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H^2-boundedness of the pullback attractor of the micropolar fluid flows with infinite delays

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Abstract
We establish the H^2-boundedness of the pullback attractor for a two-dimensional nonautonomous micropolar fluid flow with infinite delays.

MSC: 5B41; 35B65; 35Q35

Keywords: micropolar fluid flow; Pullback attractor; H^2-boundedness; infinite delays

1 Introduction

The 3D micropolar fluid model was firstly formulated by Eringen [1] and was used to describe the fluids consisting of randomly oriented particles suspended in a viscous medium. According to [1], the incompressible micropolar fluid motion can be expressed by the following system:

$$\begin{cases} \frac{\partial u}{\partial t} - (\nu + \nu_r)\Delta u - 2\nu_r \nabla \times \omega + (u \cdot \nabla)u + \nabla p = f, \\ \nabla \cdot u = 0, \\ \frac{\partial \omega}{\partial t} - (c_a + c_d)\Delta \omega + 4\nu_r \omega + (u \cdot \nabla)\omega - (c_0 + c_d - c_a)\nabla(\nabla \cdot \omega) - 2\nu_r \nabla \times u = \tilde{f}, \end{cases}$$

where $u(x,t) = (u_1, u_2, u_3)$ represents the velocity, $\omega(x,t) = (\omega_1, \omega_2, \omega_3)$ stands for the angular velocity field of rotation of particles, p is the pressure, f and $\tilde{f}$ represent the external force and moment, respectively. The positive parameters ν, ν_r, c_0, c_a, c_d are the viscous coefficients. In fact, ν is the usual Newtonian kinetic viscosity, and ν_r is the dynamics microrotation viscosity, and c_0, c_a, c_d denote the angular viscosity (see [2]). From [1, 2] we see that these equations express the balance of momentum, mass, and moment of momentum, accordingly. When microrotation effects are neglected (i.e., $\omega = 0$), the equations reduce to the incompressible Navier-Stokes equations. Therefore, the equations of micropolar fluid flows can be regarded as a generalization of the Navier-Stokes equations in the sense that they take into account the microstructure of the fluid. For physical background, we refer, for example, to [2, 3].

Due to their wide applications, the micropolar fluid flows have drawn much attention from mathematicians and physicists and have been well studied. For the theories on the existence and uniqueness of solutions of the micropolar fluid flows, we refer to [3–9].

At the same time, the long-time behavior of solutions for the micropolar fluid flows has been investigated from various aspects. Chen et al. proved the existence of H^2-compact global attractors in a bounded domain [10] and verified the existence of uniform attractors in nonsmooth domains [11]. Lukaszewicz [12] established the existence of H^1-pullback attractor for nonautonomous micropolar fluid flows in a bounded domain. As for the long-time behavior of solutions for the micropolar fluid flows on unbounded domains, Dong and Chen [13] discussed the existence and regularity of the global attractors. Later, they [14] obtained the L^2 time decay rate for global solutions of the 2D micropolar equations via the Fourier splitting method. Chen and Price [15] obtained the L^2 time decay rate for small solutions of the 3D micropolar equations via Kato's method. Zhao et al. [16] showed the existence of an H^1-uniform attractor and so on. For more theories about the micropolar fluid flows, we refer to [17–21].

There are also some efforts focused on the 2D micropolar equations with partial dissipation. Dong and Zhang [22] examined the microrotation viscosity, namely $c_a + c_d = 0$. The global regularity problem for this partial dissipation case is not trivial due to the presence of the term $\nabla \times \omega$ in the velocity equation. Dong and Zhang overcame the difficulty by making full use of the quantity $\nabla \times u - \frac{2\nu_r}{\nu+\nu_r}\omega$, which obeys a transport-diffusion equation. When the parameters $\nu = 0$ and $\nu_r \neq c_a + c_d$, the global well-posedness of the micropolar fluid equations were obtained in the framework of Besov spaces [23]. More recently, Dong et al. [24] studied the global regularity and large-time behavior of solutions to the 2D micropolar equations with only angular viscosity dissipation, in which they established the well-posedness of the solutions by fully exploiting the structure of the system and controlling the vorticity via the evolution equation of a combined quantity of the vorticity and the microrotation angular velocity; they also obtained suitable decay rates of the solution by combining diagonalization process with uniformly bounded estimates for the first derivatives of the solutions.

In this paper, we consider the special situation where the velocity component in the x_3-direction is zero and the axes of rotation of particles are parallel to the x_3-axis, that is, $u = (u_1, u_2, 0), \omega = (0, 0, \omega_3), f = (f_1, f_2, 0)$, and $\tilde{f} = (0, 0, \tilde{f}_3)$. Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with smooth boundary $\partial\Omega$ such that the following Poincaré inequality holds:

$$\text{There exists } \lambda_1 > 0 \quad \text{such that} \quad \lambda_1 \|\varphi\|^2_{L^2(\Omega)} \leq \|\nabla\varphi\|^2_{L^2(\Omega)}, \quad \forall \varphi \in H^1_0(\Omega). \tag{1.1}$$

Then, we discuss the following 2D non-autonomous incompressible micropolar fluid flows with infinite delays in Ω:

$$\begin{cases} \frac{\partial u}{\partial t} - (\nu + \nu_r)\Delta u - 2\nu_r \nabla \times \omega + (u \cdot \nabla) u + \nabla p & \\ \quad = f(t,x) + g(t, u_t), & t > \tau, x \in \Omega, \\ \frac{\partial \omega}{\partial t} - \alpha\Delta\omega + 4\nu_r\omega - 2\nu_r \nabla \times u + (u \cdot \nabla)\omega & \\ \quad = \tilde{f}(t,x) + \tilde{g}(t, \omega_t), & t > \tau, x \in \Omega, \\ \nabla \cdot u = 0, & \text{in } (\tau, +\infty) \times \Omega, \\ u = 0, \qquad \omega = 0, & \text{on } (\tau, +\infty) \times \partial\Omega, \\ (u(\tau + s, x), \omega(\tau + s, x)) = \phi(s, x), & s \in (-\infty, 0], \tau \in \mathbb{R}, x \in \Omega, \end{cases} \tag{1.2}$$

where $x = (x_1, x_2) \in \Omega$ and $\alpha = c_a + c_d$. The vector functions $g = (g_1, g_2, 0)$ and $\tilde{g} = (0, 0, \tilde{g}_3)$ are additional external forces containing some hereditary characteristics u_t and ω_t, which are defined on $(-\infty,0]$ as follows:

$$u_t = u_t(\cdot) := u(t+\cdot), \qquad \omega_t = \omega_t(\cdot) := \omega(t+\cdot), \quad t \geq \tau. \tag{1.3}$$

In addition, $\phi(s,x) = (u_\tau, \omega_\tau) = (u(\tau+s,x), \omega(\tau+s,x))$ is the initial datum in the interval of delay time $(-\infty,0]$, and

$$\nabla \times u := \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \quad \text{and} \quad \nabla \times \omega := \left(\frac{\partial \omega}{\partial x_2}, -\frac{\partial \omega}{\partial x_1}\right).$$

In the real world, delay terms appear naturally, for instance, as effects in wind tunnel experiments. Also, the delay situations may occur when we want to control the system via applying a force that considers not only the present state but also the history state of the system. However, so far, to our knowledge, there is no references discussing the micropolar fluid flows with delay in addition to [25], where the author established the global well-posedness and pullback attractors for a 2D impressible micropolar flows with infinite delays.

The main purpose of this work is to establish the H^2-boundedness of the pullback attractor $\widehat{\mathcal{A}} = \{\mathcal{A}(t) \mid t \in \mathbb{R}\}$ obtained in [25]. Before stating the main results of this paper, we give some assumptions:

(A1) **(I)** A mapping $G : [\tau, T] \times \mathcal{C}_\gamma(\widehat{H}) \mapsto (L^2(\Omega))^3$ satisfies:

(i) For any $\xi \in \mathcal{C}_\gamma(\widehat{H})$, the mapping $[\tau, T] \ni t \mapsto G(t,\xi) \in (L^2(\Omega))^3$ is measurable;

(ii) $G(\cdot, 0) = (0,0,0)$;

(iii) There exists a constant $L_G > 0$ such that, for any $t \in [\tau, T]$ and $\xi, \eta \in \mathcal{C}_\gamma(\widehat{H})$,

$$\|G(t,\xi) - G(t,\eta)\| \leq L_G \|\xi - \eta\|_\gamma.$$

(II) $F(t,x) \in L^2_{\text{loc}}(\mathbb{R}; \widehat{H})$, $2L_G < \delta_1\lambda_1 < 2\gamma$, and

$$\int_\tau^t e^{(\delta_1\lambda_1 - 2L_G)(r-\tau)} \|F(r)\|^2 \, \mathrm{d}r < +\infty, \quad \forall \tau \in \mathbb{R}, t \geq \tau.$$

(A2) **(I)** $\frac{\mathrm{d}G}{\mathrm{d}t} := (G(t,\xi))' : [\tau, T] \times \mathcal{C}_\gamma(\widehat{H}) \mapsto (L^2(\Omega))^3$ satisfies:

(i) For any $\xi(t), \xi'(t) \in \mathcal{C}_\gamma(\widehat{H})$, the mapping $t \mapsto (G(t,\xi))'$ is measurable;

(ii) $(G(\cdot, 0))' = (0,0,0)$;

(iii) There exists a constant $\tilde{L}_G > 0$ such that, for any $t \in [\tau, T]$, $\xi'(t), \eta'(t) \in \mathcal{C}_\gamma(\widehat{H})$,

$$\|(G(t,\xi))' - (G(t,\eta))'\| \leq \tilde{L}_G \|\xi' - \eta'\|_\gamma.$$

(II) $F(t,x) \in W^{1,2}_{\text{loc}}(\mathbb{R}; \widehat{H})$, $2\tilde{L}_G < \delta_1\lambda_1 < 2\gamma$, and

$$\int_\tau^t e^{(\delta_1\lambda_1 - 2\tilde{L}_G)(r-\tau)} \|F'(r)\|^2 \, \mathrm{d}r < +\infty, \quad \forall \tau \in \mathbb{R}, t \geq \tau.$$

Under the above assumptions, we have

Theorem 1.1 *Assume that* **(A1)** *and* **(A2)** *hold.*

(1) *For any bounded set* $\mathcal{B} \subset \mathcal{C}_\gamma(\widehat{H})$ *and any* $\tau \in \mathbb{R}, \epsilon > 0, t \geq \tau + 2\epsilon$, *the set* $\bigcup_{s\in[\tau+2\epsilon,t]} U(s,\tau)\mathcal{B}$ *is bounded in* $D(A) = \widehat{V} \cap (H^2(\Omega))^3$.

(2) *Let* $\widehat{\mathcal{A}} = \{\mathcal{A}(t) \mid t \in \mathbb{R}\}$ *be the pullback attractor of system* (1.2). *Then, for any* $T_1, T_2 \in \mathbb{R}$ *with* $T_1 < T_2$, *the set* $\bigcup_{t\in[T_1,T_2]} \mathcal{A}(t)$ *is bounded in* $D(A) = \widehat{V} \cap (H^2(\Omega))^3$.

We remark that García-Luengo et al. [9] proved the existence of the pullback attractor and investigated its tempered behavior for Navier-Stokes equations in bounded domains. Further, they discussed the H^2-boundedness of the pullback attractors of the Navier-Stokes equations in [26]. Recently, Zhao and Sun [25] established the existence of pullback attractors for 2D nonautonomous micropolar fluid flows with infinite delays. Motivated by [26] and following its main idea, we generalize their results to the micropolar fluid flows with infinite delays. Compared with the Navier-Stokes equations ($\omega = 0, \nu_r = 0$), the micropolar fluid flow consists of the angular velocity field ω, which leads to a different nonlinear term $B(u, w)$ and an additional term $N(u)$ in the abstract equation. In addition, the time-delay term considered in this work also increases the difficulty. Therefore, we have to obtain more delicate estimates and analysis for the solutions.

The paper is organized as follows. In Section 2, we make some preliminaries. That is, we introduce some notations and recall some known results. In Section 3, we concentrate on showing the H^2-boundedness of the pullback attractor $\widehat{\mathcal{A}}$. To this end, we first make some estimates for the Galerkin approximation solutions by mainly using the energy method. Then, we obtain a general result about $\widehat{V} \cap (H^2(\Omega))^3$-boundedness of invariant sets for the associate evolution process. Further, we have the boundedness of the pullback attractor in $\widehat{V} \cap (H^2(\Omega))^3$.

2 Preliminaries

In this section, we make some necessary preliminaries by introducing some notation and key operators. Then, we rewrite equations (1.2) in an abstract form. Finally, we recall some known results.

We denote by $L^p(\Omega)$ and $W^{m,p}(\Omega)$ the usual Lebesgue and Sobolev spaces (see [27]) endowed with norms $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, respectively:

$$\|\varphi\|_p := \left(\int_\Omega |\varphi|^p \,\mathrm{d}x\right)^{1/p} \quad \text{and} \quad \|\varphi\|_{m,p} := \left(\sum_{|\beta|\leq m} \int_\Omega \left|D^\beta \varphi\right|^p \mathrm{d}x\right)^{1/p}.$$

In particular, we denote $H^m(\Omega) := W^{m,2}(\Omega)$ and by $H_0^1(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the $H^1(\Omega)$ norm.

$$\mathcal{V} := \left\{\varphi \in \mathcal{C}_0^\infty(\Omega) \times \mathcal{C}_0^\infty(\Omega) \mid \varphi = (\varphi_1, \varphi_2), \nabla \cdot \varphi = 0\right\},$$

$H :=$ closure of $\mathcal{V}$ in $L^2(\Omega) \times L^2(\Omega)$ with norm $\|\cdot\|_H$ and dual spaceH^*,

$V :=$ closure of $\mathcal{V}$ in $H^1(\Omega) \times H^1(\Omega)$ with norm $\|\cdot\|_V$ and dual space V^*,

$\widehat{H} := H \times L^2(\Omega)$ with norm $\|\cdot\|_{\widehat{H}}$ and dual space $\widehat{H}^*$,

$\widehat{V} := V \times H_0^1(\Omega)$ with norm $\|\cdot\|_{\widehat{V}}$ and dual space $\widehat{V}^*$, where $\|\cdot\|_H$, $\|\cdot\|_V$, $\|\cdot\|_{\widehat{H}}$, and $\|\cdot\|_{\widehat{V}}$ are defined by

$$\begin{aligned}
&\|(u,v)\|_H := \left(\|u\|_2^2 + \|v\|_2^2\right)^{1/2},\\
&\|(u,v)\|_V := \left(\|u\|_{H^1}^2 + \|v\|_{H^1}^2\right)^{1/2},\\
&\|(u,v,w)\|_{\widehat{H}} := \left(\|(u,v)\|_H^2 + \|w\|_2^2\right)^{1/2},\\
&\|(u,v,w)\|_{\widehat{V}} := \left(\|(u,v)\|_V^2 + \|w\|_{H^1}^2\right)^{1/2};
\end{aligned}$$

$(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, H, or $\widehat{H}$, and $\langle\cdot,\cdot\rangle$ is the dual pairing between V and V^* or between $\widehat{V}$ and $\widehat{V}^*$. Throughout this article, we simplify the notations $\|\cdot\|_2$, $\|\cdot\|_H$, and $\|\cdot\|_{\widehat{H}}$ by the same notation $\|\cdot\|$ if there is no confusion. Furthermore, we denote

$\mathcal{C}_\gamma(\widehat{H}) := \left\{\varphi \in \mathcal{C}\left((-\infty,0];\widehat{H}\right) \mid \exists \lim_{s\to-\infty} e^{\gamma s}\varphi(s) \in \widehat{H}\right\}$ with some suitable $\gamma > 0$,

which is a Banach space with the norm

$$\|\varphi\|_\gamma := \sup_{s\in(-\infty,0]} e^{\gamma s}\|\varphi(s)\|;$$

$L^p(I;X)$:= space of strongly measurable functions on the closed interval I

with values in a Banach space X, endowed with norm

$$\|\varphi\|_{L^p(I;X)} := \left(\int_I \|\varphi\|_X^p \,\mathrm{d}t\right)^{1/p} \quad \text{for } 1 \le p < \infty,$$

$\mathcal{C}(I;X)$:= space of continuous functions on the interval I with values

in the Banach space X, endowed with the usual norm,

$L^2_{\mathrm{loc}}(I;\widehat{H})$:= space of locally integrable functions from the interval I to $\widehat{H}$,

$\mathrm{dist}_M(X,Y)$ is the Hausdorff semidistance between $X \subseteq M$ and $Y \subseteq M$ defined by

$$\mathrm{dist}_M(X,Y) = \sup_{x\in X}\inf_{y\in Y} \mathrm{dist}_M(x,y).$$

Now, we introduce three operators:

$$\begin{aligned}
&\langle Aw,\varphi\rangle := (\nu+\nu_r)(\nabla u,\nabla\Psi) + \alpha(\nabla\omega,\nabla\psi), \quad \forall w=(u,\omega)\in\widehat{V}, \forall\varphi=(\Psi,\psi)\in\widehat{V},\\
&\langle B(u,w),\varphi\rangle := \left((u\cdot\nabla)w,\varphi\right), \quad \forall u\in V, w\in\widehat{V}, \forall\varphi\in\widehat{V},\\
&N(w) := (-2\nu_r\nabla\times\omega, -2\nu_r\nabla\times u + 4\nu_r\omega), \quad \forall w=(u,\omega)\in\widehat{V}.
\end{aligned}$$

There are some useful estimations for the operators A, $B(\cdot,\cdot)$, and $N(\cdot)$ established in [3, 25, 28].

Lemma 2.1

(1) *The operator A is linear continuous both from $\widehat{V}$ to $\widehat{V}^*$ and from $D(A) = \widehat{V} \cap (H^2(\Omega))^3$ to $\widehat{H}$. Moreover, there are two positive constants c_1 and c_2 such that*

$$c_1\langle Aw,w\rangle \le \|w\|_{\widehat{V}}^2 \le c_2\langle Aw,w\rangle, \quad \forall w\in\widehat{V}. \tag{2.1}$$

In addition, for any $w \in D(A)$, we have

$$\delta\|\nabla w\|^2 \le \langle Aw, w\rangle \le \|w\|\|Aw\| \le \lambda_1^{-\frac{1}{2}}\|\nabla w\|\|Aw\|, \tag{2.2}$$

where $\delta = \min\{\nu + \nu_r, \alpha\}$, and λ_1 is the constant from (1.1).

(2) *The operator $B(\cdot,\cdot)$ is continuous from $V \times \widehat{V}$ to $\widehat{V}^*$ and satisfies the following properties:*

 (i) *For any $u \in V$ and $w \in \widehat{V}$, we have*

$$\langle B(u,w),\varphi\rangle = -\langle B(u,\varphi),w\rangle, \quad \forall \varphi \in \widehat{V}. \tag{2.3}$$

In particular,

$$\langle B(u,w),w\rangle = 0, \quad \forall u \in V, w \in \widehat{V}. \tag{2.4}$$

 (ii) *There exists a positive constant λ, which depends only on Ω, such that for any $(u,\psi,\varphi) \in V \times \widehat{V} \times \widehat{V}$, we have*

$$\left|\langle B(u,\psi),\varphi\rangle\right| \le \begin{cases} \lambda\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}}\|\nabla\varphi\|^{\frac{1}{2}}\|\nabla\psi\|, \\ \lambda\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}\|\psi\|^{\frac{1}{2}}\|\nabla\psi\|^{\frac{1}{2}}\|\nabla\varphi\|. \end{cases} \tag{2.5}$$

Moreover, if $(u,\psi,\varphi) \in V \times D(A) \times D(A)$, then

$$\left|\langle B(u,\psi),A\varphi\rangle\right| \le \lambda\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}\|\nabla\psi\|^{\frac{1}{2}}\|A\psi\|^{\frac{1}{2}}\|A\varphi\|. \tag{2.6}$$

(3) *The operator $N(\cdot)$ is continuous from $\widehat{V}$ to $\widehat{H}$. Moreover, there exists a positive constant $c(\nu_r)$ such that*

$$\|N(\psi)\| \le c(\nu_r)\|\psi\|_{\widehat{V}}, \quad \forall \psi \in \widehat{V}. \tag{2.7}$$

In addition,

$$-\langle N(\psi),A\psi\rangle \le \frac{1}{4}\|A\psi\|^2 + c^2(\nu_r)\|\psi\|^2_{\widehat{V}}, \quad \forall \psi \in D(A), \tag{2.8}$$

$$\delta_1\|\psi\|^2_{\widehat{V}} \le \langle A\psi,\psi\rangle + \langle N(\psi),\psi\rangle, \quad \forall \psi \in \widehat{V}; \tag{2.9}$$

hereinafter $\delta_1 := \min\{\nu, \alpha\}$.

According to the previous notation, we can formulate a weak version of system (1.2) as follows:

$$\begin{cases} \frac{\partial w}{\partial t} + Aw + B(u,w) + N(w) = F(t,x) + G(t,w_t), & t > \tau, \\ w|_{t=\tau} = w_\tau = (u_\tau, w_\tau) = (u(\tau+s), \omega(\tau+s)) := \phi(s), & s \in (-\infty, 0], \end{cases} \tag{2.10}$$

where $w = (u,\omega), F(t) = F(t,x) := (f(t,x), \tilde{f}(t,x))$, and $G(t,w_t) := (g(t,u_t), \tilde{g}(t,\omega_t))$.

We say that a function $w \in \mathcal{C}((-\infty,T];\widehat{H}) \cap L^2(\tau,T;\widehat{V})$ with $w_\tau = \phi(s) \in \mathcal{C}_\gamma(\widehat{H})$ is a weak solution of system (2.10) in the interval $(-\infty,T]$ if, for all $T > \tau$ and $\varphi \in \widehat{V}$, the following equation holds in the distribution sense of $\mathcal{D}'(\tau,T)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(w,\varphi) + \langle Aw,\varphi\rangle + \bigl\langle B(u,w),\varphi\bigr\rangle + \bigl\langle N(w),\varphi\bigr\rangle = \bigl\langle F(t),\varphi\bigr\rangle + \bigl(G(t,w_t),\varphi\bigr).$$

Lemma 2.2 (see [25]) *Assume that* **(A1)** *holds. Then for any given initial datum* $w_\tau := \phi(s) \in \mathcal{C}_\gamma(\widehat{H})$ *and any* $T > \tau$, *there exists a unique stable weak solution*

$$w \in \mathcal{C}\bigl((-\infty,T);\widehat{H}\bigr) \cap L^2(\tau,T;\widehat{V}), \quad w' \in L^2\bigl(\tau,T;\widehat{V}^*\bigr).$$

Moreover, for any $t \in [\tau,T]$,

$$\|w_t\|_\gamma^2 \le e^{(-\delta_1\lambda_1+2L_G)(t-\tau)}\bigl\|\phi(s)\bigr\|_\gamma^2 + \frac{2}{\delta_1}\int_\tau^t e^{(-\delta_1\lambda_1+2L_G)(t-\theta)}\bigl\|F(\theta)\bigr\|^2\,\mathrm{d}\theta, \tag{2.11}$$

$$\begin{aligned}\delta_1\int_\tau^t\bigl\|w(\theta)\bigr\|_{\widehat{V}}^2\,\mathrm{d}\theta &\le 2e^{\delta_1\lambda_1(t-\tau)}\bigl\|w(\tau)\bigr\|^2 + \frac{4}{\delta_1}e^{2L_Gt-\delta_1\lambda_1\tau}\int_\tau^t e^{(\delta_1\lambda_1-2L_G)\theta}\bigl\|F(\theta)\bigr\|^2\,\mathrm{d}\theta \\ &\quad + \frac{5}{\delta_1}e^{-\delta_1\lambda_1\tau}\int_\tau^t e^{\delta_1\lambda_1\theta}\bigl\|F(\theta)\bigr\|^2\,\mathrm{d}\theta + 2e^{2L_G(t-\tau)}\bigl\|\phi(s)\bigr\|_\gamma^2.\end{aligned} \tag{2.12}$$

In addition, if $w_\tau \in \widehat{V}$, *then the weak solution* $w \in \mathcal{C}((-\infty,T);\widehat{V}) \cap L^2(\tau,T;D(A))$.

Based on Lemma 2.2, we can define the map

$$U(t,\tau): w_\tau(\cdot) := \phi(s) \mapsto U(t,\tau;w_\tau) = U(t,\tau)\phi(s) = w_t(\cdot), \quad t \ge \tau, s \in (-\infty,0], \tag{2.13}$$

which generates a continuous process in $\mathcal{C}_\gamma(\widehat{H})$ satisfying:

- $U(s,s)$ = identity,
- $U(t,r)U(r,s) = U(t,s)$ for any $s \le r \le t$,

where w is the solution of system (2.10) corresponding to the initial datum $\phi(s) \in \mathcal{C}_\gamma(\widehat{H})$, and $w_t(s)$ is defined as in (1.3).

Lemma 2.3 *Under assumption* **(A1)**, *there exists a pullback attractor* $\widehat{\mathcal{A}} = \{\mathcal{A}(t) \mid t \in \mathbb{R}\}$ *for the process* $\{U(t,\tau)\}_{t\ge\tau}$ *that satisfies the following properties*:

- *Compactness*: *for any* $t \in \mathbb{R}$, $\mathcal{A}(t)$ *is a nonempty compact subset of* $\mathcal{C}_\gamma(\widehat{H})$;
- *Invariance*: $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$, $\forall t \ge \tau$;
- *Pullback attracting*: *for any bounded set* $\mathcal{B}$ *of* $\mathcal{C}_\gamma(\widehat{H})$, *we have*

$$\lim_{\tau\to-\infty}\mathrm{dist}_{\mathcal{C}_\gamma(\widehat{H})}\bigl(U(t,\tau)\mathcal{B},\mathcal{A}(t)\bigr) = 0, \quad \forall t \in \mathbb{R}.$$

Now, we end this section with the following lemma, which plays an important role in the proof of higher regularity of the pullback attractors.

Lemma 2.4 (*see* [29, 30]) *Let* X, Y *be Banach spaces such that* X *is reflexive and the inclusion* $X \subset Y$ *is continuous. Assume that* $\{w_n\}_{n\ge1}$ *is a bounded sequence in* $L^\infty(\tau,t;X)$ *such that* $w_n \rightharpoonup w$ *weakly in* $L^q(\tau,t;X)$ *for some* $q \in [1,+\infty)$ *and* $w \in \mathcal{C}([\tau,t];Y)$. *Then* $w(t) \in X$, *and*

$$\|w(s)\|_X \leq \liminf_{n\to+\infty}\|w_n(s)\|_{L^\infty(\tau,t;X)}, \quad \forall s\in[\tau,t].$$

3 H^2-boundedness of the pullback attractor

In this section, we concentrate on proving the H^2-boundedness of the pullback attractor $\widehat{\mathcal{A}}$.

To begin with, let us recall some properties of the operator A. According to the classical spectral theory of elliptic operators (see [31]), there exists a sequence $\{\lambda_n\}_{n=1}^\infty$ satisfying

$$0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n\leq\cdots, \quad \lambda_n\to+\infty \text{ as } n\to\infty,$$

and a sequence of elements $\{v_n\}_{n=1}^\infty \subseteq D(A)$ forming a Hilbert basis of $\widehat{H}$ and such that the span of $\{v_1,v_2,\dots,v_n,\dots\}$ is dense in $\widehat{V}$ and

$$Av_n=\lambda_n v_n, \quad \forall n\in\mathbf{N}. \tag{3.1}$$

For each $T>\tau$, denote by $w^{(m)}(t)=w^{(m)}(t;\tau,w_\tau):=\sum_{j=1}^m \beta_{m,j}(t)v_j$ the Galerkin approximation solutions of the solution $w(t)$ of system (2.10), which is the solution of the following ordinary differential equations:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(w^{(m)}(t),v_j\big)+\big\langle Aw^{(m)}(t),v_j\big\rangle+\big\langle B\big(u^{(m)}(t),w^{(m)}(t)\big),v_j\big\rangle+\big\langle N\big(w^{(m)}(t)\big),v_j\big\rangle\\&\quad=\big\langle F(t),v_j\big\rangle+\big(G\big(t,w_t^{(m)}\big),v_j\big),\quad 1\leq j\leq m, t\in(\tau,T),\end{aligned} \tag{3.2}$$

$$w_\tau^{(m)}(s)=w^{(m)}(\tau+s)=P_m\phi(s),\quad s\in(-\infty,0]. \tag{3.3}$$

Now, we verify the following results about the Galerkin approximation solutions.

Lemma 3.1 *Assume that* **(A1)** *holds. Then, for any bounded subset $\mathcal{B}$ of $\mathcal{C}_\gamma(\widehat{H})$ and any $\epsilon>0,\tau\in\mathbb{R},t>\tau+\epsilon$, we have that*

(i) *the set $\{w^{(m)}(\theta;\tau,w_\tau)\mid\theta\in[\tau+\epsilon,t],w_\tau:=\phi(s)\in\mathcal{B}\}$ is bounded in $\widehat{V}$,*
(ii) *the set $\{w^{(m)}(\cdot;\tau,w_\tau)\mid w_\tau\in\mathcal{B}\}$ is bounded in $L^2(\tau+\epsilon,t;D(A))$, and*
(iii) *the set $\{w^{(m)\prime}(\cdot;\tau,w_\tau)\mid w_\tau\in\mathcal{B}\}$ is bounded in $L^2(\tau+\epsilon,t;\widehat{H})$, where $w^{(m)\prime}(\theta)=\frac{\mathrm{d}w^m(\theta)}{\mathrm{d}\theta}$.*

Proof For any fixed bounded set $\mathcal{B}\subset\mathcal{C}_\gamma(\widehat{H}),\tau\in\mathbb{R},\epsilon>0,t>\tau+\epsilon$, and $w_\tau=\phi(s)\in\mathcal{C}_\gamma(\widehat{H})$ multiplying (3.2) by $\beta_{m,j}(t)$, summing up for j from 1 to m, and then using (2.4) and (2.9), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\big\|w^{(m)}(\theta)\big\|^2+\delta_1\big\|w^{(m)}(\theta)\big\|_{\widehat{V}}^2\\&\quad\leq\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\big\|w^{(m)}(\theta)\big\|^2+\big\langle Aw^{(m)}(\theta),w^{(m)}(\theta)\big\rangle+\big\langle N\big(w^{(m)}\big),w^{(m)}(\theta)\big\rangle\\&\quad=\big(F(\theta),w^{(m)}(\theta)\big)+\big(G\big(t,w_\theta^{(m)}\big),w^{(m)}\big)\\&\quad\leq\big\|F(\theta)\big\|\big\|w^{(m)}(\theta)\big\|+L_G\big\|w_\theta^{(m)}\big\|_\gamma\big\|w^{(m)}(\theta)\big\|\\&\quad\leq\frac{\delta_1}{2}\big\|w^{(m)}(\theta)\big\|_{\widehat{V}}^2+\frac{1}{2\delta_1}\big\|F(\theta)\big\|^2+L_G\big\|w_\theta^{(m)}\big\|_\gamma^2,\end{aligned}$$

where we also used assumption **(A1)**, the Cauchy-Schwarz inequality, the Young inequality, and the facts

$$\|w^{(m)}(\theta)\| \le \|w^{(m)}(\theta)\|_{\widehat{V}} \quad \text{and} \quad \|w^{(m)}(\theta)\| \le \sup_{s\le 0} e^{\gamma s}\|w^{(m)}(\theta+s)\| = \|w_\theta^{(m)}\|_\gamma. \tag{3.4}$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\|w^{(m)}(\theta)\|^2 + \delta_1\|w^{(m)}(\theta)\|_{\widehat{V}}^2 \le \frac{1}{\delta_1}\|F(\theta)\|^2 + 2L_G\|w_\theta^{(m)}\|_\gamma^2. \tag{3.5}$$

Integrating this inequality from τ to t, for any $t \ge \tau$, we have

$$\begin{aligned}
&\|w^{(m)}(t)\|^2 + \delta_1\int_\tau^t \|w^{(m)}(\theta)\|_{\widehat{V}}^2\,\mathrm{d}\theta \\
&\quad \le \|w^{(m)}(\tau)\|^2 + \frac{1}{\delta_1}\int_\tau^t \|F(\theta)\|^2\,\mathrm{d}\theta + 2L_G\int_\tau^t \|w_\theta^{(m)}\|_\gamma^2\,\mathrm{d}\theta.
\end{aligned} \tag{3.6}$$

Multiplying (3.2) by $\lambda_j v_j$, where λ_j is the eigenvalue associated with the eigenvector v_j, and summing from $j = 1$ to m, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\theta}\langle Aw^{(m)}(\theta), w^{(m)}(\theta)\rangle + \|Aw^{(m)}(\theta)\|^2 + \langle B(u^{(m)}, w^{(m)}), Aw^{(m)}(\theta)\rangle \\
&\quad + \langle N(w^{(m)}(\theta)), Aw^{(m)}(\theta)\rangle \\
&\quad = (F(\theta), Aw^{(m)}(\theta)) + (G(\theta, w_\theta^{(m)}), Aw^{(m)}(\theta)), \quad \theta \in (\tau, t].
\end{aligned} \tag{3.7}$$

On one hand, from (2.6), (2.8), Young's inequality, and the facts $\|\nabla u^{(m)}\| \le \|\nabla w^{(m)}\|$ and $\|u^{(m)}\| \le \|w^{(m)}\|$ it follows that

$$\begin{aligned}
&|\langle B(u^{(m)}(\theta), w^{(m)}(\theta)), Aw^{(m)}(\theta)\rangle| \\
&\quad \le \lambda\|u^{(m)}\|^{\frac{1}{2}}\|\nabla u^{(m)}\|^{\frac{1}{2}}\|\nabla w^{(m)}\|^{\frac{1}{2}}\|Aw^{(m)}\|^{\frac{1}{2}}\|Aw^{(m)}\| \\
&\quad \le \lambda\|w^{(m)}\|^{\frac{1}{2}}\|w^{(m)}\|_{\widehat{V}}\|Aw^{(m)}\|^{\frac{3}{2}} \\
&\quad \le \frac{1}{4}\|Aw^{(m)}(\theta)\|^2 + \frac{27\lambda^4}{4}\|w^{(m)}(\theta)\|^2\|w^{(m)}(\theta)\|_{\widehat{V}}^4
\end{aligned} \tag{3.8}$$

and

$$|\langle N(w^{(m)}(\theta)), Aw^{(m)}(\theta)\rangle| \le \frac{1}{4}\|Aw^{(m)}(\theta)\|^2 + c^2(\nu_r)\|w^{(m)}(\theta)\|_{\widehat{V}}^2. \tag{3.9}$$

On the other hand,

$$(F(\theta), Aw^{(m)}(\theta)) \le \|F(\theta)\|\|Aw^{(m)}(\theta)\| \le 2\|F(\theta)\|^2 + \frac{1}{8}\|Aw^{(m)}(\theta)\|^2 \tag{3.10}$$

and

$$\begin{aligned}
(G(\theta, w_\theta^{(m)}), Aw^{(m)}(\theta)) &\le \|G(\theta, w_\theta^{(m)})\|\|Aw^{(m)}(\theta)\| \le L_G\|w_\theta^{(m)}\|_\gamma\|Aw^{(m)}(\theta)\| \\
&\le 2L_G^2\|w_\theta^{(m)}\|_\gamma^2 + \frac{1}{8}\|Aw^{(m)}(\theta)\|^2.
\end{aligned} \tag{3.11}$$

Substituting (3.8)-(3.11) into (3.7) and using (2.1), we get

$$\begin{aligned}
&2\frac{\mathrm{d}}{\mathrm{d}\theta}\langle Aw^{(m)}(\theta), w^{(m)}(\theta)\rangle + \frac{1}{2}\|Aw^{(m)}(\theta)\|^2 \\
&\le \frac{27\lambda^4}{2}\|w^{(m)}(\theta)\|^2\|w^{(m)}(\theta)\|_{\widehat{V}}^4 \\
&\quad + 2c^2(\nu_r)\|w^{(m)}(\theta)\|_{\widehat{V}}^2 + 4\|F(\theta)\|^2 + 4L_G^2\|w_\theta^{(m)}\|_\gamma^2 \\
&\le \left(\frac{27c_2\lambda^4}{2}\|w^{(m)}(\theta)\|^2\|w^{(m)}(\theta)\|_{\widehat{V}}^2 + 2c_2c^2(\nu_r)\right)\langle Aw^{(m)}(\theta), w^{(m)}(\theta)\rangle \\
&\quad + 4\|F(\theta)\|^2 + 4L_G^2\|w_\theta^{(m)}\|_\gamma^2.
\end{aligned} \tag{3.12}$$

Set

$$\begin{aligned}
&H_m(\theta) := 2\langle Aw^{(m)}(\theta), w^{(m)}(\theta)\rangle, \\
&I(\theta) := 4\|F(\theta)\|^2 + 4L_G^2\|w_\theta^{(m)}\|_\gamma^2, \\
&K_m(\theta) := \frac{27c_2\lambda^4}{2}\|w^{(m)}(\theta)\|^2\|w^{(m)}(\theta)\|_{\widehat{V}}^2 + 2c_2c^2(\nu_r).
\end{aligned}$$

Then, (3.12) yields that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}H_m(\theta) \le K_m(\theta)H_m(\theta) + I(\theta). \tag{3.13}$$

Applying the Gronwall inequality to (3.13), for $\tau \le \tilde{r} \le s \le t$, we have

$$H_m(s) \le \left(H_m(\tilde{r}) + \int_\tau^t I(\theta)\,\mathrm{d}\theta\right)\exp\left\{\int_\tau^t K_m(\theta)\,\mathrm{d}\theta\right\}. \tag{3.14}$$

Integrating this inequality for $\tilde{r}$ from τ to s, we obtain

$$(s-\tau)H_m(s) \le \left(\int_\tau^s H_m(\tilde{r})\,\mathrm{d}r + (s-\tau)\int_\tau^t I(\theta)\,\mathrm{d}\theta\right)\exp\left\{\int_\tau^t K_m(\theta)\,\mathrm{d}\theta\right\}.$$

In particular, for any $\tau + \epsilon \le s \le t$, $n \ge 1$, we have

$$H_m(s) \le \left(\frac{1}{\epsilon}\int_\tau^t H_m(\tilde{r})\,\mathrm{d}r + \int_\tau^t I(\theta)\,\mathrm{d}\theta\right)\exp\left\{\int_\tau^t K_m(\theta)\,\mathrm{d}\theta\right\}. \tag{3.15}$$

By (3.6) we have

$$\begin{aligned}
\int_\tau^t K_m(\theta)\,\mathrm{d}\theta &= \int_\tau^t \left(\frac{27c_2\lambda^4}{2}\|w^{(m)}(\theta)\|^2\|w^{(m)}(\theta)\|_{\widehat{V}}^2 + 2c_2c^2(\nu_r)\right)\mathrm{d}\theta \\
&\le \frac{27c_2\lambda^4}{2}\sup_{\theta\in[\tau,t]}\|w^{(m)}(\theta)\|^2\int_\tau^t\|w^{(m)}(\theta)\|_{\widehat{V}}^2\,\mathrm{d}\theta + 2c_2c^2(\nu_r)(t-\tau) \\
&\le \frac{27c_2\lambda^4}{2\delta_1}\left(\|w^{(m)}(\tau)\|^2 + \frac{1}{\delta_1}\int_\tau^t\|F(\theta)\|^2\,\mathrm{d}\theta + 2L_G\int_\tau^t\|w_\theta^{(m)}\|_\gamma^2\,\mathrm{d}\theta\right)^2 \\
&\quad + 2c_2c^2(\nu_r)(t-\tau).
\end{aligned} \tag{3.16}$$

From (2.1) and (3.6) it follows that

$$\begin{aligned}\int_\tau^t H_m(\tilde{r})\,\mathrm{d}\tilde{r} &= 2\int_\tau^t \langle Aw^{(m)}(\tilde{r}), w^{(m)}(\tilde{r})\rangle\,\mathrm{d}\tilde{r} \le 2c_1^{-1}\int_\tau^t \|w^{(m)}(\tilde{r})\|_{\widehat{V}}^2\,\mathrm{d}\tilde{r} \\ &\le 2c_1^{-1}\delta_1^{-1}\|w^{(m)}(\tau)\|^2 + 2c_1^{-1}\delta_1^{-2}\int_\tau^t \|F(\theta)\|^2\,\mathrm{d}\theta \\ &\quad + 4c_1^{-1}\delta_1^{-1}L_G\int_\tau^t \|w_\theta^{(m)}\|_\gamma^2\,\mathrm{d}\theta. \end{aligned} \tag{3.17}$$

In addition,

$$(t-\tau)\int_\tau^t I(\theta)\,\mathrm{d}\theta = 4(t-\tau)\int_\tau^t \|F(\theta)\|^2\,\mathrm{d}\theta + 4L_G^2(t-\tau)\int_\tau^t \|w_\theta^{(m)}\|_\gamma^2\,\mathrm{d}\theta. \tag{3.18}$$

Similarly to (2.11), we have

$$\|w_\theta^{(m)}\|_\gamma^2 \le e^{-(\delta_1\lambda_1-2L_G)(\theta-\tau)}\|\phi(s)\|_\gamma^2 + \frac{2}{\delta_1}\int_\tau^\theta e^{-(\delta_1\lambda_1-2L_G)(\theta-r)}\|F(r)\|^2\,\mathrm{d}r. \tag{3.19}$$

Thus,

$$\begin{aligned}\int_\tau^t \|w_\theta^{(m)}\|_\gamma^2\,\mathrm{d}\theta &\le \frac{2}{\delta_1(\delta_1\lambda_1-2L_G)}\int_\tau^t \left[e^{(\delta_1\lambda_1-2L_G)(r-\tau)} - e^{(\delta_1\lambda_1-2L_G)(r-t)}\right]\|F(r)\|^2\,\mathrm{d}r \\ &\quad + \frac{1-e^{-(\delta_1\lambda_1-2L_G)(t-\tau)}}{\delta_1\lambda_1-2L_G}\|\phi(s)\|_\gamma^2. \end{aligned} \tag{3.20}$$

Taking (2.1), (3.15)-(3.18), and (3.20) into account, we complete the proof of assertion (i).

Now integrating (3.12) for θ between $\tau+\epsilon$ and t, we get

$$\begin{aligned}&\int_{\tau+\epsilon}^t \|Aw^{(m)}(\theta)\|^2\,\mathrm{d}\theta \\ &\quad\le 2c_1^{-1}\|w^{(m)}(\tau+\epsilon)\|_{\widehat{V}}^2 + 27\lambda^4\int_{\tau+\epsilon}^t \|w^{(m)}(\theta)\|^2\|w^{(m)}(\theta)\|_{\widehat{V}}^4\,\mathrm{d}\theta \\ &\qquad + 4c^2(\nu_r)\int_{\tau+\epsilon}^t \|w^{(m)}(\theta)\|_{\widehat{V}}^2\,\mathrm{d}\theta + 8\int_{\tau+\epsilon}^t \|F(\theta)\|^2\,\mathrm{d}\theta + 8L_G^2\int_{\tau+\epsilon}^t \|w_\theta^{(m)}\|_\gamma^2\,\mathrm{d}\theta, \end{aligned}$$

which, together with assertion (i), (3.6), and (3.20), implies assertion (ii).

Finally, multiplying (3.2) by $\beta'_{m,j}(t)$, summing them from $j=1$ to n, and replacing the variable t with θ, we obtain

$$\begin{aligned}&\|w^{(m)'}(\theta)\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\langle Aw^{(m)}(\theta), w^{(m)}(\theta)\rangle + \langle B(u^{(m)}(\theta), w^{(m)}(\theta)), w^{(m)'}(\theta)\rangle \\ &\qquad + \langle N(w^{(m)}(\theta)), w^{(m)'}(\theta)\rangle \\ &\quad = (F(\theta,x), w^{(m)'}(\theta)) + (G(\theta, w_\theta^{(m)}), w^{(m)'}(\theta)). \end{aligned} \tag{3.21}$$

Observe that

$$\begin{aligned}
&\big(F(\theta,x),w^{(m)'}(\theta)\big)+\big(G\big(\theta,w_\theta^{(m)}\big),w^{(m)'}(\theta)\big)\\
&\quad\le\big(\|F(\theta)\|+\big\|G\big(\theta,w_\theta^{(m)}\big)\big\|\big)\big\|w^{(m)'}(\theta)\big\|\\
&\quad\le 2\|F(\theta)\|^2+2L_G^2\big\|w_\theta^{(m)}\big\|_\gamma^2+\frac{1}{4}\big\|w^{(m)'}(\theta)\big\|^2.
\end{aligned}\tag{3.22}$$

By Lemma 2.1 we deduce that

$$\begin{aligned}
&\big|\big\langle B\big(u^{(m)}(\theta),w^{(m)}(\theta)\big),w^{(m)'}(\theta)\big\rangle\big|\\
&\quad\le\lambda\big\|u^{(m)}(\theta)\big\|^{\frac{1}{2}}\big\|\nabla u^{(m)}(\theta)\big\|^{\frac{1}{2}}\big\|\nabla w^{(m)}(\theta)\big\|^{\frac{1}{2}}\big\|Aw^{(m)}(\theta)\big\|^{\frac{1}{2}}\big\|w^{(m)'}(\theta)\big\|\\
&\quad\le\lambda\big\|w^{(m)}(\theta)\big\|^{\frac{1}{2}}\big\|w^{(m)}(\theta)\big\|_{\widehat{V}}\big\|Aw^{(m)}(\theta)\big\|^{\frac{1}{2}}\big\|w^{(m)'}(\theta)\big\|\\
&\quad\le\lambda^2\big\|w^{(m)}(\theta)\big\|\big\|w^{(m)}(\theta)\big\|_{\widehat{V}}^2\big\|Aw^{(m)}(\theta)\big\|+\frac{1}{4}\big\|w^{(m)'}(\theta)\big\|^2
\end{aligned}\tag{3.23}$$

and

$$\big|\big\langle N\big(w^{(m)}(\theta)\big),w^{(m)'}(\theta)\big\rangle\big|\le c^2(\nu_r)\big\|w^{(m)}(\theta)\big\|_{\widehat{V}}^2+\frac{1}{4}\big\|w^{(m)'}(\theta)\big\|^2.\tag{3.24}$$

It follows from (3.21)-(3.24) that

$$\begin{aligned}
&\big\|w^{(m)'}(\theta)\big\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\big\langle Aw^{(m)}(\theta),w^{(m)}(\theta)\big\rangle\\
&\quad=-\big\langle B\big(u^{(m)}(\theta),w^{(m)}(\theta)\big),w^{(m)'}(\theta)\big\rangle-\big\langle N\big(w^{(m)}(\theta)\big),w^{(m)'}(\theta)\big\rangle\\
&\qquad+\big(F(\theta,x),w^{(m)'}(\theta)\big)+\big(G\big(\theta,w_\theta^{(m)}\big),w^{(m)'}(\theta)\big)\\
&\quad\le\frac{3}{4}\big\|w^{(m)'}(\theta)\big\|^2+2\|F(\theta)\|^2+2L_G^2\big\|w_\theta^{(m)}\big\|_\gamma^2+\lambda^2\big\|w^{(m)}(\theta)\big\|\big\|w^{(m)}(\theta)\big\|_{\widehat{V}}^2\big\|Aw^{(m)}(\theta)\big\|\\
&\qquad+c^2(\nu_r)\big\|w^{(m)}(\theta)\big\|_{\widehat{V}}^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\big\|w^{(m)'}(\theta)\big\|^2+2\frac{\mathrm{d}}{\mathrm{d}\theta}\big\langle Aw^{(m)}(\theta),w^{(m)}(\theta)\big\rangle\\
&\quad\le 8\|F(\theta)\|^2+8L_G^2\big\|w_\theta^{(m)}\big\|_\gamma^2+4\lambda^2\big\|w^{(m)}(\theta)\big\|_{\widehat{V}}^3\big\|Aw^{(m)}(\theta)\big\|+4c^2(\nu_r)\big\|w^{(m)}(\theta)\big\|_{\widehat{V}}^2.
\end{aligned}$$

Integrating this inequality and using (2.1), we get that

$$\begin{aligned}
\int_{\tau+\epsilon}^t\big\|w^{(m)'}(\theta)\big\|^2\,\mathrm{d}\theta&\le 2c_1^{-1}\big\|w^{(m)}(\tau+\epsilon)\big\|^2+8\int_{\tau+\epsilon}^t\|F(\theta)\|^2\,\mathrm{d}\theta+8L_G^2\int_{\tau+\epsilon}^t\big\|w_\theta^{(m)}\big\|_\gamma^2\,\mathrm{d}\theta\\
&\quad+2\lambda^2\sup_{\theta\in[\tau+\epsilon,t]}\big\|w^{(m)}(\theta)\big\|_{\widehat{V}}^2\int_{\tau+\epsilon}^t\big(\big\|w^{(m)}(\theta)\big\|_{\widehat{V}}^2+\big\|Aw^{(m)}(\theta)\big\|^2\big)\,\mathrm{d}\theta\\
&\quad+4c^2(\nu_r)\int_{\tau+\epsilon}^t\big\|w^{(m)}(\theta)\big\|_{\widehat{V}}^2\,\mathrm{d}\theta,
\end{aligned}$$

which, together with (3.6), (3.20), and assertions (i)-(ii), gives assertion (iii). The proof is complete. □

Corollary 3.1 *Under the conditions of Lemma* 3.1, *for any bounded set* $\mathcal{B} \subset \mathcal{C}_\gamma(\widehat{H})$ *and any* $\tau \in \mathbb{R}, \epsilon > 0, t \geq \tau + \epsilon$, *the set* $\bigcup_{s\in[\tau+\epsilon,t]} U(s,\tau)\mathcal{B}$ *is bounded in* $\widehat{V}$.

Proof In [25], the authors proved that, for any $w_\tau = \phi(s) \in \mathcal{C}_\gamma(\widehat{H})$, the Galerkin approximation solutions $\{w^{(m)}(\cdot;\tau,w_\tau)\}_{m\geq 1}$ converge weakly to $w(\cdot;\tau,w_\tau)$ in $L^2(\tau,t;\widehat{V})$ and $w(\cdot;\tau,w_\tau) \in \mathcal{C}([\tau,t];\widehat{H})$. So Corollary 3.1 is a straightforward consequence of Lemma 2.4 and Lemma 3.1(i). □

By increasing the regularity of $F(t,x)$ and $G(t,w_t)$ properly we can improve the results of Lemma 3.1.

Lemma 3.2 *Assume that* **(A1)** *and* **(A2)** *hold. Then, for any bounded set* $\mathcal{B} \subset \mathcal{C}_\gamma(\widehat{H})$ *and any* $\tau \in \mathbb{R}, \epsilon > 0, t \geq \tau + \epsilon$, *the following properties are fulfilled*:

(iv) *the set* $\{w^{(m)'}(s;\tau,w_\tau) \mid s \in [\tau+2\epsilon,t], w_\tau = \phi(s) \in \mathcal{B}\}$ *is bounded in* $\widehat{H}$;

(v) *the set* $\{w^{(m)}(s;\tau,w_\tau) \mid s \in [\tau+2\epsilon,t], w_\tau = \phi(s) \in \mathcal{B}\}$ *is bounded in* $D(A) = \widehat{V} \cap (H^2)^3$.

Proof Without loss of generality, we consider a fixed bounded set $\mathcal{B} \subset \mathcal{C}_\gamma(\widehat{H})$. Differentiating equation (3.2) with respect to time and multiplying the resulting equation by $\beta'_{m,j}(t)$ and summing them from $j = 1$ to m, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|w^{(m)'}(t)\big\|^2 + \big\langle Aw^{(m)'}(t), w^{(m)'}(t)\big\rangle + \big\langle\big(B\big(u^{(m)}(t), w^{(m)}(t)\big)\big)', w^{(m)'}(t)\big\rangle\\&\quad + \big(N\big(w^{(m)'}(t)\big), w^{(m)'}(t)\big)\\&\quad = \big(F'(t), w^{(m)'}(t)\big) + \big(\big(G(t,w_t)\big)', w^{(m)'}(t)\big).\end{aligned} \tag{3.25}$$

In the following, we make a more detailed estimate for each term in (3.25). First, from Lemma 2.1 and the Cauchy-Schwarz inequality it is easy to see that

$$\delta_1\big\|w^{(m)'}(t)\big\|_{\widehat{V}}^2 \leq \big\langle Aw^{(m)'}(t), w^{(m)'}(t)\big\rangle + \big\langle N\big(w^{(m)'}(t)\big), w^{(m)'}(t)\big\rangle \tag{3.26}$$

and

$$\begin{aligned}&\big|\big\langle\big(B\big(u^{(m)}(t), w^{(m)}(t)\big)\big)', w^{(m)'}(t)\big\rangle\big|\\&\quad = \big|\big\langle B\big(u^{(m)'}(t), w^{(m)}(t)\big), w^{(m)'}(t)\big\rangle\big|\\&\quad \leq \lambda\big\|u^{(m)'}(t)\big\|^{\frac{1}{2}}\big\|\nabla u^{(m)'}(t)\big\|^{\frac{1}{2}}\big\|w^{(m)'}(t)\big\|^{\frac{1}{2}}\big\|\nabla w^{(m)'}(t)\big\|^{\frac{1}{2}}\big\|\nabla w^{(m)}(t)\big\|\\&\quad \leq \lambda\big\|w^{(m)'}(t)\big\|\big\|w^{(m)}(t)\big\|_{\widehat{V}}\big\|w^{(m)'}(t)\big\|_{\widehat{V}}\\&\quad \leq \delta_1^{-1}\lambda^2\big\|w^{(m)'}(t)\big\|^2\big\|w^{(m)}(t)\big\|_{\widehat{V}}^2 + \frac{\delta_1}{4}\big\|w^{(m)'}(t)\big\|_{\widehat{V}}^2.\end{aligned} \tag{3.27}$$

Then, under assumption **(A2)(I)** and (3.4), we have

$$\begin{aligned}&\big(F'(t), w^{(m)'}(t)\big) + \big(\big(G(t,w_t)\big)', w^{(m)'}(t)\big)\\&\quad \leq 2\delta_1^{-1}\big\|F'(t)\big\|^2 + 2\delta_1^{-1}\tilde{L}_G^2\big\|w_t^{(m)'}\big\|_\gamma^2 + \frac{\delta_1}{4}\big\|w^{(m)'}(t)\big\|_{\widehat{V}}^2.\end{aligned} \tag{3.28}$$

Now, taking (3.25)-(3.28) into account, we obtain

$$\begin{aligned}&\frac{d}{dt}\|w^{(m)'}(t)\|^2+\delta_1\|w^{(m)'}(t)\|_{\widehat{V}}^2\\&\quad\le 2\delta_1^{-1}\lambda^2\|w^{(m)'}(t)\|^2\|w^{(m)}(t)\|_{\widehat{V}}^2+4\delta_1^{-1}\|F'(t)\|^2+4\delta_1^{-1}\tilde{L}_G^2\|w_t^{(m)'}\|_\gamma^2.\end{aligned}$$

Replacing the variable t with θ and integrating it between r and s, we see that, for all $\tau\le r\le s\le t$,

$$\begin{aligned}&\|w^{(m)'}(s)\|^2+\delta_1\int_r^s\|w^{(m)'}(\theta)\|_{\widehat{V}}^2\,d\theta\\&\quad\le\|w^{(m)'}(r)\|^2+2\delta_1^{-1}\lambda^2\int_r^t\|w^{(m)'}(\theta)\|^2\|w^{(m)}(\theta)\|_{\widehat{V}}^2\,d\theta\\&\qquad+4\delta_1^{-1}\int_r^t\|F'(\theta)\|^2\,d\theta+4\delta_1^{-1}\tilde{L}_G^2\int_r^t\|w_\theta^{(m)'}\|_\gamma^2\,d\theta.\end{aligned}\tag{3.29}$$

Particularly, for all $\tau+2\epsilon\le r+\epsilon\le s\le t$, we have

$$\begin{aligned}\|w^{(m)'}(s)\|^2&\le\|w^{(m)'}(r)\|^2+2\delta_1^{-1}\lambda^2\sup_{\theta\in[\tau+\epsilon,t]}\|w^{(m)}(\theta)\|_{\widehat{V}}^2\int_{\tau+\epsilon}^t\|w^{(m)'}(\theta)\|^2\,d\theta\\&\quad+4\delta_1^{-1}\int_{\tau+\epsilon}^t\|F'(\theta)\|^2\,d\theta+4\delta_1^{-1}\tilde{L}_G^2\int_{\tau+\epsilon}^t\|w_\theta^{(m)'}\|_\gamma^2\,d\theta.\end{aligned}\tag{3.30}$$

Integrating this inequality with respect to r between $\tau+\epsilon$ and s, we have

$$\begin{aligned}\|w^{(m)'}(s)\|^2&\le\frac{1}{s-\tau-\epsilon}\int_{\tau+\epsilon}^t\|w^{(m)'}(r)\|^2\,dr\\&\quad+\frac{2\lambda^2}{\delta_1}\sup_{\theta\in[\tau+\epsilon,t]}\|w^{(m)}(\theta)\|_{\widehat{V}}^2\int_{\tau+\epsilon}^t\|w^{(m)'}(\theta)\|^2\,d\theta\\&\quad+4\delta_1^{-1}\int_{\tau+\epsilon}^t\|F'(\theta)\|^2\,d\theta+4\delta_1^{-1}\tilde{L}_G^2\int_{\tau+\epsilon}^t\|w_\theta^{(m)'}\|_\gamma^2\,d\theta\end{aligned}\tag{3.31}$$

for all $\tau+2\epsilon\le r+\epsilon\le s\le t$. Then, it is not difficult to get that, by using the same proof as (3.21) in [25],

$$\|w_\theta^{(m)'}\|_\gamma^2\le e^{-\delta_1\lambda_1(\theta-\tau)}\|\phi'(s)\|_\gamma^2+\frac{2}{\delta_1}\int_\tau^\theta e^{-(\delta_1\lambda_1-2\tilde{L}_G)(\theta-r)}\|F'(r)\|^2\,dr.\tag{3.32}$$

Thus, we deduce that

$$\begin{aligned}\int_\tau^t\|w_\theta^{(m)'}\|_\gamma^2\,d\theta&\le\frac{2}{\delta_1(\delta_1\lambda_1-2\tilde{L}_G)}\int_\tau^t\big[e^{(\delta_1\lambda_1-2\tilde{L}_G)(r-\tau)}-e^{(\delta_1\lambda_1-2\tilde{L}_G)(r-t)}\big]\|F'(r)\|^2\,dr\\&\quad+\frac{1-e^{-\delta_1\lambda_1(t-\tau)}}{\delta_1\lambda_1}\|\phi'(s)\|_\gamma^2,\end{aligned}\tag{3.33}$$

which, combined with assumption **(A2)(II)**, yields the boundedness of $\int_\tau^t\|w_\theta^{(m)'}\|_\gamma^2\,d\theta$. Consequently, property (iv) follows from (3.31), (3.33), assumption **(A2)**, and Lemma 3.1.

Next, we prove property (v). Multiplying (3.2) by $\lambda_j \beta_{m,j}(t)$ and summing the resulting equation from $j = 1$ to m, we obtain

$$\begin{aligned}&\big(w^{(m)'}(\theta), Aw^{(m)}(\theta)\big) + \big\|Aw^{(m)}(\theta)\big\|^2 + \big\langle B\big(u^{(m)}, w^{(m)}\big), Aw^{(m)}(\theta)\big\rangle + \big\langle N\big(w^{(m)}\big), Aw^{(m)}(\theta)\big\rangle \\ &\quad = \big(F(\theta), Aw^{(m)}(\theta)\big) + \big(G\big(\theta, w_\theta^{(m)}\big), Aw^{(m)}(\theta)\big).\end{aligned} \tag{3.34}$$

Observe that

$$\big(w^{(m)'}(\theta), Aw^{(m)}(\theta)\big) \le 2\big\|w^{(m)'}(\theta)\big\|^2 + \frac{1}{8}\big\|Aw^{(m)}(\theta)\big\|^2, \tag{3.35}$$

which, together with (3.8)-(3.11) and (3.34), gives that, for any $\theta > \tau$,

$$\begin{aligned}\big\|Aw^{(m)}(\theta)\big\|^2 \le{}& 54\lambda^4\big\|w^{(m)}(\theta)\big\|^2\big\|w^{(m)}(\theta)\big\|_{\widehat{V}}^4 + 8c^2(\nu_r)\big\|w^{(m)}(\theta)\big\|_{\widehat{V}}^2 + 16\big\|w^{(m)'}(\theta)\big\|^2 \\ &+ 16\big\|F(\theta)\big\|^2 + 16L_G^2\big\|w_\theta^{(m)}\big\|_\gamma^2.\end{aligned} \tag{3.36}$$

Since $W_{\text{loc}}^{1,2}(\mathbb{R};\widehat{H}) \hookrightarrow \mathcal{C}(\mathbb{R};\widehat{H})$ and $f \in W_{\text{loc}}^{1,2}(\mathbb{R};\widehat{H})$, so $f \in \mathcal{C}(\mathbb{R};\widehat{H})$, which, together with (3.19), (3.36), Lemmas 3.1(i) and 3.2(iv), implies property (v). The proof is complete. □

At this state, we give the proof of the main results of this paper.

Proof of Theorem 1.1 (1) According to Lemma 3.1, following the standard diagonal procedure, there exists a function $w(\cdot)$ such that (by extracting a subsequence if necessary)

$$w^{(m)}(\cdot) \rightharpoonup^* w(\cdot) \text{ weakly star in } L^\infty(\tau+\epsilon, t; \widehat{V}), \tag{3.37}$$

$$w^{(m)}(\cdot) \rightharpoonup w(\cdot) \text{ weakly in } L^2\big(\tau+\epsilon, t; D(A)\big), \tag{3.38}$$

$$w^{(m)'}(\cdot) \rightharpoonup w'(\cdot) \text{ weakly in } L^2(\tau+\epsilon, t; \widehat{H}). \tag{3.39}$$

Furthermore, $w(\cdot) \in \mathcal{C}([t+\epsilon, t]; \widehat{V})$. It follows from the uniqueness of the limit function that $w(\cdot)$ is a weak solution of system (2.10). Then, part (1) of Theorem 1.1 is a consequence of Lemmas 2.4 and 3.2.

(2) It is not difficult to see that if $\tau < T_1 - 1$ is fixed, then

$$\bigcup_{t\in[T_1,T_2]} \mathcal{A}(t) \subset \bigcup_{t\in[\tau+1,T_2]} U(t,\tau)\mathcal{A}(\tau).$$

Consequently, combining Lemma 3.2 and part (1), we obtain the boundedness result of part (2). The proof is complete. □

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Competing interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions
The authors declare that they contributed equally in this article and read and approved the final manuscript.

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