Ling et al. *Boundary Value Problems* (2017) 2017:139
DOI 10.1186/s13661-017-0871-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

# Dynamical behaviour of a generalist predator-prey model with free boundary

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**Abstract**
In this paper, we consider a free boundary problem describing the invasion of a generalist predator into a prey population. We analytically derive the conditions guaranteeing the existence and uniqueness of the classical solution by means of the Schauder fixed point theorem, and further study the long-time behaviours of these two species. Finally, we numerically investigate the dynamical behaviour during the early invasion stage. Numerical results show that generalist predators are more likely to succeed in alien invasion by reducing the threshold size of the spatial domain of initial invasion, below which invasion fails.

**Keywords:** generalist predator-prey model; free boundary; long-time behaviour

## 1 Introduction

Alien invasion has frequently been reported to cause detrimental impacts on native ecosystems functions by altering population fitness, triggering extinction and secondary extinction of native species [1]. It has been reported that about 42% of all species in the United States are at risk because of competition with or predation by an alien species [2–4]; in other parts of the world, this figure can be even higher [2, 5].

The invaders can be a specialist feeding on a particular prey population, but it can also be a generalist feeding on multiple food sources, and thus their impacts are generally unexpected. A convenient approach to understand the impact of alien invasion is to develop an appropriate mathematical modelling. In this regards, two approaches have been widely applied. One is to develop ordinary differential equations based on the mean field theory to describe the temporal population dynamics, and the other is to develop reaction-diffusion equations to describe the spatio-temporal population dynamics. While the former approach ignores the spatial aspects of alien invasion, the later suffers from the drawback that alien species can instantaneously spread over the entire spatial domain even if they start to invade in a small area.

To best describe the invasion process, free boundary problems were introduced to biology [6–10]. In fact, free boundary problems have received considerable attention in many fields, such as tumor cure [11] and wound healing [12] in medicine, vapor infiltration of pyrolytic carbon in chemistry [13], and expansion of the area infected by the virus in epidemiology [14]. To the best of our knowledge, it was first introduced to biology by Lin to describe the process of a predator invading a prey population [15]. Since then many mathematical models with free boundary have been developed to research biological population

[16–19]. For example, Du and Lin in [17] investigated a diffusive logistic model with a free boundary and proved a spreading-vanishing dichotomy. Wang in [20] studied a diffusive logistic equation with a free boundary and sign-changing coefficient and also derived a spreading-vanishing dichotomy. Additionally, Monobe and Wu in [21] introduced the free boundary into a reaction-diffusion-advection logistic model in heterogeneous environment, and obtained the long-time behaviour of the solution and the asymptotic spreading speed.

While predator-prey models with free boundary have attracted great attention, almost all of the studied models assume that the predator is a specialist, which means that the predator will certainly go to extinction in the absence of the focal prey. In reality, it might be not true since most predators are very likely to have alternative food sources [22, 23]. Predators with multiple food sources are called generalist. A question arises how the invasion of generalist predator into a local system affects the population dynamics in a predator-prey model with free boundary, which remains unexplored.

The aim of this paper is to obtain insight into the question raised above through the following generalist predator-prey model:

$$\begin{cases} u_t - u_{xx} = u(1-u) - \frac{uv}{1+au} := f(u,v), & t > 0, x > 0, \\ v_t - dv_{xx} = \frac{uv}{1+au} + \frac{bv}{1+cv} - ev := g(u,v), & t > 0, 0 < x < h(t), \\ v(t,x) = 0, & t \geq 0, x \geq h(t), \\ u_x(t,0) = v_x(t,0) = 0, & t \geq 0, \\ h'(t) = -\mu v_x(t,h(t)), & t \geq 0, \\ u(0,x) = u_0(x), & 0 \leq x < \infty, \\ h(0) = h_0, \qquad v(0,x) = v_0(x), & 0 \leq x \leq h_0, \end{cases} \tag{1}$$

where the initial values $u_0(x)$ and $v_0(x)$ are non-negative and satisfy

$$\begin{cases} u_0(x) \in C^2([0,\infty)), \qquad u_0'(0) = 0 \quad \text{and} \quad u_0(x) > 0 \quad \text{in } [0,\infty); \\ v_0(x) \in C^2([0,h_0)), \qquad v_0'(0) = v_0(h_0) \quad \text{and} \quad v_0(x) > 0 \quad \text{in } [0,h_0). \end{cases} \tag{2}$$

In this model, $u(x,t)$ and $v(x,t)$ indicate the population biomass density of the prey and predator at time $t$ and space $x$. In the absence of predator, the prey follows the logistic growth and is initially distributed in the whole domain. In the presence of predator, the prey suffers from a predation, following the Holling type II functional response where $a$ represents the handling time of prey individuals by the predator. The predator feeds on both the focal prey and other food sources, and it suffers at the same time from a background mortality rate $e$. The parameters $b$ and $c$ indicate the predator's attack rate and handling time of other preys. To mimic the invading process, we assume the predator is initially distributed only in the range $x \in [0,h_0]$. The spreading front of the predator is described by the so-called free boundary $x = h(t)$. For both populations, we assume a homogeneous Neumann boundary condition at $x = 0$, which indicates that the left boundary acts as a barrier and that the predator can only invade into the environment from the right. Since we aim for general insights, all parameters are non-dimensional, positive, and constant.

In this model we assume that the alternative food has no dynamics evolution of its own, which means that the alternative food is fixed in constant amounts, availability is significantly high, and hence unaffected due to consumption (see the references [22, 23]). This is apparently a simplification to reduce the dimension of the system from three to two, and thus allows the use of the theory of Schauder fixed point to study the consequences of the availability of alternative food. However, this simplification is justified for many arthropod predators because they can rely on plant-provided alternative food sources such as pollen or nectar, the availability of which is unlikely to be influenced by the predator's consumption [24–26]. The paper is structured as follows. In Section 2 we derive the conditions guaranteeing the existence and uniqueness of the classic solution to the model (1). In Section 3, we analyse theoretically the long term behaviour of prey and predator. In Section 4, we perform numerical analysis of the population behaviour. The paper ends with a brief conclusion.

## 2 Global existence, uniqueness and estimate of the solution

In this section, we prove the local existence and uniqueness results to problem (1) by applying contraction mapping theorem, and then we show the global existence using some suitable estimates.

**Theorem 2.1** *For any given* $(u_0, v_0)$ *satisfying* (2) *and any* $\alpha \in (0,1)$, *there is a* $T > 0$ *such that the problem* (1) *admits a unique solution*

$$(u, v, h) \in C^{(1+\alpha)/2,1+\alpha}\big(D_T^\infty\big) \times C^{(1+\alpha)/2,1+\alpha}(D_T) \times C^{1+\alpha/2}\big([0,T]\big),$$

*moreover,*

$$\|u\|_{C^{(1+\alpha)/2,1+\alpha}(D_T^\infty)} + \|v\|_{C^{(1+\alpha)/2,1+\alpha}(D_T)} + \|h\|_{C^{1+\alpha/2}[0,T]} \le C,$$

*where* $D_T = \{(t,x) \in R^2 : t \in [0,T], x \in [0,h(t))\}$, $D_T^\infty = \{(t,x) \in R^2 : t \in [0,T], x \in [0,\infty)\}$, *C and T only depend on* $h_0$, $\alpha$, $\min_{x\in[0,h_0]} u_0(x)$, $\|u_0\|_{C^2([0,\infty))}$, $\|v_0\|_{C^2([0,h_0])}$.

*Proof* The idea of this proof comes from [12]. First, we want to keep off the difficulty caused by the free boundary. Taking $\zeta(y)$ to be a function in $C^3[0,\infty)$ and to satisfy

$$\zeta(y) = 1 \quad \text{if } |y - h_0| < h_0/8,$$
$$\zeta(y) = 0 \quad \text{if } |y - h_0| > \frac{h_0}{2}, \qquad \big|\zeta'(y)\big| < 5/h_0 \quad \forall y,$$

and considering the following simple Chang case:

$$(t,x) \to (t,y), \quad \text{where } x = y + \zeta(y)\big(h(t) - h_0\big), 0 \le y < \infty,$$

we find the following. As long as

$$\big|h(t) - h_0\big| \le h_0/8,$$

the above transformation $x \to y$ is a diffeomorphism from $[0,\infty)$ onto $[0,\infty)$. Moreover, it helps us straighten the free boundary. We obtain from standard calculations that

$$\frac{\partial y}{\partial x} = \frac{1}{1+\zeta'(y)(h(t)-h_0)} \equiv \sqrt{k_1\big(h(t),y\big)},$$
$$\frac{\partial^2 y}{\partial x^2} = -\frac{\zeta''(y)(h(t)-h_0)}{[1+\zeta'(y)(h(t)-h_0)]^3} \equiv k_2\big(h(t),y\big),$$
$$-\frac{1}{h'(t)}\frac{\partial y}{\partial t} = \frac{\zeta(y)}{1+\zeta'(y)(h(t)-h_0)} \equiv k_3\big(h(t),y\big).$$

Set

$$u(t,x) = u\big(t,y+\zeta(y)\big(h(t)-h_0\big)\big) = w(t,y),$$
$$v(t,x) = v\big(t,y+\zeta(y)\big(h(t)-h_0\big)\big) = z(t,y),$$

and the free boundary problem (1) can be rewritten as

$$\begin{cases} w_t - k_1 w_{yy} - (k_2 + h'k_3)w_y = f(w,z), & 0 \le y < \infty, t > 0, \\ z_t - k_1 d z_{yy} - (k_2 d + h'k_3)z_y = g(w,z), & 0 \le y < h_0, t > 0, \\ z(y,t) = 0, & y \ge h_0, t \ge 0, \\ h'(t) = -\mu\frac{\partial z}{\partial y}, & y = h_0, t > 0, \\ \frac{\partial w}{\partial y}(0,t) = \frac{\partial z}{\partial y}(0,t) = 0, & t > 0, \\ h(0) = h_0, & \\ w(y,0) = u_0(y) := w_0(y), & 0 \le y \le h_0, \\ z(y,0) = v_0(y) := z_0(y), & 0 \le y \le l, \end{cases} \tag{3}$$

where $k_1 = k_1(h(t),y)$, $k_2 = k_2(h(t),y)$, $k_3 = k_3(h(t),y)$. It is easy to see

$$f(w,z) = \begin{cases} w(1-w) - \frac{wz}{1+aw}, & 0 < y < h_0, \\ w(1-w), & y \ge h_0. \end{cases}$$

Let $h^* = -\mu v_0'(h_0)$, $z_0(y) = 0$ for $y > h_0$, and $\delta = \min_{0\le y\le h_0} u_0(y)$ and for $0 < T \le \frac{h_0}{8(1+h^*)}$, and define

$$W_T = \big\{w \in C\big(\Delta_T^\infty\big) : w(y,0) = w_0(y), \|w - w_0\|_{C(\Delta_T^\infty)} \le \delta/4\big\},$$
$$Z_T = \left\{\begin{matrix} z \in C(\Delta_T^\infty), z(y,0) \equiv 0 \text{ for } y \ge h_0, 0 \le t \le T, \\ z_0(y) = z_0(y) \text{ for } 0 \le y \le h_0, \|z - z_0\|_{C(\Delta_T^\infty)} \le \delta/4 \end{matrix}\right\},$$
$$H_T = \big\{h \in C^1[0,T], h(0) = h_0, h'(0) = h_*, \big\|h' - h_*\big\|_{C[0,T]} \le 1\big\},$$

where $\Delta_T = [0,T] \times [0,h_0]$, $\Delta_T^\infty = [0,T] \times [0,\infty)$. Clearly, $\mathcal{D}_T = W_T \times Z_T \times H_T$ is a complete metric space with metric

$$D\big((w_1,z_1,h_1),(w_2,z_2,h_2)\big) = \|w_1 - w_2\|_{C(\Delta_T^\infty)} + \|z_1 - z_2\|_{C(\Delta_T)} + \|h_1 - h_2\|_{C(\Delta_T)}.$$

Observing that, for $h_1$ and $h_2 \in H_T$, $h_1(0) = h_2(0) = h_0$ leads to

$$\|h_1 - h_2\|_{C([0,T])} \leq T\big\|h_1' - h_2'\big\|_{C([0,T])}. \tag{4}$$

By standard $L^p$ theory and the Sobolev imbedding theorem, for any $(w, z, h) \in \mathcal{D}_T$ the following diffraction problem:

$$\begin{cases} \tilde{w}_t - k_1\tilde{w}_{yy} - (k_2 + h'k_3)\tilde{w}_y = f(w,z), & t > 0, 0 \leq y < \infty, \\ \tilde{z}_t - k_1 d\tilde{z}_{yy} - (k_2 d + h'k_3)\tilde{z}_y = g(w,z), & t > 0, 0 \leq y < h_0, \\ \tilde{w}_y(0,t) = \tilde{z}_y(0,t) = 0, \qquad z(y,t) \equiv 0, & t > 0, h_0 \leq y < \infty, \\ \tilde{w}(y,0) = w_0(y) := u_0(y), & 0 \leq y < \infty, \\ \tilde{z}(y,0) = z_0(y) := v_0(y), & 0 \leq y \leq h_0, \end{cases} \tag{5}$$

admits a unique bounded solution $(\tilde{u}, \tilde{v}) \in C^{(1+\alpha)/2,1+\alpha}(\Delta_T^\infty) \times C^{(1+\alpha)/2,1+\alpha}(\Delta_T)$ and

$$\|\tilde{w}\|_{C^{(1+\alpha)/2,1+\alpha}(\Delta_T^\infty)} \leq C_1, \tag{6}$$

$$\|\tilde{z}\|_{C^{(1+\alpha)/2,1+\alpha}(\Delta_T)} \leq C_1, \tag{7}$$

where $C_1$ is a constant depending on $\alpha$, $h_0$, $\delta$, $\|u_0\|_{C^2[0,\infty)}$ and $\|v_0\|_{C^2[0,h_0]}$.

Defining

$$\tilde{h}(t) = h_0 - \mu \int_0^t \tilde{z}_y(\tau, h_0)\,d\tau, \tag{8}$$

it follows that

$$\tilde{h}'(t) = -\mu\tilde{z}_y(t, h_0),$$

and subsequently, we have $\tilde{h}'(t) \in C^{\alpha/2}([0,T])$ and

$$\big\|\tilde{h}'(t)\big\|_{C^{(1+\alpha)/2}[0,T]} \leq C_2 := \mu C_1. \tag{9}$$

Introducing a mapping $\mathcal{F} : \mathcal{D}_T \to C(\Delta_T^\infty) \times C(\Delta_T) \times C^1([0,T])$ by $\mathcal{F}(w,z,h) = (\tilde{w}, \tilde{z}, \tilde{h})$, we can show that $\mathcal{F}$ has a fixed point, which is just a solution to the system (3).

Taking into account $\tilde{h}'(t) - h_* = \mu(\tilde{v}_y(h_0,0) - \tilde{v}_y(h_0,t))$ and using the estimates (6), (7) and (9), we have

$$\big\|\tilde{h}' - h^*\big\|_{C([0,T])} \leq \mu\big\|\tilde{h}'\big\|_{C^{\alpha/2}([0,T])} T^{\alpha/2} \leq \mu C_1 T^{\alpha/2},$$

$$\|\tilde{w} - w_0\|_{(\Delta_T^\infty)} \leq \|\tilde{w} - w_0\|_{C^{(1+\alpha)/2,0}(\Delta_T^\infty)} T^{(1+\alpha)/2} \leq C_1 T^{(1+\alpha)/2},$$

$$\|\tilde{z} - z_0\|_{(\Delta_T)} \leq \|\tilde{z} - z_0\|_{C^{(1+\alpha)/2,0}(\Delta_T)} T^{(1+\alpha)/2} \leq C_1 T^{(1+\alpha)/2}.$$

If we take $T \leq \min\{(\mu C_1)^{-2/\alpha}, C_1^{-2/(1+\alpha)}\}$, the mapping $\mathcal{F}$ maps $\mathcal{D}_T$ into itself.

Next we prove that, for $T > 0$ sufficiently small, $\mathcal{F}$ is a contraction mapping on $\mathcal{D}_T$. Let $(w_i, z_i, h_i) \in \mathcal{D}_T$, $(i = 1, 2)$ be two solutions of the problem (5), then, for $i = 1, 2$ and denoting

$(\tilde{w}_i, \tilde{z}_i, \tilde{h}_i) = \mathcal{F}(w_i, z_i, h_i)$, it follows from (6), (7) and (9) that

$$\|\tilde{w}_i\|_{C^{(1+\alpha)/2,1+\alpha}(\Delta_T^\infty)} \le C_1, \qquad \|\tilde{z}_i\|_{C^{(1+\alpha)/2,1+\alpha}(\Delta_T^\infty)} \le C_1, \qquad \|\tilde{h}_i'(t)\|_{C^{\alpha/2}[0,T]} \le C_2.$$

Setting $W = w_1 - w_2$, we find that $W(y,t)$ satisfies

$$\begin{cases} W_t - k_1(y,h_2)W_{yy} - (k_2(y,h_2) + h_2' k_3(y,h_2))W_y \\ \quad = [k_1(y,h_1) - k_1(y,h_2)]\tilde{w}_{1,yy} + [k_2(y,h_1) - k_2(y,h_2)]\tilde{w}_{1,y} \\ \qquad + [h_1' k_3(y,h_1) - h_2' k_3(yh_2)]\tilde{w}_{1,y} + f(w_1,z_1) - f(w_2,z_2), & t > 0, 0 < y < \infty, \\ W_y = (t,0) = 0, \qquad W(t,h_0) = 0, & t > 0, \\ W(0,y) = 0, & 0 \le y \le h_0. \end{cases}$$

Due to $|w_i - w_0| \le \delta/4$ and $|z_i - z_0| \le \delta/4$, $i = 1,2$, we have $1 + aw_i > 3a\delta/4$ and $1 + cz_i > 3c\delta/4$, which results in

$$\begin{aligned} &\left|\frac{w_1 z_1}{1 + aw_1} - \frac{w_2 z_2}{1 + aw_2}\right| \\ &\quad \le \frac{a|w_1 w_2| + |w_2|}{(1 + aw_1)(1 + aw_2)}|z_1 - z_2| + \frac{|z_1|}{(1 + aw_1)(1 + aw_2)}|w_1 - w_2| \\ &\quad \le \frac{16L}{9a^2\delta^2}\bigl(|z_1 - z_2| + |w_1 - w_2|\bigr) \end{aligned}$$

and

$$\left|\frac{bz_1}{1 + cz_1} - \frac{bz_2}{1 + cz_2}\right| \le \frac{b}{(1 + cz_1)(1 + cz_2)}|z_1 - z_2| \le \frac{16b}{9c^2\delta^2}|z_1 - z_2|,$$

where the constant $L$ depends on $\|u_0\|_{C([0,\infty))}$ and $\|v_0\|_{C([0,h_0])}$. This implies that $f(w,z)$ and $g(w,z)$ are local Lipschitz continuous functions of $(w,z)$. Using the $L^p$ estimates for parabolic equations and Sobolev's imbedding theorem, we obtain

$$\begin{aligned} &\|\tilde{w}_1 - \tilde{w}_2\|_{C^{(1+\alpha)/2,1+\alpha}(\Delta_T^\infty)} \\ &\quad \le C_3\bigl[\|w_1 - w_2\|_{C(\Delta_T^\infty)} + \|z_1 - z_2\|_{C(\Delta_T)} + \|h_1 - h_2\|_{C^1([0,T])}\bigr]. \end{aligned} \tag{10}$$

Similarly, we have the following estimate:

$$\begin{aligned} &\|\tilde{z}_1 - \tilde{z}_2\|_{C^{(1+\alpha)/2,1+\alpha}(\Delta_T)} \\ &\quad \le C_4\bigl[\|w_1 - w_2\|_{C(\Delta_T^\infty)} + \|z_1 - z_2\|_{C(\Delta_T)} + \|h_1 - h_2\|_{C^1([0,T])}\bigr], \end{aligned} \tag{11}$$

where $C_3$ and $C_4$ depend on $C_1$ and $C_2$, the local Lipschitz coefficients of $f$ and $g$, as well as the functions $A$, $B$ and $C$ in the definition of transformation $(t,y) \to (t,x)$. Taking the difference of the equations for $\tilde{h}_1$, $\tilde{h}_2$ results in

$$\|\tilde{h}_1' - \tilde{h}_2'\|_{C^{\alpha/2}([0,T])} \le \mu\|\tilde{z}_{1,y} - \tilde{z}_{2,y}\|_{C^{\alpha/2,0}(\Delta_T)}. \tag{12}$$

Combining inequalities (4), (10), (11) and (12), and assuming that $T \le 1$, we obtain

$$\begin{aligned}&\|\tilde{w}_1 - \tilde{w}_2\|_{C^{(1+\alpha)/2,1+\alpha}(\Delta_T^\infty)} + \|\tilde{z}_1 - \tilde{z}_2\|_{C^{(1+\alpha)/2,1+\alpha}(\Delta_T)} + \|\tilde{h}_1' - \tilde{h}_2'\|_{C^{\alpha/2}([0,T])} \\ &\quad \le C_5\bigl[\|w_1 - w_2\|_{C(\Delta_T^\infty)} + \|z_1 - z_2\|_{C(\Delta_T)} + \|h_1' - h_2'\|_{C[0,T]}\bigr],\end{aligned}$$

with $C_5$ depending on $C_3$, $C_4$, and $\mu$. Hence, denoting

$$T := \min\left\{1, \left(\frac{1}{2C_5}\right)^{2/\alpha}, (\mu C_1)^{-2/\alpha}, C_1^{-2/(1+\alpha)}, \frac{h_0}{8(1+h^*)}\right\},$$

we have

$$\begin{aligned}&\|\tilde{w}_1 - \tilde{w}_2\|_{C(\Delta_T^\infty)} + \|\tilde{z}_1 - \tilde{z}_2\|_{C(\Delta_T)} + \|\tilde{h}_1' - \tilde{h}_2'\|_{C([0,T])} \\ &\quad \le T^{(1+\alpha)/2}\bigl[\|\tilde{w}_1 - \tilde{w}_2\|_{C^{(1+\alpha)/2,1+\alpha}(\Delta_T^\infty)} + \|\tilde{z}_1 - \tilde{z}_2\|_{C^{(1+\alpha)/2,1+\alpha}(\Delta_T)}\bigr] \\ &\qquad + T^{\alpha/2}\|\tilde{h}_1' - \tilde{h}_2'\|_{C^{\alpha/2}([0,T])} \\ &\quad \le C_5 T^{\alpha/2}\bigl[\|w_1 - w_2\|_{C(\Delta_T^\infty)} + \|z_1 - z_2\|_{C(\Delta_T)} + \|h_1' - h_2'\|_{C([0,T])}\bigr] \\ &\quad \le \frac{1}{2}\bigl[\|w_1 - w_2\|_{C(\Delta_T^\infty)} + \|z_1 - z_2\|_{C(\Delta_T)} + \|h_1' - h_2'\|_{C([0,T])}\bigr].\end{aligned}$$

This shows that $\mathcal{F}$ is a contraction mapping in $\mathcal{D}_T$ if $T$ is small enough. It follows from the contraction mapping theorem that $\mathcal{F}$ has a unique fixed point $(w,z,h)$ in $\mathcal{D}_T$. That is, $(w,z,h)$ is the unique solution of the problem (3) and accordingly $(u,v,h)$ is the unique solution of the problem (1). Moreover, utilising the $L^p$ estimate, we have additional regularity of the solution $u(t,x) \in C^{(1+\alpha)/2,1+\alpha}(D_T^\infty)$, $v(t,x) \in C^{(1+\alpha)/2,1+\alpha}(D_T)$ and $h(t) \in C^{1+\alpha/2}([0,T])$. The proof of Theorem 2.1 is completed. □

It is observed that there exists a time $T$ such that the solution exists in time interval $[0,T]$. Since the global existence theorem depends on a prior estimate with respect to $h'(t)$, in what follows, we aim to derive a prior estimate for any solution of the problem (1).

**Theorem 2.2** *If $a^{-1} < e < b$, the solution $(u,v;h)$ of the free boundary problem* (1) *satisfies*

$$0 < u(t,x) \le M_1 \quad \text{for } 0 \le t \le T, 0 \le x < \infty,$$
$$0 < v(x,t) \le M_2 \quad \text{for } 0 \le t \le T, 0 \le x < h(t),$$

*and*

$$0 < h'(t) \le M_3 \quad \text{for } 0 < t \le T,$$

*where the constant $M_i$ are independent of $T$ for $i = 1,2,3$.*

*Proof* Using the strong maximum principle, we can easily see that $u > 0$ in $[0,T] \times [0,\infty)$ and $v > 0$ in $[0,T] \times [0,h(t))$ when the solution exists.

Since $u$ satisfies

$$\begin{cases} u_t - u_{xx} = u(1-u) - \frac{uv}{1+au}, & 0 < t \le T, x > 0, \\ u_x(t,0) = 0, \quad u(t,h(t)) = 0, & t \ge 0, \\ u(0,x) = u_0(x), & 0 \le x < \infty, \end{cases}$$

we get $u \le \max\{\|u_0\|_\infty, 1\} := M_1$ by the maximum principle. Similarly, as $v$ satisfies

$$\begin{cases} v_t - dv_{xx} = \frac{uv}{1+au} + \frac{bv}{1+cv} - ev, & 0 < t \le T, 0 < x < h(t), \\ v_x(t,0) = 0, & t \ge 0, \\ v(0,x) = v_0(x), & 0 \le x < \infty, \end{cases}$$

we also have $v \le \max\{\|v_0\|_\infty, \frac{1}{c}(\frac{b}{e-\frac{M_1}{1+aM_1}} - 1)\} := M_2$, where $\frac{b}{e-\frac{M_1}{1+aM_1}} - 1 > 0$ thanks to $a^{-1} < b < e$.

Considering the transformation

$$y = x/h(t), \qquad w(t,y) = u(t,x), \qquad z(t,y) = v(t,x),$$

similar to the proof of Lemma 2.1 in [27], we obtain $z_y(t,1) < 0$ for $0 < t \le T$. Thus, $h'(t) = -\mu v_x(t,h(t)) > 0$ in $(0,T]$.

Now we demonstrate that $h'(t) \le M_3$ with some $M_3$ independent of $T$. To see this, let $M$ be a positive constant,

$$\Omega_M = \{(t,x) : 0 < t < T, h(t) - M^{-1} < x < h(t)\},$$

and construct an auxiliary function

$$\omega(x,t) = M_2\bigl[2M\bigl(h(t)-x\bigr) - M^2\bigl(h(t)-x\bigr)^2\bigr].$$

In the following, we will choose $M$ to guarantee $\omega(x,t) \ge v(t,x)$ in $\Omega_M$.

Straightforward calculation shows that

$$\omega_t = 2M_2Mh'(t)\bigl(1 - M\bigl(h(t)-x\bigr)\bigr) \ge 0, \qquad -\omega_{xx} = 2M_2M^2,$$
$$\left(\frac{u}{1+au} + \frac{b}{1+cv} - e\right)v \le (1/a + b - e)M_2.$$

It follows that

$$\omega_t - \omega_{xx} \ge 2M_2M^2 \ge (1/a + b - e)M_2 \ge \left(\frac{u}{1+au} - \frac{b}{1+cv} - e\right)v,$$

if $M^2 \ge \frac{\frac{1}{a}+b-e}{2}$. On the other hand,

$$\omega\left(t, h(t) - \frac{1}{M}\right) = M_2 \ge v\left(t, h(t) - \frac{1}{M}\right), \qquad \omega\bigl(h(t),t\bigr) = 0 = v\bigl(h(t),t\bigr).$$

Next, we will further choose some $M$ such that $\omega(0,x) \geq v_0(x)$ for $x \in [h_0 - M^{-1}, h_0]$. We divide $[h_0 - M^{-1}, h_0]$ into two subsets: $[h_0 - M^{-1}, h_0 - (2M)^{-1}]$ and $[h_0 - (2M)^{-1}, h_0]$. For $x \in [h_0 - (2M)^{-1}, h_0]$,

$$\omega_x(0,x) = -2MM_2\big[1 - M(h_0 - x)\big] \leq -MM_2 \leq v_0'(x).$$

with $M \geq \frac{4\|v_0\|_{C^1([0,h_0])}}{3M_2}$. This implies

$$\omega(0,x) \geq v_0(x), \quad \text{for } x \in \big[h_0 - (2M)^{-1}, h_0\big],$$

due to $\omega(0,h_0) = v_0(h_0) = 0$. For $x \in [h_0 - (M)^{-1}, h_0 - (2M)^{-1}]$, we also have

$$\omega(0,x) \geq \frac{3}{4}M_2 \geq \|v_0\|_{C^1([0,h_0])}M^{-1} \geq v_0(x).$$

Therefore, by choosing

$$M = \max\left\{\frac{1}{h_0}, \sqrt{\frac{\frac{1}{a} + b - e}{2}}, \frac{4\|v_0\|_{C^1([0,h_0])}}{3M_2}\right\},$$

we can apply the comparison principle to $\omega$ and $v$, and conclude that $\omega \geq v$ in $\Omega_M$. Recalling the free boundary condition in (1) yields

$$h'(t) = -\mu v_x\big(t, h(t)\big) \leq -\mu\omega_x\big(t, h(t)\big) \leq M_3 := 2\mu MM_2,$$

where $M_3$ is independent of $T$. The proof is completed. □

Theorem 2.2 shows that the free boundary is strictly monotonically increasing with time $t$, which indicates that the domain invaded by the predator $v$ is gradually expanding with time.

Moreover, the other inequalities in Theorem 2.2, in which $M_i$ is independent of $T$, imply that we can extend the solution of problem (1) to the global range. The proof of the following theorem is omitted here and the interested reader can refer to that of Theorem 2.3 in [17] or Theorem 2.2 in [28].

**Theorem 2.3** *Problem* (1) *admits a unique solution* $(u, v; h)$, *which exists globally in* $[0, \infty)$ *with respect to* $t$.

## 3 Long-time behaviour of $(u, v)$

To begin with, we present the definitions of spreading or vanishing of the predator population and the related comparison principle.

**Definition 3.1** If $h_\infty = \infty$ and $\liminf_{t\to+\infty} \|v(t,\cdot)\|_{C([0,h(t)])} > 0$, we call the predator *spreading*, which means that the predator can survive and spread to the whole domain $[0, \infty)$. While if $h_\infty < \infty$ and $\liminf_{t\to+\infty} \|v(t,\cdot)\|_{C([0,h(t)])} = 0$, we call the predator *vanishing*, which means that it will be maintained in a finite region and finally goes to extinction.

**Lemma 3.1** (Comparison principle) *Assume that* $T \in (0,\infty)$, $\overline{h} \in C^1[0,T]$, $\overline{u} \in C^{1,2}(D_T)$ *and* $\overline{v} \in C(\overline{G}_T) \cap C^{1,2}(G_T)$, *where*

$$D_T = \{(t,x) \in R^2, 0 < t \le T, 0 < x < \infty\},$$
$$G_T = \{(t,x) \in R^2, 0 < t \le T, 0 < x < \overline{h}(t)\}.$$

*If* $(\overline{u},\overline{v};\overline{h})$ *satisfies*

$$\begin{cases} \overline{u}_t - \overline{u}_{xx} \ge \overline{u}(1-\overline{u}), & 0 < t \le T, 0 < x < \infty, \\ \overline{v}_t - d\overline{v}_{xx} \ge \overline{v}(\frac{1}{a} + b - e), & 0 < t \le T, 0 < x < \overline{h}(t), \\ \overline{v}(t,\overline{h}(t)) = 0, & 0 < t \le T, \\ \overline{u}_x(t,0) \le 0, \qquad \overline{v}_x(t,0) \le 0, & 0 < t \le T, \\ \overline{h}'(t) \ge -\mu\overline{v}_x(t,\overline{h}(t)), \qquad \overline{h}(0) \ge h_0, & 0 < t \le T, \\ \overline{u}(0,x) \ge u_0(x), & 0 < x < \infty, \\ \overline{v}(0,x) \ge v_0(x), & 0 \le x \le h_0, \end{cases}$$

*then the solution* $(u,v,h)$ *of problem* (1) *satisfies*

$$h(t) \le \overline{h}(t), \quad 0 < t \le T,$$
$$u(t,x) \le \overline{u}(t,x), \quad 0 < t \le T, 0 < x < \infty,$$
$$v(t,x) \le \overline{v}(t,x), \quad 0 < t \le T, 0 \le x \le \overline{h}(t).$$

In fact, $(\overline{u},\overline{v},\overline{h})$ is called an upper solution of problem (1). We continue to exhibit the following two lemmas, whose proof can be referred to Lemma 3.3 in [29] and Theorem 4.1 in [27], respectively. We omit the details here.

**Lemma 3.2** *If* $h_\infty < \infty$, *then there exists a constant* $K > 0$ *such that the solution* $(u,v,h)$ *of* (1) *satisfies*

$$\|v(t,\cdot)\|_{C^1[0,h(t)]} \le K, \quad \forall t > 1; \qquad \lim_{t\to\infty} h'(t) = 0.$$

**Lemma 3.3** *Suppose that* $a^{-1} < e < b$. *If* $h_\infty < \infty$, *then* $h_\infty \le \Lambda$, *where* $\Lambda = \frac{\pi}{2}\sqrt{d/(b-e)}$.

The following theorem presents the sufficient condition for the vanishing of predator species.

**Theorem 3.1** *Suppose* $a^{-1} < e < b$ *and* $h_0 < \Lambda$, *then there exists* $\mu_1 > 0$ *such that* $h_\infty < \infty$ *when* $\mu \le \mu_1$, *and moreover, the solution* $(u,v,h)$ *of problem* (1) *satisfies*

$$\lim_{t\to\infty} \|v(t,\cdot)\|_{C([0,h(t)])} = 0, \tag{13}$$

$$\lim_{t\to\infty} u(t,\cdot) = 1 \quad \textit{uniformly on any compact subset of } [0,\infty). \tag{14}$$

*Proof* To apply Lemma 3.1, we will construct the suitable upper solution $(\overline{u},\overline{v};\overline{h})$. Define

$$\overline{h}(t) = h_0\left(1+\delta-\frac{\delta}{2}e^{-\sigma t}\right), \quad t \geq 0,$$

$$\overline{u}(t,x) = e^t\left(e^t-1+\frac{1}{\|u_0\|_\infty}\right)^{-1}, \quad t \geq 0,$$

$$\overline{v}(t,x) = Me^{-\delta t}V\left(\frac{x}{\overline{h}(t)}\right), \quad 0 \leq x \leq \overline{h}(t),$$

where $V(y) = \cos(\frac{\pi}{2}y)$, $\delta$ is sufficiently small and meets with $h_0(1+\delta) < \Lambda$, both $\sigma$ and $M$ are positive and to be determined later. Obviously, it follows that

$$\overline{u}_t' = \overline{u}(1-\overline{u}), \quad t > 0, \qquad \overline{u}(0) = \|u_0\|_\infty \leq u_0(x).$$

Meanwhile, we carry out some direct calculations

$$\overline{v}_t - d\overline{v}_{xx} - \overline{v}\left(\frac{1}{a}+b-e\right) \geq MVe^{-\sigma t}\left[\left(\frac{\pi}{2}\right)^2\frac{d}{(1+\delta)^2h_0^2} - \left(\frac{1}{a}+b-e\right) - \sigma\right]. \tag{15}$$

Once we choose

$$\sigma := \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2\frac{d}{(1+\delta)^2h_0^2} - \left(\frac{1}{a}+b-e\right)\right],$$

we will get (15) non-negative.

By choosing $M$ large enough, one has $\overline{v}(0,x) \geq v_0(x)$ for all $x \in [0,h_0]$. Additionally, there exists $\mu_1$ such that

$$\overline{h}'(t) \geq -\mu\overline{v}_x\bigl(t,\overline{h}(t)\bigr), \quad \forall\mu \leq \mu_1.$$

According to Lemma 3.1, it follows that $h(t) \leq \overline{h}(t)$, $u(t,x) \leq \overline{u}(t,x)$ and $v(t,x) \leq \overline{v}(t,x)$. Thus $h_\infty \leq \overline{h}(\infty) = h_0(1+\delta) < \infty$. Naturally, conclusion (13) can be deducted from Proposition 3.1 in [19] and Lemma 3.2. It remains to prove (14).

Since $\lim_{t\to\infty}\bar{u}(t) = 1$, by the comparison principle $u(t,x) \leq \bar{u}(t)$ for all $t \in [0,\infty)$ and $x \in [0,\infty)$, we have $\limsup_{t\to\infty} u(t,x) \leq 1$ uniformly in $[0,\infty)$.

On the other hand, with the help of conclusion (13) and the condition that $v(t,x) \equiv 0$ for $t > 0$ and $x \notin [0,h(t))$, we see that, for any given $0 < \varepsilon_1 \ll 1$, there exists $T_\sigma$ such that $v(t,x) < \varepsilon_1$ and

$$\frac{v}{1+au} < \frac{\varepsilon_1}{1+au} < \varepsilon_1,$$

for $t > T_\sigma, x \in [0,\infty)$. For any given $\varepsilon$ and $H > 0$, let $l_\varepsilon$ and $T_\varepsilon$ be determined by Lemma A.2 in [28], where $T_\varepsilon > T_\sigma$. Then the function $u(t,x)$ satisfies

$$\begin{cases} u_t - u_{xx} \geq u(1-u-\varepsilon_1), & t > T_\varepsilon, 0 < x < l_\varepsilon, \\ u_x(t,0) = 0, \quad u(t,l_\varepsilon) > 0, & t > T_\varepsilon, \\ u(T_\varepsilon,x) > 0. \end{cases}$$

Owing to Lemma A.2 in [28], $\liminf_{t\to\infty} u(t,x) \geq (1-\varepsilon_1) - \varepsilon$ uniformly on $[0,H]$. Then from the arbitrariness of the $\varepsilon$ and $H$ one derives that $\liminf_{t\to\infty} u(t,x) \geq 1-\varepsilon_1$ uniformly in the compact subset of $[0,\infty)$. Since $\varepsilon_1 > 0$ is arbitrary, we deduce that $\liminf_{t\to\infty} u(t,x) \geq 1$ uniformly on any bounded subset of $[0,\infty)$. The proof is complete. □

The following theorem exhibits the sufficient condition for the spreading of predator species.

**Theorem 3.2** *Suppose that* $a^{-1} < e < b$ *and* $h_0 < \Lambda$. *Then there exists* $\mu_2 > 0$ *such that* $h_\infty = \infty$ *when* $\mu > \mu_2$, *and, moreover, if* $b < \min\{ce^2(1-a), (1+c)(e-\frac{1}{1+a})\}$, *then the solution* $(u,v,h)$ *of problem* (1) *satisfies*

$$\lim_{t\to\infty} u(t,x) = u^*, \qquad \lim_{t\to\infty} v(t,x) = v^*,$$

*uniformly on any compact subset of* $[0,\infty)$, *where* $(u^*,v^*)$ *is the positive solution of the following system*:

$$\begin{cases} 1-u-\frac{v}{1+au} = 0, \\ \frac{u}{1+au} + \frac{b}{1+cv} - e = 0. \end{cases}$$

*Proof* The second equation in problem (1) implies

$$v_t - dv_{xx} \geq -ev, \quad 0 < x < h(t).$$

Considering the following auxiliary problem:

$$\begin{cases} w_t - dw_{xx} = -ew, & & t > 0, 0 < x < g(t), \\ w(t,g(t)) = 0, & & t > 0, \\ w(t,x) = 0, & g'(t) = -\mu w_x(t,x), & t > 0, x = g(t), \\ w(0,x) = v_0(x), & & 0 < x < h_0, \\ g(0) = h_0. & & \end{cases}$$

The comparison principle yields $g(t) \leq h(t)$ and $w(t,x) \leq v(t,x)$ on $[0,\infty) \times [0,h(t)]$. Similar to the proof of Lemma 3.2 in [27], there exists a constant $\mu_2 > 0$ such that $g(2) \geq 2\Lambda$ for all $\mu \geq \mu_2$. Therefore,

$$h_\infty = \lim_{t\to\infty} h(t) \geq \lim_{t\to\infty} g(t) \geq g(2) \geq 2\Lambda, \quad \forall \mu \geq \mu_2,$$

from which together with Lemma 3.3 one deduces that $h_\infty = \infty$.

This idea of the remainder proof comes from [28]. We will construct the following iteration sequences.

Step 1. The constructions of $\overline{u}_1$ and $\overline{v}_1$.

Set $M = \max\{M_1, M_2\}$, where $M_i$ is determined by Theorem 2.2, $i = 1,2$. For any given $L > 0$ and $0 < \varepsilon \ll 1$, there exist $l_\varepsilon > L$ and $T_1 > 0$, according to Lemma A.3 in [28], such that

$$\begin{cases} u_t - u_{xx} \le u(1-u), & t > T_1, 0 < x < l_\varepsilon, \\ u_x(t,0) = 0, \quad u(t,l_\varepsilon) \le M, & t > T_1. \end{cases}$$

Noting that $u(T_1,x) > 0$ in $[0,l_\varepsilon]$, we deduce from Lemma A.3 in [28] that $\lim_{t\to\infty} u(t,x) \le 1+\varepsilon$ uniformly on $[0,L]$. Then the arbitrariness of $\varepsilon$ and $L$ shows

$$\limsup_{t\to\infty} u(t,x) \le 1 := \overline{u}_1 \quad \text{uniformly on the compact subset of } [0,\infty). \tag{16}$$

Consequently, for any given $L > 0$, $0 < \delta, \varepsilon \ll 1$, we combine Lemma A.5 in [28] with (16), $h_\infty = \infty$, to find that there exist $l_\varepsilon > L$ and $T_2 > T_1$ such that

$$u(t,x) \le \overline{u}_1 + \delta, \qquad h(t) > l_\varepsilon, \quad \text{for } \forall t > T_2, 0 < x < l_\varepsilon.$$

Hence, $v$ obeys

$$\begin{cases} v_t - dv_{xx} \le v(\frac{b}{1+cv} - (e - \frac{\overline{u}_1+\varepsilon}{1+a(\overline{u}_1+\varepsilon)})), & t \ge T_2, 0 < x < l_\varepsilon, \\ v_x(t,0) = 0, \quad v(t,l_\varepsilon) \le M, & t > T_2. \end{cases}$$

Since $v(T_2,x) > 0$ in $[0,l_\varepsilon]$, we can apply Lemma A.5 in [28] to deduce $\lim_{t\to\infty} v(t,x) \le \frac{1}{c}(\frac{b}{e-\frac{\overline{u}_1+\delta}{1+a(\overline{u}_1+\delta)}} - 1) + \varepsilon$ uniformly on $[0,L]$. Because of the arbitrariness of $\varepsilon$, $\delta$ and $L$, we have

$$\begin{aligned} \limsup_{t\to\infty} v(t,x) &\le \frac{1}{c}\left(\frac{b}{e - \frac{\overline{u}_1}{1+a\overline{u}_1}} - 1\right) \\ &:= \overline{v}_1 \quad \text{uniformly on the compact subset of } [0,\infty). \end{aligned} \tag{17}$$

Meanwhile, the condition that $a^{-1} < e < b$ means that $0 < \overline{v}_{<}1$.

Step 2. The constructions of $\underline{u}_1$ and $\underline{v}_1$.

Let $l_\varepsilon$ be determined by Lemma A.2 in [28]. By (17), there is $T_3 > T_2$ such that

$$v(t,x) \le \overline{v}_1 + \delta, \quad \forall t > T_3, 0 < x < l_\varepsilon.$$

Thus, $u$ satisfies

$$\begin{cases} u_t - u_{xx} \ge u(1 - \overline{v}_1 - \delta - u), & t \ge T_3, 0 < x < l_\varepsilon, \\ u_x(t,0) = 0, \quad u(t,l_\varepsilon) \ge 0, & t \ge T_3. \end{cases}$$

Due to $u(T_3,x) > 0$ in $[0,l_\varepsilon]$, it follows from Lemma A.2 in [28] that $\liminf_{t\to\infty} u(t,x) \le 1 - \overline{v}_1 - \delta - \varepsilon$ uniformly on $[0,L]$. Similarly,

$$\liminf_{t\to\infty} u(t,x) \ge 1 - \overline{v}_1 := \underline{u}_1 \quad \text{uniformly on the compact subset of } [0,\infty). \tag{18}$$

Additionally, for any given $L > 0$, $0 < \delta, \varepsilon \ll 1$, owing to Lemma A.4 in [28], (18) and $h_\infty = \infty$, we know that there exist $l_\varepsilon > L$ and $T_4 > T_3$ such that

$$u(t,x) \ge \underline{u}_1 - \delta, \qquad h(t) > l_\varepsilon, \quad \forall t > T_4, 0 < x < l_\varepsilon.$$

Therefore, $v$ satisfies

$$\begin{cases} v_t - dv_{xx} \geq v(\frac{b}{1+cv} - (e - \frac{\underline{u}_1-\delta}{1+a(\underline{u}_1-\delta)})), & t > T_4, 0 < x < l_\varepsilon, \\ v_x(t,0) = 0, \qquad v(t,l_\varepsilon) \geq 0, & t > T_4. \end{cases}$$

As $v(T_4,x) > 0$ in $[0,l_\varepsilon]$, based on Lemma A.4 in [28], one can obtain

$$\liminf_{t\to\infty} v(t,x) \geq \frac{1}{c}\left(\frac{b}{e - \frac{\underline{u}_1-\delta}{1+a(\underline{u}_1-\delta)}} - 1\right) - \varepsilon,$$

uniformly on $[0,L]$. Similarly, we have

$$\liminf_{t\to\infty} v(t,x) \geq \frac{1}{c}\left(\frac{b}{e - \frac{\underline{u}_1}{1+a\underline{u}_1}} - 1\right) := \underline{v}_1,$$

uniformly on the compact subset of $[0,\infty)$.

Step 3. The constructions of $\underline{u}_2$ and $\underline{v}_2$.

For any given $L > 0$, $0 < \delta, \varepsilon \ll 1$, let $l_\varepsilon$ be selected by Lemma A.2 in [28]. With the help of (17) and (18), one acquire that there is $T_5 > T_4$ such that

$$v(t,x) \leq \bar{v}_1 + \delta, \qquad u(t,x) \geq \underline{u}_1 - \delta, \quad \forall t > T_5, 0 < x < l_\varepsilon,$$

which makes $u$ satisfy

$$\begin{cases} u_t - u_{xx} \geq u(1 - \frac{\bar{v}_1+\delta}{1+a(\underline{u}_1-\delta)} - u), & t > T_5, 0 < x < l_\varepsilon, \\ u_x(t,0) = 0, \qquad u(t,l_\varepsilon) \geq 0, & t > T_5. \end{cases}$$

Similar to Step 2, we have

$$\liminf_{t\to\infty} u(t,x) \geq 1 - \frac{\bar{v}_1}{1 + a\underline{u}_1} := \underline{u}_2 \quad \text{uniformly on the compact subset of } [0,\infty). \tag{19}$$

Next, for any given $L > 0$, $0 < \delta, \varepsilon \ll 1$, by Lemma A.4 in [28], (19) and $h_\infty = \infty$, there exists $T_6 > T_5$ such that

$$u(t,x) \geq \underline{u}_2 - \delta, \qquad h(t) > l_\varepsilon, \quad \forall t > T_6, 0 < x < l_\varepsilon.$$

Thereupon $v$ obeys

$$\begin{cases} v_t - dv_{xx} \geq v(\frac{b}{1+cv} - (e - \frac{\underline{u}_2-\delta}{1+a(\underline{u}_2-\delta)})), & t > T_6, 0 < x < l_\varepsilon, \\ v_x(t,0) = 0, \qquad v(t,l_\varepsilon) \geq 0, & t > T_6. \end{cases}$$

Similar to Step 2, we get

$$\liminf_{t\to\infty} v(t,x) \geq \frac{1}{c}\left(\frac{b}{e - \frac{\underline{u}_2}{1+a\underline{u}_2}} - 1\right) := \underline{v}_2 \quad \text{uniformly on the compact subset of } [0,\infty). \tag{20}$$

Step 4. The constructions of $\overline{u}_2$ and $\overline{v}_2$.

For any given $L > 0$, $0 < \delta, \varepsilon \ll 1$, let $l_\varepsilon$ be selected by Lemma A.3 in [28]. In view of (16) and (20), there is $T_7 > T_6$ such that

$$u(t,x) \le \overline{u}_1 + \delta, \qquad v(t,x) \ge \underline{v}_2 - \delta, \quad \forall t > T_7, 0 < x < l_\varepsilon,$$

which implies that

$$\begin{cases} u_t - u_{xx} \ge u(1 - \frac{\underline{v}_2 - \delta}{1+a(\overline{u}_1+\delta)} - u), & t > T_7, 0 < x < l_\varepsilon, \\ u_x(t,0) = 0, \qquad u(t,l_\varepsilon) \le M, & t > T_7. \end{cases}$$

Similar to Step 1, we deduce

$$\limsup_{t\to\infty} u(t,x) \ge 1 - \frac{\underline{v}_2}{1 + a\overline{u}_1} := \overline{u}_2 \quad \text{uniformly on the compact subset of } [0,\infty). \tag{21}$$

Accordingly, for any $L > 0$, $0 < \delta, \varepsilon \ll 1$, let $l_\varepsilon$ is given by Lemma A.5 in [28]. Recalling (21) and $h_\infty = \infty$, there is $T_8 > T_7$ such that

$$u(t,x) \le \overline{u}_2 + \delta, \qquad h(t) > l_\varepsilon, \quad \forall t \ge T_8, 0 < x < l_\varepsilon.$$

So it follows that

$$\begin{cases} v_t - dv_{xx} \le v(\frac{b}{1+cv} - (e - \frac{\overline{u}_2+\delta}{1+a(\overline{u}_2+\delta)})), & t > T_8, 0 < x < l_\varepsilon, \\ v_x(t,0) = 0, \qquad v(t,l_\varepsilon) \le M, & t > T_8. \end{cases}$$

Similar to Step 1 again, it follows that

$$\limsup_{t\to\infty} v(t,x) \le \frac{1}{c}\left(\frac{b}{e - \frac{\overline{u}_2}{1+a\overline{u}_2}} - 1\right) := \overline{v}_2$$

uniformly on the compact subset of $[0,\infty)$.

Step 5. Repeating the above procedure, we can find the four sequences $\{\overline{u}_i\}$, $\{\underline{u}_i\}$, $\{\overline{v}_i\}$ and $\{\underline{v}_i\}$ satisfying

$$\underline{u}_i \le \liminf_{t\to\infty} u(t,x) \le \limsup_{t\to\infty} u(t,x) \le \overline{u}_i,$$
$$\underline{v}_i \le \liminf_{t\to\infty} v(t,x) \le \limsup_{t\to\infty} v(t,x) \le \overline{v}_i,$$

uniformly on the compact subset of $[0,\infty)$, where

$$\overline{u}_i = 1 - \frac{\underline{v}_i}{1 + a\overline{u}_{i-1}}, \qquad \underline{u}_i = 1 - \frac{\overline{v}_i}{1 + a\underline{u}_{i-1}},$$
$$\overline{v}_i = \frac{1}{c}\left(\frac{b}{e - \frac{\overline{u}_i}{1+a\overline{u}_i}} - 1\right), \qquad \underline{v}_i = \frac{1}{c}\left(\frac{b}{e - \frac{\underline{u}_i}{1+a\underline{u}_i}} - 1\right).$$

It is easy to see that $\{\overline{u}_i\}$ and $\overline{v}_i$ are monotone non-increasing and bounded, $\{\underline{u}_i\}$ and $\{\underline{v}_i\}$ are monotone non-decreasing and bounded, so all of those limits exist, which are denoted by $\overline{u}$, $\underline{u}$, $\overline{v}$ and $\underline{v}$, respectively. Let $i \to \infty$, we deduce that

$$\begin{aligned} &\overline{u} = 1 - \frac{\underline{v}}{1 + a\overline{u}}, \qquad \underline{u} = 1 - \frac{\overline{v}}{1 + a\underline{u}}, \\ &\overline{v} = \frac{1}{c}\left(\frac{b}{e - \frac{\overline{u}}{1+a\overline{u}}} - 1\right), \qquad \underline{v} = \frac{1}{c}\left(\frac{b}{e - \frac{\underline{u}}{1+a\underline{u}}} - 1\right). \end{aligned} \tag{22}$$

We now claim that $\overline{u} = \underline{u}$. In fact, we infer by (22) that

$$\frac{b}{c} \cdot \frac{\overline{u} - \underline{u}}{[e(1 + a\underline{u}) - \underline{u}][e(1 + a\overline{u}) - \overline{u}]} = (\overline{u} - \underline{u})\bigl[1 + a(\overline{u} + \underline{u}) - a\bigr]. \tag{23}$$

If $\overline{u} \neq \underline{u}$, (23) gives that

$$\frac{b}{c} \cdot \frac{1}{[e(1 + a\underline{u}) - \underline{u}][e(1 + a\overline{u}) - \overline{u}]} = 1 + a(\overline{u} + \underline{u}) - a.$$

Under the condition that $ea > 1$, the left side is less than or equal to $\frac{b}{ce^2}$, but the right side is greater than $1 - a$, which leads to a contradiction to condition that $b < ce^2(1 - a)$. Therefore, $\overline{u} = \underline{u}$, and $\overline{v} = \underline{v}$ by (22). The proof is complete. □

## 4 Numerical analysis

In this section, we conduct numerical analysis to understand the role of generality of the predator in the invasion process. The above analytical analysis shows the existence of global solutions to the system (1) and further the conditions under which the invader can either spread or vanish as time advances. While these conditions qualitatively show the long-time behaviour of the prey and predator species, they are unfortunately not sufficiently clear because it is unknown when $h_\infty$ is finite or infinite. Understanding the long-time behaviour is numerically challenging, since the spatial domain is infinitely long. To circumvent this difficulty, we consider a spatial domain of finite range and focus on the invasion process of the early stage, which is thought to play a significant role in determining the ultimate fate of the predator species.

To this aim, we restrict the prey species to living in a spatial domain $x \in [0, L]$ and $L = 15$, and impose an homogeneous Neumann boundary condition at $x = L$. To minimise the impact of the right boundary on the potential results, we consider the invasion stage from $t = 0$ to $t = 15$, at the end of which the invasive species (predator) is far from reaching the right boundary. We employ the numerical scheme in [30] to perform numerical simulation. Moreover, we assume a relatively slow diffusion process by setting $d = 1.2$ to ensure that the predator will not reach the right boundary at $t = 15$. For the initial condition of the two species, we assume that, prior to the invasion, the prey species is homogeneously distributed in the spatial domain with an equilibrium density $u(x, 0) = 1$. The predator species starts to invade in a small area with a low density $u(x) = 0.01$, $x \in (0, h_0)$.

A graphical illustration of the invasion process is presented in Figure 1. The predator remains low in density for a certain period, and during this time period the spatial domain of

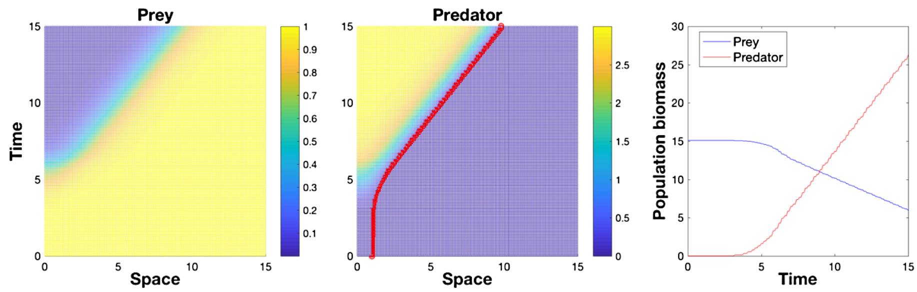


**Figure 1 The bigger predator's attack rate.** Graphical illustration of the invasion process of the generalist predator into the native prey species. The left two panels describe the spatio-temporal distribution of population density while the right-most panel shows population biomass. The bold red curve in the middle panel indicates the free boundary. Parameter values are $a = 3$, $b = 4$, $c = 1$, $e = 1$, $h_0 = 1$, $\mu = 0.8$.

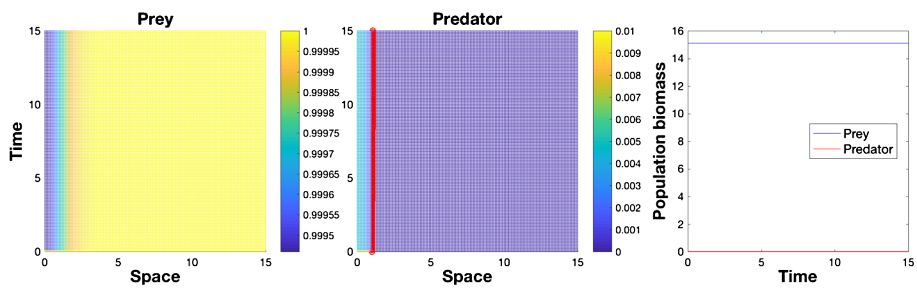


**Figure 2 The smaller predator's attack rate.** Graphical illustration of the invasion process of the generalist predator into the native prey species. The left two panels describe the spatio-temporal distribution of population density while the right-most panel shows population biomass. The bold red curve in the middle panel indicates the free boundary. Parameter values are $a = 3$, $b = 3$, $c = 1$, $e = 1$, $h_0 = 1$, $\mu = 0.8$.

the predator does not grow. After this transient period, population biomass of the predator starts to increase and the domain starts to grow linearly. Decreasing the generality of the predator by changing $b = 4$ to $b = 3$, we found that invasion is unsuccessful (Figure 2). This contrasting result shows that generalist predator is more likely to invade successfully. We tried longer simulation time but the conclusion remains unaltered. Numerical simulations also show that the size of the initial spatial domain of the predator has a significant impact on invasion. Figure 3 shows that increasing the size of the initial invading domain (*i.e.*, $h_0 = 1.2$) facilitates invasion and Figure 4 shows: the larger the better.

## 5 Conclusion

In summary, we consider an invasion scenario of a generalist predator into prey, which is formulated by a predator-prey model with free boundary. In our model (1), the so-called free boundary $x = h(t)$ characterises the change of the expanding front for the predator $v$, as well as the long-time behaviours of the predator $v$ and prey $u$ are focussed on. We conclude that the predator will gradually vanish if the limit of front function $h(t)$ is finite, while the predator will spread if the limit is infinite under some assumptions, and it further will stabilise to an equilibrium. These analytical findings indicate that the change of invasion region to predator can determine whether it successfully invade or not.

On the other hand, the numerical simulations of model (1) are also carried out. Our numerical results show that a generalist predator is more likely to succeed in invasion than

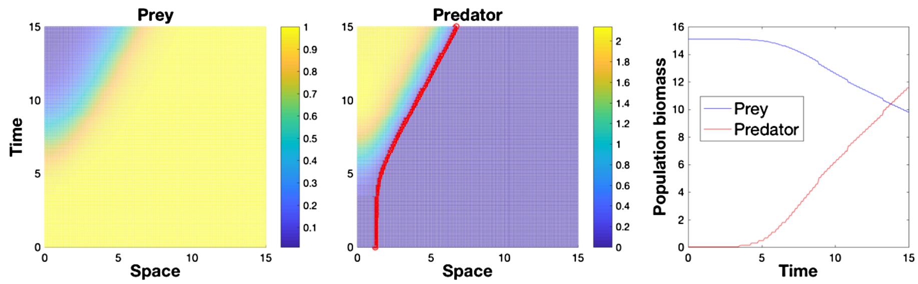


**Figure 3 The smaller size of the initial invading domain.** Graphical illustration of the invasion process of the generalist predator into the native prey species. The left two panels describe the spatio-temporal distribution of population density while the right-most panel shows population biomass. The bold red curve in the middle panel indicates the free boundary. Parameter values are $a = 3$, $b = 3$, $c = 1$, $e = 1$, $h_0 = 1.2$, $\mu = 0.8$.

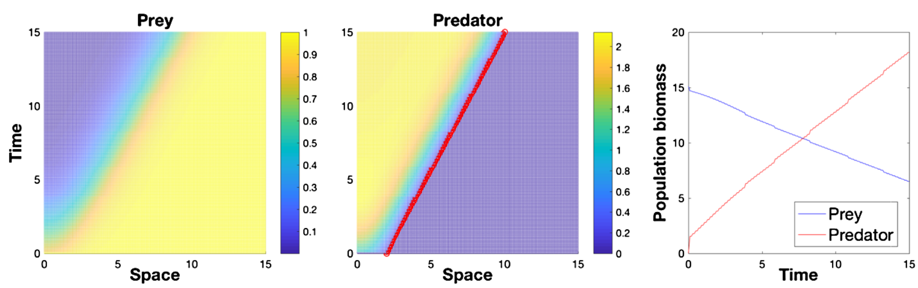


**Figure 4 The larger size of the initial invading domain.** Graphical illustration of the invasion process of the generalist predator into the native prey species. The left two panels describe the spatio-temporal distribution of population density while the right-most panel shows population biomass. The bold red curve in the middle panel indicates the free boundary. Parameter values are $a = 3$, $b = 3$, $c = 1$, $e = 1$, $h_0 = 2$, $\mu = 0.8$.

a specialist predator. There is a threshold size of the initial spatial domain for a successful invasion below which invasion can fail. Moreover, we also noticed that there is a time period during which invasive specie increases in density slowly. The existence of such a time period implies that invasion can possibly be unsuccessful if we take into account stochastic effects. We conclude that the more general the invasive species, the more likely to be successful the invasion is.

The free boundary in model (1) describes the one-dimensional environment. With regard to realities of situation, however, we realize the two or three-dimensional case to more match the reality. Naturally, there will be more challenges in mathematical analysis and numerical simulation for multi-dimensional free boundary, and we will pay more attention to these questions in future work.

An interesting extension is to explicitly consider the dynamics of the alternative food and investigate how the competition between two preys affect alien invasion. A promising extension is to make an application of the above results to test how such a kind of model with a free boundary problem can be used to solve real problems. Consideration of invasion from a distant place through the boundary of the concerned domain can be modelled accurately only with the help of spatio-temporal models with free boundary conditions.

**Acknowledgements**
Zhi Ling and Lai Zhang gratefully acknowledge the financial support by the PRC Grant NSFC 11571301.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 25 May 2017 Accepted: 11 September 2017 Published online: 25 September 2017

**References**

1. Meyerson, LA, Mooney, HA: Invasive alien species in an era of globalization. Front. Ecol. Environ. **5**(4), 199-208 (2007)
2. Pimentel, D, Zuniga, R, Morrison, D: Update on the environmental and economic costs associated with alien-invasive species in the United States. Ecol. Econ. **52**(3), 273-288 (2005)
3. Stohlgren, TJ, Schnase, JL: Risk analysis for biological hazards: what we need to know about invasive species. Risk Anal. **26**(1), 163-173 (2006)
4. Wilcove, DS, Rothstein, D, Dubow, J, Phillips, A, Lososand, E: Quantifying threats to imperiled species in the United States. Bioscience **80**(2), 607-615 (1998)
5. Armstrong, S: Rare plants protect Cape's water supplies. New Sci. **145**(1964), 8-8 (1995)
6. Cao, JF, Li, WT, Zhao, M: A prey-predator model with a free boundary and sign-changing coefficient in time-periodic environment. Discrete Contin. Dyn. Syst., Ser. B **22**(9), 3295-3316 (2017)
7. Frunzo, L, Mattei, MR: Qualitative analysis of the invasion free boundary problem in biofilms. Ric. Mat. **66**(1), 171-188 (2017)
8. Li, CL: A free boundary problem for a ratio-dependent diffusion predator-prey system. Bound. Value Probl. **2016**, 192 (2016)
9. Tian, CR, Ruan, SG: A free boundary problem for Aedes aegypti mosquito invasion. Appl. Math. Model. **46**, 203-217 (2017)
10. Zhou, L, Zhang, S, Liu, ZH: A free boundary problem of a predator-prey model with advection in heterogeneous environment. Appl. Math. Comput. **289**, 22-36 (2016)
11. Tao, YS, Chen, MJ: An elliptic-hyperbolic free boundary problem modelling cancer therapy. Nonlinearity **19**(2), 419-440 (2006)
12. Chen, XF, Friedman, A: A free boundary problem arising in a model of wound healing. SIAM J. Math. Anal. **32**(4), 778-800 (2000)
13. Merz, W, Rybka, P: A free boundary problem describing reaction-diffusion problems in chemical vapor infiltration of pyrolytic carbon. J. Math. Anal. Appl. **292**(2), 571-588 (2004)
14. Kim, KI, Lin, ZG, Zhang, QY: An *SIR* epidemic model with free boundary. Nonlinear Anal., Real World Appl. **14**(5), 1992-2001 (2013)
15. Lin, ZG: A free boundary problem for a predator-prey model. Nonlinearity **20**(8), 1883-1892 (2007)
16. Du, YH, Guo, ZM: Spreading-vanishing dichotomy in a diffusive logistic model with a free boundary, II. J. Differ. Equ. **250**(12), 4336-4366 (2011)
17. Du, YH, Lin, ZG: Spreading-vanishing dichotomy in the diffusive model with a free boundary. SIAM J. Math. Anal. **42**(1), 377-405 (2010)
18. Ling, Z, Tang, QL, Lin, ZG: A free boundary problem for two-species competitive model in ecology. Nonlinear Anal., Real World Appl. **11**(3), 1775-1781 (2010)
19. Wang, MX: On some free boundary problems of the prey-predator model. J. Differ. Equ. **256**(10), 3365-3394 (2014)
20. Wang, MX: The diffusive logistic equation with a free boundary and sign-changing coefficient. J. Differ. Equ. **258**(4), 1252-1266 (2015)
21. Monobe, H, Wu, CH: On a free boundary problem for a reaction-diffusion-advection logistic model in heterogeneous environment. J. Differ. Equ. **261**(11), 6144-6177 (2016)
22. Prasad, BSRV, Banerjee, M, Srinivasu, PDN: Dynamics of additional food provided predator-prey system with mutually interfering predators. Math. Biosci. **246**(1), 176-190 (2013)
23. van Baalen, M, Krivan, V, van Rijn, PCJ, Sabelis, MW: Alternative food, switching predators, and the persistence of predator-prey systems. Am. Nat. **157**(5), 512-524 (2001)
24. Eubanks, MD, Denno, RF: Host plants mediate omnivore-herbivore interactions and influence prey suppression. Ecology **81**(4), 936-947 (2000)
25. Rijn, PCV, Sabelis, MW: Does alternative food always enhance biological control? The effect of pollen on the interaction between western flower thrips and its predators. Transplant. Proc. **12**(4), 123-125 (1993)
26. Rijn, PCV, Tanigoshi, LK: The contribution of extrafloral nectar to survival and reproduction of the predatory mite Iphiseius degenerans on Ricinus communis. Exp. Appl. Acarol. **23**(4), 281-296 (1999)

27. Wang, MX, Zhao, JF: A free boundary problem for a predator-prey model with double free boundaries. J. Dyn. Differ. Equ. **29**(3), 957-979 (2017) doi:10.1007/s10884-015-9503-5
28. Zhang, Y, Wang, MX: A free boundary problem of the ratio-dependent prey-predator model. Appl. Anal. **94**(10), 2147-2167 (2015)
29. Guo, JS, Wu, CH: On a free boundary problem for a two-species weak competition system. J. Dyn. Differ. Equ. **24**(4), 873-895 (2012)
30. Razvan, S, Gabriel, D: Numerical approximation of a free boundary problem for a predator-prey model. Numer. Anal. Appl. **5434**, 548-555 (2009)