+1}\right), D\left(\varphi_{\mathbf{u}}^{n+1}\right)\right) \le \epsilon_4\left\|D\left(\varphi_{\mathbf{u}}^{n+1}\right)\right\|^2 + \frac{(1-\alpha)^2}{\epsilon_4}\left\|D\left(\eta_{\mathbf{u}}^{n+1}\right)\right\|^2, \tag{4.17}$$

$$\begin{aligned}
&Re\left(\sqsupset\left(\Pi_{\mathbf{u}}\left(\mathbf{u}^{n+1}\right)\right) - \partial_t\mathbf{u}^{n+1}, \varphi_{\mathbf{u}}^{n+1}\right) \\
&\quad\le Re\left\|\sqsupset\left(\Pi_{\mathbf{u}}\left(\mathbf{u}^{n+1}\right)\right) - \partial_t\mathbf{u}^{n+1}\right\| \left\|\varphi_{\mathbf{u}}^{n+1}\right\| \\
&\quad\le \epsilon_5 Re\left\|\varphi_{\mathbf{u}}^{n+1}\right\|^2 + \frac{Re}{4\epsilon_5}\left\|\sqsupset\left(\Pi_{\mathbf{u}}\left(\mathbf{u}^{n+1}\right)\right) - \partial_t\mathbf{u}^{n+1}\right\|^2.
\end{aligned} \tag{4.18}$$

For the nonlinear terms of $F_1(\varphi_{\mathbf{u}}^{n+1})$, using the Cauchy-Schwarz inequality, Young's inequality, the regularity assumption (2.3) of velocity and hypothesis **IH1**, we obtain

$$\begin{aligned}
&Rec\left(F\left(\mathbf{u}^{n+1}\right) - \mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \varphi_{\mathbf{u}}^{n+1}\right) \\
&\quad\le dRe\left\|F\left(\mathbf{u}^{n+1}\right) - \mathbf{u}^{n+1}\right\| \left\|\nabla\mathbf{u}^{n+1}\right\|_\infty \left\|\varphi_{\mathbf{u}}^{n+1}\right\| \\
&\quad\le dMRe\left\|F\left(\mathbf{u}^{n+1}\right) - \mathbf{u}^{n+1}\right\| \left\|\varphi_{\mathbf{u}}^{n+1}\right\| \\
&\quad\le \epsilon_6 Re\left\|\varphi_{\mathbf{u}}^{n+1}\right\|^2 + \frac{d^2M^2Re}{4\epsilon_6}\left\|F\left(\mathbf{u}^{n+1}\right) - \mathbf{u}^{n+1}\right\|^2,
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
&Rec\left(F\left(\varphi_{\mathbf{u}}^{n+1}\right), \mathbf{u}^{n+1}, \varphi_{\mathbf{u}}^{n+1}\right) \\
&\quad\le Red\left\|F\left(\varphi_{\mathbf{u}}^{n+1}\right)\right\| \left\|\nabla\mathbf{u}^{n+1}\right\|_\infty \left\|\varphi_{\mathbf{u}}^{n+1}\right\| \\
&\quad\le \epsilon_7 Re\left\|\varphi_{\mathbf{u}}^{n+1}\right\|^2 + \frac{Red^2M^2}{4\epsilon_7}\left\|F\left(\varphi_{\mathbf{u}}^{n+1}\right)\right\|^2 \\
&\quad\le \epsilon_7 Re\left\|\varphi_{\mathbf{u}}^{n+1}\right\|^2 + \frac{Red^2M^2}{4\epsilon_7}\left\|2\varphi_{\mathbf{u}}^n - \varphi_{\mathbf{u}}^{n-1}\right\|^2,
\end{aligned} \tag{4.20}$$

$$
\begin{aligned}
&Rec\big(\mathcal{F}(\eta_{\mathbf{u}}^{n+1}),\mathbf{u}^{n+1},\varphi_{\mathbf{u}}^{n+1}\big)\\
&\quad\le Red\big\|\mathcal{F}(\eta_{\mathbf{u}}^{n+1})\big\|\big\|\nabla\mathbf{u}^{n+1}\big\|_{\infty}\big\|\varphi_{\mathbf{u}}^{n+1}\big\|\\
&\quad\le \epsilon_8 Re\big\|\varphi_{\mathbf{u}}^{n+1}\big\|^2+\frac{Red^2M^2}{4\epsilon_8}\big\|\mathcal{F}(\eta_{\mathbf{u}}^{n+1})\big\|^2,
\end{aligned}
\tag{4.21}
$$

$$
\begin{aligned}
Rec\big(\mathcal{F}(\mathbf{u}_h^{n+1}),\eta_{\mathbf{u}}^{n+1},\varphi_{\mathbf{u}}^{n+1}\big)&\le Red\big\|\mathcal{F}(\mathbf{u}_h^{n+1})\big\|_{\infty}\big\|\nabla\eta_{\mathbf{u}}^{n+1}\big\|\big\|\varphi_{\mathbf{u}}^{n+1}\big\|\\
&\le 3RedKC_k\big\|D(\eta_{\mathbf{u}}^{n+1})\big\|\big\|\varphi_{\mathbf{u}}^{n+1}\big\|\\
&\le \epsilon_9 Re\big\|\varphi_{\mathbf{u}}^{n+1}\big\|^2+\frac{9Red^2K^2C_k^2}{4\epsilon_9}\big\|D(\eta_{\mathbf{u}}^{n+1})\big\|^2.
\end{aligned}
\tag{4.22}
$$

Combining (4.15)-(4.22), we have the following estimate of $2\alpha F_1(\varphi_{\mathbf{u}}^{n+1})$:

$$
\begin{aligned}
2\alpha F_1\big(\varphi_{\mathbf{u}}^{n+1}\big)&\le 2\alpha Re(\epsilon_5+\epsilon_6+\epsilon_7+\epsilon_8+\epsilon_9)\big\|\varphi_{\mathbf{u}}^{n+1}\big\|^2\\
&\quad+\frac{\alpha d^2M^2Re}{2\epsilon_7}\big\|2\varphi_{\mathbf{u}}^{n}-\varphi_{\mathbf{u}}^{n-1}\big\|^2+2\alpha(\epsilon_2+\epsilon_3+\epsilon_4)\big\|D(\varphi_{\mathbf{u}}^{n+1})\big\|^2\\
&\quad+\frac{dC_k^2\alpha}{2\epsilon_2}\big\|p^{n+1}-\Pi_p(p^{n+1})\big\|^2+\frac{\alpha}{2\epsilon_3}\big\|\eta_{\sigma}^{n+1}\big\|^2+\frac{2\alpha(1-\alpha)^2}{\epsilon_4}\big\|D(\eta_{\mathbf{u}}^{n+1})\big\|^2\\
&\quad+\frac{Re\alpha}{2\epsilon_5}\big\|\sqsupset\big(\Pi_{\mathbf{u}}(\mathbf{u}^{n+1})\big)-\partial_t\mathbf{u}^{n+1}\big\|^2+\frac{d^2\alpha M^2Re}{2\epsilon_6}\big\|\mathcal{F}(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\big\|^2\\
&\quad+\frac{\alpha Red^2M^2}{2\epsilon_8}\big\|\mathcal{F}(\eta_{\mathbf{u}}^{n+1})\big\|^2
\end{aligned}
\tag{4.23}
$$

Now we bound the terms of $F_2(\varphi_{\sigma}^{n+1})$ in (4.12). For the first three linear terms, applying the Cauchy-Schwarz inequality and Young's inequality, we obtain

$$
\begin{aligned}
&\lambda\big(\sqsupset\big(\Pi_{\sigma}(\sigma^{n+1})\big)-\partial_t\sigma^{n+1},\varphi_{\sigma}^{n+1}\big)\\
&\quad\le\lambda\big\|\sqsupset\big(\Pi_{\sigma}(\sigma^{n+1})\big)-\partial_t\sigma^{n+1}\big\|\big\|\varphi_{\sigma}^{n+1}\big\|\\
&\quad\le\beta_1\lambda\big\|\varphi_{\sigma}^{n+1}\big\|^2+\frac{\lambda}{4\beta_1}\big\|\sqsupset\big(\Pi_{\sigma}(\sigma^{n+1})\big)-\partial_t\sigma^{n+1}\big\|^2,
\end{aligned}
\tag{4.24}
$$

$$
\big(\eta_{\sigma}^{n+1},\varphi_{\sigma}^{n+1}\big)\le\big\|\eta_{\sigma}^{n+1}\big\|\big\|\varphi_{\sigma}^{n+1}\big\|\le\beta_2\big\|\varphi_{\sigma}^{n+1}\big\|^2+\frac{1}{4\beta_2}\big\|\eta_{\sigma}^{n+1}\big\|^2,
\tag{4.25}
$$

$$
\begin{aligned}
2\alpha\big(D(\eta_{\mathbf{u}}^{n+1}),\varphi_{\sigma}^{n+1}\big)&\le 2\alpha\big\|D(\eta_{\mathbf{u}}^{n+1})\big\|\big\|\varphi_{\sigma}^{n+1}\big\|\\
&\le\beta_3\big\|\varphi_{\sigma}^{n+1}\big\|^2+\frac{\alpha^2}{\beta_3}\big\|D(\eta_{\mathbf{u}}^{n+1})\big\|^2.
\end{aligned}
\tag{4.26}
$$

We estimate some nonlinear terms of the convection term about σ. The first nonlinear term $B(\mathcal{F}(\varphi_{\mathbf{u}}^{n+1}),\sigma^{n+1},\varphi_{\sigma}^{n+1})$ of $F_2(\varphi_{\sigma}^{n+1})$ can be rewritten as

$$
\begin{aligned}
&\lambda B\big(\mathcal{F}(\varphi_{\mathbf{u}}^{n+1}),\sigma^{n+1},\varphi_{\sigma}^{n+1}\big)\\
&\quad=\lambda\big(\mathcal{F}(\varphi_{\mathbf{u}}^{n+1})\cdot\nabla\sigma^{n+1},\varphi_{\sigma}^{n+1}\big)_h\\
&\qquad+\frac{\lambda}{2}\big(\big(\nabla\cdot\mathcal{F}(\varphi_{\mathbf{u}}^{n+1})\big)\sigma^{n+1},\varphi_{\sigma}^{n+1}\big)+\lambda\big\langle\sigma^{n+1,+}-\sigma^{n+1,-},\varphi_{\sigma}^{n+1,+}\big\rangle_{h,\mathcal{F}(\varphi_{\mathbf{u}}^{n+1})}.
\end{aligned}
\tag{4.27}
$$

Note that the term $\lambda\langle\sigma^{n+1,+} - \sigma^{n+1,-}, \varphi_\sigma^{n+1,+}\rangle_{h,F(\varphi_{\mathbf{u}}^{n+1})} = 0$ due to the continuity of σ. The other two terms on the RHS of (4.27) may be bounded by

$$\begin{aligned}
&\lambda\big(F\big(\varphi_{\mathbf{u}}^{n+1}\big)\cdot\nabla\sigma^{n+1}, \varphi_\sigma^{n+1}\big)_h \\
&\quad\le \lambda\big\|F\big(\varphi_{\mathbf{u}}^{n+1}\big)\cdot\nabla\sigma^{n+1}\big\|\big\|\varphi_\sigma^{n+1}\big\| \\
&\quad\le \lambda\sqrt{d^3}\big\|\nabla\sigma^{n+1}\big\|_\infty\big\|F\big(\varphi_{\mathbf{u}}^{n+1}\big)\big\|\big\|\varphi_\sigma^{n+1}\big\| \le \lambda\sqrt{d^3}M\big\|F\big(\varphi_{\mathbf{u}}^{n+1}\big)\big\|\big\|\varphi_\sigma^{n+1}\big\| \\
&\quad\le \beta_4\lambda\big\|\varphi_\sigma^{n+1}\big\|^2 + \frac{\lambda d^3M^2}{4\beta_4}\big\|2\varphi_{\mathbf{u}}^n - \varphi_{\mathbf{u}}^{n-1}\big\|^2, \qquad (4.28)\\
&\frac{\lambda}{2}\big(\big(\nabla\cdot F\big(\varphi_{\mathbf{u}}^{n+1}\big)\big)\sigma^{n+1}, \varphi_\sigma^{n+1}\big) \\
&\quad\le \frac{d\lambda}{2}\big\|\nabla\cdot F\big(\varphi_{\mathbf{u}}^{n+1}\big)\big\|\big\|\sigma^{n+1}\big\|_\infty\big\|\varphi_\sigma^{n+1}\big\| \\
&\quad\le \frac{\sqrt{d^3}M\lambda}{2}\big\|\nabla F\big(\varphi_{\mathbf{u}}^{n+1}\big)\big\|\big\|\varphi_\sigma^{n+1}\big\| \\
&\quad\le \frac{\sqrt{d^3}MC_k\lambda}{2}\big\|D\big(F\big(\varphi_{\mathbf{u}}^{n+1}\big)\big)\big\|\big\|\varphi_\sigma^{n+1}\big\| \\
&\quad\le \frac{2\alpha\epsilon_{10}}{10}\big\|D\big(F\big(\varphi_{\mathbf{u}}^{n+1}\big)\big)\big\|^2 + \frac{10d^3C_k^2M^2\lambda^2}{32\epsilon_{10}\alpha}\big\|\varphi_\sigma^{n+1}\big\|^2 \\
&\quad\le \alpha\epsilon_{10}\big(\big\|D\big(\varphi_{\mathbf{u}}^n\big)\big\|^2 + \big\|D\big(\varphi_{\mathbf{u}}^{n-1}\big)\big\|^2\big) + \frac{10d^3C_k^2M^2\lambda^2}{32\epsilon_{10}\alpha}\big\|\varphi_\sigma^{n+1}\big\|^2. \qquad (4.29)
\end{aligned}$$

Similarly as (4.27), we write the second nonlinear term $\lambda B(F(\eta_{\mathbf{u}}^{n+1}), \sigma^{n+1}, \varphi_\sigma^{n+1})$ of $F_2(\varphi_\sigma^{n+1})$ as

$$\begin{aligned}
&\lambda B\big(F\big(\eta_{\mathbf{u}}^{n+1}\big), \sigma^{n+1}, \varphi_\sigma^{n+1}\big) \\
&\quad= \lambda\big(F\big(\eta_{\mathbf{u}}^{n+1}\big)\cdot\nabla\sigma^{n+1}, \varphi_\sigma^{n+1}\big)_h \\
&\qquad+ \frac{\lambda}{2}\big(\big(\nabla\cdot F\big(\eta_{\mathbf{u}}^{n+1}\big)\big)\sigma^{n+1}, \varphi_\sigma^{n+1}\big) + \lambda\big\langle\sigma^{n+1,+} - \sigma^{n+1,-}, \varphi_\sigma^{n+1,+}\big\rangle_{h,F(\eta_{\mathbf{u}}^{n+1})}. \qquad (4.30)
\end{aligned}$$

Using the same method as (4.27) to estimate the three terms on the RHS of (4.30) leads to

$$\begin{aligned}
&\lambda\big(F\big(\eta_{\mathbf{u}}^{n+1}\big)\cdot\nabla\sigma^{n+1}, \varphi_\sigma^{n+1}\big)_h \\
&\quad\le \lambda\big\|F\big(\eta_{\mathbf{u}}^{n+1}\big)\cdot\nabla\sigma^{n+1}\big\|\big\|\varphi_\sigma^{n+1}\big\| \\
&\quad\le \lambda\sqrt{d^3}\big\|F\big(\eta_{\mathbf{u}}^{n+1}\big)\big\|\big\|\nabla\sigma^{n+1}\big\|_\infty\big\|\varphi_\sigma^{n+1}\big\| \\
&\quad\le \lambda\sqrt{d^3}M\big\|F\big(\eta_{\mathbf{u}}^{n+1}\big)\big\|\big\|\varphi_\sigma^{n+1}\big\| \\
&\quad\le \beta_5\lambda\big\|\varphi_\sigma^{n+1}\big\|^2 + \frac{\lambda d^3M^2}{4\beta_5}\big\|F\big(\eta_{\mathbf{u}}^{n+1}\big)\big\|^2, \qquad (4.31)\\
&\frac{\lambda}{2}\big(\big(\nabla\cdot F\big(\eta_{\mathbf{u}}^{n+1}\big)\big)\sigma^{n+1}, \varphi_\sigma^{n+1}\big) \\
&\quad\le \frac{\lambda d}{2}\big\|\nabla\cdot F\big(\eta_{\mathbf{u}}^{n+1}\big)\big\|\big\|\sigma^{n+1}\big\|_\infty\big\|\varphi_\sigma^{n+1}\big\|
\end{aligned}$$

$$\leq \frac{\lambda\sqrt{d^3}M}{2}\|\nabla\mathcal{F}(\eta_{\mathbf{u}}^{n+1})\|\|\varphi_\sigma^{n+1}\|$$

$$\leq \beta_6\lambda\|\varphi_\sigma^{n+1}\|^2+\frac{\lambda d^3M^2}{16\beta_6}\|\nabla\mathcal{F}(\eta_{\mathbf{u}}^{n+1})\|^2, \tag{4.32}$$

$$\lambda\langle\sigma^{n+1,+}-\sigma^{n+1,-},\varphi_\sigma^{n+1,+}\rangle_{h,\mathcal{F}(\eta_{\mathbf{u}}^{n+1})}=0, \tag{4.33}$$

where we have used the continuity of σ^{n+1} in (4.33).

By the same way, the third nonlinear term $\lambda B(\mathcal{F}(\mathbf{u}_h^{n+1}),\eta_\sigma^{n+1},\varphi_\sigma^{n+1})$ of $F_2(\varphi_\sigma^{n+1})$ can be written as

$$\begin{aligned}\lambda B&(\mathcal{F}(\mathbf{u}_h^{n+1}),\eta_\sigma^{n+1},\varphi_\sigma^{n+1})\\ &=\lambda(\mathcal{F}(\mathbf{u}_h^{n+1})\cdot\nabla\eta_\sigma^{n+1},\varphi_\sigma^{n+1})_h\\ &\quad+\frac{\lambda}{2}(\nabla\cdot\mathcal{F}(\mathbf{u}_h^{n+1})\eta_\sigma^{n+1},\varphi_\sigma^{n+1})+\lambda\langle\eta_\sigma^{n+1,+}-\eta_\sigma^{n+1,-},\varphi_\sigma^{n+1,+}\rangle_{h,\mathcal{F}(\mathbf{u}_h^{n+1})}.\end{aligned} \tag{4.34}$$

For the first term in (4.34), using (4.14) and hypothesis **IH1**, we can get

$$\begin{aligned}\lambda&(\mathcal{F}(\mathbf{u}_h^{n+1})\cdot\nabla\eta_\sigma^{n+1},\varphi_\sigma^{n+1})_h\\ &\leq\lambda\|\mathcal{F}(\mathbf{u}_h^{n+1})\cdot\nabla\eta_\sigma^{n+1}\|\|\varphi_\sigma^{n+1}\|\\ &\leq\sqrt{d}\lambda\|\mathcal{F}(\mathbf{u}_h^{n+1})\|_\infty\|\nabla\eta_\sigma^{n+1}\|\|\varphi_\sigma^{n+1}\|\\ &\leq 3\sqrt{d}\lambda K\|\nabla\eta_\sigma^{n+1}\|\|\varphi_\sigma^{n+1}\|\\ &\leq\beta_7\lambda\|\varphi_\sigma^{n+1}\|^2+\frac{9\lambda dK^2}{4\beta_7}\|\nabla\eta_\sigma^{n+1}\|^2.\end{aligned} \tag{4.35}$$

Making use of inverse inequality (2.8), (4.14) and hypothesis **IH1** to the second term in (4.34) yields

$$\begin{aligned}\frac{\lambda}{2}(\nabla\cdot\mathcal{F}(\mathbf{u}_h^{n+1})\eta_\sigma^{n+1},\varphi_\sigma^{n+1})&\leq\frac{\lambda}{2}d\|\nabla\mathcal{F}(\mathbf{u}_h^{n+1})\|_\infty\|\eta_\sigma^{n+1}\|\|\varphi_\sigma^{n+1}\|\\ &\leq\frac{\lambda d}{2}C_ih^{-1}\|\mathcal{F}(\mathbf{u}_h^{n+1})\|_\infty\|\eta_\sigma^{n+1}\|\|\varphi_\sigma^{n+1}\|\\ &\leq\beta_8\lambda\|\varphi_\sigma^{n+1}\|^2+\frac{9\lambda d^2K^2C_i^2h^{-2}}{16\beta_8}\|\eta_\sigma^{n+1}\|^2.\end{aligned} \tag{4.36}$$

Applying the continuity of η_σ^{n+1} to the third term on the RHS of (4.34) leads to

$$\lambda\langle\eta_\sigma^{n+1,+}-\eta_\sigma^{n+1,-},\varphi_\sigma^{n+1,+}\rangle_{h,\mathcal{F}(\mathbf{u}_h^{n+1})}=0. \tag{4.37}$$

Using $\nabla\cdot\mathbf{u}=0$ and the continuity of σ^{n+1} to the term $\lambda B(\mathcal{F}(\mathbf{u}^{n+1})-\mathbf{u}^{n+1},\sigma^{n+1},\varphi_\sigma^{n+1})$, we obtain

$$\begin{aligned}\lambda&|B(\mathcal{F}(\mathbf{u}^{n+1})-\mathbf{u}^{n+1},\sigma^{n+1},\varphi_\sigma^{n+1})|\\ &\leq\lambda|((\mathcal{F}(\mathbf{u}^{n+1})-\mathbf{u}^{n+1})\cdot\nabla\sigma^{n+1},\varphi_\sigma^{n+1})_h|\\ &\quad+\frac{\lambda}{2}|((\nabla\cdot(\mathcal{F}(\mathbf{u}^{n+1})-\mathbf{u}^{n+1})\sigma^{n+1},\varphi_\sigma^{n+1})|\end{aligned}$$

$$
\begin{aligned}
&+\lambda\left|\left\langle\sigma^{n+1,+}-\sigma^{n+1,-},\varphi_\sigma^{n+1,+}\right\rangle_{h,F(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}}\right| \\
&\le \lambda\left\|\left(F\left(\mathbf{u}^{n+1}\right)-\mathbf{u}^{n+1}\right)\cdot\nabla\sigma^{n+1}\right\|\left\|\varphi_\sigma^{n+1}\right\| \\
&\le \lambda\sqrt{d^3}\left\|\nabla\sigma^{n+1}\right\|_\infty\left\|F\left(\mathbf{u}^{n+1}\right)-\mathbf{u}^{n+1}\right\|\left\|\varphi_\sigma^{n+1}\right\| \\
&\le \beta_9\lambda\left\|\varphi_\sigma^{n+1}\right\|^2+\frac{\lambda d^3M^2}{4\beta_9}\left\|F\left(\mathbf{u}^{n+1}\right)-\mathbf{u}^{n+1}\right\|^2. \qquad (4.38)
\end{aligned}
$$

We will estimate the last five terms of $F_2(\varphi_\sigma^{n+1})$ in (4.12). Applying the Cauchy-Schwarz inequality, Young's inequality, Korn's inequality, the regularity assumption (2.3) of velocity and hypothesis **IH1**, we can obtain

$$
\begin{aligned}
\lambda\left(g_a\left(F\left(\varphi_\sigma^{n+1}\right),\nabla\mathbf{u}^{n+1}\right),\varphi_\sigma^{n+1}\right) &\le \lambda\left\|g_a\left(F\left(\varphi_\sigma^{n+1}\right),\nabla\mathbf{u}^{n+1}\right)\right\|\left\|\varphi_\sigma^{n+1}\right\| \\
&\le 4\lambda d\left\|F\left(\varphi_\sigma^{n+1}\right)\right\|\left\|\nabla\mathbf{u}^{n+1}\right\|_\infty\left\|\varphi_\sigma^{n+1}\right\| \\
&\le 4\lambda dM\left\|F\left(\varphi_\sigma^{n+1}\right)\right\|\left\|\varphi_\sigma^{n+1}\right\| \\
&\le \beta_{10}\lambda\left\|\varphi_\sigma^{n+1}\right\|^2+\frac{4\lambda d^2M^2}{\beta_{10}}\left\|2\varphi_\sigma^{n}-\varphi_\sigma^{n-1}\right\|^2, \qquad (4.39)\\
\lambda\left(g_a\left(F\left(\eta_\sigma^{n+1}\right),\nabla\mathbf{u}^{n+1}\right),\varphi_\sigma^{n+1}\right) &\le \lambda\left\|g_a\left(F\left(\eta_\sigma^{n+1}\right),\nabla\mathbf{u}^{n+1}\right)\right\|\left\|\varphi_\sigma^{n+1}\right\| \\
&\le 4\lambda d\left\|F\left(\eta_\sigma^{n+1}\right)\right\|\left\|\nabla\mathbf{u}^{n+1}\right\|_\infty\left\|\varphi_\sigma^{n+1}\right\| \\
&\le 4\lambda dM\left\|F\left(\eta_\sigma^{n+1}\right)\right\|\left\|\varphi_\sigma^{n+1}\right\| \\
&\le \beta_{11}\lambda\left\|\varphi_\sigma^{n+1}\right\|^2+\frac{4\lambda d^2M^2}{\beta_{11}}\left\|F\left(\eta_\sigma^{n+1}\right)\right\|^2, \qquad (4.40)\\
\lambda\left(g_a\left(F\left(\sigma_h^{n+1}\right),\nabla\varphi_{\mathbf{u}}^{n+1}\right),\varphi_\sigma^{n+1}\right) &\le \lambda\left\|g_a\left(F\left(\sigma_h^{n+1}\right),\nabla\varphi_{\mathbf{u}}^{n+1}\right)\right\|\left\|\varphi_\sigma^{n+1}\right\| \\
&\le 4d\lambda\left\|F\left(\sigma_h^{n+1}\right)\right\|_\infty\left\|\nabla\varphi_{\mathbf{u}}^{n+1}\right\|\left\|\varphi_\sigma^{n+1}\right\| \\
&\le 12d\lambda KC_k\left\|D\left(\varphi_{\mathbf{u}}^{n+1}\right)\right\|\left\|\varphi_\sigma^{n+1}\right\| \\
&\le 2\alpha\epsilon_{11}\left\|D\left(\varphi_{\mathbf{u}}^{n+1}\right)\right\|^2+\frac{18d^2\lambda^2K^2C_k^2}{\epsilon_{11}\alpha}\left\|\varphi_\sigma^{n+1}\right\|^2, \qquad (4.41)\\
\lambda\left(g_a\left(F\left(\sigma_h^{n+1}\right),\nabla\eta_{\mathbf{u}}^{n+1}\right),\varphi_\sigma^{n+1}\right) &\le \lambda\left\|g_a\left(F\left(\sigma_h^{n+1}\right),\nabla\eta_{\mathbf{u}}^{n+1}\right)\right\|\left\|\varphi_\sigma^{n+1}\right\| \\
&\le 4d\lambda\left\|F\left(\sigma_h^{n+1}\right)\right\|_\infty\left\|\nabla\eta_{\mathbf{u}}^{n+1}\right\|\left\|\varphi_\sigma^{n+1}\right\| \\
&\le 12d\lambda KC_k\left\|D\left(\eta_{\mathbf{u}}^{n+1}\right)\right\|\left\|\varphi_\sigma^{n+1}\right\| \\
&\le \beta_{12}\lambda\left\|\varphi_\sigma^{n+1}\right\|^2+\frac{36d^2\lambda K^2C_k^2}{\beta_{12}}\left\|D\left(\eta_{\mathbf{u}}^{n+1}\right)\right\|^2, \qquad (4.42)
\end{aligned}
$$

$$
\begin{aligned}
&\lambda\left(g_a\left(F\left(\sigma^{n+1}\right)-\sigma^{n+1},\nabla\mathbf{u}^{n+1}\right),\varphi_\sigma^{n+1}\right) \\
&\le \lambda\left\|g_a\left(F\left(\sigma^{n+1}\right)-\sigma^{n+1},\nabla\mathbf{u}^{n+1}\right)\right\|\left\|\varphi_\sigma^{n+1}\right\| \\
&\le 4\lambda d\left\|F\left(\sigma^{n+1}\right)-\sigma^{n+1}\right\|\left\|\nabla\mathbf{u}^{n+1}\right\|_\infty\left\|\varphi_\sigma^{n+1}\right\| \\
&\le 4\lambda dM\left\|F\left(\sigma^{n+1}\right)-\sigma^{n+1}\right\|\left\|\varphi_\sigma^{n+1}\right\| \\
&\le \beta_{13}\lambda\left\|\varphi_\sigma^{n+1}\right\|^2+\frac{4d^2M^2\lambda}{\beta_{13}}\left\|F\left(\sigma^{n+1}\right)-\sigma^{n+1}\right\|^2. \qquad (4.43)
\end{aligned}
$$

Combining inequalities (4.24)-(4.43), we obtain the estimate for $F_2(\varphi_\sigma^{n+1})$

$$\begin{aligned} F_2\big(\varphi_\sigma^{n+1}\big) \le{}& \big\|\varphi_\sigma^{n+1}\big\|^2\lambda\bigg[\beta_1+\beta_4+\beta_5+\cdots+\beta_{13}+\frac{5d^3C_k^2M^2\lambda}{16\epsilon_{10}\alpha} \\ &+\frac{18d^2\lambda K^2C_k^2}{\epsilon_{11}\alpha}\bigg]+(\beta_2+\beta_3)\big\|\varphi_\sigma^{n+1}\big\|^2+\frac{4\lambda^2d^2M^2}{\beta_{10}}\big\|2\varphi_\sigma^n-\varphi_\sigma^{n-1}\big\|^2 \\ &+\alpha\epsilon_{10}\big(\big\|D\big(\varphi_{\mathbf{u}}^n\big)\big\|^2+\big\|D\big(\varphi_{\mathbf{u}}^{n-1}\big)\big\|^2\big)+2\alpha\epsilon_{11}\big\|D\big(\varphi_{\mathbf{u}}^{n+1}\big)\big\|^2 \\ &+\frac{\lambda d^3M^2}{4\beta_4}\big\|2\varphi_{\mathbf{u}}^n-\varphi_{\mathbf{u}}^{n-1}\big\|^2+\frac{\lambda}{4\beta_1}\big\|\sqsupset\big(\Pi_\sigma\big(\sigma^{n+1}\big)\big)-\partial_t\sigma^{n+1}\big\|^2+\frac{1}{4\beta_2}\big\|\eta_\sigma^{n+1}\big\|^2 \\ &+\frac{\alpha^2}{\beta_3}\big\|D\big(\eta_{\mathbf{u}}^{n+1}\big)\big\|^2+\frac{\lambda d^3M^2}{4\beta_5}\big\|F\big(\eta_{\mathbf{u}}^{n+1}\big)\big\|^2+\frac{\lambda d^3M^2}{16\beta_6}\big\|\nabla F\big(\eta_{\mathbf{u}}^{n+1}\big)\big\|^2 \\ &+\frac{9\lambda dK^2}{4\beta_7}\big\|\nabla\eta_\sigma^{n+1}\big\|^2+\frac{9\lambda d^2K^2C_i^2h^{-2}}{16\beta_8}\big\|\eta_\sigma^{n+1}\big\|^2 \\ &+\frac{\lambda d^3M^2}{4\beta_9}\big\|F\big(\mathbf{u}^{n+1}\big)-\mathbf{u}^{n+1}\big\|^2+\frac{4\lambda d^2M^2}{\beta_{11}}\big\|F\big(\eta_\sigma^{n+1}\big)\big\|^2 \\ &+\frac{36d^2\lambda K^2C_k^2}{\beta_{12}}\big\|D\big(\eta_{\mathbf{u}}^{n+1}\big)\big\|^2 \end{aligned} \tag{4.44}$$

Plugging estimate (4.23) of $2\alpha F_1(\varphi_{\mathbf{u}}^{n+1})$ and estimate (4.44) of $F_2(\varphi_\sigma^{n+1})$ into (4.12) yields

$$\begin{aligned} &\alpha Re\big\|\varphi_{\mathbf{u}}^{l+1}\big\|^2+\alpha Re\big\|2\varphi_{\mathbf{u}}^{l+1}-\varphi_{\mathbf{u}}^l\big\|^2+\frac{\lambda}{2}\big\|\varphi_\sigma^{l+1}\big\|^2+\frac{\lambda}{2}\big\|2\varphi_\sigma^{l+1}-\varphi_\sigma^l\big\|^2 \\ &\quad+8\alpha(1-\alpha)\triangle t\sum_{n=0}^{l}\big\|D\big(\varphi_{\mathbf{u}}^{n+1}\big)\big\|^2+2\triangle t\sum_{n=0}^{l}\big\|\varphi_\sigma^{n+1}\big\|^2 \\ &\le 2\alpha Re\big\|\varphi_{\mathbf{u}}^0\big\|^2+\lambda\big\|\varphi_\sigma^0\big\|^2+2\triangle t\sum_{n=0}^{l}\big[2\alpha Re(\epsilon_5+\epsilon_6+\cdots+\epsilon_9)\big]\big\|\varphi_{\mathbf{u}}^{n+1}\big\|^2 \\ &\quad+2\triangle t\sum_{n=0}^{l}\bigg[\frac{\alpha d^2M^2Re}{2\epsilon_7}+\frac{\lambda d^3M^2}{4\beta_4}\bigg]\big\|2\varphi_{\mathbf{u}}^n-\varphi_{\mathbf{u}}^{n-1}\big\|^2+2\triangle t\sum_{n=0}^{l}\big[2\alpha(\epsilon_2 \\ &\quad+\epsilon_3+\epsilon_4+\epsilon_{11})\big]\big\|D\big(\varphi_{\mathbf{u}}^{n+1}\big)\big\|^2+2\triangle t\sum_{n=0}^{l}\alpha\epsilon_{10}\big(\big\|D\big(\varphi_{\mathbf{u}}^n\big)\big\|^2+\big\|D\big(\varphi_{\mathbf{u}}^{n-1}\big)\big\|^2\big) \\ &\quad+2\triangle t\sum_{n=0}^{l}\bigg[\lambda\bigg(\beta_1+\beta_4\ldots+\beta_{13}+\frac{5d^3C_k^2M^2\lambda}{16\epsilon_{10}\alpha}+\frac{18d^2\lambda K^2C_k^2}{\epsilon_{11}\alpha}\bigg)\bigg]\big\|\varphi_\sigma^{n+1}\big\|^2 \\ &\quad+2\triangle t\sum_{n=0}^{l}(\beta_2+\beta_3)\big\|\varphi_\sigma^{n+1}\big\|^2+2\triangle t\sum_{n=0}^{l}\frac{4\lambda d^2M^2}{\beta_{10}}\big\|2\varphi_\sigma^n-\varphi_\sigma^{n-1}\big\|^2 \\ &\quad+2\triangle t\sum_{n=0}^{l}\frac{dC_k^2\alpha}{2\epsilon_2}\big\|p^{n+1}-\Pi_p\big(p^{n+1}\big)\big\|^2+2\triangle t\sum_{n=0}^{l}\frac{Re\alpha}{2\epsilon_5}\big\|\sqsupset\big(\Pi_{\mathbf{u}}\big(\mathbf{u}^{n+1}\big)\big)-\partial_t\mathbf{u}^{n+1}\big\|^2 \\ &\quad+2\triangle t\sum_{n=0}^{l}\frac{\lambda}{4\beta_1}\big\|\sqsupset\big(\Pi_\sigma\big(\sigma^{n+1}\big)\big)-\partial_t\sigma^{n+1}\big\|^2+2\triangle t\sum_{n=0}^{l}\frac{4d^2M^2\lambda}{\beta_{13}}\big\|F\big(\sigma^{n+1}\big) \\ &\quad-\sigma^{n+1}\big\|^2+2\triangle t\sum_{n=0}^{l}\bigg[\frac{\alpha d^2M^2Re}{2\epsilon_6}+\frac{\lambda d^3M^2}{4\beta_9}\bigg]\big\|F\big(\mathbf{u}^{n+1}\big)-\mathbf{u}^{n+1}\big\|^2 \end{aligned}$$

$$+ 2\triangle t\sum_{n=0}^{l}\left(\frac{\alpha}{2\epsilon_3}+\frac{1}{4\beta_2}\right)\|\eta_\sigma^{n+1}\|^2 + 2\triangle t\sum_{n=0}^{l}\left[\frac{\alpha Re d^2M^2}{2\epsilon_8}+\frac{\lambda d^3M^2}{4\beta_5}\right]\|F(\eta_{\mathbf{u}}^{n+1})\|^2$$
$$+ 2\triangle t\sum_{n=0}^{l}\left[\frac{\alpha(1-\alpha)^2}{2\epsilon_4}+\frac{\alpha^2}{\beta_3}+\frac{9\alpha Re d^2K^2C_k^2}{2\epsilon_9}+\frac{36d^2\lambda K^2C_k^2}{\beta_{12}}\right]\|D(\eta_{\mathbf{u}}^{n+1})\|^2$$
$$+ 2\triangle t\sum_{n=0}^{l}\frac{4\lambda d^2M^2}{\beta_{11}}\|F(\eta_\sigma^{n+1})\|^2$$
$$+ 2\triangle t\sum_{n=0}^{l}\frac{9\lambda dK^2}{4\beta_7}\|\nabla\eta_\sigma^{n+1}\|^2 + 2\triangle t\sum_{n=0}^{l}\frac{9\lambda d^2K^2C_i^2h^{-2}}{16\beta_8}\|\eta_\sigma^{n+1}\|^2. \tag{4.45}$$

With the following choices: $\epsilon_2=\epsilon_3=\epsilon_4=\epsilon_{10}=\epsilon_{11}=\frac{1-\alpha}{6}$, $\epsilon_5=\epsilon_6=\epsilon_7=\epsilon_8=\epsilon_9=\frac{1}{20}$, $\beta_1=\beta_4=\cdots=\beta_{13}=\frac{1}{26\lambda}$, $\beta_2=\beta_3=\frac{1}{26}$, $\mathbf{u}_h^0=\Pi_{\mathbf{u}}(\mathbf{u}_0)(\Rightarrow\varphi_{\mathbf{u}}^0=0)$, $\sigma_h^0=\Pi_\sigma(\sigma_0)(\Rightarrow\varphi_\sigma^0=0)$, substituting these into (4.45) yields

$$\alpha Re\|\varphi_{\mathbf{u}}^{l+1}\|^2+\alpha Re\|2\varphi_{\mathbf{u}}^{l+1}-\varphi_{\mathbf{u}}^{l}\|^2+\frac{\lambda}{2}\|\varphi_\sigma^{l+1}\|^2+\frac{\lambda}{2}\|2\varphi_\sigma^{l+1}-\varphi_\sigma^{l}\|^2$$
$$+ 4\alpha(1-\alpha)\triangle t\sum_{n=0}^{l}\|D(\varphi_{\mathbf{u}}^{n+1})\|^2+\triangle t\sum_{n=0}^{l}\|\varphi_\sigma^{n+1}\|^2$$
$$\le \alpha Re\triangle t\sum_{n=0}^{l}\|\varphi_{\mathbf{u}}^{n+1}\|^2+\triangle t\sum_{n=0}^{l}\left[20d^2M^2Re+13\lambda^2d^3M^2\right]\|2\varphi_{\mathbf{u}}^{n}-\varphi_{\mathbf{u}}^{n-1}\|^2$$
$$+ \triangle t\sum_{n=0}^{l}\left[\frac{15d^3C_k^2M^2\lambda^2}{4\alpha(1-\alpha)}+\frac{216d^2\lambda^2K^2C_k^2}{\alpha(1-\alpha)}\right]\|\varphi_\sigma^{n+1}\|^2+208\lambda^2d^2M^2\triangle t\sum_{n=0}^{l}\|2\varphi_\sigma^{n}$$
$$-\varphi_\sigma^{n-1}\|^2+\triangle t\sum_{n=0}^{l}\frac{6\alpha dC_k^2}{1-\alpha}\|p^{n+1}-\Pi_p(p^{n+1})\|^2+20Re\alpha\triangle t\sum_{n=0}^{l}\|\beth(\Pi_{\mathbf{u}}(\mathbf{u}^{n+1}))$$
$$-\partial_t\mathbf{u}^{n+1}\|^2+13\lambda^2\triangle t\sum_{n=0}^{l}\|\beth(\Pi_\sigma(\sigma^{n+1}))-\partial_t\sigma^{n+1}\|^2$$
$$+ 208d^2M^2\lambda^2\triangle t\sum_{n=0}^{l}\|F(\sigma^{n+1})-\sigma^{n+1}\|^2+\left[\frac{6\alpha}{1-\alpha}+13\right]\triangle t\sum_{n=0}^{l}\|\eta_\sigma^{n+1}\|^2$$
$$+ \left[20\alpha d^2M^2Re+13\lambda^2d^3M^2\right]\triangle t\sum_{n=0}^{l}\|F(\mathbf{u}^{n+1})-\mathbf{u}^{n+1}\|^2+\left[6\alpha(1-\alpha)\right.$$
$$\left.+ 52\lambda\alpha^2+180\alpha Re d^2K^2C_k^2+1872d^2\lambda^2K^2C_k^2\right]\triangle t\sum_{n=0}^{l}\|D(\eta_{\mathbf{u}}^{n+1})\|^2$$
$$+ \left[20\alpha Re d^2M^2+13\lambda^2d^3M^2\right]\triangle t\sum_{n=0}^{l}\|F(\eta_{\mathbf{u}}^{n+1})\|^2$$
$$+ 208\lambda^2d^2M^2\triangle t\sum_{n=0}^{l}\|F(\eta_\sigma^{n+1})\|^2$$
$$+ 117\lambda^2dK^2\triangle t\sum_{n=0}^{l}\|\nabla\eta_\sigma^{n+1}\|^2+\frac{117\lambda^2d^2K^2C_i^2h^{-2}}{4}\triangle t\sum_{n=0}^{l}\|\eta_\sigma^{n+1}\|^2. \tag{4.46}$$

We now apply the approximation properties (2.4)-(2.6) to the terms on the RHS of (4.46). Using elements of order k for velocity, elements of order m for stress, and elements of order q for pressure, we have

$$\triangle t\sum_{n=0}^{l}\frac{6\alpha dC_k^2}{1-\alpha}\left\|p^{n+1}-\Pi_p\left(p^{n+1}\right)\right\|^2\leq\frac{6\alpha dC_k^2C_{ip}^2}{1-\alpha}h^{2q+2}|||p|||_{0,q+1}^2, \tag{4.47}$$

$$\left[\frac{6\alpha}{1-\alpha}+13\right]\triangle t\sum_{n=0}^{l}\left\|\eta_\sigma^{n+1}\right\|^2\leq\left[\frac{6\alpha}{1-\alpha}+13\right]C_{ip}^2h^{2m+2}|||\sigma|||_{0,m+1}^2, \tag{4.48}$$

$$\begin{aligned}&\left[6\alpha(1-\alpha)+52\lambda\alpha^2+d^2K^2C_k^2\left(180\alpha Re+1872\lambda^2\right)\right]\triangle t\sum_{n=0}^{l}\left\|D\left(\eta_{\mathbf{u}}^{n+1}\right)\right\|^2\\&\quad\leq\left[6\alpha(1-\alpha)+52\lambda\alpha^2+d^2K^2C_k^2\left(180\alpha Re+1872\lambda^2\right)\right]C_{ip}^2h^{2k}|||\mathbf{u}|||_{0,k+1}^2,\end{aligned} \tag{4.49}$$

$$\begin{aligned}&\left[20\alpha Red^2M^2+13\lambda^2d^3M^2\right]\triangle t\sum_{n=0}^{l}\left\|F\left(\eta_{\mathbf{u}}^{n+1}\right)\right\|^2\\&\quad\leq\left[20\alpha Red^2M^2+13\lambda^2d^3M^2\right]C_{ip}^2h^{2k+2}|||\mathbf{u}|||_{0,k+1}^2,\end{aligned} \tag{4.50}$$

$$\frac{13\lambda^2d^3M^2}{4}\triangle t\sum_{n=0}^{l}\left\|\nabla F\left(\eta_{\mathbf{u}}^{n+1}\right)\right\|^2\leq\frac{13\lambda^2d^3M^2}{4}C_{ip}^2h^{2k}|||\mathbf{u}|||_{0,k+1}^2, \tag{4.51}$$

$$208\lambda^2d^2M^2\triangle t\sum_{n=0}^{l}\left\|F\left(\eta_\sigma^{n+1}\right)\right\|^2\leq 208\lambda^2d^2M^2C_{ip}^2h^{2m+2}|||\sigma|||_{0,m+1}^2, \tag{4.52}$$

$$\begin{aligned}&117\lambda^2dK^2\triangle t\sum_{n=0}^{l}\left\|\nabla\eta_\sigma^{n+1}\right\|^2+\frac{117\lambda^2d^2K^2C_i^2h^{-2}}{4}\triangle t\sum_{n=0}^{l}\left\|\eta_\sigma^{n+1}\right\|^2\\&\quad\leq 117\lambda^2dK^2\left[1+\frac{dC_i^2}{4}\right]C_{ip}^2h^{2m}|||\sigma|||_{0,m+1}^2.\end{aligned} \tag{4.53}$$

In view of the truncation error (2.11) and the interpolation properties (2.4), we can obtain

$$\begin{aligned}&\left\|\beth\left(\Pi_{\mathbf{u}}\left(\mathbf{u}^{n+1}\right)\right)-\partial_t\mathbf{u}^{n+1}\right\|\\&\quad=\left\|\beth\left(\Pi_{\mathbf{u}}\left(\mathbf{u}(t_{n+1})\right)\right)-\partial_t\mathbf{u}(t_{n+1})\right\|\\&\quad\leq\left\|\Pi_{\mathbf{u}}\left(\beth\left(\mathbf{u}(t_{n+1})\right)-\partial_t\mathbf{u}(t_{n+1})\right)\right\|+\left\|\partial_t\left(\Pi_{\mathbf{u}}\left(\mathbf{u}(t_{n+1})\right)-\mathbf{u}(t_{n+1})\right)\right\|\\&\quad\leq C_T(\triangle t)^{3/2}\left\|\partial_t^3\mathbf{u}\right\|_{L^2(t_{n-2},t_n;L^2(\Omega)^d)}+C_{ip}\frac{h^k}{\sqrt{\triangle t}}\left\|\partial_t\mathbf{u}\right\|_{L^2(0,T;H^{k+1}(\Omega)^d)}.\end{aligned} \tag{4.54}$$

Then we have

$$\begin{aligned}&20Re\alpha\triangle t\sum_{n=0}^{l}\left\|\beth\left(\Pi_{\mathbf{u}}\left(\mathbf{u}^{n+1}\right)\right)-\partial_t\mathbf{u}^{n+1}\right\|^2+13\lambda^2\triangle t\sum_{n=0}^{l}\left\|\beth\left(\Pi_\sigma\left(\sigma^{n+1}\right)\right)-\partial_t\sigma^{n+1}\right\|^2\\&\quad\leq 40Re\alpha C_T^2\triangle t^4\left\|\partial_t^3\mathbf{u}\right\|_{L^2(0,T;L^2(\Omega)^d)}^2+26\lambda^2C_T^2\triangle t^4\left\|\partial_t^3\sigma\right\|_{L^2(0,T;L^2(\Omega)^{d\times d})}^2\\&\qquad+40Re\alpha C_{ip}^2h^{2k}\left\|\partial_t\mathbf{u}\right\|_{L^2(0,T;H^{k+1}(\Omega)^d)}^2+26\lambda^2C_{ip}^2h^{2m}\left\|\partial_t\sigma\right\|_{L^2(0,T;H^{m+1}(\Omega)^{d\times d})}^2.\end{aligned} \tag{4.55}$$

Similarly, using the truncation error (2.12), we get

$$\begin{aligned}
&\big[20\alpha d^2M^2Re+13\lambda^2d^3M^2\big]\triangle t\sum_{n=0}^{l}\big\|F\big(\mathbf{u}^{n+1}\big)-\mathbf{u}^{n+1}\big\|^2\\
&\quad+208d^2M^2\lambda^2\triangle t\sum_{n=0}^{l}\big\|F\big(\sigma^{n+1}\big)-\sigma^{n+1}\big\|^2\\
&\quad\le\big[20\alpha d^2M^2Re+13\lambda^2d^3M^2\big]C_T^2\triangle t^4\big\|\partial_t^2\mathbf{u}\big\|^2_{L^2(0,T;L^2(\Omega)^d)}\\
&\quad+208d^2M^2\lambda^2C_T^2\triangle t^4\big\|\partial_t^2\sigma\big\|^2_{L^2(0,T;L^2(\Omega)^{d\times d})}. \qquad (4.56)
\end{aligned}$$

Combining inequalities (4.47)-(4.56) with (4.46) yields

$$\begin{aligned}
&\alpha Re\big\|\varphi_{\mathbf{u}}^{l+1}\big\|^2+\alpha Re\big\|2\varphi_{\mathbf{u}}^{l+1}-\varphi_{\mathbf{u}}^{l}\big\|^2+\frac{\lambda}{2}\big\|\varphi_\sigma^{l+1}\big\|^2+\frac{\lambda}{2}\big\|2\varphi_\sigma^{l+1}-\varphi_\sigma^{l}\big\|^2\\
&\quad+4\alpha(1-\alpha)\triangle t\sum_{n=0}^{l}\big\|D\big(\varphi_{\mathbf{u}}^{n+1}\big)\big\|^2+\triangle t\sum_{n=0}^{l}\big\|\varphi_\sigma^{n+1}\big\|^2\\
&\le\alpha Re\triangle t\sum_{n=0}^{l}\big\|\varphi_{\mathbf{u}}^{n+1}\big\|^2+\big[20d^2M^2Re+13\lambda^2d^3M^2\big]\triangle t\sum_{n=0}^{l}\big\|2\varphi_{\mathbf{u}}^{n}\\
&\quad-\varphi_{\mathbf{u}}^{n-1}\big\|^2+\triangle t\sum_{n=0}^{l}\left[\frac{15d^3C_k^2M^2\lambda^2}{4\alpha(1-\alpha)}+\frac{216d^2\lambda^2K^2C_k^2}{\alpha(1-\alpha)}\right]\big\|\varphi_\sigma^{n+1}\big\|^2\\
&\quad+208\lambda^2d^2M^2\triangle t\sum_{n=0}^{l}\big\|2\varphi_\sigma^{n}-\varphi_\sigma^{n-1}\big\|^2+\frac{6\alpha dC_k^2C_{ip}^2}{1-\alpha}h^{2q+2}|||p|||^2_{0,q+1}\\
&\quad+\left[\frac{6\alpha}{1-\alpha}+13+208\lambda^2d^2M^2\right]C_{ip}^2h^{2m+2}|||\sigma|||^2_{0,m+1}+\bigg[6\alpha(1-\alpha)+52\lambda\alpha^2\\
&\quad+180\alpha Red^2K^2C_k^2+1872d^2\lambda^2K^2C_k^2+\frac{13\lambda^2d^3M^2}{4}\bigg]C_{ip}^2h^{2k}|||\mathbf{u}|||^2_{0,k+1}\\
&\quad+\big[20\alpha Red^2M^2+13\lambda^2d^3M^2\big]C_{ip}^2h^{2k+2}|||\mathbf{u}|||^2_{0,k+1}+117\lambda^2K^2d\bigg[1\\
&\quad+\frac{dC_i^2}{4}\bigg]C_{ip}^2h^{2m}|||\sigma|||^2_{0,m+1}+40Re\alpha C_{ip}^2h^{2k}\|\partial_t\mathbf{u}\|^2_{L^2(0,T;H^{k+1}(\Omega)^d)}\\
&\quad+\big[20\alpha d^2M^2Re+13\lambda^2d^3M^2\big]C_T^2\triangle t^4\big\|\partial_t^2u\big\|^2_{L^2(0,T;L^2(\Omega)^d)}\\
&\quad+40Re\alpha C_T^2\triangle t^4\big\|\partial_t^3\mathbf{u}\big\|^2_{L^2(0,T;L^2(\Omega)^d)}+26\lambda^2C_{ip}^2h^{2m}\|\partial_t\sigma\|^2_{L^2(0,T;H^{m+1}(\Omega)^{d\times d})}\\
&\quad+26\lambda^2C_T^2\triangle t^4\big\|\partial_t^3\sigma\big\|^2_{L^2(0,T;L^2(\Omega)^{d\times d})}\\
&\quad+208d^2M^2\lambda^2C_T^2\triangle t^4\big\|\partial_t^2\sigma\big\|^2_{L^2(0,T;L^2(\Omega)^{d\times d})}. \qquad (4.57)
\end{aligned}$$

In order to use the discrete Gronwall Lemma 2.3, here we set

$$\begin{aligned}
&a_l=\alpha Re\big\|\varphi_{\mathbf{u}}^{l+1}\big\|^2+\alpha Re\big\|2\varphi_{\mathbf{u}}^{l+1}-\varphi_{\mathbf{u}}^{l}\big\|^2+\frac{\lambda}{2}\big\|\varphi_\sigma^{l+1}\big\|^2+\frac{\lambda}{2}\big\|2\varphi_\sigma^{l+1}-\varphi_\sigma^{l}\big\|^2,\\
&b_n=4\alpha(1-\alpha)\big\|D\big(\varphi_{\mathbf{u}}^{n+1}\big)\big\|^2+\big\|\varphi_\sigma^{n+1}\big\|^2,
\end{aligned}$$

$$a_n = \alpha Re\left\|\varphi_{\mathbf{u}}^{n+1}\right\|^2 + \alpha Re\left\|2\varphi_{\mathbf{u}}^{n+1} - \varphi_{\mathbf{u}}^{n}\right\|^2 + \frac{\lambda}{2}\left\|\varphi_{\sigma}^{n+1}\right\|^2 + \frac{\lambda}{2}\left\|2\varphi_{\sigma}^{n+1} - \varphi_{\sigma}^{n}\right\|^2,$$

$$\gamma_n = \max\left\{1, \frac{d^2M^2}{\alpha}\left(20 + \frac{13\lambda^2 d}{Re}\right), \frac{d^2C_k^2\lambda}{\alpha(1-\alpha)}\left(\frac{15dM^2}{2} + 432K^2\right), 416\lambda d^2M^2\right\},$$

$$c_n = 0, \qquad \zeta_n = \frac{1}{1 - \triangle t\gamma_n},$$

H = other (non-summing) terms on the RHS of (4.57).

For $\gamma_n \triangle t \le \frac{1}{2}$, using the discrete Gronwall lemma to (4.57) yields Theorem 4.1. □

We will deduce that the induction assumption **IH1** (3.3) is right for any $n = 0, 1, 2, \ldots, N$, by mathematical induction.

Lemma 4.2 *Let* $(\mathbf{u}_h^l, p_h^l, \sigma_h^l) \in X_h \times Q_h \times S_h$ *satisfy* (3.1a)-(3.1c) *for each* $l \in \{0, 1, 2, \ldots, N\}$. *There is a bounded constant* K *such that*

$$\left\|\sigma_h^l\right\|_\infty \le K, \qquad \left\|\mathbf{u}_h^l\right\|_\infty \le K. \tag{4.58}$$

Proof Since $\|\sigma_h^0\|_\infty = \|\Pi_\sigma(\sigma_0)\|_\infty \le \|\sigma_0\|_\infty \le M \le K$. Now we assume that (4.58) holds true for $n = 0, 1, 2, \ldots, l$. By interpolation properties, inverse estimates, the regularity assumption (2.3) of σ, and result (4.3), we have that

$$\begin{aligned}
\left\|\sigma_h^{l+1}\right\|_\infty &\le \left\|\left(\sigma_h^{l+1} - \Pi_\sigma\left(\sigma^{l+1}\right)\right) + \left(\Pi_\sigma\left(\sigma^{l+1}\right) - \sigma^{l+1}\right) + \sigma^{l+1}\right\|_\infty \\
&\le \left\|\varphi_\sigma^{l+1}\right\|_\infty + \left\|\eta_\sigma^{l+1}\right\|_\infty + \left\|\sigma^{l+1}\right\|_\infty \\
&\le Ch^{-\frac{d}{2}}\left\|\varphi_\sigma^{l+1}\right\| + Ch^{-\frac{d}{2}}\left\|\eta_\sigma^{l+1}\right\| + M \\
&\le C\left(\triangle t^2 h^{-\frac{d}{2}} + h^{k-\frac{d}{2}} + h^{m-\frac{d}{2}} + h^{q+1-\frac{d}{2}}\right) + M.
\end{aligned} \tag{4.59}$$

We can see that the expression $C(\triangle t^2 h^{-\frac{d}{2}} + h^{k-\frac{d}{2}} + h^{m-\frac{d}{2}} + h^{q+1-\frac{d}{2}})$ is independent of l. Hence, if we set $k, m \ge \frac{d}{2}$, $q + 1 \ge \frac{d}{2}$, and choose h, $\triangle t$ such that

$$h^{k-\frac{d}{2}}, h^{m-\frac{d}{2}}, h^{q+1-\frac{d}{2}} \le \frac{1}{C}, \qquad \triangle t \le \frac{1}{C}h^{\frac{d}{4}},$$

then from (4.59)

$$\left\|\sigma_h^{l+1}\right\|_\infty \le 4 + M = K.$$

Similarly, we get $\|\mathbf{u}_h^{l+1}\|_\infty \le 4 + M = K$. □

Theorem 4.3 *Under the conditions of Theorem* 4.1 *and* $\triangle t \le C_0 h^{\frac{d}{4}}$, *we have*

$$\begin{aligned}
&\alpha Re\left\|\mathbf{u}^l - \mathbf{u}_h^l\right\|^2 + \frac{\lambda}{2}\left\|\sigma^l - \sigma_h^l\right\|^2 + \triangle t\sum_{n=0}^{l}\left\|\sigma^{n+1} - \sigma_h^{n+1}\right\|^2 \\
&\quad + 4\alpha(1-\alpha)\triangle t\sum_{n=0}^{l}\left\|D\left(\mathbf{u}^{n+1} - \mathbf{u}_h^{n+1}\right)\right\|^2 \le W(\triangle t, h).
\end{aligned} \tag{4.60}$$

Proof We add both sides of (4.3) with

$$\begin{aligned} \text{Extraterms} &= \alpha Re\|\eta_{\mathbf{u}}^{l+1}\|^2 + \alpha Re\|2\eta_{\mathbf{u}}^{l+1} - \eta_{\mathbf{u}}^{l}\|^2 + \frac{\lambda}{2}\|2\eta_{\sigma}^{l+1} - \eta_{\sigma}^{l}\|^2 \\ &\quad + \frac{\lambda}{2}\|\eta_{\sigma}^{l+1}\|^2 + \triangle t\sum_{n=0}^{l}\bigl[4\alpha(1-\alpha)\|D(\eta_{\mathbf{u}}^{n+1})\|^2 + \|\eta_{\sigma}^{n+1}\|^2\bigr], \end{aligned} \tag{4.61}$$

and apply the triangle inequality for the left-hand side. Noticing that the upcoming terms are already contained in the RHS of the model, we obtain the a priori error estimate (4.60). □

Theorem 4.4 *Under the conditions of Theorem* 4.3, *for any* $0 \le l \le N-1$, *there is a positive constant* C_1 *independent of* $\triangle t$ *and h such that*

$$\beta^2\triangle t\sum_{n=0}^{l}\|p^{n+1} - p_h^{n+1}\|^2 \le C_1 W(\triangle t, h). \tag{4.62}$$

Proof As $V_h \subset X_h$, for all $\mathbf{v}_h \in V_h$, we have from (4.6a)

$$\begin{aligned} &Re\bigl(\beth(e_{\mathbf{u}}^{n+1}), \mathbf{v}_h\bigr) + \bigl(e_{\sigma}^{n+1}, D(\mathbf{v}_h)\bigr) + (1-\alpha)\bigl(\nabla(e_{\mathbf{u}}^{n+1}), \nabla(\mathbf{v}_h)\bigr) \\ &\quad + Rec\bigl(\digamma(e_{\mathbf{u}}^{n+1}), \mathbf{u}^{n+1}, \mathbf{v}_h\bigr) + Rec\bigl(\digamma(\mathbf{u}_h^{n+1}), e_{\mathbf{u}}^{n+1}, \mathbf{v}_h\bigr) \\ &= \bigl(p^{n+1} - \lambda_h^{n+1}, \nabla\cdot\mathbf{v}_h\bigr) + Re\bigl(\beth(\mathbf{u}^{n+1}) - \partial_t\mathbf{u}^{n+1}, \mathbf{v}_h\bigr) \\ &\quad + Rec\bigl(\digamma(\mathbf{u}^{n+1}) - \mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{v}_h\bigr), \end{aligned} \tag{4.63}$$

where λ_h^{n+1} is an approximation to p^{n+1}. Dividing by $\|\nabla\mathbf{v}_h\|$, using the Cauchy-Schwarz inequality and the Poincaré inequality lead to

$$\begin{aligned} Re\frac{|(\beth(e_{\mathbf{u}}^{n+1}), \mathbf{v}_h)|}{\|\nabla(\mathbf{v}_h)\|} &\le \|e_{\sigma}^{n+1}\| + (1-\alpha)\|\nabla(e_{\mathbf{u}}^{n+1})\| \\ &\quad + ReC_p\sqrt{d}\|\digamma(e_{\mathbf{u}}^{n+1})\|\|\nabla\mathbf{u}^{n+1}\|_\infty + Re\sqrt{d}\|\digamma(\mathbf{u}_h^{n+1})\|_\infty\|e_{\mathbf{u}}^{n+1}\| \\ &\quad + \sqrt{d}\|p^{n+1} - \lambda_h^{n+1}\| + ReC_p\|\beth(\mathbf{u}^{n+1}) - \partial_t\mathbf{u}^{n+1}\| \\ &\quad + ReC_p\|\digamma(\mathbf{u}^{n+1}) - \mathbf{u}^{n+1}\|\|\nabla\mathbf{u}^{n+1}\|_\infty. \end{aligned} \tag{4.64}$$

Applying Lemma 2.2 and the regularity assumption (2.3) and taking the supremum over $\mathbf{v}_h \in V_h$ yield

$$\begin{aligned} Re\|\beth(e_{\mathbf{u}}^{n+1})\|_{X_h'} &\le \|e_{\sigma}^{n+1}\| + (1-\alpha)\|\nabla(e_{\mathbf{u}}^{n+1})\| + ReC_pM\sqrt{d}\|\digamma(e_{\mathbf{u}}^{n+1})\| \\ &\quad + 3ReK\sqrt{d}\|e_{\mathbf{u}}^{n+1}\| + \sqrt{d}\|p^{n+1} - \lambda_h^{n+1}\| \\ &\quad + ReC_p\|\beth(\mathbf{u}^{n+1}) - \partial_t\mathbf{u}^{n+1}\| + ReC_pM\|\digamma(\mathbf{u}^{n+1}) - \mathbf{u}^{n+1}\|. \end{aligned} \tag{4.65}$$

Splitting $p^{n+1} - p_h^{n+1} = (p^{n+1} - \lambda_h^{n+1}) + (\lambda_h^{n+1} - p_h^{n+1})$, we get from (4.63) that

$$\begin{aligned}
&\big(\lambda_h^{n+1} - p_h^{n+1}, \nabla \cdot \mathbf{v}_h\big) \\
&\quad = -\big(p^{n+1} - \lambda_h^{n+1}, \nabla \cdot \mathbf{v}_h\big) + Re\big(\sqsupset\big(e_{\mathbf{u}}^{n+1}\big), \mathbf{v}_h\big) \\
&\qquad + \big(e_\sigma^{n+1}, D(\mathbf{v}_h)\big) + (1-\alpha)\big(\nabla\big(e_{\mathbf{u}}^{n+1}\big), \nabla(\mathbf{v}_h)\big) + Rec\big(F\big(e_{\mathbf{u}}^{n+1}\big), \mathbf{u}^{n+1}, \mathbf{v}_h\big) \\
&\qquad + Rec\big(F\big(\mathbf{u}_h^{n+1}\big), e_{\mathbf{u}}^{n+1}, \mathbf{v}_h\big) - Re\big(\sqsupset\big(\mathbf{u}^{n+1}\big) - \partial_t \mathbf{u}^{n+1}, \mathbf{v}_h\big) \\
&\qquad + Rec\big(F\big(\mathbf{u}^{n+1}\big) - \mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{v}_h\big).
\end{aligned} \tag{4.66}$$

Combining the inf-sup condition (2.7) with (4.64)-(4.65), we have

$$\begin{aligned}
\beta\big\|\lambda_h^{n+1} - p_h^{n+1}\big\| &\le 2\sqrt{d}\big\|\lambda_h^{n+1} - p^{n+1}\big\| + 2\big\|e_\sigma^{n+1}\big\| + 2(1-\alpha)\big\|\nabla\big(e_{\mathbf{u}}^{n+1}\big)\big\| \\
&\quad + 2\sqrt{d}ReC_p^2 M\big\|\nabla\big(F\big(e_{\mathbf{u}}^{n+1}\big)\big)\big\| + 2ReC_p\big\|\sqsupset\big(\mathbf{u}^{n+1}\big) - \partial_t \mathbf{u}^{n+1}\big\| \\
&\quad + 6\sqrt{d}ReKC_p\big\|\nabla e_{\mathbf{u}}^{n+1}\big\| + 2ReC_p M\big\|F\big(\mathbf{u}^{n+1}\big) - \mathbf{u}^{n+1}\big\|.
\end{aligned} \tag{4.67}$$

Applying the triangle inequality to (4.67) yields

$$\begin{aligned}
\beta\big\|p^{n+1} - p_h^{n+1}\big\| &\le (1 + 2\sqrt{d})\big\|\lambda_h^{n+1} - p^{n+1}\big\| + 2\sqrt{d}ReC_p^2 M\big\|\nabla\big(F\big(e_{\mathbf{u}}^{n+1}\big)\big)\big\| \\
&\quad + 2ReC_p\big\|\sqsupset\big(\mathbf{u}^{n+1}\big) - \partial_t \mathbf{u}^{n+1}\big\| + [2 - 2\alpha + 6\sqrt{d}ReKC_p]\big\|\nabla\big(e_{\mathbf{u}}^{n+1}\big)\big\| \\
&\quad + 2\big\|e_\sigma^{n+1}\big\| + 2ReC_p M\big\|F\big(\mathbf{u}^{n+1}\big) - \mathbf{u}^{n+1}\big\|.
\end{aligned} \tag{4.68}$$

Applying $(a_1 + \cdots + a_6)^2 \le 6(a_1^2 + \cdots + a_6^2)$ to above equation, summing (4.68) with respect to n from 0 to l, and multiplying both sides of the equation by $\triangle t$ yield

$$\begin{aligned}
&\beta^2 \triangle t \sum_{n=0}^{l} \big\|p^{n+1} - p_h^{n+1}\big\|^2 \\
&\quad \le 6(1 + 2\sqrt{d})^2 \triangle t \sum_{n=0}^{l} \big\|\lambda_h^{n+1} - p^{n+1}\big\|^2 \\
&\qquad + 6(2 - 2\alpha + 6\sqrt{d}ReKC_p)^2 C_k^2 \triangle t \sum_{n=0}^{l} \big\|D\big(e_{\mathbf{u}}^{n+1}\big)\big\|^2 \\
&\qquad + 24dRe^2 C_p^4 M^2 C_k^2 \triangle t \sum_{n=0}^{l} \big\|D\big(F\big(e_{\mathbf{u}}^{n+1}\big)\big)\big\|^2 \\
&\qquad + 24Re^2 C_p^2 \triangle t \sum_{n=0}^{l} \big\|\sqsupset\big(\mathbf{u}^{n+1}\big) - \partial_t \mathbf{u}^{n+1}\big\|^2 \\
&\qquad + 24\triangle t \sum_{n=0}^{l} \big\|e_\sigma^{n+1}\big\|^2 + 24Re^2 C_p^2 M^2 \triangle t \sum_{n=0}^{l} \big\|F\big(\mathbf{u}^{n+1}\big) - \mathbf{u}^{n+1}\big\|^2.
\end{aligned} \tag{4.69}$$

Making use of the approximation property (2.5) of pressure, the error estimate $|||\sigma - \sigma_h|||_{0,0}^2$ and $|||D(\mathbf{u} - \mathbf{u}_h)|||_{0,0}^2$ in Theorem 4.3, the truncation errors (2.11) and (2.12) of the temporal discretion, we can derive the required result (4.62). □

If the domain $\Omega \subset \mathbb{R}^2$, then we can use the MINI elements $(P_1 b, P_1)$ pair, which satisfies the discrete inf-sup condition (2.7), to approximate the velocity $\mathbf{u}$ and pressure p and the P_1 discontinuous element to approximate the stress σ, that is, $k = 1$, $q = 1$, $m = 1$, we have the following convergence result.

Corollary 4.5 *Under the conditions of Theorem* 4.3 *and using the pair* $(P_1 b, P_1, P_1 dc)$ *elements to approximate* $(\mathbf{u}, p, \sigma)$, *there is a positive constant* C_2 *independent of* $\triangle t$ *and* h *such that*

$$\begin{aligned} &\alpha Re\|\mathbf{u}^l - \mathbf{u}_h^l\|^2 + \frac{\lambda}{2}\|\sigma^l - \sigma_h^l\|^2 + \triangle t\sum_{n=0}^{l}\|\sigma^{n+1} - \sigma_h^{n+1}\|^2 \\ &\quad + 4\alpha(1-\alpha)\triangle t\sum_{n=0}^{l}\|D(\mathbf{u}^{n+1} - \mathbf{u}_h^{n+1})\|^2 + \triangle t\sum_{n=0}^{l}\|p^{n+1} - p_h^{n+1}\|^2 \\ &\le C_2(\triangle t^4 + h^2). \end{aligned}$$

Corollary 4.6 *If the domain* $\Omega \subset \mathbb{R}^d$, $d = 2,3$, *and making use of Taylor-Hood* (P_2, P_1) *elements to approximate velocity* $\mathbf{u}$ *and pressure* p, *and* P_2 *discontinuous element for* σ, *that is*, $k = 2$, $q = 1$, $m = 2$, *we have*

$$\begin{aligned} &\alpha Re\|\mathbf{u}^l - \mathbf{u}_h^l\|^2 + \frac{\lambda}{2}\|\sigma^l - \sigma_h^l\|^2 + \triangle t\sum_{n=0}^{l}\|\sigma^{n+1} - \sigma_h^{n+1}\|^2 \\ &\quad + 4\alpha(1-\alpha)\triangle t\sum_{n=0}^{l}\|D(\mathbf{u}^{n+1} - \mathbf{u}_h^{n+1})\|^2 + \triangle t\sum_{n=0}^{l}\|p^{n+1} - p_h^{n+1}\|^2 \\ &\le C_3(\triangle t^4 + h^4). \end{aligned}$$

5 Numerical experiments

In this section, some numerical tests are performed by using FreeFem++ [41] to confirm our theoretical analysis.

5.1 Analytical solution

A known analytical solution example is used to verify theoretical convergence rates of the linearized scheme. We choose the final time $T = 0.1$ and computer domain $\Omega = [0,1]^2$. Same as [8, 39], the right-hand side function is added to the constitutive equation (1.1b) such that the analytical solutions $(\mathbf{u}, p, \sigma)$ are taken as follows:

$$\begin{aligned} &u_1(x,y) = 10x^2(x-1)^2 y(y-1)(2y-1)\cos(t), \\ &u_2(x,y) = -10x(x-1)(2x-1)y^2(y-1)^2\cos(t), \\ &p(x,y) = 10(2x-1)(2y-1)\cos(t), \\ &\sigma = 2\alpha D(\mathbf{u}), \qquad \mathbf{u} = (u_1, u_2), \end{aligned}$$

with the parameter $\lambda = 1.0$, $\alpha = 0.5$, $a = 0$, $Re = 1.0$. It is easy to see that the known solution of velocity is divergence-free. The source term $\mathbf{f}$, initial and boundary conditions are chosen to correspond to the exact solution. The spatial discretization is effected via the

Figure 1 A uniform mesh with $h = \sqrt{2}/8$.

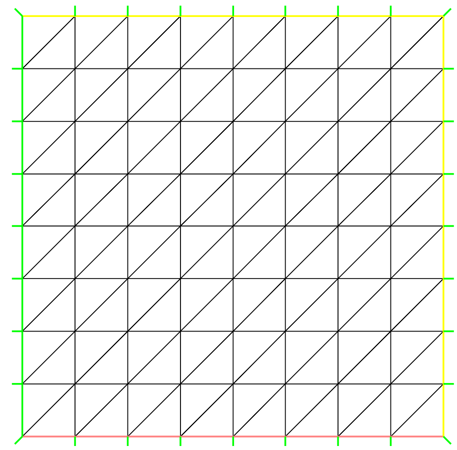


Table 1 Errors and CPU performance of the BDF2-LE scheme by using(P_1b, P_1, P_1dc) finite element for $T = 0.1$ and $\Delta t = 0.1h^2$

$1/h$	$\|\|\|e_\sigma\|\|\|_{\infty,0}$	$\|\|\|e_\sigma\|\|\|_{0,0}$	$\|e_{\mathbf{u}}\|_{\infty,0}$	$\|\|\|\nabla e_u\|\|\|_{0,0}$	$\|\|\|e_p\|\|\|_{0,0}$	CPU
4	0.0668476	0.020227	0.0144429	0.0680567	0.0802753	3.932
6	0.0329998	0.0102078	0.00709982	0.0423537	0.0363262	16.068
8	0.0197219	0.00603989	0.00417817	0.0309046	0.0204888	47.252
12	0.00961065	0.00283642	0.00192262	0.0201723	0.00904258	223.86
16	0.0058987	0.00166652	0.00109412	0.0150118	0.00501535	698.801
24	0.00311194	0.000809214	0.000489492	0.00995059	0.00214579	3,569.78
32	0.00205163	0.000499604	0.000275582	0.00744755	0.00116269	11,565.2
order	1.67091	1.78025	1.90816	1.06096	2.03938	

Table 2 Errors and CPU performance of the BDF2-LE scheme by using (P_2, P_1, P_2dc) finite element for $T = 0.1$ and $\Delta t = 0.1h^2$

$1/h$	$\|\|\|e_\sigma\|\|\|_{\infty,0}$	$\|\|\|e_\sigma\|\|\|_{0,0}$	$\|e_{\mathbf{u}}\|_{\infty,0}$	$\|\|\|\nabla e_u\|\|\|_{0,0}$	$\|\|\|e_p\|\|\|_{0,0}$	CPU
4	0.0092267	0.00275575	0.00172818	0.0144507	0.0803158	9.064
6	0.00315424	0.000907616	0.00050577	0.0069492	0.0362654	36.925
8	0.00151397	0.000411656	0.000206825	0.00401875	0.0204507	109.762
12	0.000570131	0.000139673	5.93583e–005	0.00182162	0.00903151	533.443
16	0.000296275	6.74014e–005	2.59447e–005	0.00103212	0.00501414	1,732.81
24	0.000122468	2.56494e–005	1.1174e–005	0.000462319	0.00214968	8,761.63
32	6.67788e–005	1.34476e–005	8.7483e–006	0.000262501	0.00116617	29,263.6
order	2.3616	2.55223	2.50376	1.93062	2.03803	

pairs (P_1b, P_1, P_1dc) and (P_2, P_1, P_2dc) to approximate velocity, pressure and stress tensor on a uniform triangular grid (see Figure 1 for $h = \sqrt{2}/8$), respectively.

Tables 1 and 2 are numerical results of the BDF2-LE scheme (3.1a)-(3.1c) by using (P_1b, P_1, P_1dc) elements and (P_2, P_1, P_2dc) elements, respectively. We see that $|||D(e_{\mathbf{u}})|||_{0,0}$ error has optimal convergence rate; however, $|||e_\sigma|||_{\infty,0}$, $|||e_\sigma|||_{0,0}$ and $|||e_{\mathbf{u}}|||_{\infty,0}$ errors are not optimal, while $|||e_p|||_{0,0}$ is super-convergence for (P_1b, P_1, P_1dc) elements.

Choosing $\triangle t = 0.05h$ and using (P_2, P_1, P_2dc) elements, we present the results in Table 3 to verify time convergence order. It is easy to see that the time convergence order is two.

In order to test the computational efficiency, we compared the CPU time of the BDF2-LE scheme (Scheme 3.1) with the classical fully implicit BDF2 scheme (Scheme 5.1).

Table 3 Errors and CPU performance of the BDF2-LE scheme by using (P_2, P_1, P_2dc) finite element with $\Delta t = 0.05h$ and $T = 0.1$

Δt	$\|\|\|e_\sigma\|\|\|_{\infty,0}$	$\|\|\|e_\sigma\|\|\|_{0,0}$	$\|\|e_{\mathbf{u}}\|\|_{\infty,0}$	$\|\|\|\nabla e_u\|\|\|_{0,0}$	$\|\|\|e_p\|\|\|_{0,0}$	CPU
1/120	3.09191e–3	8.76703e–4	5.05062e–4	6.74744e–3	0.0352161	16.61
1/160	1.47401e–3	3.98486e–4	2.06665e–4	3.92193e–3	0.0199588	37.535
1/240	5.53103e–4	1.35618e–4	5.93415e–5	1.78953e–3	8.87254e–3	125.627
1/320	2.87945e–4	6.56174e–5	2.58288e–5	1.01788e–3	4.94498e–3	317.612
1/480	1.19708e–4	2.50901e–5	1.11776e–5	4.57881e–4	2.12907e–3	1,124.82
order	2.35658	2.57344	2.78518	1.93818	2.02091	

Table 4 Errors and CPU performance of the BDF2-nonlinear scheme by using (P_1b, P_1, P_1dc) finite element for $T = 0.1$ and $\Delta t = 0.1h^2$

$1/h$	$\|\|\|e_\sigma\|\|\|_{\infty,0}$	$\|\|\|e_\sigma\|\|\|_{0,0}$	$\|\|e_{\mathbf{u}}\|\|_{\infty,0}$	$\|\|\|\nabla e_u\|\|\|_{0,0}$	$\|\|\|e_p\|\|\|_{0,0}$	CPU
4	0.0668514	0.0202271	0.0144429	0.0680566	0.0802749	12.651
6	0.0330003	0.0102078	0.00709982	0.0423537	0.0363262	55.302
8	0.0197219	0.00603989	0.00417817	0.0309046	0.0204888	113.306
12	0.00961065	0.00283642	0.00192262	0.0201723	0.00904258	557.295
16	0.0058987	0.00166652	0.00109412	0.0150118	0.00501535	1,759.45
24	0.00311194	0.000809214	0.000489492	0.00995059	0.00214579	9,120.06
32	0.00205163	0.000499604	0.000275582	0.00744755	0.00116269	29,252.6
order	1.67093	1.78026	1.90816	1.06096	2.03938	

Scheme 5.1 (BDF2 fully implicit scheme) Given $\mathbf{u}_h^{-1} = \mathbf{u}_h^0 \in V_h$, $\sigma_h^{-1} = \sigma_h^0 \in S_h$, find $(\mathbf{u}_h^{n+1}, p_h^{n+1}, \sigma_h^{n+1}) \in X_h \times Q_h \times S_h$ for $n = 0, 1, 2, \dots, N-1$ such that

$$\begin{aligned}
&Re\big(\beth(\mathbf{u}_h^{n+1}), \mathbf{v}_h\big) + Rec\big(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h\big) + \big(\sigma_h^{n+1}, D(\mathbf{v}_h)\big) \\
&\quad + 2(1-\alpha)\big(D(\mathbf{u}_h^{n+1}), D(\mathbf{v}_h)\big) - \big(p_h^{n+1}, \nabla \cdot \mathbf{v}_h\big) = \big(\mathbf{f}^{n+1}, \mathbf{v}_h\big), \\
&\big(q_h, \nabla \cdot \mathbf{u}_h^{n+1}\big) = 0, \\
&\lambda\big(\beth(\sigma_h^{n+1}), \tau_h\big) + \big(\sigma_h^{n+1}, \tau_h\big) + \lambda B\big(\mathbf{u}_h^{n+1}, \sigma_h^{n+1}, \tau_h\big) \\
&\quad - 2\alpha\big(D(\mathbf{u}_h^{n+1}), \tau_h\big) + \lambda\big(g_a\big(\sigma_h^{n+1}, \nabla \mathbf{u}_h^{n+1}\big), \tau_h\big) = 0
\end{aligned}$$

for all $(\mathbf{v}_h, q_h, \tau_h) \in X_h \times Q_h \times S_h$.

Unlike the BDF2-LE, the classical fully implicit BDF2 presented in Scheme 5.1 requires to solve a nonlinear problem at each time level. We employ the Newton iterative method. When relative nonlinear residual is less than 10^{-8}, the Newton iteration is stopped. The results of Scheme 5.1 are presented in Tables 4 and 5.

Comparing Tables 1-2 with Tables 4-5, respectively, we find that two numerical schemes have the same level of accuracy, while the BDF2-LE scheme can save significant CPU time for both (P_1b, P_1, P_1dc) elements and (P_2, P_1, P_2dc) elements.

5.2 4-to-1 planar contraction flow

Numerical simulations of viscoelastic flow through a planar or axisymmetric contraction have been widely studied in [42, 43]. Here the case of planar flow through a contraction geometry with a ratio of 4:1 with respect to upstream and downstream channel widths is considered. The contraction angle is fixed $3\pi/2$, and the channel lengths are sufficiently long to impose a fully developed Poiseuille flow in the inflow and outflow channels. The

Table 5 Errors and CPU performance of a fully implicit BDF2 scheme by using (P_2, P_1, P_2dc) finite element for $T = 0.1$,and $\triangle t = 0.1h^2$

$1/h$	$\|\|\|e_\sigma\|\|\|_{\infty,0}$	$\|\|\|e_\sigma\|\|\|_{0,0}$	$\|\|e_{\mathbf{u}}\|\|_{\infty,0}$	$\|\|\|\nabla e_u\|\|\|_{0,0}$	$\|\|\|e_p\|\|\|_{0,0}$	CPU
4	0.00922671	0.00275576	0.00172819	0.0144507	0.0803158	29.361
6	0.00315424	0.000907616	0.00050577	0.0069492	0.0362654	90.013
8	0.00151397	0.000411656	0.000206825	0.00401875	0.0204507	279.113
12	0.000570131	0.000139673	5.93583e–005	0.00182162	0.00903151	1,416.41
16	0.000296275	6.74015e–005	2.59467e–005	0.00103212	0.00501414	4,432.56
24	0.000122468	2.56494e–005	1.08e–005	0.000462319	0.00214968	22,743.7
32	6.67788e–005	1.34476e–005	8.74507e–006	0.000262501	0.00116617	75,954.1
order	2.3616	2.55223	2.49823	1.93062	2.03803	

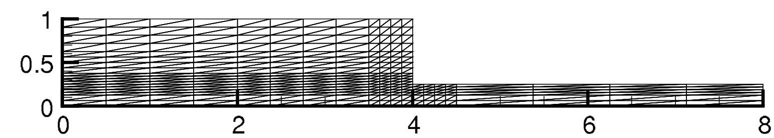


Figure 2 Plot of 4:1 contraction domain geometry and sample contraction mesh.

geometry of the computational domain is illustrated in Figure 2. The lower left corner of the domain corresponds to $x = y = 0$.

The computations of the mesh are also shown in Figure 2 with $\triangle x_{\min} = 0.0625$ and $\triangle y_{\min} = 0.015625$. We denote $\Gamma_{\text{in}} = \{(x,y) : x = 0, 0 \leq y \leq 1\}$ and $\Gamma_{\text{out}} = \{(x,y) : x = 8, 0 \leq y \leq 0.25\}$. On this domain the velocity boundary conditions are

$$\mathbf{u} = \begin{bmatrix} \frac{1}{32}(1-y^2) \\ 0 \end{bmatrix} \quad \text{on } \Gamma_{\text{in}}, \qquad \mathbf{u} = \begin{bmatrix} 2(\frac{1}{16}-y^2) \\ 0 \end{bmatrix} \quad \text{on } \Gamma_{\text{out}}. \tag{5.1}$$

For stress tensor σ on Γ_{in},

$$\begin{aligned} \sigma_{11} &= \frac{-\alpha\lambda(a+1)(-y/16)^2}{(a^2-1)\lambda^2(-y/16)^2-1}, \\ \sigma_{12} = \sigma_{21} &= \frac{-\alpha(-y/16)}{(a^2-1)\lambda^2(-y/16)^2-1}, \\ \sigma_{22} &= \frac{-\alpha\lambda(a-1)(-y/16)^2}{(a^2-1)\lambda^2(-y/16)^2-1}. \end{aligned} \tag{5.2}$$

Symmetry conditions are imposed on the bottom of the computational domain. Besides, the parameters Re, α, λ and a are chosen to be 1, 8/9, 0.7 and 1, respectively.

We performed the following study: starting from rest, we measured the time that the approximation solution reaches a steady state by using (P_1b, P_1, P_1dc) elements. The criterion to stop this process is the following:

$$\max\left\{ \frac{\|\mathbf{u}_h^{n+1}-\mathbf{u}_h^n\|}{\|\mathbf{u}_h^{n+1}\|}, \frac{\|\sigma_h^{n+1}-\sigma_h^n\|}{\|\sigma_h^{n+1}\|} \right\} \leq 10^{-5},$$

where $n+1$, n denote t_{n+1}, t_n, respectively.

We plot the evolution of the kinetic energy $0.5 * \|\mathbf{u}_h^{n+1}\|^2$ and $0.5 * \|\sigma_h^{n+1}\|^2$ using time step $\triangle t = 0.01$ until it reaches its steady state in Figure 3 and Figure 4, respectively. We

Figure 3 Evolution of $0.5 * \|u_h^{n+1}\|^2$ in time with time step $\Delta t = 0.01$ by using $(P_1 b, P_1, P_1 dc)$ element.

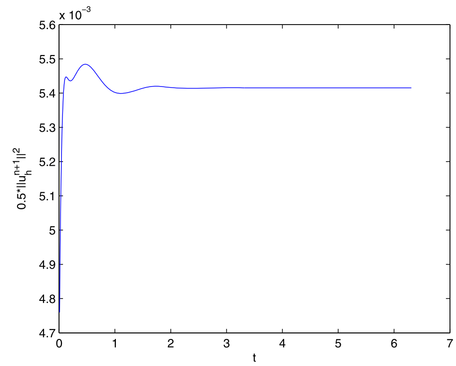


Figure 4 Evolution of $0.5 * \|\sigma_h^{n+1}\|^2$ in time with time step $\Delta t = 0.01$ by using $(P_1 b, P_1, P_1 dc)$ element.

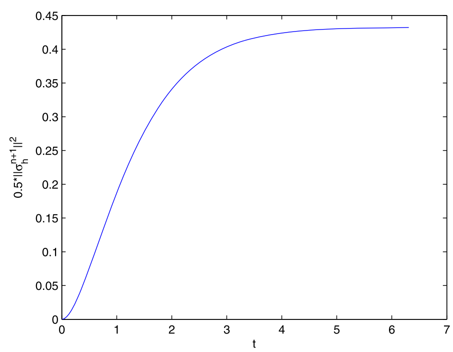
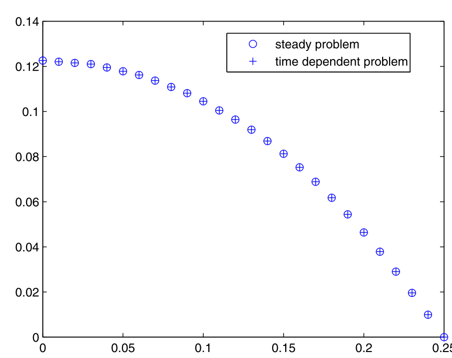


Figure 5 Horizontal velocity u_1 near reentrant corner ($x = 4.0625$). The mark 'o' indicates results for the steady problem and '+' indicates results for the time-dependent problem at final time $t = 6.31$ with time step $\Delta t = 0.01$ and $(P_1 b, P_1, P_1 dc)$ element.

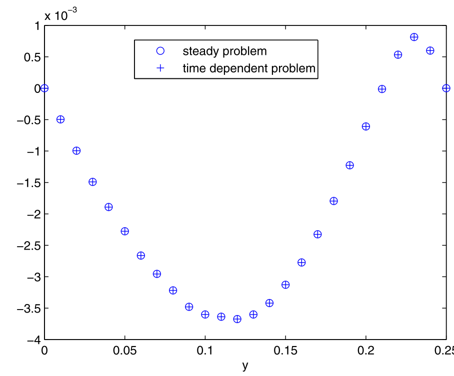


observe that it converges towards a steady state, while the kinetic energy of velocity has some oscillations at the beginning.

Figures 5 and 6 present the horizontal and vertical velocities near the reentrance corner along the vertical line $x = 4.0625$. We observe that the horizontal velocity is almost continuous, while the vertical velocity has high gradients near $y = 0.23$. However, we find that the solutions of the time-dependent problem can converge to the solutions of the steady problem.

We plot the streamlines of velocity for the steady problem and the time-dependent problem at final time $t = 6.31$ in Figure 7 and Figure 8, respectively. It is easy to observe that the two figures are almost alike.

Figure 6 Vertical velocity u_2 near reentrant corner ($x = 4.0625$). The mark 'o' indicates results for the steady problem and '+' indicates results for the time-dependent problem at final time $t = 6.31$ with time step $\Delta t = 0.01$ and $(P_1 b, P_1, P_1 dc)$ element.



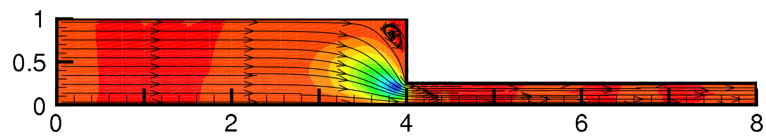


Figure 7 Streamlines of velocity contours for u: steady problem.

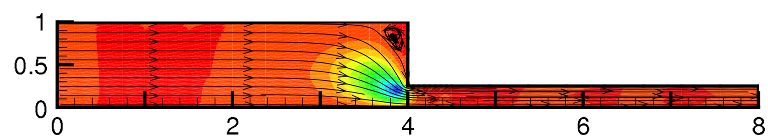


Figure 8 Streamlines of velocity contours for u: time-dependent problem at final time $t = 6.31$.

6 Conclusions and discussions

In this work, we have applied the BDF2-LE time-stepping scheme with Galerkin finite element to solve the time-dependent viscoelastic fluid flow in $\mathbb{R}^d$, $d = 2, 3$. We establish the stability analysis and a priori error estimates. Some numerical tests are provided to support the theoretical results and to demonstrate the effectiveness of the method.

Also, our analysis can be easily extended to the BDF2-LE decoupled scheme and other nonlinear viscoelastic fluid flow.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The study was carried out in collaboration among all authors. YZZ carried out the main theorem and wrote the paper; CX revised and checked the paper; and JQZ checked the article. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang, 471023, China. [2]Faculty of Mathematics and Physics Education, Luoyang Institute of Science and Technology, Luoyang, 471023, China.

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