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# On the existence of solutions for some infinite coefficient-symmetric Caputo-Fabrizio fractional integro-differential equations

Dumitru Baleanu[1,2*], Asef Mousalou[3] and Shahram Rezapour[3]

*Correspondence:
dumitru@cankaya.edu.tr
[1]Department of Mathematics, Cankaya University, Ogretmenler Cad., Balgat, Ankara, 06530, Turkey
[2]Institute of Space Sciences, Magurele-Bucharest, Bucharest, Romania
Full list of author information is available at the end of the article

**Abstract**
By mixing the idea of 2-arrays, continued fractions, and Caputo-Fabrizio fractional derivative, we introduce a new operator entitled the infinite coefficient-symmetric Caputo-Fabrizio fractional derivative. We investigate the approximate solutions for two infinite coefficient-symmetric Caputo-Fabrizio fractional integro-differential problems. Finally, we analyze two examples to confirm our main results.

**MSC:** 30B70; 34A08

**Keywords:** continued fraction; contraction map; fractional integro-differential equation; infinite Caputo-Fabrizio fractional derivative

## 1 Introduction

Fractional calculus has many real-world applications in various fields of science and engineering [1–10]. During the recent years, the researchers started to think how to enlarge the range of fractional calculus by constructing operators with different nonlocal kernels. For example, a new nonlocal derivative without singular kernel was introduced in [11]. After that, this new fractional operator was utilized to get more information from solving different fractional differential equations corresponding to complex phenomena (the reader can see, for example, [11–20], and the references therein). Let use consider $b > 0$ and $x \in H^1(0,b)$ together with $\alpha \in (0,1)$. For a function $x$, Caputo and Fabrizio defined its fractional derivative (CF) of order $\alpha$ as ${}^{\mathrm{CF}}C^{\alpha}x(p) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^p \exp(\frac{-\alpha}{1-\alpha}(p-w))x'(w)\,dw$, where $t \geq 0$, and $M(\alpha)$ is such that $M(0) = M(1) = 1$ [11]. The corresponding fractional integral of order $\alpha$ for the function $x$ is ${}^{\mathrm{CF}}I^{\alpha}x(p) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}x(p) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^p x(w)\,dw$ whenever $0 < \alpha < 1$ [21]. Also, the values of the function $M$ were found as $M(\alpha) = \frac{2}{2-\alpha}$ for all $0 \leq \alpha \leq 1$ [21]. Taking into account the results mentioned, for a given function $x$, its fractional CF of order $\alpha$ becomes ${}^{\mathrm{CF}}C^{\alpha}x(p) = \frac{1}{1-\alpha}\int_0^p \exp(-\frac{\alpha}{1-\alpha}(p-w))x'(w)\,dw$ for $t \geq 0$ and $0 < \alpha < 1$ [21]. In this way a new type of fractional calculus was established. The aim of the manuscript is to propose a new operator named the infinite coefficient-symmetric Caputo-Fabrizio fractional derivative and to study some its properties.

## 2 Basic tools and new fractional operators

We further introduce some basic notation.

**Lemma 2.1** ([21]) *Let us consider the equation* ${}^{\mathrm{CF}}C^{\alpha}x(t) = y(t)$ *such that* $x(0) = c$ *and* $0 < \alpha < 1$. *The solutions of this equation has the form* $x(p) = c + a_\alpha(y(p) - y(0)) + b_\alpha \int_0^p y(z)\,dz$, *where* $a_\alpha = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} = 1 - \alpha$ *and* $b_\alpha = \frac{2\alpha}{(2-\alpha)M(\alpha)} = \alpha$.

Let $\varepsilon > 0$. We consider a metric space $(Z, d_1)$, a selfmap $G$ on $Z$, and a mapping $\alpha : Z \times Z \to [0, \infty)$. As a result, we say that $G$ is $\alpha$-admissible whenever $\alpha(t,s) \geq 1$ implies $\alpha(Gt, Gs) \geq 1$ [22]. An element $z_0 \in Z$ is called an $\varepsilon$-fixed point of $G$ if $d(Gz_0, z_0) \leq \varepsilon$. We say that $G$ possess the approximate fixed point property if $G$ possesses an $\varepsilon$-fixed point for all $\varepsilon > 0$ [22]. Denote by $\mathcal{R}$ the set of all continuous mappings $j : [0,\infty)^5 \to [0,\infty)$ satisfying $j(1,1,1,2,0) = j(1,1,1,0,2) := l \in (0,1)$, $j(\mu t_1, \mu t_2, \mu t_3, \mu t_4, \mu t_5) \leq \mu j(t_1, t_2, t_3, t_4, t_5)$ for all $(t_1, t_2, t_3, t_4, t_5) \in [0,\infty)^5$ and $\mu \geq 0$ and also $j(t_1, t_2, t_3, 0, t_4) \leq j(s_1, s_2, s_3, 0, s_4)$ and $j(t_1, t_2, t_3, t_4, 0) \leq j(s_1, s_2, s_3, s_4, 0)$ whenever $t_1, \ldots, t_4, s_1, \ldots, s_4 \in [0,\infty)$ with $t_k < s_k$ for $k = 1,2,3,4$ [22]. Next, we recall that $G$ is called a generalized $\alpha$-contractive mapping if there exists $j \in \mathcal{R}$ such that $\alpha(t,s)d_1(Gt, Gs) \leq j(d_1(t_1, s_1), d_1(t_1, Gt_1), d_1(s_1, Gs_1), d_1(t_1, Gs_1), d_1(s_1, Gt_1))$ for all $t_1, s_1 \in Z$ [22]. We need the following key result.

**Theorem 2.2** ([22]) *Suppose that there exists* $t_0 \in Z$ *such that* $\alpha(t_0, Gt_0) \geq 1$. *Then G possesses an approximate fixed point, where* $(Z, d)$ *is a metric space,* $\alpha : Z \times Z \to [0, \infty)$ *denotes a mapping, and G represents a generalized* $\alpha$*-contractive and* $\alpha$*-admissible selfmap on* $Z$.

Let $\{L_{i,2^i}\}_{i\geq 1}$ be a sequence of operators on a set. For reduction and approximation in large and infinite potential-driven flow networks, there is a method of using 2-arrays and continued fractions (see [23] and [24]). In fact, it is sufficient to arrange the operators $\{L_{i,2^i}\}_{i\geq 1}$ symmetrically on a 2-array, and by using a continued fraction we make a new operator $L_N$ from the operators $L_{i,2^i}$, where $N$ is a natural number (see [23] and [24]). First, we arrange the operators $L_{i,2^i}$ on a 2-array (tree) as in Figure 1 (see [23]).

Now, using a finite continued fraction, consider the new operator $L_N$ defined by

$$L_N = \cfrac{1}{\cfrac{1}{L_{11} + \cfrac{1}{\cfrac{1}{L_{21} + \cdots + \cfrac{1}{\frac{1}{L_{N1}} + \frac{1}{L_{N2}}}} + \cfrac{1}{L_{22} + \cdots + \cfrac{1}{\frac{1}{L_{N3}} + \frac{1}{L_{N4}}}}}} + \cfrac{1}{L_{12} + \cfrac{1}{\cfrac{1}{L_{23} + \cdots + \cfrac{1}{\frac{1}{L_{N2N-3}} + \frac{1}{L_{N2N-2}}}} + \cfrac{1}{L_{24} + \cdots + \cfrac{1}{\frac{1}{L_{N2N-1}} + \frac{1}{L_{N2N}}}}}}}.$$

Here, we replace symmetrically the operators $L_{ij}$ with ${}^{\mathrm{CF}}C^{\alpha}$ for $j$ odd (the upper branch) and ${}^{\mathrm{CF}}C^{\beta}$ for $j$ even (the lower branch) as in Figure 2.

**Figure 1 An *N* generation tree network composed of the operators $L_{i,2^i}$.**

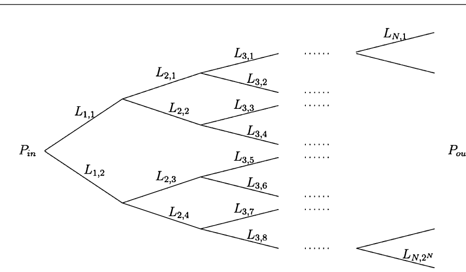


**Figure 2 A symmetric generation tree network composed of the operators ${}^{\mathrm{CF}}C^{\alpha}$ and ${}^{\mathrm{CF}}C^{\beta}$.**

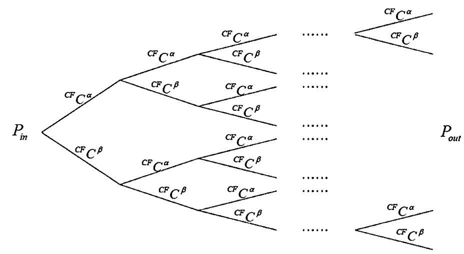
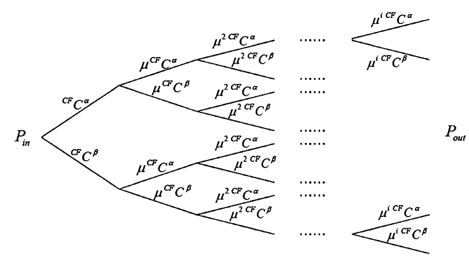



**Figure 3 A coefficient-symmetric generation tree composed of the operators ${}^{\mathrm{CF}}C^{\alpha}$ and ${}^{\mathrm{CF}}C^{\beta}$.**




Put

$$ {}^{\mathrm{CF}}\mathbb{C}_{1}^{(\alpha,\beta)}=\frac{1}{\frac{1}{{}^{\mathrm{CF}}C^{\alpha}}+\frac{1}{{}^{\mathrm{CF}}C^{\beta}}},\qquad {}^{\mathrm{CF}}\mathbb{C}_{2}^{(\alpha,\beta)}=\frac{1}{\frac{1}{{}^{\mathrm{CF}}C^{\alpha}+\frac{1}{\frac{1}{{}^{\mathrm{CF}}C^{\alpha}}+\frac{1}{{}^{\mathrm{CF}}C^{\beta}}}}+\frac{1}{{}^{\mathrm{CF}}C^{\beta}+\frac{1}{\frac{1}{{}^{\mathrm{CF}}C^{\alpha}}+\frac{1}{{}^{\mathrm{CF}}C^{\beta}}}}} $$

and

$$ {}^{\mathrm{CF}}\mathbb{C}_{3}^{(\alpha,\beta)} = \frac{1}{\frac{1}{{}^{\mathrm{CF}}C^{\alpha}+\frac{1}{\frac{1}{{}^{\mathrm{CF}}C^{\alpha}+\frac{1}{\frac{1}{{}^{\mathrm{CF}}C^{\alpha}}+\frac{1}{{}^{\mathrm{CF}}C^{\beta}}}}+\frac{1}{{}^{\mathrm{CF}}C^{\beta}+\frac{1}{\frac{1}{{}^{\mathrm{CF}}C^{\alpha}}+\frac{1}{{}^{\mathrm{CF}}C^{\beta}}}}}}+\frac{1}{{}^{\mathrm{CF}}C^{\beta}+\frac{1}{\frac{1}{{}^{\mathrm{CF}}C^{\alpha}+\frac{1}{\frac{1}{{}^{\mathrm{CF}}C^{\alpha}}+\frac{1}{{}^{\mathrm{CF}}C^{\beta}}}}+\frac{1}{{}^{\mathrm{CF}}C^{\beta}+\frac{1}{\frac{1}{{}^{\mathrm{CF}}C^{\alpha}}+\frac{1}{{}^{\mathrm{CF}}C^{\beta}}}}}}}. $$

Continuing this process, we can define the new operator ${}^{\mathrm{CF}}\mathbb{C}_{N}^{(\alpha,\beta)}$. Now, we define the infinite symmetric CF fractional derivative by ${}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\alpha,\beta)}=\lim_{N\to\infty}{}^{\mathrm{CF}}\mathbb{C}_{N}^{(\alpha,\beta)}$. A simple calculation shows that ${}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\alpha,\beta)}=({}^{\mathrm{CF}}C^{\alpha}\,{}^{\mathrm{CF}}C^{\beta})^{\frac{1}{2}}$. Similarly, we can define the infinite symmetric CF fractional integral ${}^{\mathrm{CF}}\mathbb{I}_{\infty}^{(\alpha,\beta)}$ by

$$ {}^{\mathrm{CF}}\mathbb{I}_{\infty}^{(\alpha,\beta)}=\frac{1}{\frac{1}{{}^{\mathrm{CF}}I^{\alpha}+\frac{1}{\frac{1}{{}^{\mathrm{CF}}I^{\alpha}+\cdots}+\frac{1}{{}^{\mathrm{CF}}I^{\beta}+\cdots}}}+\frac{1}{{}^{\mathrm{CF}}I^{\beta}+\frac{1}{\frac{1}{{}^{\mathrm{CF}}I^{\alpha}+\cdots}+\frac{1}{{}^{\mathrm{CF}}I^{\beta}+\cdots}}}}. $$

Let $\mu\geq 0$, $\mu\neq 2$. Putting $\mu^{i-1}\,{}^{\mathrm{CF}}C^{\alpha}$ on the upper branch and $\mu^{i-1}\,{}^{\mathrm{CF}}C^{\beta}$ on the lower branch in the $i$th stage as in Figure 3, we can make the infinite coefficient-symmetric CF fractional derivative as a generalization for last case.

In fact, we define

$$ {}^{\mathrm{CF}}\mathbb{C}_{(\mu,\infty)}^{(\alpha,\beta)}=\frac{1}{\frac{1}{{}^{\mathrm{CF}}C^{\alpha}+\frac{1}{\frac{1}{\mu\,{}^{\mathrm{CF}}C^{\alpha}+\cdots}+\frac{1}{\mu\,{}^{\mathrm{CF}}C^{\beta}+\cdots}}}+\frac{1}{{}^{\mathrm{CF}}C^{\beta}+\frac{1}{\frac{1}{\mu\,{}^{\mathrm{CF}}C^{\alpha}+\cdots}+\frac{1}{\mu\,{}^{\mathrm{CF}}C^{\beta}+\cdots}}}}, $$

and so

$$ {}^{\mathrm{CF}}\mathbb{C}^{(\alpha,\alpha)}_{(\mu,\infty)} = \frac{1}{\frac{1}{{}^{\mathrm{CF}}C^{\alpha}+\mu\,{}^{\mathrm{CF}}\mathbb{C}^{(\alpha,\alpha)}_{(\mu,\infty)}} + \frac{1}{{}^{\mathrm{CF}}C^{\alpha}+\mu\,{}^{\mathrm{CF}}\mathbb{C}^{(\alpha,\alpha)}_{(\mu,\infty)}}}. $$

This implies that

$$ (*) \qquad {}^{\mathrm{CF}}\mathbb{C}^{(\alpha,\beta)}_{(\mu,\infty)} = \frac{1}{2-\mu}\,{}^{\mathrm{CF}}C^{\alpha}. $$

## 3 Results

To show our results, we recall below two lemmas [15] under the assumption that $x, y \in H^1(0,1)$.

**Lemma 3.1** ([15]) *If there exists a real number* $K_1$ *such that* $|x(p) - y(p)| \leq K_1$ *for all* $p \in [0,1]$, *then* $|{}^{\mathrm{CF}}C^{\alpha}x(p) - {}^{\mathrm{CF}}C^{\alpha}y(p)| \leq \frac{2-\alpha}{(1-\alpha)^2}K_1$ *for all* $p \in [0,1]$.

**Lemma 3.2** ([15]) *Assume that* $x(0) = y(0)$ *and there exists a real number* $K_1$ *such that* $|x(p) - y(p)| \leq K_1$ *for* $p \in [0,1]$. *Then* $|{}^{\mathrm{CF}}C^{\alpha}x(p) - {}^{\mathrm{CF}}C^{\alpha}y(p)| \leq \frac{1}{(1-\alpha)^2}K_1$ *for all* $p \in [0,1]$.

Let $x, y \in C_{\mathbb{R}}[0,1]$.

**Lemma 3.3** ([15]) *If there is* $K_1 \geq 0$ *such that* $|x(p) - y(p)| \leq K_1$ *for all* $p \in [0,1]$, *then* $|{}^{\mathrm{CF}}I^{\alpha}x(p) - {}^{\mathrm{CF}}I^{\alpha}y(p)| \leq K_1$ *for* $p \in [0,1]$.

Now we are ready to show our main results. Using Lemmas 3.1 and 3.2, we obtain the next key results.

**Lemma 3.4** *Let* $x, y \in H^1$. *If there exists a real number* $K_1$ *such that* $|x(p) - y(p)| \leq K_1$ *for all* $p \in [0,1]$, *then* $|{}^{\mathrm{CF}}\mathbb{C}^{(\alpha,\alpha)}_{\infty}x(p) - {}^{\mathrm{CF}}\mathbb{C}^{(\alpha,\alpha)}_{\infty}y(p)| \leq \frac{2-\alpha}{(1-\alpha)^2}K_1$ *for all* $p \in [0,1]$.

**Lemma 3.5** *Let* $x, y \in H^1$ *with* $x(0) = y(0)$ *and* $K_1 \in \mathbb{R}$. *If* $|x(p) - y(p)| \leq K_1$ *for* $p \in [0,1]$, *then* $|{}^{\mathrm{CF}}\mathbb{C}^{(\alpha,\alpha)}_{\infty}x(p) - {}^{\mathrm{CF}}\mathbb{C}^{(\alpha,\alpha)}_{\infty}y(p)| \leq \frac{1}{(1-\alpha)^2}K_1$ *for all* $p \in [0,1]$.

Using Lemmas 3.4 and 3.5 and (*), we get the following results.

**Lemma 3.6** *Let* $x, y \in H^1$. *If there exists a real number* $K_1$ *such that* $|x(p) - y(p)| \leq K_1$ *for all* $p \in [0,1]$, *then* $|{}^{\mathrm{CF}}\mathbb{C}^{(\alpha,\alpha)}_{(\mu,\infty)}x(p) - {}^{\mathrm{CF}}\mathbb{C}^{(\alpha,\alpha)}_{(\mu,\infty)}y(p)| \leq \frac{(2-\alpha)}{(2-\mu)(1-\alpha)^2}K_1$ *for all* $p \in [0,1]$.

**Lemma 3.7** *Let* $x, y \in H^1$ *with* $x(0) = y(0)$ *and* $K_1 \in \mathbb{R}$. *If* $|x(p) - y(p)| \leq K_1$ *for all* $p \in [0,1]$, *then* $|{}^{\mathrm{CF}}\mathbb{C}^{(\alpha,\alpha)}_{(\mu,\infty)}x(p) - {}^{\mathrm{CF}}\mathbb{C}^{(\alpha,\alpha)}_{(\mu,\infty)}y(p)| \leq \frac{1}{(2-\mu)(1-\alpha)^2}K_1$ *for all* $p \in [0,1]$.

**Lemma 3.8** *Let* $x, y \in C_{\mathbb{R}}[0,1]$. *Let* $K_1$ *be a real number such that* $|x(p) - y(p)| \leq K_1$ *for all* $p \in [0,1]$, *then* $|{}^{\mathrm{CF}}\mathbb{I}^{(\alpha,\alpha)}_{\infty}x(p) - {}^{\mathrm{CF}}\mathbb{I}^{(\alpha,\alpha)}_{\infty}y(p)| \leq K_1$ *for all* $p \in [0,1]$.

Using Lemma 2.1, we can prove the next key result.

**Lemma 3.9** *Let* $\alpha \in (0,1)$ *and* $c \in \mathbb{R}$. *The unique solution of the problem*

$$ {}^{\mathrm{CF}}\mathbb{C}^{(\alpha,\alpha)}_{\infty}x(p) = y(p) $$

*with boundary condition* $x(0) = c$ *is given by* $x(p) = c + a_{\alpha}(y(p) - y(0)) + b_{\alpha}\int_0^t y(s)\,ds$.

Also, using Lemma 2.1 and (*), we can prove the next key result.

**Lemma 3.10** *Let $\alpha \in (0,1)$ and $c \in \mathbb{R}$. The unique solution of the problem*

$$ {}^{\mathrm{CF}}\mathbb{C}^{(\alpha,\alpha)}_{(\mu,\infty)}x(p) = y(p) $$

*with boundary condition $x(0) = c$ is given by*

$$ x(p) = c + a_\alpha(2-\mu)\big(y(p) - y(0)\big) + b_\alpha(2-\mu)\int_0^p y(s)\,ds. $$

Let $I = [0,1]$, and let $\gamma, \lambda : [0,1] \times [0,1] \to [0,\infty)$ be two continuous maps such that $\sup_{p\in I} |\int_0^p \lambda(p,r)\,dr| < \infty$ and $\sup_{p\in I} |\int_0^p \gamma(p,r)\,dr| < \infty$. We introduce the following maps $\phi$ and $\varphi$ defined by $(\phi u)(p) = \int_0^p \gamma(p,r)u(r)\,dr$ and $(\varphi u)(p) = \int_0^p \lambda(p,r)u(r)\,dr$, respectively. Let us consider $\gamma_0 = \sup |\int_0^p \gamma(p,r)\,dr|$ and $\lambda_0 = \sup |\int_0^p \lambda(p,r)\,dr|$, respectively. Let $\eta(p) \in L^\infty(I)$ with $\eta^* = \sup_{p\in I} |\eta(p)|$. We further are going to investigate the infinite CF fractional integro-differential problem, namely

$$ \begin{aligned} {}^{\mathrm{CF}}\mathbb{C}^{(\alpha,\alpha)}_{\infty}u_1'(r) &= \mu\big({}^{\mathrm{CF}}\mathbb{C}^{(\beta,\beta)}_{\infty}u_1'(r) + {}^{\mathrm{CF}}\mathbb{C}^{(\gamma,\gamma)}_{\infty}u_1'(r)\big) \\ &\quad + f'\big(r, u_1'(r), (\phi u_1')(r), (\varphi u_1')(r), {}^{\mathrm{CF}}\mathbb{I}^{(\theta,\theta)}_{\infty}u_1'(r), {}^{\mathrm{CF}}\mathbb{C}^{(\delta,\delta)}_{\infty}u_1'(r)\big) \end{aligned} \tag{1} $$

with $u_1'(0) = 0$. Here $\alpha, \beta, \gamma, \theta, \delta \in (0,1)$, and $\mu \geq 0$.

**Theorem 3.11** *Let $f' : [0,1] \times \mathbb{R}^5 \to \mathbb{R}$ be a continuous function satisfying*

$$ \begin{aligned} &\big|f'(r, x_1, y_1, w_1, u_1, u_2) - f'\big(r, x_1', y_1', w_1', v_1, v_2\big)\big| \\ &\quad \leq \eta(r)\big(|x_1 - x_1'| + |y_1 - y_1'| + |w_1 - w_1'| + |u_1 - v_1| + |u_2 - v_2|\big) \end{aligned} $$

*for all $r \in I$ and $x_1, y_1, w_1, x_1', y_1', w_1', u_1, u_2, v_1, v_2 \in \mathbb{R}$. If $\Delta = [\eta^*(2 + \gamma_0 + \lambda_0 + \frac{1}{(1-\delta)^2}) + \mu(\frac{1}{(1-\gamma)^2} + \frac{1}{(1-\beta)^2})] < 1$, then problem (1) possesses an approximate solution.*

*Proof* Let $H^1$ be equipped with $d(u_1', v_1') = \|u_1' - v_1'\|$, where $\|u_1'\| = \sup_{t\in I} |u_1'(t)|$. Now, consider the selfmap $F : H^1 \to H^1$ defined by

$$ \begin{aligned} \big(Fu_1'\big)(r) &= a_\alpha\big[\mu\big({}^{\mathrm{CF}}\mathbb{C}^{(\beta,\beta)}_{\infty}u_1'(r) + {}^{\mathrm{CF}}\mathbb{C}^{(\gamma,\gamma)}_{\infty}u_1'(r)\big) \\ &\quad + f'\big(r, u_1'(r), (\phi u_1')(r), (\varphi u_1')(r), {}^{\mathrm{CF}}\mathbb{I}^{(\theta,\theta)}_{\infty}u_1'(r), {}^{\mathrm{CF}}\mathbb{C}^{(\delta,\delta)}_{\infty}u_1'(r)\big)\big] \\ &\quad + b_\alpha\int_0^r \big[\mu\big({}^{\mathrm{CF}}\mathbb{C}^{(\beta,\beta)}_{\infty}u_1'(s) + {}^{\mathrm{CF}}\mathbb{C}^{(\gamma,\gamma)}_{\infty}u_1'(s)\big) \\ &\quad + f'\big(s, u_1'(s), (\phi u_1')(s), (\varphi u_1')(s), {}^{\mathrm{CF}}\mathbb{I}^{(\theta,\theta)}_{\infty}u_1'(r), {}^{\mathrm{CF}}\mathbb{C}^{(\delta,\delta)}_{\infty}u_1'(s)\big)\big]\,ds \end{aligned} $$

for all $r \in I$ and $u_1', v_1' \in H^1$, where $a_\alpha$ and $b_\alpha$ have the meaning given in Lemma 3.9. Now, utilizing Lemmas 3.5 and 3.8, we get

$$ \begin{aligned} &\big|\big(Fu_1'\big)(r) - \big(Fv_1'\big)(r)\big| \\ &\quad \leq a_\alpha\big(\mu\big|\big({}^{\mathrm{CF}}\mathbb{C}^{(\beta,\beta)}_{\infty}u_1'(r) + {}^{\mathrm{CF}}\mathbb{C}^{(\gamma,\gamma)}_{\infty}u_1'(r)\big) - \big({}^{\mathrm{CF}}\mathbb{C}^{(\beta,\beta)}_{\infty}v_1'(r) + {}^{\mathrm{CF}}\mathbb{C}^{(\gamma,\gamma)}_{\infty}v_1'(r)\big)\big| \end{aligned} $$

$$
\begin{aligned}
&+ \big|f'\big(r, u_1'(r), (\phi u_1')(r), (\varphi u_1')(r), {}^{\mathrm{CF}}\mathbb{I}_{\infty}^{(\theta,\theta)} u_1'(r), {}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\delta,\delta)} u_1'(r)\big) \\
&- f'\big(r, v_1'(t), (\phi v_1')(r), (\varphi v_1')(r), {}^{\mathrm{CF}}\mathbb{I}_{\infty}^{(\theta,\theta)} v_1'(r), {}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\delta,\delta)} v_1'(r)\big)\big|\big) \\
&+ b_\alpha \int_0^r \big[\mu\big|\big({}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\beta,\beta)} u_1'(s) + {}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\gamma,\gamma)} u_1'(s)\big) - \big({}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\beta,\beta)} v_1'(s) + {}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\gamma,\gamma)} v_1'(s)\big)\big| \\
&+ \big|f'\big(s, u_1'(r), (\phi u_1')(s), (\varphi u_1')(s), {}^{\mathrm{CF}}\mathbb{I}_{\infty}^{(\theta,\theta)} u_1'(s), {}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\delta,\delta)} u_1'(s)\big) \\
&- f'\big(s, v_1'(s), (\phi v_1')(s), (\varphi v_1')(s), {}^{\mathrm{CF}}\mathbb{I}_{\infty}^{(\theta,\theta)} v_1'(s), {}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\delta,\delta)} v_1'(s)\big)\big|\big]\,ds \\
\le\ & a_\alpha \mu\big[\big|{}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\beta,\beta)}\big(u_1'(r) - v_1'(r)\big)\big| \\
&+ \big|{}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\gamma,\gamma)}\big(u_1'(r) - v_1'(r)\big)\big|\big] + a_\alpha\big|\eta(r)\big|\big[\big|u_1'(r) - v_1'(r)\big| + \big|(\phi u_1')(r) - (\phi v_1')(r)\big| \\
&+ \big|(\varphi u_1')(r) - (\varphi v_1')(r)\big| + \big|{}^{\mathrm{CF}}\mathbb{I}_{\infty}^{(\theta,\theta)} u_1'(r) - {}^{\mathrm{CF}}\mathbb{I}_{\infty}^{(\theta,\theta)} v_1'(r)\big| \\
&+ \big|{}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\delta,\delta)} u_1'(r) - {}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\delta,\delta)} v_1'(r)\big|\big] \\
&+ b_\alpha \int_0^r \big[\mu\big(\big|{}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\beta,\beta)}\big(u_1'(s) - v_1'(s)\big)\big| + \big|{}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\gamma,\gamma)}\big(u_1'(s) - v_1'(s)\big)\big|\big) \\
&+ \big|\eta(s)\big|\big(\big|u_1'(s) - v_1'(s)\big| \\
&+ \big|(\phi u_1')(s) - (\phi v_1')(s)\big| + \big|(\varphi u_1')(s) - (\varphi v_1')(s)\big| + \big|{}^{\mathrm{CF}}\mathbb{I}_{\infty}^{(\theta,\theta)} u_1'(s) - {}^{\mathrm{CF}} I_{\infty}^{(\theta,\theta)} v_1'(s)\big| \\
&+ \big|{}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\delta,\delta)} u_1'(s) - {}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\delta,\delta)} v_1'(s)\big|\big)\big]\,ds \\
\le\ & \left[\eta^*\left(2 + \gamma_0 + \lambda_0 + \frac{1}{(1-\delta)^2}\right) + \mu\left(\frac{1}{(1-\gamma)^2} + \frac{1}{(1-\beta)^2}\right)\right][a_\alpha + b_\alpha]\big\|u_1' - v_1'\big\|
\end{aligned}
$$

for all $r \in I$ and $u_1', v_1' \in H^1$. Hence,

$$
\big\|Fu_1' - Fv_1'\big\| \le \left[\eta^*\left(2 + \gamma_0 + \lambda_0 + \frac{1}{(1-\delta)^2}\right) + \mu\left(\frac{1}{(1-\gamma)^2} + \frac{1}{(1-\beta)^2}\right)\right]\big\|u_1' - v_1'\big\|
$$

for all $u_1', v_1' \in H^1$. Consider the mappings $j : [0,\infty)^5 \to [0,\infty)$ and $\alpha : H^1 \times H^1 \to [0,\infty)$ defined by $j(t_1,t_2,t_3,t_4,t_5) = \Delta t_1$ and $\alpha(t,s) = 1$ for all $t, s \in H^1$. We can check that $j \in \mathcal{R}$ and $F$ is a generalized $\alpha$-contraction. From Theorem 2.2 we conclude that $F$ possesses an approximate fixed point, which is an approximate solution for problem (1). □

Let $c$ be a real number, and $k$, $s$, and $q$ bounded functions on $I = [0,1]$ with $M_1 = \sup_{p\in I}|k(p)| < \infty$, $M_2 = \sup_{p\in I}|s(p)| < \infty$, and $M_3 = \sup_{t\in I}|q(p)| < \infty$. We investigate the infinite coefficient-symmetric CF fractional integro-differential problem

$$
\begin{aligned}
{}^{\mathrm{CF}}\mathbb{C}_{(\mu,\infty)}^{(\alpha,\alpha)} x(p) &= \lambda k(p)\,{}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\delta,\delta)} x(p) + \rho s(p)\,{}^{\mathrm{CF}}\mathbb{I}_{\infty}^{(\theta,\theta)} x(p) \\
&\quad + \int_0^p f\big(w, x(w), (\varphi x)(w), q(w)\,{}^{\mathrm{CF}}\mathbb{C}_{(m,\infty)}^{(\gamma,\gamma)} x(w)\big)\,dw
\end{aligned}
\tag{2}
$$

with $x(0) = c$, where $\lambda, \rho \ge 0$ and $\alpha, \gamma, \delta, \theta \in (0,1)$.

**Theorem 3.12** *Let $\xi_1, \xi_2, \xi_3 \ge 0$, and let $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be a bounded integrable function satisfying $|f(p,x_1,y_1,w_1) - f(p,x_1',y_1',w_1')| \le \xi_1|x_1 - x_1'| + \xi_2|y_1 - y_1'| + \xi_3|w_1 - w_1'|$ for all $p \in I$ and $x_1, y_1, w_1, v_1, x_1', y_1', w_1' \in \mathbb{R}$. If $\Delta = |2 - \mu|[\lambda\frac{M_1}{(1-\delta)^2} + \rho M_2 + \xi_1 + \xi_2\gamma_0 + \xi_3\frac{M_3}{(1-\gamma)^2|2-m|}] < 1$, then problem* (2) *admits an approximate solution.*

*Proof* Let $H^1$ be equipped with $d(x,y) = \|x-y\|$, where $\|x\| = \sup_{t\in I}|x(t)|$. Now, consider the selfmap $\mathcal{F}: H^1 \to H^1$ defined by

$$\begin{aligned}(\mathcal{F}x)(p) &= x(0) + (2-\mu)a_\alpha\Big[\big(\lambda k(p)^{\mathrm{CF}}\mathbb{C}_\infty^{(\delta,\delta)}x(p) + \rho s(p)^{\mathrm{CF}}\mathbb{I}_\infty^{(\theta,\theta)}x(p) \\ &\quad + \int_0^p f\big(w,x(w),(\varphi x)(w),q(w)^{\mathrm{CF}}\mathbb{C}_{(m,\infty)}^{(\gamma,\gamma)}x(w)\big)\,dw\Big] \\ &\quad + b_\alpha(2-\mu)\int_0^p\Big[\lambda k(w)^{\mathrm{CF}}\mathbb{C}_\infty^{(\delta,\delta)}x(w) + \rho s(w)^{\mathrm{CF}}\mathbb{I}_\infty^{(\theta,\theta)}x(w)\,dw \\ &\quad + \int_0^w f\big(r,x(r),(\varphi x)(r),q(r)^{\mathrm{CF}}\mathbb{C}_{(m,\infty)}^{(\gamma,\gamma)}x(r)\big)\,dr\Big]\,dw\end{aligned}$$

for all $p \in I$ and $x,y \in H^1$, where $a_\alpha$ and $b_\alpha$ are given in Lemma 3.10. As a result, utilizing Lemmas 3.5, 3.7, and 3.8, we get

$$\begin{aligned}&\bigg|\Big[\lambda k(p)^{\mathrm{CF}}\mathbb{C}_\infty^{(\delta,\delta)}x(p) + \rho s(p)^{\mathrm{CF}}\mathbb{I}_\infty^{(\theta,\theta)}x(p) + \int_0^p f\big(w,x(w),(\varphi x)(w),q(w)^{\mathrm{CF}}\mathbb{C}_{(m,\infty)}^{(\gamma,\gamma)}x(w)\big)\,dw\Big] \\ &\quad - \Big[\lambda k(p)^{\mathrm{CF}}\mathbb{C}_\infty^{(\delta,\delta)}y(p) + \rho s(w)^{\mathrm{CF}}\mathbb{I}_\infty^{(\theta,\theta)}y(p) \\ &\quad + \int_0^p f\big(w,y(w),(\varphi y)(w),q(w)^{\mathrm{CF}}\mathbb{C}_{(m,\infty)}^{(\gamma,\gamma)}y(w)\big)\,dw\Big]\bigg| \\ &\quad \le \Big[\lambda\frac{M_1}{(1-\delta)^2} + \rho M_2\Big]\|x-y\| + \xi_1\|x-y\| + \xi_2\gamma_0\|x-y\| + \xi_3\frac{M_3}{(1-\gamma)^2|2-m|}\|x-y\| \\ &\quad \le \Big[\lambda\frac{M_1}{(1-\delta)^2} + \rho M_2 + \xi_1 + \xi_2\gamma_0 + \xi_3\frac{M_3}{(1-\gamma)^2|2-m|}\Big]\|x-y\|\end{aligned}$$

for all $p \in I$ and $x,y \in H^1$. As a result, we get

$$\begin{aligned}&\big|(\mathcal{F}x)(p) - (\mathcal{F}x)(p)\big| \\ &\quad \le a_\alpha|2-\mu|\Big[\lambda\frac{M_1}{(1-\delta)^2} + \rho M_2 + \xi_1 + \xi_2\gamma_0 + \xi_3\frac{M_3}{(1-\gamma)^2|2-m|}\Big]\|x-y\| \\ &\qquad + b_\alpha|2-\mu|\int_0^p\Big[\lambda\frac{M_1}{(1-\delta)^2} + \rho M_2 + \xi_1 + \xi_2\gamma_0 + \xi_3\frac{M_3}{(1-\gamma)^2|2-m|}\Big]\|x-y\|\,ds \\ &\quad \le |2-\mu|\Big[\lambda\frac{M_1}{(1-\delta)^2} + \rho M_2 + \xi_1 + \xi_2\gamma_0 + \xi_3\frac{M_3}{(1-\gamma)^2|2-m|}\Big]\|x-y\|\end{aligned}$$

for all $p \in I$ and $x,y \in H^1$. Now we consider the mappings $j: [0,\infty)^5 \to [0,\infty)$ and $\alpha: H^1 \times H^1 \to [0,\infty)$ defined by $\alpha(t,s) = 1$ and $j(t_1,t_2,t_3,t_4,t_5) = \frac{\Delta}{3}(t_1 + 2t_2)$. We can check that $j \in \mathcal{R}$ and $\mathcal{F}$ is a generalized $\alpha$-contraction. With the help of Theorem 2.2, we conclude that $\mathcal{F}$ possesses an approximate fixed point, which represents an approximate solution for the investigated problem (2). □

The next step is to study two applications to describe the reported results.

**Example 1** Let us define $\eta \in L^\infty([0,1])$ and $\gamma,\lambda: [0,1]\times[0,1] \to [0,\infty)$ by $\eta(p) = \frac{\pi}{e^{(p+12)}}$, $\gamma(p,s) = e^{p-s}$ and $\lambda(p,s) = \ln(5^{\sin(\pi p - s)})$. Then, we have $\eta^* = \frac{\pi}{e^{12}}$, $\gamma_0 \le e$, and $\lambda_0 \le \ln 5$. Let us consider $\alpha = \frac{1}{5}$, $\mu = \frac{1}{20}$, $\beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$, $\theta = \frac{3}{4}$, and $\delta = \frac{3}{5}$. Consider the problem

$$\begin{aligned} {}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\frac{1}{5},\frac{1}{5})} u_1'(p) &= \frac{1}{20}\big({}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\frac{1}{4},\frac{1}{4})} u_1'(p) + {}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\frac{1}{2},\frac{1}{2})} u_1'(p)\big) \\ &\quad + e^{-\pi(t+12)}\bigg[p + u_1'(p) + \int_0^p e^{p-s} u_1'(s)\,ds \\ &\quad + \int_0^p \ln\big(5^{\sin(\pi p-s)}\big) u_1'(s)\,ds + {}^{\mathrm{CF}}\mathbb{I}_{\infty}^{(\frac{3}{4},\frac{3}{4})} u_1'(p) + {}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\frac{3}{5},\frac{3}{5})} u_1'(p)\bigg] \end{aligned} \tag{3}$$

with $u_1'(0) = 0$. Considering $f(p,x,y,w,u_1,u_2) = e^{-\pi(p+12)}(p+x+y+w+u_1+u_2)$, we note that $\Delta = [\eta^*(2 + \gamma_0 + \lambda_0 + \frac{1}{(1-\delta)^2}) + \mu(\frac{1}{(1-\gamma)^2(1-\beta)^2})] < 0/4447 < 1$. Now, by Theorem 3.11 problem (3) admits an approximate solution.

**Example 2** Consider the function $\lambda : [0,1] \times [0,1] \to [0,\infty)$ by $\lambda(p,s) = \frac{e^{2p-s}}{e}$. Thus, $\lambda_0 \leq e$. Let us consider $\mu = 3$, $m = \frac{1}{2}$, $\alpha = \frac{1}{4}$, $\delta = \frac{1}{4}$, $\theta = \frac{1}{2}$, $\gamma = \frac{1}{2}$, $\lambda = \frac{1}{200}$, $\rho = \frac{1}{122}$, $\xi_1 = \frac{1}{320}$, $\xi_2 = \frac{1}{40}$, and $\xi_3 = \frac{1}{119}$. Let $k(t) = \frac{2-p}{p+1}$, $s(p) = \sin p$ and $q(p) = \tan^{-1}(p)$. Then, $M_1 = \sup_{p\in[0,1]} |k(p)| = 2$, $M_2 = \sup_{t\in[0,1]} |s(p)| = 1$, and $M_3 = \sup_{t\in[0,1]} |q(p)| = \frac{\pi}{2}$. As a next step, we consider the problem

$$\begin{aligned} {}^{\mathrm{CF}}\mathbb{C}_{(\mu,\infty)}^{(\frac{1}{4},\frac{1}{4})} x(p) &= \frac{1}{200} k(p)\, {}^{\mathrm{CF}}\mathbb{C}_{\infty}^{(\frac{1}{4},\frac{1}{4})} x(p) + \frac{1}{122} s(p)\, {}^{\mathrm{CF}}\mathbb{I}_{\infty}^{(\frac{1}{2},\frac{1}{2})} x(p) \\ &\quad + \int_0^p \bigg[\frac{2}{56}s + \frac{1}{320}x(s) + \frac{1}{40}\int_0^s \frac{e^{2s-r}}{e} x(r)\,dr \\ &\quad + \frac{1}{119}\tan^{-1}(s)\, {}^{\mathrm{CF}}\mathbb{C}_{(m,\infty)}^{(\frac{1}{2},\frac{1}{2})} x(s)\bigg]\,ds \end{aligned} \tag{4}$$

with $x(0) = 0$. Considering $f(p,x_1,y_1,w_1) = \frac{2}{56}p + \xi_1 x_1 + \xi_2 y_1 + \xi_3 w_1$ for all $p \in I$ and $x_1, y_1, w_1, v \in \mathbb{R}$, we note that

$$\Delta = |2-\mu|\bigg[\lambda\frac{M_1}{(1-\delta)^2} + \rho M_2 + \xi_1 + \xi_2\gamma_0 + \xi_3\frac{M_3}{(1-\gamma)^2|2-m|}\bigg] < 0.111 < 1.$$

Now, by Theorem 3.12, problem (4) admits an approximate solution.

## 4 Conclusion

Fractional derivatives with nonsingular kernels started to be utilized from both theoretical and applied viewpoints. Particularly, the fractional Caputo-Fabrizio derivative was applied to models possessing memory effect of exponential type. Therefore, new generalizations of this operator should be investigated and applied to the dynamics of real-world problems. In this manuscript, we suggested a new operator called the infinite coefficient-symmetric CF fractional derivative. Besides, its properties were investigated, and two examples clearly show the advantages of the newly introduced concept.

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**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

**Author details**
[1]Department of Mathematics, Cankaya University, Ogretmenler Cad., Balgat, Ankara, 06530, Turkey. [2]Institute of Space Sciences, Magurele-Bucharest, Bucharest, Romania. [3]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.

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