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Remarks on L^2 decay of solutions for the third-grade non-Newtonian fluid flows in $\mathbb{R}^3$

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Abstract

This paper is concerned with the improved L^2 decay for solutions of a class of the third-grade non-Newtonian fluid flows in $\mathbb{R}^3$. By developing the classic Fourier splitting methods, we prove the non-uniform decay of solutions when $u_0 \in L^2(\mathbb{R}^3)$ and improve algebraic decay rates of solutions as $(1+t)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{2})-\frac{1}{2}}$ when the initial data satisfy some moment condition. The results extend the previous result by Zhao (Nonlinear Anal., Real World Appl. 15:229-238, 2014).

Keywords: non-Newtonian fluid flows; improved L^2 decay

1 Introduction

Mathematical models for fluid dynamics have been attracting more and more attention in theoretical and computational studies. Naiver-Stokes equations [2] are generally accepted as proving an accurate model for the incompressible motion of viscous fluids in many practical situations, where the constitutive relation is linear due to the Stokes hypotheses. However, in industrial application, some fluids which exhibit the nonlinear constitutive relation, such as liquid crystals, some polymers, some oils and so on (refer to [3, 4]), cannot be modeled by the classic Navier-Stokes equations. In order to explain a lot of non-standard features such as normal stress effects, rod climbing, shear-thinning and shear-thickening, Rivlin and Ericksen [5] introduced a class of n-grade non-Newtonian fluid flows where the stress tensor is a polynomial of degree n. The basic constitutive relation is given by

$$\mathcal{T} = -pI + F(A_1, A_2, A_3, \dots, A_n). \tag{1.1}$$

$\mathcal{T}$ is the Cauchy stress tensor, p is the indeterminate part of the stress and F is an isotropic polynomial of degree n. $A_1, A_2, A_3, \dots, A_n$ are the first n Rivlin-Erickson tensors defined recursively by

$$A_1 = A = 2D, \qquad A_{k+1} = \frac{d}{dt}A_k + L^t A_k + A_k L,$$

where

$$\frac{d}{dt} = \partial_t + u \cdot \nabla u$$

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denotes the material derivative and

$$L = (\partial_j u_i)_{i,j}, \qquad L^t = (\partial_i u_j)_{i,j}, \qquad D = \frac{1}{2}(\partial_i u_j + \partial_j u_i)_{i,j}.$$

In this study, we consider the case of third-grade non-Newtonian fluid flows where the constitutive law is given

$$\mathcal{T} = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 + \beta |A_1|^2 A_1, \tag{1.2}$$

where μ is the viscosity, $\alpha_1, \alpha_2, \beta$ are material constants. These coefficients satisfy the following restriction conditions (for a more detailed thermodynamic analysis of model (1.2), one can refer to [6–8]):

$$\mu \geq 0, \qquad \alpha_1 \geq 0, \qquad \beta \geq 0 \quad \text{and} \quad |\alpha_1 + \alpha_2|^2 \leq 24\mu\beta. \tag{1.3}$$

In particular, when

$$\mu > 0, \qquad \alpha_1 = 0, \qquad \beta > 0 \quad \text{and} \quad \alpha = \alpha_2 \quad \text{with } |\alpha|^2 \leq 24\mu\beta, \tag{1.4}$$

the constitutive relation (1.2), together with the equation of motions

$$\frac{d}{dt} u = \operatorname{div} \mathcal{T},$$

yields the following third-grade non-Newtonian fluid flows:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + J(u) + K(u) + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(x,0) = u_0, \end{cases} \tag{1.5}$$

where

$$J(u) = -\alpha \operatorname{div}\big(A(u)^2\big),$$
$$K(u) = -\beta \operatorname{div}\big(\big|A(u)\big|^2 A(u)\big)$$

and

$$A(u) = (\partial_i u_j + \partial_j u_i)_{i,j}.$$

When $J(u) = K(u) = 0$, the third-grade non-Newtonian fluid flows (1.5) reduce the classic Navier-Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(x,0) = u_0. \end{cases} \tag{1.6}$$

For the classic Navier-Stokes equations, it should be mentioned that the problem on the regularity or finite time singularity for the weak solution still remains unsolved. However, the third-grade non-Newtonian fluid flows (1.5) considered here can be viewed as a singular perturbation of the Navier-Stokes equations. The presence of the nonlinear dissipation $K(u)$ in (1.5) more or less decreases the singularity of system (1.5). Hamza1 and Paicu [9] actually proved the existence and uniqueness of the weak solution of Cauchy problem of (1.5).

Proposition 1.1 (Hamza1 and Paicu [9]) *Let $u_0 \in L^2(\mathbb{R}^3)$ and $|\alpha| < \sqrt{2\mu\beta}$, then the third-grade non-Newtonian fluid flows* (1.5) *has a unique global weak solution such that*

$$u \in L^\infty\big(0,\infty;L^2\big(\mathbb{R}^3\big)\big) \cap L^2\big(0,\infty;H^1\big(\mathbb{R}^3\big)\big) \cap L^4\big(0,\infty;W^{1,4}\big(\mathbb{R}^3\big)\big), \quad \forall T > 0 \tag{1.7}$$

and

$$\big\|u(t)\big\|_{L^2}^2 + C\int_0^t \big\|\nabla u(s)\big\|_{L^2}^2\,ds + C\int_0^t \big\|\nabla u(s)\big\|_{L^4}^4\,ds \le \|u_0\|_{L^2}^2 \quad \textit{for } t > 0. \tag{1.8}$$

The aim of this study is to investigate the algebraic L^2 decay for the solutions of the third-grade non-Newtonian fluid flows (1.5). When the initial velocity $u_0 \in L^2(\mathbb{R}^3)$, we first investigate that the L^2 norm of the weak solution tends to zero but not uniformly, that is, there are solutions with arbitrarily slow decay. When $u_0 \in L^2(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$ for $1 \le r < 2$ and satisfies $\int_{\mathbb{R}^3} |xu_0(x)|^r\,dx < \infty$, we prove the improved algebraic L^2 decay rates of the weak solutions as $(1+t)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{2})-\frac{1}{2}}$. Compared with the time decay problem of the classic Navier-Stokes equations [10–13], on the one hand, the additional difficulty on the decay estimates of the third-grade non-Newtonian fluid flows (1.5) is to investigate the $L^p - L^q$ estimates of the heat semigroup acting on the nonlinear terms. One may also refer to some interesting time decay results of some non-Newtonian flows by several authors [14, 15] and [16].

Let us end this Introduction by the notations. Throughout this paper, C stands for a generic positive constant which may vary from line to line. $L^p(\mathbb{R}^3)$ with $1 \le p \le \infty$ denotes the usual Lebesgue space of all L^p integral functions associated with the norm

$$\|f\|_{L^p(\mathbb{R}^3)} = \begin{cases} (\int_{\mathbb{R}^3} |f(x)|^p\,dx)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}^3} |f(x)|, & p = \infty, \end{cases}$$

and $H^s(\mathbb{R}^3)$ with $s \in \mathbb{R}$ the fractional Sobolev space with

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}^3} |\xi|^{2s}|\hat{f}|^2\,d\xi\right)^{1/2},$$

where $\hat{f}$ or $F[f]$ denotes the Fourier transformation of $f(x)$.

2 Non-uniform L^2 decay of weak solutions

In this section, we plan to investigate the non-uniform L^2 decay of weak solutions of the zero-forced third-grade non-Newtonian fluid flows (1.5) when the initial data only lie in L^2. More precisely, we will show the following results.

Theorem 2.1 *Suppose that $u(x,t)$ is a weak solution of the zero-forced third-grade non-Newtonian fluid flows* (1.5) *with the initial data $u_0 \in L^2(\mathbb{R}^3)$, then the solution $u(x,t)$ has the following non-uniform asymptotic behavior*:

$$\|u(t)\|_2 \to 0, \quad \text{as } t \to \infty. \tag{2.1}$$

Remark 2.2 Our result here shows that the weak solutions of non-Newtonian fluid flows (1.5) decay arbitrarily slowly. Just like the classic Navier-Stokes equations [17], the solution in Theorem 2.1 also exhibits a special asymptotic behavior for this sort of non-Newtonian flows (1.5). However, compared with the Navier-Stokes equations, the additional difficulty here is that some more explicit estimates for nonlinear terms should be done.

Proof of Theorem 2.1 Since the third-grade non-Newtonian fluid flows (1.5) have a uniquely strong solution and the weak solutions are regular, we can deal with our problem directly by weak solutions although some regular derivation for solutions is required.

Multiplying both sides of the third-grade non-Newtonian fluid flows (1.5) by u and integrating on $\mathbb{R}^3$, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \mu\|\nabla u\|_2^2 - \beta\int_{\mathbb{R}^3} \operatorname{div}\bigl(|A(u)|^2 A(u)\bigr) u\,dx = \alpha\int_{\mathbb{R}^3} \operatorname{div}\bigl(A(u)^2\bigr) u\,dx, \tag{2.2}$$

where we used the following properties:

$$\int_{\mathbb{R}^3} (u\cdot\nabla u)u\,dx = 0.$$

Since

$$-\beta\int_{\mathbb{R}^3} \operatorname{div}\bigl(|A(u)|^2 A(u)\bigr) u\,dx = \frac{\beta}{2}\int_{\mathbb{R}^3} |A(u)|^4\,dx$$

and

$$\left|\alpha\int_{\mathbb{R}^3} \operatorname{div}\bigl(A(u)^2\bigr) u\,dx\right| \le \alpha\left|\int_{\mathbb{R}^3} \bigl(A(u)^2\bigr)\nabla u\,dx\right| \le \alpha\|A(u)\|_4^2\|\nabla u\|_2,$$

where we used Hölder's inequality in the last line.

Noting that

$$\varepsilon_0 = 1 - \sqrt{\frac{\alpha^2}{2\mu\beta}} > 0,$$

and applying Young's inequality, we have

$$\left|\alpha\int_{\mathbb{R}^3} \operatorname{div}\bigl(A(u)^2\bigr) u\,dx\right| \le \mu(1-\varepsilon_0)\|\nabla u\|_2^2 + \frac{\beta(1-\varepsilon_0)}{2}\|A(u)\|_4^4.$$

We now rewrite (2.2) as

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \mu\varepsilon_0\|\nabla u\|_2^2 + \frac{\beta\varepsilon_0}{2}\|A(u)\|_4^4 \le 0$$

or

$$\frac{d}{dt}\|u(t)\|_2^2 + 2\mu\varepsilon_0\|\nabla u(t)\|_2^2 \le 0. \tag{2.3}$$

Applying the Plancherel theorem to (2.3) gives

$$\frac{d}{dt}\|\widehat{u}(t)\|_2^2 + 2\mu\varepsilon_0\int_{\mathbb{R}^3}|\xi|^2\left|\widehat{u}(t)\right|^2\,d\xi \le 0. \tag{2.4}$$

Letting

$$B(t) = \left\{\xi \,\middle|\, |\xi| \le g^{-\frac{1}{2}}(t)\right\},$$

where the smooth function $g(t)$ will be chosen later, we have

$$\begin{aligned}
&2\mu\varepsilon_0\int_{\mathbb{R}^3}|\xi|^2\left|\widehat{u}(t)\right|^2\,d\xi \\
&\quad = 2\mu\varepsilon_0\int_{B(t)}|\xi|^2\left|\widehat{u}(t)\right|^2\,d\xi + 2\mu\varepsilon_0\int_{B(t)^c}|\xi|^2\left|\widehat{u}(t)\right|^2\,d\xi \\
&\quad \ge 2\mu\varepsilon_0\int_{B(t)^c}|\xi|^2\left|\widehat{u}(t)\right|^2\,d\xi \\
&\quad \ge 2\mu\varepsilon_0 g^{-1}(t)\int_{B(t)^c}\left|\widehat{u}(t)\right|^2\,d\xi \\
&\quad \ge 2\mu\varepsilon_0 g^{-1}(t)\|\widehat{u}\|_2^2 - 2\mu\varepsilon_0 g^{-1}(t)\int_{B(t)}\left|\widehat{u}(t)\right|^2\,d\xi.
\end{aligned}$$

The substitution of the above inequality into (2.4) gives

$$\frac{d}{dt}\|\widehat{u}(t)\|_2^2 + 2\mu\varepsilon_0 g^{-1}(t)\|\widehat{u}(t)\|_2^2 \le 2\mu\varepsilon_0 g^{-1}(t)\int_{B(t)}\left|\widehat{u}(\xi,t)\right|^2\,d\xi. \tag{2.5}$$

Now we need to estimate the right-hand side of (2.5). To do so, taking the Fourier transform of the third-grade non-Newtonian fluid flows (1.5) yields

$$\widehat{\partial_t u} + \mu|\xi|^2\widehat{u} = G(\xi,t), \tag{2.6}$$

where

$$G(\xi,t) = -\widehat{u\cdot\nabla u} - \widehat{J(u)} - \widehat{K(u)} - i\xi\widehat{P}.$$

We need to estimate $G(\xi,t)$ one by one. By integrating by parts, we have

$$\begin{aligned}
|\widehat{u\cdot\nabla u}| &= \left|\int_{\mathbb{R}^3}\sum_{j=1}^{3}\partial_{x_j}(u_j u)e^{-i\xi\cdot x}\,dx\right| \\
&= \left|\int_{\mathbb{R}^3}\sum_{j=1}^{3}u_j u\xi_j e^{-i\xi\cdot x}\,dx\right| \le C|\xi|\|u\|_2^2
\end{aligned}$$

and

$$\big|\widehat{J(u)}\big| = \big|\alpha \operatorname{div}\widehat{\big(A^2(u)\big)}\big| \le C|\xi|\big\|A(u)\big\|_2^2 \le C|\xi|\|\nabla u\|_2^2.$$

For $K(u)$, similarly,

$$\begin{aligned}\big|-\beta\widehat{K(u)}\big| &= \big|\beta \operatorname{div}\widehat{\big(\big|A(u)\big|^2 A(u)\big)}\big| \le C|\xi|\big\|A(u)\big\|_3^3 \\ &\le C|\xi|\big(\|\nabla u\|_4^4 + \|\nabla u\|_2^2\big),\end{aligned}$$

where we have used the interpolation inequality.

In order to estimate the pressure term in $G(\xi,t)$, taking the divergence of (1.5) yields

$$-\Delta P = \operatorname{div}(u\cdot\nabla u) - \alpha \operatorname{div}\operatorname{div}\big(A^2(u)\big) - \beta \operatorname{div}\operatorname{div}\big(\big|A(u)\big|^2 A(u)\big), \tag{2.7}$$

and applying the Fourier transform to (2.7) and Young's inequality gives

$$\begin{aligned}|\xi|^2\widehat{P} &\le C|\xi|^2\big(\|u\|_2^2 + \alpha\big\|A(u)\big\|_2^2 + \beta\big\|A(u)\big\|_3^3\big) \\ &\le C|\xi|^2\big(\|u\|_2^2 + \alpha\big\|A(u)\big\|_2^2 + \beta\big\|A(u)\big\|_4^4 + \beta\big\|A(u)\big\|_2^2\big) \\ &\le C|\xi|^2\big(\|u\|_2^2 + \|\nabla u\|_2^2 + \|\nabla u\|_4^4\big)\end{aligned}$$

or

$$|\widehat{P}| \le C\big(\|u\|_2^2 + \|\nabla u\|_2^2 + \|\nabla u\|_4^4\big).$$

Then we have

$$G(\xi,t) \le C|\xi|\big(\|u\|_2^2 + \|\nabla u\|_2^2 + \|\nabla u\|_4^4\big).$$

The solution of (2.6) is given by

$$\widehat{u}(\xi,t) = e^{-\mu|\xi|^2 t}\widehat{u}_0(\xi) + \int_0^t e^{-\mu|\xi|^2(t-s)}G(\xi,s)\,ds,$$

so we have

$$\big|\widehat{u}(\xi,t)\big| \le \big|e^{-\mu|\xi|^2 t}\widehat{u}_0(\xi)\big| + C|\xi|\int_0^t \big(\|u\|_2^2 + \|\nabla u\|_2^2 + \|\nabla u\|_4^4\big)\,ds.$$

Thanks to (1.8), we have

$$\int_0^\infty \|\nabla u\|_2^2\,ds + \int_0^\infty \|\nabla u\|_4^4\,ds < C.$$

Hence we rewrite the estimation $|\widehat{u}(\xi,t)|$ as

$$\big|\widehat{u}(\xi,t)\big| \le C\big|e^{-\mu|\xi|^2 t}\widehat{u}_0(\xi)\big| + C|\xi|\bigg(\int_0^t \|u\|_2^2\,ds + 1\bigg).$$

Thus we have

$$\begin{aligned}
&\int_{B(t)} \left|\widehat{u}(\xi,t)\right|^2 d\xi \\
&\quad\le C\int_{B(t)} \left|e^{-\mu|\xi|^2 t}\widehat{u}_0(\xi)\right|^2 d\xi + C\int_{B(t)} |\xi|^2\left(\int_0^t \|u\|_2^2\,ds\right)^2 d\xi + C\int_{B(t)} |\xi|^2\,d\xi \\
&\quad\le C\left\|e^{\mu\Delta t}u_0\right\|_2^2 + C\int_0^{g^{-\frac{1}{2}}(t)} r^4\left(\int_0^t \|u\|_2^2\,ds\right)^2 dr + Cg^{-\frac{5}{2}}(t) \\
&\quad\le C\left\|e^{\mu\Delta t}u_0\right\|_2^2 + Cg^{-\frac{5}{2}}(t)\left(\int_0^t \|u\|_2^2\,ds\right)^2 + Cg^{-\frac{5}{2}}(t).
\end{aligned}$$

Plugging the above inequality into (2.5) gives

$$\begin{aligned}
&\frac{d}{dt}\left\|\widehat{u}(t)\right\|^2 + 2\mu\varepsilon_0 g^{-1}(t)\|\widehat{u}\|_2^2 \\
&\quad\le Cg^{-1}(t)\left\|e^{\mu\Delta t}u_0\right\|_2^2 + Cg^{-\frac{7}{2}}(t)\left(\int_0^t \|u\|_2^2\,ds\right)^2 + Cg^{-\frac{7}{2}}(t).
\end{aligned}$$

Now choosing

$$g(t) = \frac{2\mu\varepsilon_0(t+1)}{m}, \quad m > 5$$

and multiplying both sides above by $(t+1)^m$ gives

$$\begin{aligned}
&\frac{d}{dt}\left((t+1)^m\left\|\widehat{u}(t)\right\|_2^2\right) \\
&\quad\le C(t+1)^{m-1}\left\|e^{\mu\Delta t}u_0\right\|_2^2 + C(t+1)^{m-\frac{7}{2}}\left(\int_0^t \|u\|_2^2\,ds\right)^2 + C(t+1)^{m-\frac{7}{2}}.
\end{aligned} \tag{2.8}$$

Integrating with respect to time and applying the Plancherel theorem leads to

$$\begin{aligned}
\left\|u(t)\right\|_2^2 &\le C(t+1)^{-m}\int_0^t (s+1)^{m-1}\left\|e^{\mu\Delta s}u_0\right\|_2^2 ds \\
&\quad + C(t+1)^{-m}\int_0^t (s+1)^{m-\frac{7}{2}}\left(\int_0^s \|u\|_2^2\,d\tau\right)^2 ds + C(t+1)^{-\frac{5}{2}}.
\end{aligned} \tag{2.9}$$

Thanks to Proposition 1.1,

$$\left\|u(t)\right\|_2^2 \le \|u_0\|_2^2, \quad t \ge 0.$$

Then we have from (2.9) and the Plancherel theorem

$$\left\|\widehat{u}(\xi,t)\right\|_2^2 \le C(t+1)^{-m}\int_0^t (s+1)^{m-1}\left\|e^{\mu\Delta s}u_0\right\|_2^2 ds + C(t+1)^{-\frac{1}{2}} + C(t+1)^{-\frac{5}{2}}. \tag{2.10}$$

Since

$$\left\|e^{\mu\Delta t}u_0\right\|_2 \to 0, \quad t \to \infty,$$

which implies

$$(t+1)^{-m}\int_0^t (s+1)^{m-1}\left\|e^{\mu\Delta s}u_0\right\|_2^2\,ds \to 0, \quad t\to\infty.$$

Thus we conclude from (2.10) that

$$\left\|u(t)\right\|_2 \to 0, \quad t\to\infty,$$

which completes the proof of Theorem 2.1. □

3 Improved L^2 decay of weak solutions

In this section, under the non-uniform decay in the previous section, we will further investigate an improved L^2 decay of weak solutions under some additional moment condition on the initial data. More precisely, we will show the following more explicit decay rate.

Theorem 3.1 *Under the same condition in Theorem* 2.1 *together with* $u_0 \in L^2(\mathbb{R}^3)\cap L^r(\mathbb{R}^3)$ *for some* $1\le r<2$ *and*

$$\int_{\mathbb{R}^3}\left|xu_0(x)\right|^r dx < \infty,$$

then the solution $u(x,t)$ *has the following uniform algebraic* L^2 *decay rates*:

$$\left\|u(t)\right\|_2 \le C(1+t)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{2})-\frac{1}{2}} \quad \text{for } t>1. \tag{3.1}$$

Remark 3.2 It should be mentioned that Zhao et al. [1] (see also [18, 19]) have proved the time decay rate of solutions to the third-grade non-Newtonian fluid flows (1.5)

$$\left\|u(t)\right\|_2 \le C(1+t)^{-\frac{3}{4}} \quad \text{for } t>1$$

with the initial data $u_0 \in L^2(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)$. Our results here are obviously more explicit and extend the previous results.

Proof of Theorem 3.1 We first recall the well-known (L^r, L^q) estimates of heat equation in a three-dimensional whole space.

Lemma 3.3 (*Fujigaki and Miyakawa* [20]) *Suppose* $1\le r\le q\le\infty, k\ge 0$ *and* $f\in L^2(\mathbb{R}^3)\cap L^r(\mathbb{R}^3)$, *then we have the following* $L^r - L^q$ *estimate of* 3*D linear heat equation*:

$$\left\|\nabla^k e^{\mu\Delta t}f\right\|_q \le Ct^{-\frac{k}{2}-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})}\|f\|_r, \tag{3.2}$$

and the modified $L^r - L^q$ *estimate*

$$\left\|\nabla^k e^{\mu\Delta t}u\right\|_q \le Ct^{-\frac{k}{2}-\frac{3}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}}\left(\int_{\mathbb{R}^3}\left|xu(x)\right|^r dx\right)^{\frac{1}{r}}. \tag{3.3}$$

By the same argument as that in (2.9), we have

$$\|u\|_2^2 \le C(t+1)^{-m}\int_0^t (s+1)^{m-1}\left\|e^{\mu\Delta s}u_0\right\|_2^2\,ds + C(t+1)^{-m}\int_0^t (s+1)^{m-\frac{7}{2}}\left(\int_0^s \|u\|_2^2\,d\tau\right)^2 ds + C(t+1)^{-\frac{5}{2}} \tag{3.4}$$

and apply Lemma 3.3 and $\|u(t)\|_{L^2} \le \|u_0\|_{L^2}$. Using the same argument as that in (2.9), we have

$$\begin{aligned}\|u\|_2^2 &\le C(t+1)^{-m}\int_0^t (s+1)^{m-1}s^{-3(\frac{1}{r}-\frac{1}{2})-1}\,ds + C(t+1)^{-\frac{1}{2}} + C(t+1)^{-\frac{5}{2}}\\ &\le C(t+1)^{-3(\frac{1}{r}-\frac{1}{2})-1} + C(t+1)^{-\frac{1}{2}} + C(t+1)^{-\frac{5}{2}}\\ &\le C(t+1)^{-\frac{1}{2}}, \quad t>1.\end{aligned} \tag{3.5}$$

Inserting the estimate of (3.5) into the second term on the right-hand of (3.4) again, we have

$$\begin{aligned}\|u\|_2^2 &\le C(t+1)^{-m}\int_0^t (s+1)^{m-1}s^{-3(\frac{1}{r}-\frac{1}{2})-1}\,ds\\ &\quad + C(t+1)^{-m}\int_0^t (s+1)^{m-\frac{7}{2}}\left(\int_0^s (\tau+1)^{-\frac{1}{2}}\,d\tau\right)^2 ds + C(t+1)^{-\frac{5}{2}}\\ &\le C(t+1)^{-3(\frac{1}{r}-\frac{1}{2})-1} + C(t+1)^{-\frac{3}{2}} + C(t+1)^{-\frac{5}{2}}\\ &\le C(t+1)^{-3(\frac{1}{r}-\frac{1}{2})-1} + C(t+1)^{-\frac{3}{2}} \quad \text{for } t>1.\end{aligned}$$

When $\frac{3}{2} \le r < 2$, then

$$3\left(\frac{1}{r}-\frac{1}{2}\right)+1 \le \frac{3}{2},$$

then

$$\left\|u(t)\right\|_2 \le C(t+1)^{-\frac{3}{2}(\frac{1}{r}-\frac{1}{2})-\frac{1}{2}} \quad \text{for } t>1,$$

which derives the result of Theorem 3.1.

When $1 \le r < \frac{3}{2}$, then

$$\left\|u(t)\right\|_2^2 \le C(t+1)^{-\frac{3}{2}} \quad \text{for } t>1,$$

which implies

$$\int_0^s \|u\|_2^2\,d\tau \le \int_0^\infty \|u\|_2^2\,d\tau = \int_0^1 \|u\|_2^2\,d\tau + C\int_1^\infty (\tau+1)^{-\frac{3}{2}}\,d\tau \le \|u_0\|_2^2 + C \le C.$$

Thus we repeat the same action from (3.4) to derive

$$\|u\|_2^2 \le C(t+1)^{-m}\int_0^t (s+1)^{m-1}\left\|e^{\mu\Delta s}u_0\right\|_2^2\,ds$$

$$
\begin{aligned}
&+ C(t+1)^{-m} \int_0^t (s+1)^{m-\frac{7}{2}}\, ds + C(t+1)^{-\frac{5}{2}} \\
&\le C(t+1)^{-3(\frac{1}{r}-\frac{1}{2})-1} + C(t+1)^{-\frac{5}{2}} \\
&\le C(t+1)^{-3(\frac{1}{r}-\frac{1}{2})-1}, \quad t > 1. \qquad (3.6)
\end{aligned}
$$

Hence the proof of Theorem 3.1 is completed. □

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Authors' contributions
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