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Dependence of eigenvalues of $2m$th-order spectral problems

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Abstract

A regular $2m$th-order spectral problem with self-adjoint boundary conditions is considered in this paper. The continuous dependence of eigenvalues and normalized eigenfunctions on the problem is researched. The derivative formulas of eigenvalues with respect to the given parameters are obtained: endpoints, boundary conditions, coefficients and the weight function. These are of both theoretical and computational importance.

Keywords: $2m$th-order spectral problem; eigenvalues; Frechet derivative; dependence of eigenvalues on the problem

1 Introduction

Spectral problems of differential operators arise in many different physical applications. There is a vast amount of research papers on the boundary value problems (BVPs) for ordinary differential operators. A substantial part of the previous work focuses on second-order problems; see [1–7]. For a regular Sturm-Liouville differential equation of the form

$$-(py')' + qy = \lambda wy \tag{1.1}$$

with different boundary conditions (BCs), Pöschel and Trubowitz use the Dirichlet BCs and consider the mth eigenvalue. It takes the form $\lambda = \lambda_n(q)$, where $q \in \mathbf{L}^2(a,b)$, $p = w = 1$ in [1]. They prove that λ is Frechet differentiable and they give the expression of its derivative. Dauge and Helffer (see [2, 3]) consider equation (1.1) with Neumann BCs and the coefficients $p, q, w \in C^\infty$ on the interval $[a,b]$ with $p(t) \geq k > 0$. They prove that, as a function of the endpoint b, the Neumann eigenvalues are C^1 functions. In [4] and [5], Kong and Zettl study differential equation (1.1) with general self-adjoint BCs. They find that the corresponding eigenvalues of this problem are C^1 functions of all its data and they give the derivative formulas. Battle in [6] studies a more general second-order problem. Further, in [7], the authors prove the dependence of the kth eigenvalues on all its data.

There are also many papers that deal with the fourth-order spectral problems. Ge *et al.* (see [8, 9]) consider the fourth-order differential equation

$$(py'')'' + qy = \lambda wy \tag{1.2}$$

with self-adjoint BCs. They study the continuity and differentiability of eigenvalues. Their method is similar to that of [4] and [5]. In [10–18], the authors treat fourth-order equations of the form

$$\left(py''\right)'' - \left(sy'\right)' + qy = \lambda wy, \tag{1.3}$$

and they introduce different methods to compute the eigenvalues and eigenfunctions associated with the above-mentioned fourth-order differential equation.

The dependence of parameters on the $2m$th spectral problem has drawn less attention than the Sturm-Liouville equations and fourth-order differential equations. In [17], Yang, Wang and Gao study the $2k$th-order differential equation of the form

$$ly = \sum_{r=0}^{k}(-1)^r\left(p_{k-r}y^{(r)}\right)^{(r)} = \lambda wy \tag{1.4}$$

with separated self-adjoint BCs, rather than the coupled self-adjoint BCs. They consider the eigenvalues as a function of one endpoint, and show that the eigenvalues of $2k$th-order spectral problems depend not only continuously but also smoothly on endpoints. In [19], Greenberg and Marletta discuss some numerical methods for self-adjoint boundary eigenvalues problems of (1.4).

In this paper, we deal with a $2m$th-order spectral problem with more complicated BCs which consist of separated self-adjoint BCs and coupled self-adjoint BCs. This kind of spectral problems is widely used in many research areas, since the dependence of eigenvalues is closely related to the computation of eigenvalues. Based on the theories on the Hilbert space and complex variable theory, we study the continuity and differentiability properties of eigenvalues and eigenfunctions of the $2m$th-order spectral problem. In accordance with [20], we pose the conditions $1/p_m, p_{m-1}, \dots, p_0, w \in L^1(a,b)$ so that each of the initial value problems has a unique solution. Naimark, in [21], shows that these local integrable properties of the coefficients and weight functions imply not only the existence but also the uniqueness of locally absolutely continuous solutions. First, by the extension of $1/p_m, p_{m-1}, \dots, p_0, w$, we establish a new space and norm to study the continuity both of eigenvalues and normalized eigenfunctions. The isolated eigenvalues, being continuous functions, of all the parameters are proved. Based on proofs of the unique solution of the initial value problem, which is dependent continuously on all variables, we get the continuity of normalized eigenfunctions.

Next, we fix all but one of $\mathbf{A}, \mathbf{B}, a, b, 1/p_m, p_{m-1}, \dots, p_0, w$ and we study the continuous dependence of eigenvalues on the problem and show that the eigenvalues are C^1 functions of the BCs, the endpoints, coefficients and the weight function. By means of computation, we obtain its derivative formulas. Here differentiability with respect to $1/p_m, p_{m-1}, \dots, p_0, q$ or w is the Frechet derivative in the Banach space $L^1(a,b)$. Our proof is fundamental, but it is useful in spectral analysis.

Following this introduction, we give some general results and notations in Section 2. In Section 3, we give the continuous dependence of the eigenvalues and corresponding eigenfunctions. Section 4 establishes the differentiability of the eigenvalues on all its parameters.

2 Preliminaries

We characterize a $2m$th-order spectral problem consisting of differential equations as follows:

$$ly = \sum_{i=0}^{m} (-1)^i \bigl(p_i(x)y^{(i)}\bigr)^{(i)} = \lambda w(x)y, \quad x \in \bigl(a', b'\bigr), \tag{2.1}$$

where $-\infty \le a' < b' \le \infty, \lambda \in \mathbb{C}$ is the spectral parameter. The coefficient functions $p_i, 0 \le i \le m$ and weight function $w(x)$ satisfy the following basic conditions:

$$p_m, p_i, w : \bigl(a', b'\bigr) \to \mathbb{R}, \qquad 1/p_m, p_i \in L^1_{\text{loc}}\bigl(a', b'\bigr), \quad w > 0 \text{ a.e. on } \bigl(a', b'\bigr), \tag{2.2}$$

where $i = 0, 1, 2, \ldots, m-1$, and $L^1_{\text{loc}}(a', b')$ denotes the complex-valued Lebesgue integrable functions on all compact subintervals of (a', b'). Any self-adjoint differential equation with sufficiently smooth real valued coefficients can be written in this form, so equation (2.1) is a natural starting point. This equation (2.1) is a wide type of equation which has been researched by many scholars. For example, $m = 1$ implies the classical Sturm-Liouville equation, and when $m = 2$, (2.1) reduces to the beam equation which is one of the important models in engineering (see [22] for details).

We introduce the quasi-derivatives (up to order $2m$) of a function y, which are the functions $y^{[0]} = y, y^{[1]}, \ldots, y^{[2m]}$, given by

$$\begin{aligned} &y^{[k]} = y^{(k)}, \quad k = 0, 1, 2, \ldots, m-1, \\ &y^{[m]} = p_m y^{(m)}, \\ &y^{[k]} = \bigl(y^{[k-1]}\bigr)' - p_{2m-k} y^{(2m-k)}, \quad k = m+1, m+2, \ldots, 2m, \end{aligned} \tag{2.3}$$

where $y^{(k)}$ is the usual kth derivative (see [7], p.185). Then the differential equation (2.1) can be simplified as

$$ly = (-1)^m y^{[2m]} = \lambda w y. \tag{2.4}$$

The differential expression l on (a', b') is defined for all functions y such that $y^{[0]}, y^{[1]}, \ldots, y^{[2k-1]}$ exist and are absolutely continuous over compact subintervals of (a', b').

Let

$$I = [a, b], \quad a' < a < b < b', \tag{2.5}$$

and the BC be of the form

$$\mathbf{A} \begin{pmatrix} y(a) \\ y^{[1]}(a) \\ \vdots \\ y^{[2m-2]}(a) \\ y^{[2m-1]}(a) \end{pmatrix} + \mathbf{B} \begin{pmatrix} y(b) \\ y^{[1]}(b) \\ \vdots \\ y^{[2m-2]}(b) \\ y^{[2m-1]}(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \tag{2.6}$$

where the complex $2m \times 2m$ matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy

$$\text{the } 2m \times 4m \text{ matrix } (\mathbf{A}|\mathbf{B}) \text{ has full rank} \tag{2.7}$$

and

$$\mathbf{AQA}^* = \mathbf{BQB}^*, \qquad \mathbf{Q} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.8}$$

Here by $\mathbf{A}^*$, we mean the complex conjugate transpose of matrix $\mathbf{A}$.

Remark 2.1 The differential equation (2.1) together with BCs (2.6)-(2.8) are said to be a self-adjoint $2m$th-order spectral problem. Thanks to [21], we know that the basic conditions (2.2) ensure that the differential expression l is regular on $[a,b]$. The problem we study can be identified with a self-adjoint operator in the Hilbert space $L^2_\omega(a,b)$ (see [22]). So the spectrum of the problem corresponds to the spectrum of the operator and there exist an infinite but countable number of real eigenvalues.

It is shown in [23] that the self-adjoint BCs are of three types: separated, coupled and mixed. Here we study three forms of the separated self-adjoint BCs, one form of the coupled real self-adjoint BCs and one form of the coupled complex self-adjoint BCs.

1. Separated self-adjoint BCs
   (1) Separated self-adjoint BCs (I). We have

$$\begin{aligned} &y(a)\cos\alpha - y^{[1]}(a)\sin\alpha = 0, \\ &y^{[2]}(a)\cos\alpha - y^{[3]}(a)\sin\alpha = 0, \\ &\vdots \\ &y^{[2m-2]}(a)\cos\alpha - y^{[2m-1]}(a)\sin\alpha = 0, \quad \alpha \in [0,\pi); \end{aligned} \tag{2.9}$$

$$\begin{aligned} &y(b)\cos\beta - y^{[1]}(b)\sin\beta = 0, \\ &y^{[2]}(b)\cos\beta - y^{[3]}(b)\sin\beta = 0, \\ &\vdots \\ &y^{[2m-2]}(b)\cos\beta - y^{[2m-1]}(b)\sin\beta = 0, \quad \beta \in (0,\pi]. \end{aligned} \tag{2.10}$$

   (2) Separated self-adjoint BCs (II). We have

$$\begin{aligned} &y(a)\cos\varphi - y^{[m]}(a)\sin\varphi = 0, \\ &y^{[1]}(a)\cos\varphi + y^{[m+1]}(a)\sin\varphi = 0, \\ &\vdots \\ &y^{[m-1]}(a)\cos\varphi - (-1)^{m-1}y^{[2m-1]}(a)\sin\varphi = 0, \quad \varphi \in [0,\pi); \end{aligned} \tag{2.11}$$

$$
\begin{aligned}
&y(b)\cos\psi - y^{[m]}(b)\sin\psi = 0,\\
&y^{[1]}(b)\cos\psi + y^{[m+1]}(b)\sin\psi = 0,\\
&\vdots\\
&y^{[m-1]}(b)\cos\psi - (-1)^{m-1}y^{[2m-1]}(b)\sin\psi = 0, \quad \psi\in(0,\pi].
\end{aligned}
\tag{2.12}
$$

(3) Separated self-adjoint BCs (III). We have

$$
\begin{aligned}
&y(a)\cos\eta - y^{[2m-1]}(a)\sin\eta = 0,\\
&y^{[1]}(a)\cos\eta - y^{[2m-2]}(a)\sin\eta = 0,\\
&\vdots\\
&y^{[m-1]}(a)\cos\eta - y^{[m]}(a)\sin\eta = 0, \quad \eta\in[0,\pi);
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
&y(b)\cos\tau - y^{[2m-1]}(b)\sin\tau = 0,\\
&y^{[1]}(b)\cos\tau - y^{[2m-2]}(b)\sin\tau = 0,\\
&\vdots\\
&y^{[m-1]}(b)\cos\tau - y^{[m]}(b)\sin\tau = 0, \quad \tau\in(0,\pi].
\end{aligned}
\tag{2.14}
$$

2. Coupled real self-adjoint BCs. We have

$$\mathbf{Y}(b) = \mathbf{K}\mathbf{Y}(a), \tag{2.15}$$

where $\mathbf{K}$ satisfies

$$\mathbf{KQK}^* = I, \quad \mathbf{K} = (k_{ij}), k_{ij}\in\mathbb{R}\ (i,j = 1,2,\dots,2m). \tag{2.16}$$

3. Coupled complex self-adjoint BCs. We have

$$\mathbf{Y}(b) = e^{i\theta}\mathbf{K}\mathbf{Y}(a), \tag{2.17}$$

where $\mathbf{K}$ fulfills (2.16), and $\theta\in(-\pi,0)\cup(0,\pi)$.

Remark 2.2 A function y is said to be a solution of the $2m$th-order differential equation (2.1), if $y^{[0]}, y^{[1]},\dots,y^{[2m-1]}\in AC_{\mathrm{loc}}(a',b')$, and y satisfies equation (2.1) a.e. on (a',b').

3 Continuity of eigenvalues and eigenfunctions

It is easy to see that all but one of the parameters being fixed can determine a $2m$th-order spectral problem. Then we fix all but one of $\mathbf{A},\mathbf{B},a,b,1/p_m,p_{m-1},\dots,p_0,w$, so we can study the continuity properties of the eigenvalues and eigenfunctions on that parameter. First we give a definition of $\mathbf{W}$.

Definition 3.1 Assume (2.2), (2.5), (2.7) and (2.8) hold, then the $2m$th-order spectral problem reads

$$\mathbf{W} = \bigl\{\omega = (\mathbf{A},\mathbf{B},a,b,1/p_m,p_{m-1},\dots,p_0,w)\bigr\}. \tag{3.1}$$

In the case of separated BCs (2.9) and (2.10), we take the form

$$\mathbf{W}_{s_1} = \{\omega = (\alpha, \beta, a, b, 1/p_m, p_{m-1}, \dots, p_0, w)\}. \tag{3.2}$$

For BCs (2.11) and (2.12), we take the form

$$\mathbf{W}_{s_2} = \{\omega = (\varphi, \psi, a, b, 1/p_m, p_{m-1}, \dots, p_0, w)\}. \tag{3.3}$$

For BCs (2.13) and (2.14), we take the form

$$\mathbf{W}_{s_3} = \{\omega = (\eta, \tau, a, b, 1/p_m, p_{m-1}, \dots, p_0, w)\}. \tag{3.4}$$

For the coupled BCs (2.16) and (2.17), we take the form

$$\mathbf{W}_{c} = \{\omega = (\theta, \mathbf{K}, a, b, 1/p_m, p_{m-1}, \dots, p_0, w)\}. \tag{3.5}$$

If $\theta = 0$ in (3.5), we denote the BCs by

$$\mathbf{W}_{rc} = \{\omega = (\mathbf{K}, a, b, 1/p_m, p_{m-1}, \dots, p_0, w)\}. \tag{3.6}$$

By Definition 3.1, we conclude that each of the $\omega \in \mathbf{W}$ defines a unique spectral problem. In order to investigate the continuous dependence and differentiability of eigenvalues and corresponding eigenfunctions on the problem, we need to compare the different spectrum problems determined by different ω. Since the values of $1/p_m, p_{m-1}, \dots, p_0, w$, being zeros outside the interval $[a, b]$, do not change the spectrum determined by $\boldsymbol{\omega}$, we can extend $1/p_m, p_{m-1}, \dots, p_0, w$ to the larger interval (a', b').

Definition 3.2 We define the new set

$$\tilde{\mathbf{W}} = \{\tilde{\boldsymbol{\omega}} = (\mathbf{A}, \mathbf{B}, a, b, \widetilde{1/p_m}, \widetilde{p_{m-1}}, \dots, \widetilde{p_0}, \widetilde{w})\}, \tag{3.7}$$

where

$$\widetilde{1/p_m} = \begin{cases} 1/p_m, & x \in [a, b] \\ 0, & x \in (a', b') \backslash [a, b] \end{cases} \tag{3.8}$$

i.e., $\widetilde{1/p_m}$ is the extension of $1/p_m$ to (a', b') that is equal to zero on $(a', b') \backslash [a, b]$, and $\widetilde{p_{m-1}}, \dots, \widetilde{p_0}$ and $\widetilde{w}$ have similar meanings.

Definition 3.3 A Banach space X is defined as

$$\begin{aligned} X = {} & M_{2m,2m}(\mathbb{C}) \times M_{2m,2m}(\mathbb{C}) \times \mathbb{R} \times \mathbb{R} \\ & \times \underbrace{L^1(a', b') \times L^1(a', b') \times \cdots \times L^1(a', b')}_{m+2}, \end{aligned} \tag{3.9}$$

and the norm is given by

$$\|\omega\| = \|\widetilde{\omega}\| = \|\mathbf{A}\| + \|\mathbf{B}\| + |a| + |b| + \int_{a'}^{b'} \left(|\widetilde{1/p_m}| + \sum_{i=0}^{m-1} |\widetilde{p_i}| + |\widetilde{w}| \right), \tag{3.10}$$

where $\|\cdot\|$ is any fixed matrix norm.

Remark 3.1 It is with respect to this space $\mathbf{X}$ that we study the dependence of the eigenvalues and eigenfunctions of a regular $2m$th-order spectral problem on its parameters. Since $1/p_m, p_{m-1}, \dots, p_0, w \in \mathbf{L}^1_{\text{loc}}(a,b)$ only, $\mathbf{W}$ is not a subset of $\mathbf{X}$. By means of the extension of $1/p_m, p_{m-1}, \dots, p_0, w$, we obtain $\widetilde{1/p_m}, \widetilde{p_{m-1}}, \dots, \widetilde{p_0}, \widetilde{w} \in \mathbf{L}^1(a',b')$, thus $\tilde{\mathbf{W}}$ is a subset of the Banach space $\mathbf{X}$.

Lemma 3.1 (cf. [24]) *Suppose that* (2.2) *holds and* $c \in (a',b')$, $m_i \in \mathbb{C}$ $(i = 0,1,2,\dots,2m-1)$. *Consider the following initial value problem*:

$$\begin{cases} \sum_{i=0}^{m} (-1)^i (p_i(x) y^{(i)})^{(i)} = \lambda w y, \\ y^{[0]}(c) = m_0, \qquad y^{[1]}(c) = m_1, \\ y^{[2]}(c) = m_2, \qquad \dots, \qquad y^{[2m-2]} = m_{2m-2}, \qquad y^{[2m-1]}(c) = m_{2m-1}, \end{cases}$$

then the unique solution $y = y(\cdot, m_0, m_1, \dots, m_{2m-1}, 1/p_m, p_{m-1}, p_{m-2}, \dots, p_1, p_0, w)$ *is a continuous function of all its variables. More precisely, given any* $\varepsilon > 0$ *and any subinterval* I, *there exists a* $\delta > 0$. *If*

$$|c - c_0| + \sum_{i=0}^{2m-1} |m_i - m_{i_0}| + \int_a^b \left(|1/p_m - 1/p_{m_0}| + \sum_{i=0}^{m-1} |p_i - p_{i_0}| + |w - w_0| \right) ds < \delta,$$

then, for all $x \in I$,

$$\begin{aligned} &\big| y^{[0]}(x, c, m_0, m_1, \dots, m_{2m-1}, 1/p_m, p_{m-1}, p_{m-2}, \dots, p_0, w) \\ &\quad - y^{[0]}(x, c_0, m_{0_0}, m_{1_0}, \dots, m_{(2m-1)_0}, 1/p_{m_0}, p_{(m-1)_0}, p_{(m-2)_0}, \dots, p_{0_0}, w_0) \big| < \varepsilon, \\ &\big| y^{[1]}(x, c, m_0, m_1, \dots, m_{2m-1}, 1/p_m, p_{m-1}, p_{m-2}, \dots, p_0, w) \\ &\quad - y^{[1]}(x, c_0, m_{0_0}, m_{1_0}, \dots, m_{(2m-1)_0}, 1/p_{m_0}, p_{(m-1)_0}, p_{(m-2)_0}, \dots, p_{0_0}, w_0) \big| < \varepsilon, \\ &\vdots \\ &\big| y^{[2m-1]}(x, c, m_0, m_1, \dots, m_{2m-1}, 1/p_m, p_{m-1}, p_{m-2}, \dots, p_0, w) \\ &\quad - y^{[2m-1]}(x, c_0, m_{0_0}, m_{1_0}, \dots, m_{(2m-1)_0}, 1/p_{m_0}, p_{(m-1)_0}, p_{(m-2)_0}, \dots, p_{0_0}, w_0) \big| < \varepsilon. \end{aligned}$$

Next, we introduce the characteristic function $\Delta(\boldsymbol{\omega}, \boldsymbol{\lambda})$ *which characterizes the eigenvalues of BVPs* (2.1) *and* (2.6) *as roots of* $\Delta(\boldsymbol{\omega}, \boldsymbol{\lambda})$.

Let (2.2) *hold and* $\Phi(\cdot, a, 1/p_m, p_{m-1}, \dots, p_0, w)$ *denote the fundamental matrix solution of the following problem*:

$$\mathbf{Y}' = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1/p_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & p_{m-1} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & (-1)^{m-1}p_2 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & (-1)^m p_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ (-1)^{m+1}(p_0 - \lambda\omega) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{Y}$$

$$on\ (a', b'),$$

and $\Phi(\cdot, a, 1/p_m, p_{m-1}, \dots, p_0, w)$ *satisfy the initial condition* $\Phi(a, \lambda) = \mathbf{I_{2m}}$. *Here* $\mathbf{I_{2m}}$ *is the* $2m \times 2m$ *identity matrix. Then* $\lambda \in \mathbb{C}$ *is an eigenvalue of the spectral problems* (2.1) *and* (2.6) *if and only if*

$$\Delta(\boldsymbol{\omega}, \boldsymbol{\lambda}) = \det\bigl[\mathbf{A} + \mathbf{B}\boldsymbol{\Phi}(b, a, 1/p_m, p_{m-1}, \dots, p_0, w)\bigr] = 0.$$

The function $\Delta(\boldsymbol{\omega}, \boldsymbol{\lambda}) = \det[\mathbf{A} + \mathbf{B}\boldsymbol{\Phi}(b, a, 1/p_m, p_{m-1}, \dots, p_0, w)]$ *is called the characteristic function of the problems* (2.1) *and* (2.6).

Next we show that the isolated eigenvalues of a regular $2m$*th-order spectral problem are continuous functions of all the parameters. This theorem is a special case of Kong, Wu, Zettl* (*see* [7], *Theorem* 3.4) *for the continuity of isolated eigenvalues, in which a more generalized* m*th-order differential equation is defined using quasi-derivatives.*

Theorem 3.1 *Suppose that* $\boldsymbol{\omega_0} = (a_0, b_0, \mathbf{A}_0, \mathbf{B}_0, 1/p_{m_0}, p_{(m-1)_0}, \dots, p_{0_0}, w_0) \in \mathbf{W}$, *and* $\mu = \lambda(\omega_0)$ *is an eigenvalue of the spectral problems* (2.1) *and* (2.6)-(2.8), *being determined by* ω_0. *We see that* $\lambda(\omega)$ *is continuous at* $\boldsymbol{\omega_0}$, *which means that for any* $\varepsilon > 0$, *there exists a* $\delta > 0$. *For any* $\omega \in \mathbf{W}$ *that satisfies*

$$\begin{aligned} \|\omega - \omega_0\| &= \|\mathbf{A} - \mathbf{A}_0\| + \|\mathbf{B} - \mathbf{B}_0\| + |a - a_0| + |b - b_0| \\ &\quad + \int_{a'}^{b'} \Biggl[|\widetilde{1/p_m} - \widetilde{1/p_{m_0}}| + \sum_{i=0}^{m-1} |\widetilde{p}_i - \widetilde{p_{i_0}}| + |\widetilde{w} - \widetilde{w_0}| \Biggr] ds < \delta, \end{aligned} \tag{3.11}$$

it is implied that

$$\bigl|\lambda(\omega) - \lambda(\boldsymbol{\omega_0})\bigr| < \varepsilon. \tag{3.12}$$

Proof This proof is similar to that of Kong, Wu, Zettl [7], Theorem 3.4, so we omit the details. □

Remark 3.2 From Theorem 3.1, we infer that, for any fixed eigenvalue μ associated with $\boldsymbol{\omega} = \boldsymbol{\omega}_0$, there is a continuous eigenvalue branch $\lambda(\boldsymbol{\omega})$, such that $\lambda(\boldsymbol{\omega}_0) = \mu$. However, this does not imply that, for a fixed m, the mth eigenvalue $\lambda_m(\boldsymbol{\omega})$ is always continuous in $\boldsymbol{\omega}$. In what follows, for any $\boldsymbol{\omega} \in \mathbf{W}$, each eigenvalue $\lambda(\boldsymbol{\omega})$ of the $2m$th-order spectral problems (2.1) and (2.6)-(2.8), as a function of $\boldsymbol{\omega}$, is embedded in one of the continuous eigenvalue branches.

The following lemma characterizes the unique solution of any initial value problem of equation (2.1) which is dependent continuously on all parameters in the norm $\mathbf{L}^1$.

As a consequence of Theorem 3.1 and Lemma 3.1, we obtain the following.

Lemma 3.2 *Assume that* $\omega_0 = (a_0, b_0, \mathbf{A}_0, \mathbf{B}_0, 1/p_{m_0}, p_{(m-1)_0}, p_{(m-2)_0}, \dots, p_{0_0}, w_0) \in \mathbf{W}$ *and* $\lambda = \lambda(\boldsymbol{\omega})$ *is an eigenvalue of the* $2m$*th-order spectral problems* (2.1) *and* (2.6)-(2.8). *If the multiplicity of* $\lambda(\omega_0)$ *is* 1, *then we see that there exists an* $M \subset \mathbf{W}$, *where* M *is a neighborhood of* ω_0, *such that the multiplicity of* $\lambda = \lambda(\omega)$ *is* 1 *for each* $\omega \in M$.

Proof Given any solution y of (2.1) and eigenfunction $u(\cdot, \boldsymbol{\omega})$ of the spectral problem, we denote by

$$\mathbf{Y} = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ \vdots \\ y^{[2m-2]} \\ y^{[2m-1]} \end{pmatrix} \quad \text{and} \quad \mathbf{U} = \begin{pmatrix} u^{[0]} \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix} \tag{3.13}$$

the vector solution and vector eigenfunction. The multiplicities of the eigenvalues are l ($l = 1, 2, \dots, 2m$). Clearly, Lemma 3.2 holds for the multiplicity of the eigenvalues equal to 1. If the eigenvalues are multiplicity l ($l = 2, 3, \dots, 2m$), we give a proof of coupled real BCs and the other two BCs are similar. For $\omega_0 = \{\mathbf{K}_0, a_0, b_0, 1/p_{m_0}, p_{(m-1)_0}, p_{(m-2)_0}, \dots, p_{0_0}, w_0\} \in \mathbf{W}_{rc}$, assume that the multiplicity of $\lambda(\omega_0)$ is l. If the conclusion does not hold, then there exists some sequence $\{\omega_k\} \subset \mathbf{W}_{rc}$ such that $\omega_k \to \omega_0$, and for $k \in \mathbb{M}$, $\lambda(\omega_k)$ is an eigenvalue with multiplicity l. Now we choose $v_1, \dots, v_l \in \mathbb{R}^m$, which are linearly independent vectors, and $\mathbf{U}_1(\cdot, \omega_k), \dots, \mathbf{U}_l(\cdot, \omega_k)$ are solutions of (2.1) with $\lambda = \lambda(\omega_k)$ satisfying the initial conditions

$$\mathbf{U}_j(a, \omega_k) = v_j, \quad j = 1, \dots, l.$$

Then $\mathbf{U}_1(\cdot, \omega_k), \dots, \mathbf{U}_l(\cdot, \omega_k)$ are all vector eigenfunctions, and they satisfy

$$\mathbf{U}_j(b, \omega_k) = \mathbf{K}_k \mathbf{U}_j(a, \omega_k), \quad k \in \mathbb{M}, j = 1, \dots, l. \tag{3.14}$$

Applying Lemma 3.1 and Theorem 3.1 as $k \to \infty$ in (3.14), we obtain

$$\mathbf{Y}_j(b, \omega_0) = \mathbf{K}_0 \mathbf{Y}_j(a, \omega_0), \quad k \in \mathbb{M}, j = 1, \dots, l, \tag{3.15}$$

where $\mathbf{Y}_j = \lim_{k\to\infty} \mathbf{U}_j$ ($j = 1, \dots, l$). Thus $\mathbf{Y}_1, \dots, \mathbf{Y}_l$ ($2 \le l \le 2m$) are l linearly independent eigenfunctions of $\lambda(\omega_0)$. This is impossible since the multiplicity of $\lambda(\omega_0)$ is 1. This finishes the proof of Lemma 3.2. □

Definition 3.4 A normalized eigenfunction u of a $2m$th-order spectral problem means an eigenfunction u satisfying

$$\int_a^b \left|u(s)\right|^2 w(s)\,ds = 1. \tag{3.16}$$

Following the above definition, we notice that these eigenfunctions are not uniquely determined by (3.16). In the situation of the multiplicity of the eigenvalue being 1, there exists a unique real eigenfunction up to sign, and for double eigenvalues, there exist a pair of normalized linearly independent eigenfunctions, and for a multiplicity l ($l = 3, 4, \dots, 2m$) eigenvalue, the number of the normalized linearly independent eigenfunctions is l.

Theorem 3.2 *Assume all the assumptions of Theorem* 3.1 *are true.*

(i) *If the multiplicity of the eigenvalue* $\lambda(\omega_0)$ *is* 1 *for some* $\omega_0 \in \mathbf{W}$, *and* $u = u(\cdot,\omega_0)$ *is a corresponding normalized eigenfunction of* $\lambda(\omega_0)$, *then for* $\omega \in \mathbf{W}$, *we see that there exists a normalized eigenfunction* $u = u(\cdot,\omega)$ *corresponding to* $\lambda(\omega)$, *as* $\omega \to \omega_0$, *and*

$$\begin{aligned} &u(\cdot,\omega) \to u(\cdot,\omega_0), \\ &u'(\cdot,\omega) \to u'(\cdot,\omega_0), \\ &\vdots \\ &u^{[2m-2]}(\cdot,\omega) \to u^{[2m-2]}(\cdot,\omega_0), \\ &u^{[2m-1]}(\cdot,\omega) \to u^{[2m-1]}(\cdot,\omega_0), \end{aligned} \tag{3.17}$$

hold uniformly on arbitrarily compact subinterval I *of* (a',b').

(ii) *If the multiplicity of eigenvalue* $\lambda(\boldsymbol{\omega_0})$ *is* l ($l = 2, 3, \dots, 2m$), $u_j = u_j(\cdot,\omega_0)$ ($j = 1, 2, \dots, l$) *is any* l *normalized eigenfunction of* $\lambda(\omega_0)$, *and suppose that the multiplicity of the eigenvalue* $\lambda(\boldsymbol{\omega})$ *is* l ($l = 2, 3, \dots, 2m$) *for each* $\boldsymbol{\omega} \in \mathbf{M}$, *where* $\mathbf{M}$ *is some neighborhood of* $\omega_0 \in \mathbf{W}$, *then there exist* l *normalized eigenfunctions* $u_j = u_j(\cdot,\omega)$ ($j = 1, 2, \dots, l$) *of* $\lambda(\omega)$, *as* $\omega \to \omega_0$. *We have*

$$\begin{aligned} &u_j^{[0]}(\cdot,\omega) \to u_j^{[0]}(\cdot,\omega_0), \\ &u_j^{[1]}(\cdot,\omega) \to u_j^{[1]}(\cdot,\omega_0), \\ &\vdots \\ &u_j^{[2m-2]}(\cdot,\omega) \to u_j^{[2m-2]}(\cdot,\omega_0), \\ &u_j^{[2m-1]}(\cdot,\omega) \to u_j^{[2m-1]}(\cdot,\omega_0), \end{aligned} \tag{3.18}$$

uniformly on arbitrarily compact subinterval I *of* (a',b').

Proof Firstly, we prove that there are eigenfunctions $u(\cdot,\omega)$ satisfying (3.17) uniformly on I. Suppose (3.14) holds, y is a solution of (2.1) and $u(\cdot,\omega)$ is an eigenfunction.

(i) Now we consider the case that the multiplicity of $\lambda(\omega_0)$ is 1. Using Lemma 3.2, we see that there exists a neighborhood $\mathbf{M}$ of ω_0 such that the multiplicity of $\lambda(\omega)$ is 1 for all

$\omega \in \mathbf{M}$. We select an eigenfunction $u = u(\cdot, \omega)$ of $\lambda(\omega)$ for all $\omega \in \mathbf{M}$, and they satisfy

$$\left\|\mathbf{U}(c,\omega)\right\| = \sum_{i=0}^{2m-1} \left|u^{[i]}(c,\omega)\right| = 1,$$

where $c \in (a_0, b_0)$ is an arbitrary point, and $u(t, \boldsymbol{\omega}) > 0$ when t is near c.

It is sufficient to illustrate that

$$\mathbf{U}(c,\boldsymbol{\omega}) \to \mathbf{U}(c,\boldsymbol{\omega}_0) \quad \text{as } \boldsymbol{\omega} \to \omega_0 \text{ in } \mathbf{W}. \tag{3.19}$$

Then by Lemma 3.1 and Theorem 3.1, the above limit is uniformly convergent on $[a_0, b_0]$. Indeed, if (3.19) is not fulfilled, we select a sequence $\omega_k \to \omega_0$ in $\mathbf{W}$ such that

$$\mathbf{U}(c,\mathbf{W}_k) \to \mathbf{Y} \quad \text{as } \omega \to \omega_0, \tag{3.20}$$

since the normalization at c, $\mathbf{Y}$ and $\mathbf{U}(c,\boldsymbol{\omega}_0)$ are two linearly independent vectors in $\mathbb{C}^4$. Let $\mathbf{Z}$ be the vector solution of (2.1) at $\omega = \omega_0$, which satisfies the initial condition $\mathbf{Z}(c) = \mathbf{Y}$, and $\lambda = \lambda(\omega_0)$ is determined by $\mathbf{Z}(c) = \mathbf{Y}$. It follows from Lemma 3.1 that $\mathbf{U}(t,\omega_k) \to \mathbf{Z}(t)$ uniformly on any compact subinterval of (a', b') as $k \to \infty$. Noticing that $\omega_k \to \omega_0$ implies that $a_k \to a_0$ and $b_k \to b_0$, we deduce

$$\mathbf{U}(a_k,\omega_k) \to \mathbf{Z}(a_0) \quad \text{and} \quad \mathbf{U}(b_k,\omega_k) \to \mathbf{Z}(b_0) \quad \text{as } k \to \infty.$$

So $\mathbf{U}(\cdot,\omega_k)$ $(k = 1, 2, \ldots)$, satisfy the **BC**,

$$\mathbf{A}_k\mathbf{U}(a_k,\omega_k) + \mathbf{B}_k\mathbf{U}(b_k,\omega_k) = 0,$$

by taking limits as $k \to \infty$. We infer

$$\mathbf{A}_0\mathbf{Z}(a_0) + \mathbf{B}_0\mathbf{Z}(b_0) = 0.$$

Hence $\mathbf{Z}$ is a vector eigenfunction relating to the eigenvalue $\lambda(\omega_0)$. It is easy to see that $\mathbf{Z}$ is linearly independent to $\mathbf{U}(\cdot,\boldsymbol{\omega}_0)$ since their Cauchy data at c are independent. This is impossible since the multiplicity of $\lambda(\omega_0)$ is 1.

(ii) Suppose that the multiplicity of the eigenvalue $\lambda(\boldsymbol{\omega})$ is l $(l = 2, 3, \ldots, 2m)$ for $\boldsymbol{\omega}$ in some neighborhood $\mathbf{M}$ of $\omega_0 \in \mathbf{W}$. For $\omega \in \mathbf{M}$, let l linearly independent eigenfunctions of $\lambda(\omega)$ be $u_1, u_2, \ldots, u_l$ $(l = 2, 3, \ldots, 2m)$, which satisfy the same initial condition at a_0, where a_0 is determined by $\omega_0, \omega_0 = (a_0, b_0, \mathbf{A}_0, \mathbf{B}_0, 1/p_{m_0}, p_{(m-1)_0}, p_{(m-2)_0}, \ldots, p_{0_0}, w_0) \in \mathbf{W}$. By the definition of ω_0, we know that a_0 varies as ω_0, *i.e.*, given any ω_0, there is an a_0 corresponding to it. We choose $u(\cdot,\omega) = \sum_{j=1}^{l} d_j u_j$ $(l = 2, 3, \ldots, 2m)$, which is an arbitrary linear combination of l linearly independent eigenfunctions. It is easy to see that $u(\cdot,\omega) = \sum_{j=1}^{l} d_j u_j$ $(l = 2, 3, \ldots, 2m)$ is also an eigenfunction of $\lambda(\omega)$ which satisfies an arbitrary initial condition at a_0. Now we consider the coupled BCs, and $-\pi < \theta \leq \pi$ and $u(\cdot,\omega) = \sum_{j=1}^{l} d_j u_j, \omega \to \omega_0$ satisfying

$$\left\|\mathbf{U}(a_0,\omega)\right\| = \sum_{i=0}^{2m-1} \left|u(a_0,\omega)\right| = 1,$$

and $u(t, \boldsymbol{\omega}) > 0$ when t is near a_0.

It is sufficient to illustrate that

$$\mathbf{U}(a_0,\boldsymbol{\omega}) \to \mathbf{U}(a_0,\boldsymbol{\omega}_0) \quad \text{as } \boldsymbol{\omega} \to \omega_0 \text{ in } \mathbf{W}. \tag{3.21}$$

By Lemma 3.1 and Theorem 3.1, we find the above limit is convergent uniformly on $[a_0, b_0]$. If (3.21) is not fulfilled, then we select a sequence $\boldsymbol{\omega}_k \to \omega_0$ in $\mathbf{W}$ such that

$$\mathbf{V}_k := \mathbf{U}(a_0,\omega_0) - \mathbf{U}(a_0,\omega_k) \to \mathbf{V}_0 \neq 0 \quad \text{as } \omega \to \omega_0. \tag{3.22}$$

Suppose that $\mathbf{Z}_k, \mathbf{Y}_k$ and $\mathbf{Y}$ are the vector solutions of (2.1) satisfying the initial conditions

$$\mathbf{Z}_k(a_0) = \mathbf{U}\big(a_0, \mathbf{Y}_k(a_0) = \mathbf{V}_k, \omega_k\big), \qquad \mathbf{Y}(a_0) = \mathbf{V}_0, \quad k \in \mathbb{N},$$

with the same $\omega = \omega_0$, respectively, then the uniqueness of the initial value problem implies

$$\mathbf{Y}_k = \mathbf{U}(\cdot,\omega_0) - \mathbf{Z}_k \quad \text{on } [a_0, b_0].$$

Then the BC (2.6) implies

$$\begin{aligned}
\mathbf{Y}_k(b_0) &= \mathbf{U}(b_0,\omega_0) - \mathbf{Z}_k(b_0) = \mathbf{U}(b_0,\omega_0) - \mathbf{U}(b_k,\omega_k) + \mathbf{U}(b_k,\omega_k) - \mathbf{Z}_k(b_0) \\
&= e^{i\theta_0}\mathbf{K}_0\mathbf{U}(a_0,\omega_0) - e^{i\theta_k}\mathbf{K}_k\mathbf{U}(a_0,\omega_k) + \mathbf{U}(b_k,\omega_k) - \mathbf{Z}_k(b_0) \\
&= e^{i\theta_0}\mathbf{K}_0\big[\mathbf{U}(a_0,\omega_0) - \mathbf{U}(a_0,\omega_k)\big] + e^{i\theta_0}\mathbf{K}_0\mathbf{U}(a_0,\omega_k) \\
&\quad - e^{i\theta_k}\mathbf{K}_k\mathbf{U}(a_k,\omega_k) + \mathbf{U}(b_k,\omega_k) - \mathbf{Z}_k(b_0) \\
&= e^{i\theta_0}\mathbf{K}_0\mathbf{Y}_k(a_0) + e^{i\theta_0}\mathbf{K}_0\mathbf{U}(a_0,\omega_k) - e^{i\theta_k}\mathbf{K}_k\mathbf{U}(a_k,\omega_k) \\
&\quad + \mathbf{U}(b_k,\omega_k) - \mathbf{Z}_k(b_0),
\end{aligned} \tag{3.23}$$

taking limits as $k \to \infty$ in (3.23). By Lemma 3.1, noticing $\mathbf{Y}(a_0) = \mathbf{V}_0 \neq 0$, we deduce

$$\mathbf{Y}(b_0) = e^{i\theta_0}\mathbf{K}_0\mathbf{Y}(a_0).$$

So $\mathbf{Y}$ is a nontrivial vector eigenfunction of the eigenvalue $\lambda(\omega_0)$. The eigenfunction $u(\cdot,\omega) = \sum_{j=1}^{l} d_j u_j$ ($l = 2, 3, \ldots, 2m$) of $\lambda(\omega)$ is an arbitrary linear combination of l linearly independent eigenfunctions. Moreover, $\|\mathbf{U}(a_0,\boldsymbol{\omega})\| = 1$, and there is a constant $s \neq 0$ such that $\mathbf{Y} = s\mathbf{U}(\cdot,\omega_0)$ for some d_j ($j = 1, 2, \ldots, l$), and $\mathbf{V}_0 = \mathbf{Y}(a_0) = s\mathbf{U}(a_0,\omega_0)$. As $k \to \infty$ in (3.22), we obtain

$$\mathbf{U}(a_0,\omega_0) - \lim_{k\to\infty} \mathbf{U}(a_0,\omega_k) = \mathbf{V}_0 = s\mathbf{U}(a_0,\omega_0),$$

that is,

$$\lim_{k\to\infty} \mathbf{U}(a_0,\omega_k) = (1-s)\mathbf{U}(a_0,\omega_0).$$

Noticing that the signs of $u(x,\omega_k)$ and $u(x,\omega_0)$ are the same when x is near a_0, so $1 - s > 0$ and we also have

$$\lim_{k\to\infty} \big\|\mathbf{U}(a_0,\omega_k)\big\| = (1-s)\big\|\mathbf{U}(a_0,\omega_0)\big\|.$$

This contradicts

$$\|\mathbf{V}(a_0,\omega_k)\| = \|\mathbf{U}(a_0,\omega_0)\| = 1.$$

The proof of separated conditions is similar to that of coupled conditions, we can use the same method to the proof and have hence omitted it.

The above discussion illustrates that, for each self-adjoint boundary problem and each eigenvalue $\lambda(\omega)$, we can choose the eigenfunction $u(\cdot,\omega)$ and its quasi-derivative $u^{[k]}$ ($k = 1,2,\ldots,2m-1$) convergent uniformly with ω on any compact subinterval of (a',b'). Then we normalize the eigenfunctions to end the proof. □

4 Differentiability of eigenvalues on the problems

The results of the previous section will be the key to prove the differentiability of eigenvalues. In this section, our aim is to illustrate that the isolated eigenvalues are differentiable with respect to all the data. For this purpose, we will make use of the definition of Frechet derivatives and we recall Lemma 3.2 of [5] here.

Definition 4.1 Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces. A map $\mathbf{T}:\mathbf{X}\to\mathbf{Y}$ is a linear map, $x\in\mathbf{X}$ is a given point, if it is satisfied by a bounded linear operator $d\mathbf{T}_x:\mathbf{X}\to\mathbf{Y}$, for $h\in\mathbf{X}$ and as $h\to 0$, and

$$\left|\mathbf{T}(x+h)-\mathbf{T}(x)-d\mathbf{T}_x\right| = o(h),$$

we call the map $\mathbf{T}$ differentiable at x.

Lemma 4.1 *We suppose that u and v are two solutions of* (2.1) *with different* $\lambda=\mu$ *and* $\lambda=\nu$. *Then*

$$\begin{aligned}[u,v]_a^b &:= \left[(-1)^m\sum_{i=0}^{2m-1}(-1)^{2m+1-i}u^{[i]}\bar{v}^{[2m-i-1]}\right]_a^b \\ &= (\mu-\nu)\int_a^b u\bar{v}w\,ds. \end{aligned}\tag{4.1}$$

Proof Since u and v are solutions of (2.1) with $\lambda=\mu$, $\lambda=\nu$, respectively, we have

$$\begin{aligned}(\mu-\nu)\int_a^b u\bar{v}w\,ds &= \left(w^{-1}lu,v\right)-\left(u,w^{-1}lv\right) \\ &= \int_a^b\left[\sum_{i=0}^m(-1)^i\left(p_i(x)u^{(i)}\right)^{(i)}\bar{v}-u\sum_{i=0}^m(-1)^i\left(p_i(x)\bar{v}^{(i)}\right)^{(i)}\right]ds.\end{aligned}$$

The integration by parts directly leads to the desired result. □

Lemma 4.2 ([5], Lemma 3.2) *We suppose that* $f\in\mathbf{L}_{\mathrm{loc}}(a',b')$ *is a real valued function. We deduce that*

$$\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}f(s)\,ds = f(x)\quad a.e.\ \left(a',b'\right).$$

In the sequel of this paper, we will investigate the differentiability of the eigenvalue as one of the components of ω, as the other components of ω are considered to be the same.

Theorem 4.1 (Eigenvalue-eigenfunction differential equation for special case of separated BVPs) *Assume* (2.2) *and* $\boldsymbol{\omega} = (\mathbf{A}, \mathbf{B}, a, b, 1/p_m, p_{m-1}, \dots, p_0, w) \in \mathbf{W}$ *hold. We suppose further that either* (i) *the multiplicity of* $\lambda(\omega)$ *is* 1 *in some neighborhood* $M \subset \mathbf{W}$ *of* ω, *or* (ii) $\lambda(\omega)$ *is an eigenvalue of multiplicity* l ($l = 2, 3, \dots, 2m$) *for each* $\omega \in M$, $M \subset \mathbf{W}$.

(1) *Consider the spectral problems* (2.1), (2.9) *and* (2.10) *with* $0 \le \alpha < \pi$ *and* $\beta = \pi$. *Let* $\lambda = \lambda(b)$ *be a function of endpoint* b, *and* $u = u(\cdot, b)$ *the corresponding eigenfunction. We see that* λ *is differentiable a.e. and it satisfies*:

if m *is even, then*

$$\lambda'(b) = \sum_{i=1}^{[\frac{m}{2}]} \bigl(2u^{[2i-1]}(b)u^{[2(n-i)+1]}(b) + p_{2i-1}(b)\bigl|u^{[2i-1]}(b)\bigr|^2\bigr) \quad \text{a.e. in } \bigl(a, b'\bigr); \tag{4.2}$$

if m *is odd, then*

$$\lambda'(b) = -\sum_{i=1}^{[\frac{m}{2}]} \bigl(2u^{[2i-1]}(b)u^{[2(n-i)+1]}(b) - p_{2i-1}(b)\bigl|u^{[2i-1]}(b)\bigr|^2\bigr) - \frac{|u^{[m]}(b)|^2}{p_m(b)} \quad \text{a.e. in } \bigl(a, b'\bigr). \tag{4.3}$$

(2) *Consider the spectral problems* (2.1), (2.11) *and* (2.12) *with* $0 \le \varphi < \pi$ *and* $\psi = \pi$. *Let* $\lambda = \lambda(b)$ *be a function of endpoint* b, *and* $u = u(\cdot, b)$ *the corresponding eigenfunction. Then* λ *is differentiable a.e. and it satisfies*

$$\bigl(p_m\lambda'\bigr)(b) = -\bigl|u^{[m]}(b)\bigr|^2 \quad \text{a.e. in } \bigl(a, b'\bigr). \tag{4.4}$$

(3) *Consider the spectral problems* (2.1), (2.13) *and* (2.14) *with* $0 \le \eta < \pi$ *and* $\tau = \pi$. *Let* $\lambda = \lambda(b)$ *be a function of endpoint* b, *and* $u = u(\cdot, b)$ *the corresponding eigenfunction. We see that* λ *is differentiable a.e. and it satisfies*

$$\bigl(p_m\lambda'\bigr)(b) = -\bigl|u^{[m]}(b)\bigr|^2 \quad \text{a.e. in } \bigl(a, b'\bigr). \tag{4.5}$$

Proof (1) For sufficiently small h in (4.1), we denote by $\mu = \lambda(b), u = u(\cdot, b)$ and $\nu = \lambda(b + h), v = u(\cdot, b + h)$ the corresponding eigenvalues and eigenfunctions, respectively. Since $[u, v](a) = 0, u(b, b) = 0$ and $u^{[2j]}(b, b) = 0$ ($j = 0, 1, \dots, m - 1$), we infer

$$\begin{aligned}&\bigl(\lambda(b) - \lambda(b + h)\bigr) \int_a^b u(s, b)u(s, b + h)w(s)\,ds \\ &\quad= (-1)^m \sum_{i=1}^{m} u^{[2i-1]}(b, b)u^{[2(n-i)]}(b, b + h).\end{aligned} \tag{4.6}$$

By normalizing the eigenfunction and applying Theorem 3.2, we deduce that

$$\int_a^b u(s, b)u(s, b + h)w(s)\,ds \to \int_a^b u^2(s, b)w(s)\,ds = 1, \quad h \to 0. \tag{4.7}$$

When m is even, by (4.6), we obtain

$$\begin{aligned}\big(\lambda(b)-\lambda(b+h)\big)\int_a^b u(s,b)u(s,b+h)w(s)\,ds &= (-1)^m\sum_{i=1}^{\frac{m}{2}} u^{[2i-1]}(b,b)u^{[2(m-i)]}(b,b+h)\\ &\quad + (-1)^m\sum_{\frac{m}{2}+1}^{m} u^{[2i-1]}(b,b)u^{[2(m-i)]}(b,b+h).\end{aligned}\tag{4.8}$$

For $i=1,2,\ldots,m/2$, $u^{[2(m-i)]}(b,b)=0$, we obtain

$$\begin{aligned}u^{[2(m-i)]}(b,b+h) &= u^{[2(m-i)]}(b,b+h)-u^{[2(m-i)]}(b+h,b+h)\\ &= -\int_b^{b+h}\big(u^{[2(m-i)]}\big)'(s,b+h)\,ds.\end{aligned}\tag{4.9}$$

It follows from Theorem 3.2 that as $h\to 0$, $u_m^{[j]}(\cdot,b+h)\longrightarrow u_m^{[j]}(\cdot,b)$ $(j=0,1,2,\ldots,2m-1)$ uniformly hold on any compact subinterval of (a',b'), by (4.9) and Lemma 4.2, we get

$$\lim_{h\to 0}\frac{(-1)^m u^{[2(m-i)]}(b,b+h)}{h} = (-1)^{m+1}\big(u^{[2(m-i)]}\big)'(b,b),\quad \text{a.e. in } \big(a,b'\big).$$

Since $(u^{[2(m-i)]})' = u^{[2(m-i)+1]}-(-1)^{m-2i+1}p_{2i-1}y^{(2i-1)}$, we conclude that

$$\lim_{h\to 0}\frac{(-1)^m u^{[2(m-i)]}(b,b+h)}{h} = (-1)^{m+1}u^{[2(m-i)+1]}(b,b)-p_{2i-1}(b)u^{[2i-1]}(b,b).\tag{4.10}$$

In a similar way, for $i=n/2+1,n/2+2,\ldots,n$, $u^{[2(m-i)]}(b,b)=0$, we have

$$\begin{aligned}u^{[2(m-i)]}(b,b+h) &= u^{(2m-2i)}(b,b+h)-u^{(2m-2i)}(b+h,b+h)\\ &= -\int_b^{b+h}u^{(2m-2i+1)}(s,b+h)\,ds.\end{aligned}\tag{4.11}$$

Now (4.11) and Lemma 4.2 imply that

$$\lim_{h\to 0}\frac{u^{[2(m-i)]}(b,b+h)}{h} = -u^{[2(m-i)+1]}(b,b),\quad \text{a.e. in } \big(a,b'\big).\tag{4.12}$$

We divide (4.8) by h, combine (4.7), (4.10) and (4.12), and as $h\to 0$, we get

$$\begin{aligned}-\lambda'(b) &= \sum_{i=1}^{\frac{m}{2}} u^{[2i-1]}(b,b)\big(-u^{[2(m-i)+1]}(b,b)-p_{2i-1}(b)u^{2i-1}(b,b)\big)\\ &\quad -\sum_{\frac{m}{2}+1}^{m} u^{[2i-1]}(b,b)u^{[2(m-i)+1]}(b,b)\\ &= -\sum_{i=1}^{\frac{m}{2}}\big(2u^{[2i-1]}(b,b)u^{[2(m-i)+1]}(b,b)+p_{2i-1}(b)\big(u^{[2i-1]}\big)^2(b,b)\big).\end{aligned}$$

So, we have (4.2);

if m is odd, by (4.6) we infer

$$\begin{aligned}
&\big(\lambda(b)-\lambda(b+h)\big)\int_a^b u(s,b)u(s,b+h)w(s)\,ds \\
&\quad=(-1)^m\sum_{i=1}^{\frac{m-1}{2}}u^{[2i-1]}(b,b)u^{[2(m-i)]}(b,b+h)+(-1)^m u^{[m]}(b,b)u^{[m-1]}(b,b+h) \\
&\qquad+(-1)^m\sum_{\frac{m+3}{2}}^{m}u^{[2i-1]}(b,b)u^{[2(m-i)]}(b,b+h),
\end{aligned}\tag{4.13}$$

for $i=m-1, u^{(m-1)}(b,b)=0$; we get

$$\begin{aligned}
u^{[m-1]}(b,b+h)&=u^{(m-1)}(b,b+h)-u^{(m-1)}(b+h,b+h) \\
&=-\int_b^{b+h}u^{(n)}(s,b+h)\,ds \\
&=-\int_b^{b+h}\frac{p_m u^{(n)}(s,b+h)}{p_m(s)}\,ds=-\int_b^{b+h}\frac{u^{[m]}(s,b+h)}{p_m(s)}\,ds.
\end{aligned}\tag{4.14}$$

Applying Lemma 4.1, Lemma 4.2 and (4.14), we get

$$\lim_{h\to 0}\frac{u^{[m-1]}(b,b+h)}{h}=-\frac{u^{[m]}(b,b)}{p_m(b)},\quad \text{a.e. in } \big(a,b'\big).\tag{4.15}$$

We divide (4.13) by h, as $h\to 0$ and use (4.7), (4.10), (4.12) and (4.13) to get

$$\begin{aligned}
-\lambda'(b)&=\sum_{i=1}^{\frac{m-1}{2}}u^{[2i-1]}(b,b)\big(u^{[2(m-i)+1]}(b,b)-p_{2i-1}(b)u^{[2i-1]}(b,b)\big)+\frac{(u^{[m]})^2(b,b)}{p_m(b)} \\
&\quad+\sum_{\frac{m+3}{2}}^{m}u^{[2i-1]}(b,b)u^{[2(m-i)+1]}(b,b+h) \\
&=\sum_{i=1}^{\frac{m-1}{2}}\big(2u^{[2i-1]}(b,b)u^{[2(m-i)+1]}(b,b)-p_{2i-1}(b)\big(u^{[2i-1]}\big)^2(b,b)\big) \\
&\quad+\frac{(u^{[m]})^2(b,b)}{p_m(b)}.
\end{aligned}$$

So, we obtain (4.3).

(2) For sufficiently small h in (4.1), we denote by $\mu=\lambda(b), u=u(\cdot,b)$ and $\nu=\lambda(b+h), \nu=u(\cdot,b+h)$ the corresponding eigenvalues and eigenfunctions, respectively. Since $[u,\nu](a)=0, u(b,b)=0$ and $u^{[2j]}(b,b)=0$ $(j=0,1,\dots,m-1)$, we infer

$$\begin{aligned}
&\big(\lambda(b)-\lambda(b+h)\big)\int_a^b u(s,b)u(s,b+h)w(s)\,ds \\
&\quad=(-1)^m\sum_{i=n}^{2m-1}(-1)^{2m+1-i}u^{[i]}(b,b)u^{[2m-i-1]}(b,b+h)
\end{aligned}$$

$$= -u^{[m]}(b,b)u^{[m-1]}(b,b+h) \\ + (-1)^m \sum_{i=n+1}^{2m-1} (-1)^{2m+1-i} u^{[i]}(b,b)u^{[2m-i-1]}(b,b+h), \tag{4.16}$$

for $i = m+1, m+2, \dots, 2m-1, u^{[2m-i-1]}(b,b) = 0$. Now proceeding with the proof of (4.12), we have

$$\lim_{h\to 0} \frac{u^{[2m-i-1]}(b,b+h)}{h} = -u^{[2m-i]}(b,b), \quad \text{a.e. in } (a,b'). \tag{4.17}$$

Dividing (4.16) by h, using (4.7) (4.15) and (4.17), as $h \to 0$ we get

$$-\lambda'(b) = \frac{(u^{[m]})^2(b,b)}{p_m(b)} + \sum_{i=n+1}^{2m-1} (-1)^{m-i} u^{[i]}(b,b)u^{[2m-i]}(b,b).$$

By $u^{[j]}(b,b) = 0$ ($j = 0,1,\dots,m-1$), we obtain $-\lambda'(b) = \frac{|u^{[m]}(b)|^2}{p_m(b)}$ and (4.4) holds. The proof of (4.5) is similar to the proof of (4.4), so we omit it. □

Corollary 4.1 *Suppose* (2.2) *holds and we consider the BVPs* (2.1) *and* (2.11)-(2.14) *with* $0 \le \varphi < \pi, \psi = \pi$, *or* $0 \le \eta < \pi, \tau = \pi$. *Let* $\lambda = \lambda(b)$ *be a function of endpoint* b, *and* $u = u(\cdot,b)$ *the corresponding eigenfunction. If* $p_m \ge 0$, *a.e., then* $\lambda(b)$ *is strictly decreasing on* (a,b').

Theorem 4.2 (Eigenvalue-eigenfunction differential equation for special case of separated BVPs) *Assume* (2.2) *and* $\boldsymbol{\omega} = (\mathbf{A},\mathbf{B},a,b,1/p_m,p_{m-1},\dots,p_0,w) \in \mathbf{W}$ *hold. We suppose further that either* (i) *the multiplicity of* $\lambda(\omega)$ *is* 1 *in some neighborhood* $M \subset \mathbf{W}$ *of* ω, *or* (ii) $\lambda(\omega)$ *is an eigenvalue of multiplicity* l ($l = 2,3,\dots,2m$) *for each* $\omega \in M$, $M \subset \mathbf{W}$.

(1) *Consider the spectral problems* (2.1), (2.9) *and* (2.10) *with* $0 \le \alpha < \pi$ *and* $\beta = \frac{\pi}{2}$. *Let* $\lambda = \lambda(b)$ *be a function of endpoint* b, *and* $u = u(\cdot,b)$ *the corresponding eigenfunction. We see that* λ *is differentiable a.e. and it satisfies*:

if m *is even, then*

$$\lambda'(b) = |u(b)|^2 \big(p_0(b) - \lambda(b)w(b)\big) - \frac{|u^{[m]}(b)|^2}{p_m(b)} \\ - \sum_{i=1}^{\frac{m-2}{2}} \big(2u^{[2i]}(b)u^{[2m-2i]}(b) - p_{2i}(b)|u^{[2i]}(b)|^2\big) \quad \textit{a.e. in } (a,b'); \tag{4.18}$$

if m *is odd, then*

$$\lambda'(b) = |u(b)|^2 \big(p_0(b) - \lambda(b)w(b)\big) \\ + \sum_{i=1}^{\frac{m-1}{2}} \big(2u^{[2i]}(b)u^{[2(m-i)]}(b) + p_{2i}(b)|u^{[2i]}(b)|^2\big) \quad \textit{a.e. in } (a,b'). \tag{4.19}$$

(2) *Consider the spectral problems* (2.1), (2.11) *and* (2.12) *with* $0 \le \varphi < \pi$ *and* $\psi = \pi/2$. *Let* $\lambda = \lambda(b)$ *be a function of endpoint* b, *and* $u = u(\cdot,b)$ *the corresponding*

eigenfunction. We see that λ *is differentiable a.e. and it satisfies*

$$\lambda'(b) = \left|u(b)\right|^2\big(p_0(b) - \lambda(b)w(b)\big) + \sum_{i=1}^{m-1} p_i(b)\left|u^{[i]}(b)\right|^2 \quad a.e.\ in\ \big(a, b'\big); \tag{4.20}$$

(3) *Consider the spectral problems* (2.1), (2.13) *and* (2.14) *with* $0 \le \eta < \pi$ *and* $\tau = \pi/2$. *Let* $\lambda = \lambda(b)$ *be a function of endpoint* b, *and* $u = u(\cdot, b)$ *the corresponding eigenfunction. Then* λ *is differentiable a.e. and it satisfies*

$$\lambda'(b) = \left|u(b)\right|^2\big(p_0(b) - \lambda(b)w(b)\big) + \sum_{i=1}^{m-1} p_i(b)\left|u^{[i]}(b)\right|^2 \quad a.e.\ in\ \big(a, b'\big). \tag{4.21}$$

Proof (1) For sufficiently small h in (4.1), we denote by $\mu = \lambda(b), u = u(\cdot, b), \nu = \lambda(b+h), v = u(\cdot, b+h)$ the corresponding eigenvalues and eigenfunctions, respectively. Since $[u, v](a) = 0$ and $u^{[2j+1]}(b, b) = 0$ $(j = 0, 1, \dots, m-1)$, we have

$$\begin{aligned} &\int_a^b u(s,b)u(s,b+h)w(s)\,ds \\ &\quad = (-1)^m \sum_{i=0}^{m-1} (-1)^{2m+1-2i} u^{[2i]}(b,b)u^{[2(m-i)-1]}(b,b+h) \\ &\quad = (-1)^{m+1} \sum_{i=0}^{m-1} u^{[2i]}(b,b)u^{[2(m-i)-1]}(b,b+h). \end{aligned} \tag{4.22}$$

If m is even, we get

$$\begin{aligned} &\big(\lambda(b) - \lambda(b+h)\big)\int_a^b u(s,b)u(s,b+h)w(s)\,ds \\ &\quad = -(-1)^m u(b,b)u^{[2m-1]}(b,b+h) - (-1)^m \sum_{i=1}^{\frac{m-2}{2}} u^{[2i]}(b,b)u^{[2(m-i)-1]}(b,b+h) \\ &\qquad - u^{[m]}(b,b)u^{[m-1]}(b,b+h) - \sum_{\frac{m+2}{2}}^{m-1} u^{[2i]}(b,b)u^{[2(m-i)-1]}(b,b+h), \end{aligned} \tag{4.23}$$

for $i = 2m-1, u^{[2m-1]}(b,b) = 0$; we infer

$$\begin{aligned} (-1)^m u^{[2m-1]}(b,b+h) &= (-1)^m u^{[2m-1]}(b,b+h) - (-1)^m u^{[2m-1]}(b+h,b+h) \\ &= -\int_b^{b+h} (-1)^m \big(u^{[2m-1]}\big)'(s,b+h)\,ds. \end{aligned}$$

By $u^{[2m]} = (u^{[2m-1]})' + (-1)^m p_0 y$ and (2.4), we obtain

$$\begin{aligned} &(-1)^m u^{[2m-1]}(b,b+h) \\ &\quad = -\int_b^{b+h} (-1)^m \big(u^{[2m]}(s,b+h) - (-1)^m p_0(s)u(s,b+h)\big)\,ds \end{aligned}$$

$$
\begin{aligned}
&= \int_b^{b+h} \big[p_0(s)u(s,b+h) - \lambda(b+h)u(s,b+h)w(s)\big]\,ds \\
&= \int_b^{b+h} p_0(s)u(s,b)\,ds + \int_b^{b+h} p_0(s)\big[u(s,b+h) - u(s,b)\big]\,ds - \lambda(b+h) \\
&\quad \times \int_b^{b+h} u(s,b)w(s)\,ds + \lambda(b+h)\int_b^{b+h}\big[u(s,b) - u(s,b+h)\big]w(s)\,ds.
\end{aligned} \tag{4.24}
$$

Applying Theorem 3.2, as $h \to 0$, $u_m^{[j]}(\cdot, b+h) \longrightarrow u_m^{[j]}(\cdot, b)$ $(j = 0,1,2,\ldots,2m-1)$ uniformly hold on the compact subinterval of (a', b'). We get by (4.24) and Lemma 4.2

$$
\lim_{h\to 0} \frac{(-1)^m u^{[2m-1]}(b,b+h)}{h} = u(b,b)\big[p_0(b) - \lambda(b)w(b)\big] \quad \text{a.e. in } \big(a, b'\big), \tag{4.25}
$$

for $i = 1,2,\ldots,(m-2)/2$, $u^{[2(m-i)-1]}(b,b) = 0$. We obtain

$$
\begin{aligned}
u^{[2(m-i)-1]}(b,b+h) &= u^{[2(m-i)-1]}(b,b+h) - u^{[2(m-i)-1]}(b+h,b+h) \\
&= -\int_b^{b+h} \big(u^{[2(m-i)-1]}\big)'(s,b+h)\,ds.
\end{aligned} \tag{4.26}
$$

Applying Lemma 4.2 and (4.26), we get

$$
\lim_{h\to 0} \frac{(-1)^m u^{[2(m-i)-1]}(b,b+h)}{h} = (-1)^{m+1}\big(u^{[2(m-i)-1]}\big)'(b,b) \quad \text{a.e. in } \big(a, b'\big).
$$

By $(u^{[2(m-i)-1]})' = u^{[2(m-i)]} - (-1)^{m-2i} p_{2i} y^{(2i)}$, we conclude that

$$
\begin{aligned}
\lim_{h\to 0} \frac{(-1)^m u^{[2(m-i)-1]}(b,b+h)}{h} &= (-1)^{m+1} u^{[2(m-i)]}(b,b) + p_{2i}(b)u^{[2i]}(b,b) \\
&= -u^{[2(m-i)]}(b,b) + p_{2i}(b)u^{[2i]}(b,b).
\end{aligned} \tag{4.27}
$$

Similarly, for $i = (m+2)/2, (m+4)/2, \ldots, m-1$, then $u^{[2(m-i)-1]}(b,b) = 0$, hence

$$
\lim_{h\to 0} \frac{u^{[2(m-i)-1]}(b,b+h)}{h} = -u^{[2(m-i)]}(b,b). \tag{4.28}
$$

Dividing (4.23) by h, as $h \to 0$ and using (4.5), (4.10), (4.25), (4.27) and (4.28), we get

$$
\begin{aligned}
-\lambda'(b) &= -u^2(b,b)\big[p_0(b) - \lambda(b)w(b)\big] + \frac{(u^{[m]})^2(b,b)}{p_m(b)} \\
&\quad + \sum_{i=1}^{\frac{m-2}{2}} \big(2u^{[2i]}(b,b)u^{[2(m-i)]}(b,b) - p_{2i}(b)\big(u^{[2i]}\big)^2(b,b)\big).
\end{aligned}
$$

Then we obtain (4.17);

if m is odd, we get

$$
\big[\lambda(b) - \lambda(b+h)\big]\int_a^b u(s,b)u(s,b+h)w(s)\,ds
$$

$$= -(-1)^m \left[u(b,b)u^{[2m-1]}(b,b+h) + \sum_{i=1}^{\frac{m-1}{2}} u^{[2i]}(b,b)u^{[2(m-i)-1]}(b,b+h) \right]$$
$$- (-1)^m \sum_{i=\frac{m+1}{2}}^{m-1} u^{[2i]}(b,b)u^{[2(m-i)-1]}(b,b+h). \tag{4.29}$$

Dividing (4.29) by h, as $h \to 0$ and using (4.7) (4.25), (4.27) and (4.28), we conclude that

$$-\lambda'(b) = -u^2(b,b)\big[p_0(b) - \lambda(b)w(b)\big]$$
$$- \sum_{i=1}^{\frac{m-1}{2}} \big(2u^{[2i]}(b,b)u^{[2(m-i)]}(b,b) + p_{2i}(b)\big(u^{[2i]}\big)^2(b,b)\big).$$

Then we obtain (4.19).

For sufficiently small h in (4.1), we denote by $\mu = \lambda(b), u = u(\cdot,b)$ and $\nu = \lambda(b+h), v = u(\cdot,b+h)$ the corresponding eigenvalues and eigenfunctions, respectively. Since $[u,v](a) = 0, u(b,b) = 0$ and $u^{[j]}(b,b) = 0$ $(j = m, m+1, \ldots, 2m-1)$, we have

$$\big(\lambda(b) - \lambda(b+h)\big) \int_a^b u(s,b)u(s,b+h)w(s)\,ds$$
$$= (-1)^m \sum_{i=0}^{m-1} (-1)^{2m+1-i} u^{[i]}(b,b)u^{[2m-i-1]}(b,b+h)$$
$$= -(-1)^m u(b,b)u^{[2m-1]}(b,b+h)$$
$$+ (-1)^m \sum_{i=1}^{m-1} (-1)^{2m+1-i} u^{[i]}(b,b)u^{[2m-i-1]}(b,b+h), \tag{4.30}$$

for $i = 1, 2, \ldots, m-1$, $u^{[2m-i-1]}(b,b) = 0$; hence

$$u^{[2m-i-1]}(b,b+h) = u^{[2m-i-1]}(b,b+h) - u^{[2m-i-1]}(b+h,b+h)$$
$$= -\int_b^{b+h} \big(u^{[2m-i-1]}\big)'(s,b+h)\,ds. \tag{4.31}$$

By Lemma 4.2 and (4.31), we get

$$\lim_{h\to 0} \frac{(-1)^m u^{[2m-i-1]}(b,b+h)}{h} = (-1)^{m+1}\big(u^{[2m-i-1]}\big)'(b,b) \quad \text{a.e. in } \big(a,b'\big).$$

Combining $(u^{[2m-i-1]})' = u^{[2m-i]} - (-1)^{m-i} p_i y^{(i)}$, we get

$$\lim_{h\to 0} \frac{(-1)^m u^{[2m-i-1]}(b,b+h)}{h}$$
$$= (-1)^{m+1} u^{[2m-i]}(b,b) - (-1)^{2m-i+1} p_i(b)u^{[i]}(b,b)$$
$$= -(-1)^{2m-i+1} p_i(b)u^{[i]}(b,b). \tag{4.32}$$

We divide (4.30) by h, as $h \to 0$, and use (4.7) (4.25) and (4.32) to get

$$-\lambda'(b) = -u^2(b,b)\big[p_0(b) - \lambda(b)w(b)\big] - \sum_{i=1}^{m-1} p_i(b)\big(u^{[i]}\big)^2(b,b).$$

We obtain (4.20). The proof of (4.21) is similar to that of (4.20), so we omit it. □

Theorem 4.3 (Eigenvalue-eigenfunction differential equation for separated BVPs) *Assume* (2.2) *and* $\boldsymbol{\omega} = (\mathbf{A}, \mathbf{B}, a, b, 1/p_m, p_{m-1}, \ldots, p_0, w) \in \mathbf{W}$ *hold. Consider the BVPs* (2.1), (2.9) *and* (2.10) *with* $0 \leq \alpha < \pi$ *and* $0 < \beta \leq \pi$, *or BVP* (2.1), (2.11) *and* (2.12) *with* $0 \leq \varphi < \pi$ *and* $0 < \psi \leq \pi$, *or BVP* (2.1), (2.13) *and* (2.14) *with* $0 \leq \eta < \pi$ *and* $0 < \tau \leq \pi$. *We suppose further that either* (i) *the multiplicity of* $\lambda(\omega)$ *is* 1 *in some neighborhood* $M \subset \mathbf{W}$ *of* ω, *or* (ii) $\lambda(\omega)$ *is an eigenvalue of multiplicity* l ($l = 2, 3, \ldots, 2m$) *for each* $\omega \in M$, $M \subset \mathbf{W}$.

(1) *Let* $\lambda = \lambda(a)$ *be a function of endpoint* a, *and* $u = u(\cdot, a)$ *the corresponding eigenfunction. We see that* λ *is differentiable a.e. and it satisfies*

$$\begin{aligned} \lambda'(a) &= \big|u(a)\big|^2\big[p_0(b) - \lambda(b)w(b)\big] - \frac{|u^{[m]}(a)|^2}{p_m(a)} \\ &\quad - \sum_{i=1}^{m-1}\big(p_i(a)\big|u^{[i]}(a)\big|^2 - (-1)^{m-i}2u^{[i]}(a)u^{[2m-i]}(a)\big) \quad \textit{a.e. in } \big(a', b\big), \end{aligned} \tag{4.33}$$

particularly, if $p_m, p_{m-1}, \ldots, p_0$ *and* w *are continuous at* a *and* $p_m(a) \neq 0$; *then* (4.33) *holds at* a.

(2) *Let* $\lambda = \lambda_m$ *and* $u = u_m$. *We see that* λ *is differentiable a.e. and it satisfies*

$$\begin{aligned} \lambda'(b) &= \big|u(b)\big|^2\big[p_0(b) - \lambda(b)w(b)\big] - \frac{|u^{[m]}(b)|^2}{p_m(b)} \\ &\quad - \sum_{i=1}^{m-1}\big(p_i(b)\big|u^{[i]}(b)\big|^2 - (-1)^{m-i}2u^{[i]}(b)u^{[2m-i]}(b)\big) \quad \textit{a.e. in } \big(a, b'\big), \end{aligned} \tag{4.34}$$

particularly, if $p_m, p_{m-1}, \ldots, p_0$ *and* w *are continuous at* b *and* $p_m(b) \neq 0$; *then* (4.34) *holds at* b.

Proof The proofs of (4.33) and (4.34) are similar, so we will prove (4.34) only. For sufficiently small h in (4.1), we denote by $\mu = \lambda(b), u = u(\cdot, b)$ and $\nu = \lambda(b+h), v = u(\cdot, b+h)$ the corresponding eigenvalues and eigenfunctions, respectively. Since $[u, v](a) = 0$, we have

$$\begin{aligned} &\big(\lambda(b) - \lambda(b+h)\big)\int_a^b u(s,b)u(s,b+h)w(s)\,ds \\ &\quad = (-1)^m \sum_{i=0}^{2m-1}(-1)^{2m+1-i}u^{[i]}(b,b)u^{[2m-i-1]}(b,b+h) \\ &\quad = -(-1)^m u(b,b)u^{[2m-1]}(b,b+h) \\ &\qquad + (-1)^m \sum_{i=1}^{m-1}(-1)^{2m+1-i}u^{[i]}(b,b)u^{[2m-i-1]}(b,b+h) \end{aligned}$$

$$- u^{[m]}(b,b)u^{[m-1]}(b,b+h) + (-1)^m \sum_{i=n+1}^{2m-1} (-1)^{2m+1-i} u^{[i]}(b,b)u^{[2m-i-1]}(b,b+h). \tag{4.35}$$

We divide (4.35) by h and take the limit $h \to 0$, use (4.5) (4.17) (4.25) and (4.32), and apply the continuity of λ at b in Theorem 3.2 and Lemma 4.2, so we get (4.34). □

Theorem 4.4 (Eigenvalue-eigenfunction differential equation for coupled BVPs) *Assume that* $\boldsymbol{\omega} = (\mathbf{A}, \mathbf{B}, a, b, 1/p_m, p_{m-1}, \ldots, p_0, w) \in \mathbf{W}$ *and* (2.2) *hold. We suppose further that either* (i) *the multiplicity of* $\lambda(\omega)$ *is* 1 *in some neighborhood* $M \subset \mathbf{W}$ *of* ω, *or* (ii) $\lambda(\omega)$ *is an eigenvalue of multiplicity* l ($l = 2, 3, \ldots, 2m$) *for each* $\omega \in M$, $M \subset \mathbf{W}$. *Consider the spectral problems* (2.1), (2.16) *and* (2.17) *with* $-\pi < \theta \leq \pi$.

(1) *Let* $\lambda = \lambda(a)$ *be a function of endpoint* a, *and* $u = u(\cdot, a)$ *the corresponding eigenfunction. We see that* λ *is differentiable a.e. and it satisfies*:

if m is even, then

$$\lambda'(a) = -\left|u(a)\right|^2 \left(p_0(a) - \lambda(a)w(a)\right) + 2\operatorname{Re} \sum_{i=1}^{m-1} (-1)^i u^{[i]}(a)\overline{u}^{[2m-i]}(a) + \frac{u^{[m]}(a)}{p_m(a)} - \sum_{i=1}^{m-1} p_i(a)\left|\overline{u}^{[i]}(a)\right|^2; \tag{4.36}$$

if m is odd, then

$$\lambda'(a) = -\left|u(a)\right|^2 \left(p_0(a) - \lambda(a)w(a)\right) - 2\operatorname{Re} \sum_{i=1}^{m-1} (-1)^i u^{[i]}(a)\overline{u}^{[2m-i]}(a) + \frac{u^{[m]}(a)}{p_m(a)} - \sum_{i=1}^{m-1} p_i(a)\left|\overline{u}^{[i]}(a)\right|^2, \tag{4.37}$$

particularly, if $p_m, p_{m-1}, \ldots, p_0$ *and* w *are continuous at* $a \in (a', b]$ *and* $p_m(a) \neq 0$, *then equations* (4.36) *and* (4.37) *hold at* a.

(2) *Let* $\lambda = \lambda(b)$ *be a function of endpoint* b, *and* $u = u(\cdot, b)$ *the corresponding eigenfunction. We see that* λ *is differentiable a.e. and it satisfies*:

if m is even, then

$$\lambda'(b) = \left|u(b)\right|^2 \left(p_0(b) - \lambda(b)w(b)\right) - 2\operatorname{Re} \sum_{i=1}^{m-1} (-1)^i u^{[i]}(b)\overline{u}^{[2m-i]}(b) - \frac{u^{[m]}(b)}{p_m(b)} + \sum_{i=1}^{m-1} p_i(b)\left|\overline{u}^{[i]}(b)\right|^2; \tag{4.38}$$

if m is odd, then

$$\lambda'(b) = \left|u(b)\right|^2 \left(p_0(b) - \lambda(b)w(b)\right) + 2\operatorname{Re} \sum_{i=1}^{m-1} (-1)^i u^{[i]}(b)\overline{u}^{[2m-i]}(b) - \frac{u^{[m]}(b)}{p_m(b)} + \sum_{i=1}^{m-1} p_i(b)\left|\overline{u}^{[i]}(b)\right|^2, \tag{4.39}$$

particularly, if $p_m, p_{m-1}, \dots, p_0$ and w are continuous at $b \in [a, b')$ and $p_m(b) \neq 0$, then equations (4.38) *and* (4.39) *hold at b.*

Proof The proofs of (4.36) and (4.37) are similar to (4.38) and (4.39), respectively, so we prove (4.38) and (4.39) only. For sufficiently small h in (4.1), we denote by $\mu = \lambda(b), \nu = \lambda(b+h)$ and $u = u(\cdot, b), v = u(\cdot, b+h)$ the eigenvalues and eigenfunctions.

$$\bigl[\lambda(b) - \lambda(b+h)\bigr] \int_a^b u\bar{v}w\, ds = \Biggl[(-1)^m \sum_{i=0}^{2m-1} (-1)^{2m+1-i} u^{[i]} \bar{v}^{[2m-i-1]}\Biggr]_a^b.$$

If m is even,

$$\begin{aligned}
&\bigl(\lambda(b) - \lambda(b+h)\bigr) \int_a^b u\bar{v}w\, ds \\
&\quad = -\begin{pmatrix} \bar{v}^{[2m-1]}, & -\bar{v}^{[2m-2]}, & \cdots, & \bar{v}^{[1]}, & -\bar{v} \end{pmatrix}(b) \begin{pmatrix} cu \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}(b) \\
&\qquad + \begin{pmatrix} \bar{v}^{[2m-1]}, & -\bar{v}^{[2m-2]}, & \cdots, & \bar{v}^{[1]}, & -\bar{v} \end{pmatrix}(a) \begin{pmatrix} cu \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}(a),
\end{aligned}$$

$$\begin{aligned}
&\begin{pmatrix} u^{[2m-1]}, & -u^{[2m-2]}, & u^{[2m-3]}, & \cdots, & -u^{[2]}, & u^{[1]} & -u \end{pmatrix}(b) \\
&\quad = \begin{pmatrix} u, & u^{[1]}, & u^{[2]}, & \cdots, & u^{[2m-3]}, & u^{[2m-2]}, & u^{[2m-1]} \end{pmatrix}(b) \\
&\qquad \times \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \\
&\quad = e^{i\theta} \begin{pmatrix} u, & u^{[1]}, & u^{[2]}, & \cdots, & u^{[2m-3]}, & u^{[2m-2]}, & u^{[2m-1]} \end{pmatrix}(a)\mathbf{K}^T \\
&\qquad \times \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$= e^{i\theta}\left(u^{[2m-1]}, \quad -u^{[2m-2]}, \quad u^{[2m-3]}, \quad \cdots, \quad -u^{[2]}, \quad u^{[1]}, \quad -u\right)(a)$$

$$\times \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \mathbf{K}^T \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

$$= e^{i\theta}\left(u^{[2m-1]}, \quad -u^{[2m-2]}, \quad u^{[2m-3]}, \quad \cdots, \quad -u^{[2]}, \quad u^{[1]}, \quad -u\right)(a)\mathbf{K}^{-1},$$

hence

$$\begin{aligned}&\left(u^{[2m-1]}, \quad -u^{[2m-2]}, \quad \cdots, \quad u^{[1]}, \quad -u\right)(a)\\ &\quad = e^{-i\theta}\left(u^{[2m-1]}, \quad -u^{[2m-2]}, \quad \cdots, \quad u^{[1]}, \quad -u\right)(b)\mathbf{K},\end{aligned}$$

or

$$\begin{aligned}&\left(\overline{u}^{[2m-1]}, \quad -\overline{u}^{[2m-2]}, \quad \cdots, \quad \overline{u}^{[1]}, \quad -\overline{u}\right)(a)\\ &\quad = e^{i\theta}\left(\overline{u}^{[2m-1]}, \quad -\overline{u}^{[2m-2]}, \quad \cdots, \quad \overline{u}^{[1]}, \quad -\overline{u}\right)(b)\mathbf{K}.\end{aligned} \tag{4.40}$$

Then

$$\begin{aligned}&\big(\lambda(b) - \lambda(b+h)\big)\int_a^b u\overline{v}w\,ds\\
&\quad = -\left(\overline{v}^{[2m-1]}, \quad -\overline{v}^{[2m-2]}, \quad \cdots, \quad \overline{v}^{[1]}, \quad -\overline{v}\right)(b)\begin{pmatrix} u \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}(b)\\
&\qquad + \left(\overline{v}^{[2m-1]}, \quad -\overline{v}^{[2m-2]}, \quad \cdots, \quad \overline{v}^{[1]}, \quad -\overline{v}\right)(b+h)\begin{pmatrix} u \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}\\
&\quad = \Big[\left(\overline{v}^{[2m-1]}, \quad -\overline{v}^{[2m-2]}, \quad \cdots, \quad \overline{v}^{[1]}, \quad -\overline{v}\right)(b+h)\\
&\qquad - \left(\overline{v}^{[2m-1]}, \quad -\overline{v}^{[2m-2]}, \quad \cdots, \quad \overline{v}^{[1]}, \quad -\overline{v}\right)(b)\Big]\\
&\qquad \times \begin{pmatrix} u \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}(b).\end{aligned} \tag{4.41}$$

Now proceeding as in Theorem 4.1 and Theorem 4.2, we get

$$\begin{aligned}
&\lim_{h\to 0}\frac{\overline{v}^{[2m-1]}(b+h)-\overline{v}^{[2m-1]}(b)}{h}=\bigl(\lambda(b)w(b)-p_0(b)\bigr)\overline{u}(b),\\
&\lim_{h\to 0}\frac{\overline{v}^{[2m-i-1]}(b+h)-\overline{v}^{[2m-i-1]}(b)}{h}=\overline{u}^{[2m-i]}(b)-(-1)^{m-i}p_i(b)u^{[i]}(b),\quad i=1,2,\ldots,m-1,\\
&\lim_{h\to 0}\frac{\overline{v}^{[m-1]}(b+h)-\overline{v}^{[m-1]}(b)}{h}=\frac{\overline{u}^{[m]}(b)}{p_m(b)},\\
&\lim_{h\to 0}\frac{\overline{v}^{[i]}(b+h)-\overline{v}^{[i]}(b)}{h}=\overline{u}^{[i+1]}(b),\quad i=1,2,\ldots,m-1.
\end{aligned}$$

Dividing (4.41) by h, as $h\to 0$, we obtain

$$\begin{aligned}
-\lambda'(b)={}&\Bigl((\lambda w-p_0)\overline{u},\quad -\overline{u}^{[2m-1]}-p_1\overline{u}^{[1]},\quad -\overline{u}^{[2m-2]}-p_2\overline{u}^{[2]},\quad\cdots,\\
&\overline{u}^{[m]}/p_m,\cdots,\overline{u}^{[2]},-\overline{u}^{[1]}\Bigr)(b)\\
&\times\begin{pmatrix} u\\ u^{[1]}\\ u^{[2]}\\ \vdots\\ u^{[m]}\\ \vdots\\ u^{[2m-2]}\\ u^{[2m-1]}\end{pmatrix}(b)\\
={}&\bigl|u(b)\bigr|^2\bigl(\lambda(b)w(b)-p_0(b)\bigr)+2\operatorname{Re}\sum_{i=1}^{m-1}(-1)^i u^{[i]}(b)\overline{u}^{[2m-i]}(b)\\
&+\frac{u^{[m]}(b)}{p_m(b)}-\sum_{i=1}^{m-1}p_i(b)\bigl|\overline{u}^{[i]}(b)\bigr|^2,
\end{aligned}$$

and (4.38) holds.

If m is odd, we infer

$$\begin{aligned}
&\bigl(\lambda(b)-\lambda(b+h)\bigr)\int_a^b u\bar{v}w\,ds\\
&=-\Bigl[\Bigl(\overline{v}^{[2m-1]},\quad -\overline{v}^{[2m-2]},\quad\cdots,\quad \overline{v}^{[1]},\quad -\overline{v}\Bigr)(b+h)\\
&\quad-\Bigl(\overline{v}^{[2m-1]},\quad -\overline{v}^{[2m-2]},\quad\cdots,\quad \overline{v}^{[1]},\quad -\overline{v}\Bigr)(b)\Bigr]\\
&\quad\times\begin{pmatrix} u\\ u^{[1]}\\ \vdots\\ u^{[2m-2]}\\ u^{[2m-1]}\end{pmatrix}(b)
\end{aligned}\tag{4.42}$$

and

$$\lim_{h\to 0}\frac{\overline{v}^{[2m-1]}(b+h)-\overline{v}^{[2m-1]}(b)}{h}=\big(p_0(b)-\lambda(b)w(b)\big)\overline{u}(b),$$
$$\lim_{h\to 0}\frac{\overline{v}^{[2m-i-1]}(b+h)-\overline{v}^{[2m-i-1]}(b)}{h}=\overline{u}^{[2m-i]}(b)-(-1)^{m-i}p_i(b)u^{[i]}(b),\quad i=1,2,\dots,m-1,$$
$$\lim_{h\to 0}\frac{\overline{v}^{[m-1]}(b+h)-\overline{v}^{[m-1]}(b)}{h}=\frac{\overline{u}^{[m]}(b)}{p_m(b)},$$
$$\lim_{h\to 0}\frac{\overline{v}^{[i]}(b+h)-\overline{v}^{[i]}(b)}{h}=\overline{u}^{[i+1]}(b),\quad i=1,2,\dots,m-1.$$

We divide (4.42) by h, and take the limit as $h\to 0$ to obtain

$$\begin{aligned}-\lambda'(b)={}&-\Big((p_0-\lambda w)\overline{u},\quad -\overline{u}^{[2m-1]}+p_1\overline{u}^{[1]},\quad \overline{u}^{[2m-2]}+p_2\overline{u}^{[2]},\quad\cdots,\\ &-\overline{u}^{[m]}/p_m,\cdots,\overline{u}^{[2]},-\overline{u}^{[1]}\Big)(b)\\ &\times\begin{pmatrix}u\\ u^{[1]}\\ u^{[2]}\\ \vdots\\ u^{[m]}\\ \vdots\\ u^{[2m-2]}\\ u^{[2m-1]}\end{pmatrix}(b)\\ ={}&\big|u(b)\big|^2\big(\lambda(b)w(b)-p_0(b)\big)-2\operatorname{Re}\sum_{i=1}^{m-1}(-1)^i u^{[i]}(b)\overline{u}^{[2m-i]}(b)\\ &+\frac{u^{[m]}(b)}{p_m(b)}-\sum_{i=1}^{m-1}p_i(b)\big|\overline{u}^{[i]}(b)\big|^2,\end{aligned}$$

and (4.39) holds. □

Theorem 4.5 *For the BVPs* (2.1) *and* (2.9)-(2.17), *let* $\boldsymbol{\omega}=(\mathbf{A},\mathbf{B},a,b,1/p_m,p_{m-1},\dots,p_0,w)\in\mathbf{W}$, *and* $\lambda=\lambda(\omega)$ *and* $u=u(\cdot,\omega)$ *be the eigenvalue and normalized eigenfunction. We suppose further that either* (i) *the multiplicity of* $\lambda(\omega)$ *is* 1 *in some neighborhood* $M\subset\mathbf{W}$ *of* ω, *or* (ii) $\lambda(\omega)$ *is an eigenvalue of multiplicity* l ($l=2,3,\dots,2m$) *for each* $\omega\in M$, $M\subset\mathbf{W}$. *Then* λ *is continuously differentiable with respect to each variable of* ω *in the appropriate sense. Their derivatives are given as follows.*

1. *Let* $\lambda=\lambda(\alpha)$ *be a function of* α, *and* $u=u(\cdot,\alpha)$ *the corresponding eigenfunction. We see that* λ *is differentiable and it satisfies*:
   if m is even, then

$$\lambda'(\alpha)=2\operatorname{Re}\sum_{i=1}^{\frac{m-1}{2}}\big(u^{[2i]}(a)\overline{u}^{[2m-2i]}(a)+u^{[2i+1]}(a)\overline{u}^{[2m-2i-1]}(a)\big);\tag{4.43}$$

if m is odd, then

$$\lambda'(\alpha) = -2\,\mathrm{Re}\sum_{i=0}^{\frac{m-3}{2}}\big(u^{[2i]}(a)\overline{u}^{[2m-2i-2]}(a) + u^{[2i+1]}(a)\overline{u}^{[2m-2i-1]}(a)\big) - \big|u^{[m-1]}(a)\big|^2 - \big|u^{[m]}(a)\big|^2. \tag{4.44}$$

2. *Let $\lambda = \lambda(\beta)$ be a function of β, and $u = u(\cdot,\beta)$ the corresponding eigenfunction. We see that λ is differentiable and it satisfies*: *if m is even, then*

$$\lambda'(\beta) = -2\,\mathrm{Re}\sum_{i=1}^{\frac{m-1}{2}}\big(u^{[2i]}(b)\overline{u}^{[2m-2i]}(b) + u^{[2i+1]}(b)\overline{u}^{[2m-2i-1]}(b)\big); \tag{4.45}$$

if m is odd, then

$$\lambda'(\beta) = 2\,\mathrm{Re}\sum_{i=0}^{\frac{m-3}{2}}\big(u^{[2i]}(b)\overline{u}^{[2m-2i-2]}(b) + u^{[2i+1]}(b)\overline{u}^{[2m-2i-1]}(b)\big) + \big|u^{[m-1]}(b)\big|^2 + \big|u^{[m]}(b)\big|^2. \tag{4.46}$$

3. *Let $\lambda = \lambda(\varphi)$ be a function of φ, and $u = u(\cdot,\varphi)$ the corresponding eigenfunction. We see that λ is differentiable and it satisfies*:
   if m is even, then

$$\lambda'(\varphi) = -2\,\mathrm{Re}\sum_{i=0}^{\frac{m-2}{2}}\big(u^{[i]}(a)\overline{u}^{[m-i-1]}(a) - u^{[m+i]}(a)\overline{u}^{[2m-i-1]}(a)\big); \tag{4.47}$$

if m is odd, then

$$\lambda'(\varphi) = -2\,\mathrm{Re}\sum_{i=0}^{\frac{m-3}{2}}\big(u^{[i]}(a)\overline{u}^{[m-i-1]}(a) + u^{[m+i]}(a)\overline{u}^{[2m-i-1]}(a)\big) - \big|u^{[\frac{m-1}{2}]}(a)\big|^2 - \big|u^{[\frac{3m-1}{2}]}(a)\big|^2. \tag{4.48}$$

4. *Let $\lambda = \lambda(\psi)$ be a function of ψ, and $u = u(\cdot,\psi)$ the corresponding eigenfunction. We see that λ is differentiable and it satisfies*:
   if m is even, then

$$\lambda'(\psi) = 2\,\mathrm{Re}\sum_{i=0}^{\frac{m-2}{2}}\big(u^{[i]}(b)\overline{u}^{[m-i-1]}(b) - u^{[m+i]}(b)\overline{u}^{[2m-i-1]}(b)\big); \tag{4.49}$$

if m is odd, then

$$\lambda'(\psi) = 2\,\mathrm{Re}\sum_{i=0}^{\frac{m-3}{2}}\big(u^{[i]}(b)\overline{u}^{[m-i-1]}(b) + u^{[m+i]}(b)\overline{u}^{[2m-i-1]}(b)\big) + \big|u^{[\frac{m-1}{2}]}(b)\big|^2 + \big|u^{[\frac{3m-1}{2}]}(b)\big|^2. \tag{4.50}$$

5. *Let* $\lambda = \lambda(\eta)$ *be a function of* η*, and* $u = u(\cdot,\eta)$ *the corresponding eigenfunction. We see that* λ *is differentiable and it satisfies*:

$$\lambda'(\eta) = (-1)^{(n)} \sum_{i=0}^{m-1} \left(\left| u^{[i]}(a) \right|^2 + \left| u^{[m+i]}(a) \right|^2 \right). \tag{4.51}$$

6. *Let* $\lambda = \lambda(\tau)$ *be a function of* τ*, and* $u = u(\cdot,\tau)$ *the corresponding eigenfunction. We see that* λ *is differentiable and it satisfies*

$$\lambda'(\tau) = (-1)^{(n+1)} \sum_{i=0}^{m-1} \left(\left| u^{[i]}(b) \right|^2 + \left| u^{[m+i]}(b) \right|^2 \right). \tag{4.52}$$

7. *Let* $\lambda = \lambda(\theta)$ *be a function of* θ*, and* $u = u(\cdot,\theta)$ *the corresponding eigenfunction. We see that* λ *is differentiable and it satisfies*

$$\lambda'(\theta) = (-1)^m 2 \operatorname{Im} \sum_{i=0}^{m-1} (-1)^i u^{[i]}(b) \overline{u}^{[2m-i-1]}(b). \tag{4.53}$$

8. *Let* $\lambda = \lambda(\mathbf{K})$ *be a function of matrix* $\mathbf{K}$*, and* $u = u(\cdot,\mathbf{K})$ *the corresponding eigenfunction. We suppose that* $\mathbf{K}$ *fulfills* (2.16)*. Then we see that* λ *is differentiable, and its Frechet derivative is*

$$\begin{aligned} d\lambda_K(\mathbf{H}) = (-1)^{m+1} &\left(\overline{u}^{[2m-1]}, \quad -\overline{u}^{[2m-2]}, \quad \cdots, \quad \overline{u}^{[1]}, \quad -\overline{u} \right)(b) \\ &\times \mathbf{H}\mathbf{K}^{-1} \begin{pmatrix} u \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix} (b). \end{aligned} \tag{4.54}$$

9. *Consider* λ *as a function of* $p_0 \in \mathbf{L}^1(a,b)$*. We see that* λ *is differentiable and its Frechet derivative is*

$$d\lambda_{p_0}(h) = \int_a^b |u|^2 h. \tag{4.55}$$

10. *Consider* λ *as a function* $w \in \mathbf{L}^1(a,b)$*. We see that* λ *is differentiable and its Frechet derivative is*

$$d\lambda_w(h) = -\lambda \int_a^b |u|^2 h. \tag{4.56}$$

Proof Since the proofs of (4.43)-(4.50) are similar we prove (4.45) and (4.46) only. Assume $\beta \neq \pi/2$. When $h \in \mathbb{R}$ is small enough, we denote by $u = u(\cdot,\beta)$ and $v = u(\cdot,\beta+h)$ the normalized real valued eigenfunctions of $\mu = \lambda(\beta)$ and $\nu = \lambda(\beta+h)$. By (2.1) we obtain

$$\left[\lambda(\beta) - \lambda(\beta+h) \right] \int_a^b uvw\, ds = \left[(-1)^m \sum_{i=0}^{2m-1} (-1)^{2m+1-i} u^{[i]} \bar{v}^{[2m-i-1]} \right]_a^b.$$

If m is even,

$$
\begin{aligned}
&\big(\lambda(\beta)-\lambda(\beta+h)\big)\int_a^b uvw\,ds \\
&\quad=\big[-u\overline{v}^{[2m-1]}+u^{[1]}\overline{v}^{[2m-2]}-u^{[2]}\overline{v}^{[2m-3]}+\cdots+u^{[2m-3]}\overline{v}^{[2]}-u^{[2m-2]}\overline{v}^{[1]}+u^{[2m-1]}\overline{v}\big]_a^b.
\end{aligned}
$$

By the BCs, we conclude

$$
\big[-u\overline{v}^{[2m-1]}+u^{[1]}\overline{v}^{[2m-2]}-u^{[2]}\overline{v}^{[2m-3]}+\cdots+u^{[2m-3]}\overline{v}^{[2]}-u^{[2m-2]}\overline{v}^{[1]}+u^{[2m-1]}\overline{v}\big](a)=0.
$$

Then

$$
\begin{aligned}
&\big(\lambda(\beta)-\lambda(\beta+h)\big)\int_a^b uvw\,ds \\
&\quad=-u(b)\overline{v}^{[2m-1]}(b)+u^{[1]}(b)\overline{v}^{[2m-2]}(b)-u^{[3]}(b)\overline{v}^{[2m-3]}(b)+\cdots \\
&\qquad+u^{[m-1]}(b)\overline{v}^{[m]}(b)-u^{[m]}(b)\overline{v}^{[m-1]}(b) \\
&\qquad+\cdots+u^{[2m-3]}(b)\overline{v}^{[2]}(b)-u^{[2m-2]}(b)\overline{v}^{[1]}(b)+u^{[2m-1]}(b)\overline{v}(b) \\
&\quad=-\tan\beta u^{[1]}(b)\overline{v}^{[2m-1]}(b)+\tan(\beta+h)u^{[1]}(b)\overline{v}^{[2m-1]}(b)-\tan\beta u^{[3]}(b)\overline{v}^{[2m-3]}(b)+\cdots \\
&\qquad-\tan\beta u^{[m-1]}(b)\overline{v}^{[m+1]}(b)+\tan(\beta+h)u^{[m-1]}(b)\overline{v}^{[m+1]}(b)-\cdots \\
&\qquad+\tan(\beta+h)u^{[2m-3]}(b)\overline{v}^{[3]}(b) \\
&\qquad-\tan\beta u^{[2m-2]}(b)\overline{v}^{[1]}(b)+\tan(\beta+h)u^{[2m-1]}(b)\overline{v}^{[1]}(b) \\
&\quad=\big[\tan(\beta+h)-\tan(\beta)\big]\big(u^{[1]}(b)\overline{v}^{[2m-1]}(b)+u^{[3]}(b)\overline{v}^{[2m-3]}(b)+\cdots+u^{[m-1]}(b)\overline{v}^{[m+1]}(b) \\
&\qquad+u^{[m+1]}(b)\overline{v}^{[m-1]}(b)+\cdots+u^{[2m-3]}(b)\overline{v}^{[3]}(b)+u^{[2m-1]}(b)\overline{v}^{[1]}(b)\big).
\end{aligned}
\tag{4.57}
$$

We divide both sides of (4.57) by h, as $h\to 0$, and we have

$$
\begin{aligned}
-\lambda'(\beta)={}&\sec^2\beta\big(u^{[1]}(b)\overline{u}^{[2m-1]}(b)+u^{[2]}(b)\overline{u}^{[2m-3]}(b)+\cdots+u^{[m-1]}(b)\overline{u}^{[m+1]}(b) \\
&+u^{[m+1]}(b)\overline{u}^{[m-1]}(b)+\cdots+u^{[2m-3]}(b)\overline{u}^{[3]}(b)+u^{[2m-1]}(b)\overline{u}^{[1]}(b)\big) \\
={}&\tan^2\beta u^{[1]}(b)\overline{u}^{[2m-1]}(b)+\tan^2\beta u^{[3]}(b)\overline{u}^{[2m-3]}(b)+\cdots \\
&+\tan^2\beta u^{[m-1]}(b)\overline{u}^{[m+1]}(b) \\
&+\tan^2\beta u^{[m+1]}(b)\overline{u}^{[m-1]}(b)+\cdots+\tan^2\beta u^{[2m-3]}(b)\overline{u}^{[3]}(b) \\
&+\tan^2\beta u^{[2m-1]}(b)\overline{u}^{[1]}(b) \\
&+u^{[1]}(b)\overline{u}^{[2m-1]}(b)+u^{[3]}(b)\overline{u}^{[2m-3]}(b)+\cdots \\
&+u^{[m-1]}(b)\overline{u}^{[m+1]}(b)+u^{[m+1]}(b)\overline{u}^{[m-1]}(b) \\
&+\cdots+u^{[2m-3]}(b)\overline{u}^{[3]}(b)+u^{[2m-1]}(b)\overline{u}^{[1]}(b) \\
={}&u(b)\overline{u}^{[2m-2]}(b)+u^{[2]}(b)\overline{u}^{[2m-4]}(b)+\cdots+u^{[m-2]}(b)\overline{u}^{[m]}(b) \\
&+u^{[m]}(b)\overline{u}^{[m-2]}(b)+\cdots \\
&+u^{[2m-2]}(b)\overline{u}(b)+u^{[1]}(b)\overline{u}^{[2m-1]}(b)+u^{[3]}(b)\overline{u}^{[2m-3]}(b)
\end{aligned}
$$

$$
\begin{aligned}
&+ \cdots + u^{[m-1]}(b)\overline{u}^{[m+1]}(b) \\
&+ u^{[m+1]}(b)\overline{u}^{[m-1]}(b) + \cdots + u^{[2m-3]}(b)\overline{u}^{[3]}(b) + u^{[2m-1]}(b)\overline{u}^{[1]}(b) \\
&= 2\operatorname{Re}\sum_{i=1}^{\frac{m-1}{2}}\bigl(u^{[2i]}(b)\overline{u}^{[2m-2i]}(b) + u^{[2i+1]}(b)\overline{u}^{[2m-2i-1]}(b)\bigr),
\end{aligned}
$$

and (4.45) holds;

if m is odd,

$$
\begin{aligned}
&\bigl(\lambda(\beta) - \lambda(\beta+h)\bigr)\int_a^b uvw\,ds \\
&\quad = \bigl[u\overline{v}^{[2m-1]} - u^{[1]}\overline{v}^{[2m-2]} + u^{[2]}\overline{v}^{[2m-3]} - \cdots - u^{[2m-3]}\overline{v}^{[2]} + u^{[2m-2]}\overline{v}^{[1]} - u^{[2m-1]}\overline{v}\bigr]_a^b.
\end{aligned}
$$

By the BCs, we infer that

$$
\bigl[u\overline{v}^{[2m-1]} - u^{[1]}\overline{v}^{[2m-2]} + u^{[2]}\overline{v}^{[2m-3]} - \cdots - u^{[2m-3]}\overline{v}^{[2]} + u^{[2m-2]}\overline{v}^{[1]} - u^{[2m-1]}\overline{v}\bigr](a) = 0.
$$

Then

$$
\begin{aligned}
&\bigl(\lambda(\beta) - \lambda(\beta+h)\bigr)\int_a^b uvw\,ds \\
&\quad = u(b)\overline{v}^{[2m-1]}(b) - u^{[1]}(b)\overline{v}^{[2m-2]}(b) + u^{[2]}(b)\overline{v}^{[2m-3]}(b) + \cdots \\
&\qquad + u^{[m-1]}(b)\overline{v}^{[m]}(b) - u^{[m]}(b)\overline{v}^{[m-1]}(b) \\
&\qquad + \cdots - u^{[2m-3]}(b)\overline{v}^{[2]}(b) + u^{[2m-2]}(b)\overline{v}^{[1]}(b) - u^{[2m-1]}(b)\overline{v}(b) \\
&\quad = \tan\beta u^{[1]}(b)\overline{v}^{[2m-1]}(b) - \tan(\beta+h)u^{[1]}(b)\overline{v}^{[2m-1]}(b) + \tan\beta u^{[3]}(b)\overline{v}^{[2m-3]}(b) - \cdots \\
&\qquad + \tan\beta u^{[m]}(b)\overline{v}^{[m]}(b) - \tan(\beta+h)u^{[m]}(b)\overline{v}^{[m]}(b) + \cdots \\
&\qquad - \tan(\beta+h)u^{[2m-3]}(b)\overline{v}^{[3]}(b) \\
&\qquad + \tan\beta u^{[2m-2]}(b)\overline{v}^{[1]}(b) - \tan(\beta+h)u^{[2m-1]}(b)\overline{v}^{[1]}(b) \\
&\quad = -\bigl[\tan(\beta+h) - \tan(\beta)\bigr]\bigl(u^{[1]}(b)\overline{v}^{[2m-1]}(b) + u^{[3]}(b)\overline{v}^{[2m-3]}(b) \\
&\qquad + \cdots + u^{[m-2]}(b)\overline{v}^{[m+2]}(b) \\
&\qquad + u^{[m]}(b)\overline{v}^{[m]}(b) + \cdots + u^{[2m-3]}(b)\overline{v}^{[3]}(b) + u^{[2m-1]}(b)\overline{v}^{[1]}(b)\bigr).
\end{aligned}
\tag{4.58}
$$

We divide both sides of (4.58) by h, as $h \to 0$, and we obtain

$$
\begin{aligned}
\lambda'(\beta) = {}& \sec^2\beta\bigl(u^{[1]}(b)\overline{u}^{[2m-1]}(b) + u^{[3]}(b)\overline{u}^{[2m-3]}(b) + \cdots + u^{[m-2]}(b)\overline{u}^{[m+2]}(b) \\
& + u^{[m]}(b)\overline{u}^{[m]}(b) + \cdots + u^{[2m-3]}(b)\overline{u}^{[3]}(b) + u^{[2m-1]}(b)\overline{u}^{[1]}(b)\bigr) \\
= {}& \tan^2\beta u^{[1]}(b)\overline{u}^{[2m-1]}(b) + \tan^2\beta u^{[3]}(b)\overline{u}^{[2m-3]}(b) + \cdots + \tan^2\beta u^{[m-2]}(b)\overline{u}^{[m+2]}(b) \\
& + \tan^2\beta u^{[m]}(b)\overline{u}^{[m]}(b) + \cdots + \tan^2\beta u^{[2m-3]}(b)\overline{u}^{[3]}(b) + \tan^2\beta u^{[2m-1]}(b)\overline{u}^{[1]}(b) \\
& + u^{[1]}(b)\overline{u}^{[2m-1]}(b) + u^{[3]}(b)\overline{u}^{[2m-3]}(b) + \cdots + u^{[m-2]}(b)\overline{u}^{[m+2]}(b) + u^{[m]}(b)\overline{u}^{[m]}(b) \\
& + \cdots + u^{[2m-3]}(b)\overline{u}^{[3]}(b) + u^{[2m-1]}(b)\overline{u}^{[1]}(b)
\end{aligned}
$$

$$
\begin{aligned}
&= u(b)\overline{u}^{[2m-2]}(b) + u^{[2]}(b)\overline{u}^{[2m-4]}(b) + \cdots \\
&\quad + u^{[m-3]}(b)\overline{u}^{[m+1]}(b) + u^{[m-1]}(b)\overline{u}^{[m-1]}(b) \\
&\quad + u^{[m+1]}(b)\overline{u}^{[m-3]}(b) + \cdots + u^{[2m-2]}(b)\overline{u}(b) \\
&\quad + u^{[1]}(b)\overline{u}^{[2m-1]}(b) + u^{[3]}(b)\overline{u}^{[2m-3]}(b) + \cdots \\
&\quad + u^{[m-2]}(b)\overline{u}^{[m+2]}(b) + u^{[m]}(b)\overline{u}^{[m]}(b) + \cdots \\
&\quad + u^{[2m-3]}(b)\overline{u}^{[3]}(b) + u^{[2m-1]}(b)\overline{u}^{[1]}(b) \\
&= 2\operatorname{Re}\sum_{i=0}^{\frac{m-3}{2}}\bigl(u^{[2i]}(b)\overline{u}^{[2m-2i-2]}(b) + u^{[2i+1]}(b)\overline{u}^{[2m-2i-1]}(b)\bigr) \\
&\quad + \bigl|u^{[m-1]}(b)\bigr|^2 + \bigl|u^{[m](b)}\bigr|^2,
\end{aligned}
$$

and (4.46) holds. This finishes the proof.

Next, we prove (4.53) and (4.54). Firstly, we show (4.53) is true. Let $\mu = \lambda(\theta), u = u(\cdot,\theta)$ and $\nu = \lambda(\theta + h), v = u(\cdot,\theta + h)$ be the corresponding eigenvalues and eigenfunctions, respectively. When $h \in \mathbb{R}$ is small enough, we apply (4.40) to infer

$$
\begin{aligned}
&\bigl(\lambda(\theta) - \lambda(\theta + h)\bigr)\int_a^b uvw\,ds \\
&= \left[(-1)^m\sum_{i=0}^{2m-1}(-1)^{2m+1-i}u^{[i]}\overline{v}^{[2m-i-1]}\right]_a^b \\
&= (-1)^{m+1}\begin{pmatrix}\overline{v}^{[2m-1]}, & -\overline{v}^{[2m-2]}, & \cdots, & \overline{v}^{[1]}, & -\overline{v}\end{pmatrix}(b)\begin{pmatrix}u\\ u^{[1]}\\ \vdots\\ u^{[2m-2]}\\ u^{[2m-1]}\end{pmatrix}(b) \\
&\quad - (-1)^{m+1}\begin{pmatrix}\overline{v}^{[2m-1]}, & -\overline{v}^{[2m-2]}, & \cdots, & \overline{v}^{[1]}, & -\overline{v}\end{pmatrix}(a)\begin{pmatrix}u\\ u^{[1]}\\ \vdots\\ u^{[2m-2]}\\ u^{[2m-1]}\end{pmatrix}(a) \\
&= (-1)^{m+1}e^{i\theta}\begin{pmatrix}\overline{v}^{[2m-1]}, & -\overline{v}^{[2m-2]}, & \cdots, & \overline{v}^{[1]}, & -\overline{v}\end{pmatrix}(b)\mathbf{K}\begin{pmatrix}u\\ u^{[1]}\\ \vdots\\ u^{[2m-2]}\\ u^{[2m-1]}\end{pmatrix}(a) \\
&\quad - (-1)^{m+1}e^{i(\theta+h)}\begin{pmatrix}\overline{v}^{[2m-1]}, & -\overline{v}^{[2m-2]}, & \cdots, & \overline{v}^{[1]}, & -\overline{v}\end{pmatrix}(b)\mathbf{K}\begin{pmatrix}u\\ u^{[1]}\\ \vdots\\ u^{[2m-2]}\\ u^{[2m-1]}\end{pmatrix}(a)
\end{aligned}
$$

$$= (-1)^m e^{i\theta} \left(\overline{v}^{[2m-1]}, \quad -\overline{v}^{[2m-2]}, \quad \cdots, \quad \overline{v}^{[1]}, \quad -\overline{v}\right)(b)\mathbf{K} \times \begin{pmatrix} u \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}(a)\left(e^{ih}-1\right). \tag{4.59}$$

We divide both sides of (4.59) by h, as $h \to 0$, and we get

$$\begin{aligned} \lambda'(\theta) &= (-1)^{m+1} i e^{i\theta} \left(\overline{u}^{[2m-1]}, \quad -\overline{u}^{[2m-2]}, \quad \cdots, \quad \overline{u}^{[1]}, \quad -\overline{u}\right)(b)\mathbf{K} \begin{pmatrix} u \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}(a) \\ &= (-1)^{m+1} i \left(\overline{u}^{[2m-1]}, \quad -\overline{u}^{[2m-2]}, \quad \cdots, \quad \overline{u}^{[1]}, \quad -\overline{u}\right)(b) \begin{pmatrix} u \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}(b) \\ &= (-1)^m 2 \operatorname{Im} \sum_{i=0}^{m-1} (-1)^i u^{[i]}(b) \overline{u}^{[2m-i-1]}(b), \end{aligned}$$

and (4.53) holds.

Then we turn to establishing (4.54). Let $u = u(\cdot, \mathbf{K})$, $v = u(\cdot, \mathbf{K}+\mathbf{H})$ for $\mathbf{K}$ and $\mathbf{K}+\mathbf{H}$ satisfy (2.16). Proceeding similar to the argument above, we have

$$\begin{aligned} &\left(\lambda(\mathbf{K}) - \lambda(\mathbf{K}+\mathbf{H})\right) \int_a^b uvw\,ds \\ &= \left[(-1)^m \sum_{i=0}^{2m-1} (-1)^{2m+1-i} u^{[i]} \overline{v}^{[2m-i-1]} \right]_a^b \\ &= (-1)^{m+1} \left(\overline{v}^{[2m-1]}, \quad -\overline{v}^{[2m-2]}, \quad \cdots, \quad \overline{v}^{[1]}, \quad -\overline{v}\right)(b) \begin{pmatrix} u \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}(b) \\ &\quad - (-1)^{m+1} \left(\overline{v}^{[2m-1]}, \quad -\overline{v}^{[2m-2]}, \quad \cdots, \quad \overline{v}^{[1]}, \quad -\overline{v}\right)(a) \begin{pmatrix} u \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}(a) \end{aligned}$$

$$= (-1)^{m+1}e^{i\theta}\left(\overline{v}^{[2m-1]},\quad -\overline{v}^{[2m-2]},\quad \cdots,\quad \overline{v}^{[1]},\quad -\overline{v}\right)(b)\mathbf{K}\begin{pmatrix} u \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}(a)$$

$$-(-1)^{m+1}e^{i\theta}\left(\overline{v}^{[2m-1]},\quad -\overline{v}^{[2m-2]},\quad \cdots,\quad \overline{v}^{[1]},\quad -\overline{v}\right)(b)(\mathbf{K}+\mathbf{H})\begin{pmatrix} u \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}(a)$$

$$= (-1)^{m}e^{i\theta}\left(\overline{v}^{[2m-1]},\quad -\overline{v}^{[2m-2]},\quad \cdots,\quad \overline{v}^{[1]},\quad -\overline{v}\right)(b)\mathbf{H}\begin{pmatrix} u \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}(a)$$

$$= (-1)^{m}\left(\overline{v}^{[2m-1]},\quad -\overline{v}^{[2m-2]},\quad \cdots,\quad \overline{v}^{[1]},\quad -\overline{v}\right)(b)\mathbf{H}\mathbf{K}^{-1}\begin{pmatrix} u \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}(b)$$

$$= (-1)^{m}\left(\overline{u}^{[2m-1]},\quad \overline{u}^{[2m-2]},\quad \cdots,\quad \overline{u}^{[1]},\quad -\overline{u}\right)(b)\mathbf{H}\mathbf{K}^{-1}\begin{pmatrix} u \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}(b)$$

$$= (-1)^{m}\left(\overline{v}^{[2m-1]}-\overline{u}^{[2m-1]},\quad \overline{u}^{[2m-2]}-\overline{v}^{[2m-2]},\quad \cdots,\quad \overline{v}^{[1]}-\overline{u}^{[1]},\quad \overline{u}-\overline{v}\right)(b)\mathbf{H}\mathbf{K}^{-1}$$
$$\times\begin{pmatrix} u \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}(b).$$

Then

$$\lambda(\mathbf{K}+\mathbf{H})-\lambda(\mathbf{K}) = (-1)^{m+1}\left(\overline{u}^{[2m-1]},\quad -\overline{u}^{[2m-2]},\quad \cdots,\quad \overline{u}^{[1]},\quad -\overline{u}\right)(b)$$
$$\times\mathbf{H}\mathbf{K}^{-1}\begin{pmatrix} u \\ u^{[1]} \\ \vdots \\ u^{[2m-2]} \\ u^{[2m-1]} \end{pmatrix}(b)+o(\mathbf{H}),$$

and (4.54) follows.

To show (4.55), we let $u = u(\cdot, q), v = u(\cdot, q + h)$ where $h \in \mathbf{L}^1(a, b)$. Using (2.1) and integration by parts, then

$$\big[\lambda(p_0) - \lambda(p_0 + h)\big] \int_a^b u\bar{v}w\,ds = \left[(-1)^m \sum_{i=0}^{2m-1} (-1)^{2m+1-i} u^{[i]} \bar{v}^{[2m-i-1]}\right]_a^b - \int_a^b u\bar{v}h\,ds.$$

For all BCs, we have

$$\left[(-1)^m \sum_{i=0}^{2m-1} (-1)^{2m+1-i} u^{[i]} \bar{v}^{[2m-i-1]}\right]_a^b = 0.$$

An application of Lemma 3.1 and Theorem 3.1 implies

$$\big[\lambda(p_0 + h) - \lambda(p_0)\big]\big(1 + o(h)\big) = \int_a^b |u|^2 h\,ds + o(h).$$

Consequently,

$$\lambda(p_0 + h) - \lambda(p_0) = \left[\int_a^b |u|^2 h\,ds + o(h)\right]\big(1 + o(h)\big)^{-1} = \int_a^b |u|^2 h\,ds + o(h),$$

as $h \to 0$ in $\mathbf{L}^1(a, b)$, and (4.55) is proved. □

In a similar manner, we can deduce (4.56), here we omit its details.

Remark 4.1 The continuity of the nth eigenvalue of the self-adjoint BCs for $2m$th-order spectral problems is more complicated. For regular Sturm-Liouville problems, Everitt, Möller and Zettl [25] show that the nth eigenvalue is not a continuous function of the BCs with separated boundary conditions, and similar results are obtained by Kong, Wu and Zettl [26] for general BCs.

5 Conclusions

The dependence of eigenvalues on parameters is one important branch of spectral theories of differential operators. It offers theoretical support of approximal calculation of eigenvalues, and the derivatives with respect to the parameters give information of monotonicity of the eigenvalues on the given parameters.

In this paper, with the separated, real coupled and complex coupled BCs, we give the continuity of eigenvalues and eigenfunctions with respect to parameters. The formulas of their derivatives with respect to all the parameters are computed in detail. Our results are new, and the work established in this paper is of quite a general nature and covers a variety of special cases involved in the problem. As for the discontinuity of the nth eigenvalue, it is still an open problem till now, and we will work on it later.

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Competing interests
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