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Global existence of weak solutions to a three-dimensional fractional model in magneto-viscoelastic interactions

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Abstract
In this paper, we prove the existence of global weak solutions for a model described by the fractional Heisenberg equation for the magnetization field and the viscoelastic integro-differential equation for the displacements. We study the three-dimensional case. The demonstration of the existence of weak solution is based on the method of Faedo-Galerkin; and to get the convergence of the nonlinear terms, we introduce the commutator structure.

Keywords: the three-dimensional periodical fractional Heisenberg equation; equation of viscoelasticity; weak solution; commutator estimates

1 Introduction

There is a wide class of materials, although they are naturally insensitive to magnetic fields, which have been rendered magneto-sensitive by the incorporation of particles of iron or magnetite. The resulting composite materials are often divided into two categories according to the forms of response. These materials, which are often characterized by their low mechanical stiffness and their usually isotropic distribution, react to magnetic stimuli with strong deformation. Typical deformation models include elongation, rotation and torsion, winding and bending. The high deformation capacity or the response to the magnetic field by modification of the mechanical properties, such as rigidity, made these materials a promising field for the industry. It is difficult to classify these materials as solids or viscous liquids, while their mechanical properties are highly dependent on the test conditions. A composite material can show all the characteristics of a glassy solid or an elastic rubber or a viscous fluid, depending on the temperature and the time of the scale of measurement. These materials are generally described as viscoelastic materials, a generic term that emphasizes their intermediate position between elastic solids and viscous fluids [1]. Viscoelasticity is undoubtedly one of the properties which clearly shows the complexity and the particularities of these materials. At low temperature, or at high measurement frequencies, a body can be glassy, and at high or low temperatures the same material can be difficult to chew and be able to withstand extensions without permanent deformation. At even higher temperatures, they have a permanent deformation under the load, and the material behaves like a very viscous liquid. In a frequency range or at intermediate temperatures, commonly referred to as the glass transition range, the material behaves neither as the rubber nor as the glass. It appears in an intermediate form, it is viscoelastic and can dissipate a considerable amount of energy during deformation.

In this paper we are concerned with a system describing the evolution of a magneto-viscoelastic material. Our investigation has its starting point in the work of Carillo *et al.* [2], where the authors proposed a three-dimensional evolutive model and established the existence of weak solutions. We intend to study the existence of global weak solutions for a three-dimensional fractional model.

Let us now describe the model equations. The nonlinear coupled system describing the dynamics in $Q = (0,T) \times \Omega$ (Ω is a bounded open set of $\mathbb{R}^3$) is given by

$$\dot{\mathbf{m}} = \nu \mathbf{m} \times \mathbf{H}_{\text{eff}} - \mu \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}), \tag{1}$$

$$\rho \ddot{\mathbf{u}} - \operatorname{div}\left(\mathcal{S}(\mathbf{u}) + \frac{1}{2}\mathcal{L}(\mathbf{m})\right) = 0. \tag{2}$$

The first equation (1), well known in the literature, is the Landau-Lifshitz equation. The unknown $\mathbf{m}$, the magnetization vector, is a map from Ω to S^2 (the unit sphere of $\mathbb{R}^3$) and $\dot{\mathbf{m}}$ is its derivative with respect to time. The symbol $\times$ denotes the vector cross product in $\mathbb{R}^3$. $\nu \in \mathbb{R}$ and $\mu > 0$ are some physical constants, and μ is known as the Gilbert damping parameter. $\mathbf{H}_{\text{eff}}$ represents the effective field, and in this paper we take

$$\mathbf{H}_{\text{eff}} = -a\Lambda^{2\alpha}\mathbf{m} - \boldsymbol{\ell}(\mathbf{m},\mathbf{u}), \tag{3}$$

where a is a positive constant, $\Lambda = (-\Delta)^{\frac{1}{2}}$ denotes the square root of the Laplacian which can be defined via Fourier transformation [3]. In this paper we are interested in the case $\alpha \in (1, \frac{3}{2})$. We denote by $\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ the deformation tensor ϵ where, as a common praxis, $\partial_i u_j$ stands for $\frac{\partial u_j}{\partial x_i}$ and by $m_i, i = 1,2,3$, the components of $\mathbf{m}$. The components of the vector $\boldsymbol{\ell}(\mathbf{m},\mathbf{u})$ are given by

$$\ell_i = \lambda_{ijkl} m_j \epsilon_{kl}(\mathbf{u}),$$

where $\lambda_{ijkl} = \lambda_1\delta_{ijkl} + \lambda_2\delta_{ij}\delta_{kl} + \lambda_3(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ with $\delta_{ijkl} = 1$ if $i = j = k = l$ and $\delta_{ijkl} = 0$ otherwise. Note that we adopt the Einstein summation convention for repeated indices.

The second equation (2) describes the evolution of the displacement $\mathbf{u}$, ρ is a positive constant and the tensor $\mathcal{L}(\mathbf{m})$ is given by

$$\mathcal{L}_{kl} = \lambda_{ijkl} m_i m_j,$$

and we take (see [2])

$$S(\mathbf{u}) = \mathcal{G}(0)\nabla\mathbf{u} + \int_0^t \dot{\mathcal{G}}(t-s)\nabla\mathbf{u}(s)\,\mathrm{d}s,$$

where the tensor field $\mathcal{G}(t) = \{\mathcal{G}_{ijkl}(t)\}, t \in [0,T]$, is assumed to check the following properties: for any $t \in [0,T]$ and for any symmetric tensor e_{ij},

$$\mathcal{G}_{ijkl} = \mathcal{G}_{klij} = \mathcal{G}_{jikl}, \tag{H_1}$$

$$\mathcal{G}_{ijkl} \in C^2[0,T], \tag{H_2}$$

$$\dot{\mathcal{G}}_{ijkl} e_{ij} e_{kl} \le 0, \tag{H_3}$$

$$\ddot{\mathcal{G}}_{ijkl} e_{ij} e_{kl} \ge 0, \tag{H_4}$$

$$\mathcal{G}_{ijkl} e_{ij} e_{kl} \ge \beta \sum |e_{ij}|^2 \quad \text{for some } \beta > 0 \tag{H_5}$$

and

$$\mathcal{G}_{ijkl} \epsilon_{ij} \epsilon_{kl} \le \tau \sum |\nabla \mathbf{u}|^2 \quad \text{for some } \tau > 0. \tag{H_6}$$

Many studies have been done on the fractional Landau-Lifshitz equation, we quote here for example [4] where the existence of weak solutions under periodical boundary condition was proved for equation of a reduced model for thin-film micromagnetics. When $\nu = 0$ and $\mathbf{H}_{\text{eff}} = \Lambda^{2\alpha}\mathbf{m}$, (1) can be regarded as a generalization of the harmonic map heat flow to the fractional order which was studied in [5] under both conditions $\alpha \in (0,1)$ and $\alpha > \frac{d}{4}$. In [6] the main purpose is to consider the well-posedness of the fractional Landau-Lifshitz equation without Gilbert damping. There are also other authors who have studied the magneto-elasticity coupling. In [7] they studied the three-dimensional case and established the existence of weak solutions taking into account three terms of the total free energy. Existence and uniqueness of solutions were proved in [8] for a simplified model, and in [9] they treated a one-dimensional penalty problem and studied the gradient flow of the associated type Ginzburg-Landau functional. They proved the existence and uniqueness of a classical solution which tends asymptotically for subsequences to a stationary point of the energy functional. Global existence of weak solutions to a fractional model in magneto-elasticity was proved in 1D [10] and in 3D [11] with a behavior without memory (which is the subject of current work), the demonstration is based on the method of Faedo-Galerkin and on a commutator structure to ensure the convergence of nonlinear terms. Note that, in [11], the tensor $\mathcal{S}(\mathbf{u})$ has for expression $S_{kl} = \sigma_{ijkl}\epsilon_{ij}(\mathbf{u})$, where the elasticity tensor $\sigma = (\sigma_{ijkl})$, which is independent of time and space, is assumed to satisfy the symmetry property $\sigma_{ijkl} = \sigma_{klij} = \sigma_{jikl}$, whereas in this work $\mathcal{S}(\mathbf{u})$ is expressed as a function of an integral with respect to time, which leads to additional difficulties. The first one is the existence of a local solution, which we prove by a fixed point argument, and the second one is to establish *a priori* estimate in order to extend the local solution, which requires more techniques compared to the work realized in [11].

For the dynamics of magneto-viscoelastic materials, we quote [2] where the authors study the general three-dimensional case and establish a theorem for the existence of weak solutions. The existence is proved by compactness of the approximated penalty problem.

In the last three decades, many scientific studies have shown the importance of fractional calculus and its applications in mathematics and in many other applied scientific branches. Several theoretical and experimental studies show that some systems, in particular magnetic systems, are governed by non-integer partial differential equations. The use of traditional integer order differential equations is therefore not appropriate. Inspired by contributions from [5, 6, 12], in this paper we aim at going further by coupling a fractional order equation with a viscoelasticity equation as a kind of generalization of the work [2].

Throughout, we make use of the following notation. For Ω an open bounded domain of $\mathbb{R}^3$, we denote by $\mathbf{L}^p(\Omega) = (L^p(\Omega))^3$ and $\mathbf{H}^1(\Omega) = (H^1(\Omega))^3$ the classical Hilbert spaces equipped with the usual norm denoted by $\|\cdot\|_{\mathbf{L}^p(\Omega)}$ and $\|\cdot\|_{\mathbf{H}^1(\Omega)}$ (in general, the product functional spaces $(X)^3$ are all simplified to $\mathbf{X}$). For all $s > 0$, $W^{s,p}$ denotes the usual Sobolev space consisting of all f such that

$$\|f\|_{W^{s,p}} := \left\|\mathcal{F}^{-1}\left(1 + |\cdot|^2\right)^{\frac{s}{2}}(\mathcal{F}f)(\cdot)\right\|_{L^p} < \infty,$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ its inverse. Let $\dot{W}^{s,p}$ denote the corresponding homogeneous Sobolev space. When $p = 2$, $W^{s,p}$ corresponds to the usual Sobolev space H^s, and in this case we have

$$\|f\|_{\dot{H}^s} := \left\|\Lambda^s f\right\|_{L^2}.$$

The rest of the paper is divided as follows. In the next section we present the model on which we will work. Section 3 is devoted to the definition of weak solutions and the main result. In Section 4 we prove a global existence result, and we conclude in the last section by some remarks and perspectives.

2 The model

The right-hand side of equation (1) consists of two terms, a dissipative processes represented by the μ-term and a gyromagnetic precession (the ν-term) which takes its name from the ν constant called the gyromagnetic ratio. This constant is measured experimentally via the precession of the spin vector and it is equal to, for a particle or system, the ratio of its magnetic moment to its angular momentum. Considering the difficulty represented by equation (1), we shall adopt a model without precession (i.e. $\nu = 0$) which corresponds in some ways to a weak magnetic moment. More precisely, we will study global existence of weak solutions in the spatial domain $\Omega = (0, 2\pi)^d$ with periodic boundary conditions for the magnetization vector. We take $d = 3$ and for simplicity we let $\mu = 1$, which will not affect the result essentially. The generic point of Ω is denoted by $x = (x_1, x_2, x_3)$. We take the following system:

$$\begin{cases} \dot{\mathbf{m}} = -\mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}), \\ \rho\ddot{\mathbf{u}} - \operatorname{div}(\mathcal{S}(\mathbf{u}) + \frac{1}{2}\mathcal{L}(\mathbf{m})) = 0, \end{cases} \tag{4}$$

where $\mathbf{H}_{\text{eff}}$ is given by (3). Now, we propose an equivalent equation for the first equation of (4)

$$\mathbf{m} \times \dot{\mathbf{m}} = \mathbf{m} \times \mathbf{H}_{\text{eff}}. \tag{5}$$

Indeed, let us first recall the two properties: for all $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

and

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}.$$

One uses the fact that $|\mathbf{m}| = 1$ and, consequently, $\mathbf{m} \cdot \dot{\mathbf{m}} = 0$.

We start from the first equation of (4), and by using the property of the double vector product, we have

$$\dot{\mathbf{m}} = -(\mathbf{m} \cdot \mathbf{H}_{\mathrm{eff}})\mathbf{m} + \mathbf{H}_{\mathrm{eff}},$$

which implies, since $(\mathbf{m} \cdot \mathbf{H}_{\mathrm{eff}})\mathbf{m} \times \mathbf{m} = 0$,

$$\mathbf{m} \times \dot{\mathbf{m}} = \mathbf{m} \times \mathbf{H}_{\mathrm{eff}}.$$

Conversely, from (5) we have

$$\mathbf{m} \times (\dot{\mathbf{m}} - \mathbf{H}_{\mathrm{eff}}) = 0.$$

Hence there exists a multiplier $r : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ such that

$$\dot{\mathbf{m}} - \mathbf{H}_{\mathrm{eff}} = r\mathbf{m},$$

and in order to find r, it is enough to multiply the last equation by $\mathbf{m}$

$$\mathbf{m} \cdot \dot{\mathbf{m}} - \mathbf{m} \cdot \mathbf{H}_{\mathrm{eff}} = r|\mathbf{m}|^2.$$

Therefore $r = -\mathbf{m} \cdot \mathbf{H}_{\mathrm{eff}}$, which finishes the proof.

Now we can replace system (4) with the following:

$$\begin{cases} \mathbf{m} \times \dot{\mathbf{m}} = \mathbf{m} \times \mathbf{H}_{\mathrm{eff}}, \\ \rho\ddot{\mathbf{u}} - \operatorname{div}(\mathcal{S}(\mathbf{u}) + \frac{1}{2}\mathcal{L}(\mathbf{m})) = 0. \end{cases} \tag{6}$$

As initial conditions, we assume

$$\mathbf{u}(\cdot,0) = \mathbf{u}_0, \qquad \dot{\mathbf{u}}(\cdot,0) = \mathbf{u}_1, \qquad \mathbf{m}(\cdot,0) = \mathbf{m}_0, \qquad |\mathbf{m}_0| = 1 \quad \text{in } \Omega, \tag{7}$$

with a boundary condition for the displacement vector

$$\mathbf{u} = 0 \quad \text{on } \Sigma := \partial\Omega \times (0,T). \tag{8}$$

3 Global existence of weak solutions

3.1 Weak solution

Now we give a definition of the solution in the weak sense of problem (6)-(7)-(8).

Definition 3.1 Let $\mathbf{m}_0 \in \mathbf{H}^\alpha(\Omega)$, $|\mathbf{m}_0| = 1$ a.e., $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and $\mathbf{u}_1 \in \mathbf{L}^2(\Omega)$. We say that the pair $(\mathbf{m},\mathbf{u})$ is a weak solution of problem (6)-(7)-(8) if:

- for all $T > 0$, $\mathbf{m} \in L^\infty(0,T;\mathbf{H}^\alpha(\Omega))$, $\dot{\mathbf{m}} \in L^2(0,T;\mathbf{L}^2(\Omega))$, $|\mathbf{m}| = 1$ a.e., $\mathbf{u} \in L^2(0,T;\mathbf{H}_0^1(\Omega))$ and $\dot{\mathbf{u}} \in L^2(0,T;\mathbf{L}^2(\Omega))$;
- for all $\boldsymbol{\varphi} \in \mathbf{C}^\infty(\overline{Q})$ and $\boldsymbol{\psi} \in \mathbf{H}_0^1(Q)$, we have:

$$\int_Q (\dot{\mathbf{m}} \times \mathbf{m}) \cdot \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t + a\int_Q \Lambda^\alpha\mathbf{m} \cdot \Lambda^\alpha(\mathbf{m} \times \boldsymbol{\varphi})\,\mathrm{d}x\,\mathrm{d}t + \int_Q \bigl(\boldsymbol{\ell}(\mathbf{m},\mathbf{u}) \times \mathbf{m}\bigr) \cdot \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{9}$$

$$-\rho\int_Q \dot{\mathbf{u}}\cdot\dot{\boldsymbol{\psi}}\,\mathrm{d}x\,\mathrm{d}t+\int_Q\left(\mathcal{S}(\mathbf{u})+\frac{1}{2}\mathcal{L}(\mathbf{m})\right)\cdot\nabla\boldsymbol{\psi}\,\mathrm{d}x\,\mathrm{d}t=0;$$

- $\mathbf{m}(0,x)=\mathbf{m}_0(x)$ and $\mathbf{u}(0,x)=\mathbf{u}_0(x)$ in the trace sense;
- for all $T>0$, we have

$$\begin{aligned}&\int_Q|\dot{\mathbf{m}}|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{a}{2}\int_\Omega\left|\Lambda^\alpha\mathbf{m}(T)\right|^2\mathrm{d}x+\frac{\rho}{2}\int_\Omega\left|\dot{\mathbf{u}}(T)\right|^2\mathrm{d}x+\frac{\beta}{4}\int_\Omega\left|\nabla\mathbf{u}(T)\right|^2\mathrm{d}x\\&\quad\le\frac{a}{2}\int_\Omega\left|\Lambda^\alpha\mathbf{m}_0\right|^2\mathrm{d}x+\frac{\rho}{2}\int_\Omega|\mathbf{u}_1|^2\,\mathrm{d}x+\frac{3\tau}{4}\int_\Omega|\nabla\mathbf{u}_0|^2\,\mathrm{d}x+C(\Omega,\beta,\lambda),\end{aligned}\tag{10}$$

where $C(\Omega,\beta,\lambda)$ is a positive constant which depends only on Ω, β and λ.

Remark 1 We will show in Section 4 (Section 4.2) that $\Lambda^\alpha(\mathbf{m}\times\boldsymbol{\varphi})$ is in $\mathbf{L}^2(Q)$, and then it will be clear that the second term in (9) is well defined.

3.2 Main result

The main result of this paper is the following.

Theorem 3.2 *Let* $\alpha\in(1,\frac{3}{2})$, $\mathbf{m}_0\in\mathbf{H}^\alpha(\Omega)$ *such that* $|\mathbf{m}_0|=1$ *a.e.*, $\mathbf{u}_0\in\mathbf{H}^1_0(\Omega)$ *and* $\mathbf{u}_1\in\mathbf{L}^2(\Omega)$. *Then there exists at least a weak solution for the problem in the sense of Definition* 3.1.

The proof of Theorem 3.2 will be given in Section 5.

4 Proof of the main result

This section is dedicated to constructing the global weak solutions for the fractional problem (6)-(7)-(8) via the Faedo-Galerkin/penalty method. In particular, the global existence theorem (3.2) for the problem considered will be proved.

4.1 The penalty problem

Let $\varepsilon>0$ be a fixed parameter. We construct approximated solutions $(\mathbf{m}^\varepsilon,\mathbf{u}^\varepsilon)$ converging, as $\varepsilon\to0$, to a solution $(\mathbf{m},\mathbf{u})$ of the problem. Then we consider the following problem:

$$\begin{cases}\dot{\mathbf{m}}^\varepsilon+a\Lambda^{2\alpha}\mathbf{m}^\varepsilon+\boldsymbol{\ell}(\mathbf{m}^\varepsilon,\mathbf{u}^\varepsilon)+\frac{|\mathbf{m}^\varepsilon|^2-1}{\varepsilon}\mathbf{m}^\varepsilon=0,\\ \rho\ddot{\mathbf{u}}^\varepsilon-\operatorname{div}(\mathcal{S}(\mathbf{u}^\varepsilon)+\frac{1}{2}\mathcal{L}(\mathbf{m}^\varepsilon))=0,\end{cases}\tag{11}$$

in $Q=\Omega\times(0,T)$. Note that the last term of the first equation in (11) has been introduced in order to represent the constraint $|\mathbf{m}|=1$ in the limit $\varepsilon\to0$.

System (11) is supplemented with initial and boundary conditions

$$\mathbf{u}^\varepsilon(\cdot,0)=\mathbf{u}_0,\qquad\dot{\mathbf{u}}^\varepsilon(\cdot,0)=\mathbf{u}_1,\qquad\mathbf{m}^\varepsilon(\cdot,0)=\mathbf{m}_0,\qquad|\mathbf{m}_0|=1\quad\text{a.e. in }\Omega,\tag{12}$$

$$\mathbf{u}^\varepsilon=0\quad\text{on }\Sigma.\tag{13}$$

We construct approximate solutions of (11)-(12)-(13) by using the Galerkin method: let $\{f_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $L^2(\Omega)$ consisting of all the eigenfunctions for the operator $\Lambda^{2\alpha}$ (the existence of such a basis can be proved as in [13], Ch. II)

$$\Lambda^{2\alpha}f_i=\alpha_if_i,\quad i=1,2,\ldots$$

under periodic boundary conditions, and let $\{g_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $L^2(\Omega)$ consisting of all the eigenfunctions for the operator $-\Delta$

$$\begin{cases} -\Delta g_i = \beta_i g_i, & i = 1,2,\dots, \\ g_i = 0 & \text{on } \partial\Omega. \end{cases}$$

We consider the following problem in $Q = \Omega \times (0,T)$:

$$\begin{cases} \dot{\mathbf{m}}^{\varepsilon,N} + a\Lambda^{2\alpha}\mathbf{m}^{\varepsilon,N} + \boldsymbol{\ell}(\mathbf{m}^{\varepsilon,N},\mathbf{u}^{\varepsilon,N}) + \frac{|\mathbf{m}^{\varepsilon,N}|^2-1}{\varepsilon}\mathbf{m}^{\varepsilon,N} = 0, \\ \rho\ddot{\mathbf{u}}^{\varepsilon,N} - \operatorname{div}(\mathcal{S}(\mathbf{u}^{\varepsilon,N}) + \frac{1}{2}\mathcal{L}(\mathbf{m}^{\varepsilon,N})) = 0, \end{cases} \tag{14}$$

with initial and boundary conditions

$$\begin{aligned} &\mathbf{u}^{\varepsilon,N}(\cdot,0) = \mathbf{u}^N(\cdot,0), \qquad \dot{\mathbf{u}}^{\varepsilon,N}(\cdot,0) = \dot{\mathbf{u}}^N(\cdot,0), \qquad \mathbf{m}^{\varepsilon,N}(\cdot,0) = \mathbf{m}^N(\cdot,0) \quad \text{in } \Omega, \\ &\mathbf{u}^{\varepsilon,N} = 0 \quad \text{on } \Sigma = \partial\Omega \times (0,T) \end{aligned}$$

and

$$\begin{aligned} &\int_\Omega \mathbf{u}^N(x,0)g_i(x)\,\mathrm{d}x = \int_\Omega \mathbf{u}_0(x)g_i(x)\,\mathrm{d}x, \qquad \int_\Omega \dot{\mathbf{u}}^N(x,0)g_i(x)\,\mathrm{d}x = \int_\Omega \mathbf{u}_1(x)g_i(x)\,\mathrm{d}x, \\ &\int_\Omega \mathbf{m}^N(x,0)f_i(x)\,\mathrm{d}x = \int_\Omega \mathbf{m}_0(x)f_i(x)\,\mathrm{d}x. \end{aligned}$$

We are looking for approximate solutions $(\mathbf{m}^{\varepsilon,N},\mathbf{u}^{\varepsilon,N})$ to (14) under the form

$$\mathbf{m}^{\varepsilon,N} = \sum_{i=1}^N \mathbf{a}_i(t)f_i(x), \qquad \mathbf{u}^{\varepsilon,N} = \sum_{i=1}^N \mathbf{b}_i(t)g_i(x),$$

where $\mathbf{a}_i$ and $\mathbf{b}_i$ are $\mathbb{R}^3$-valued vectors.

If we multiply each scalar equation of the first equation of (14) by f_i and the second by g_i and integrate in Ω, we get to a system of integro-differential equations in the unknown $(\mathbf{a}_i(t),\mathbf{b}_i(t))$, $i = 1,2,\dots,N$, that we can write in the form (based on ideas exploited in [2])

$$\begin{cases} \mathbf{m}^{\varepsilon,N} = -\int_0^t (a\Lambda^{2\alpha}\mathbf{m}^{\varepsilon,N}(s) + \boldsymbol{\ell}(\mathbf{m}^{\varepsilon,N}(s),\mathbf{u}^{\varepsilon,N}(s)) + \frac{|\mathbf{m}^{\varepsilon,N}(s)|^2-1}{\varepsilon}\mathbf{m}^{\varepsilon,N}(s))\,\mathrm{d}s + \mathbf{m}^N(0), \\ \dot{\mathbf{u}}^{\varepsilon,N} = \frac{1}{\rho}\int_0^t \operatorname{div}(\mathcal{S}(\mathbf{u}^{\varepsilon,N}(s)) + \frac{1}{2}\mathcal{L}(\mathbf{m}^{\varepsilon,N}(s)))\,\mathrm{d}s + \dot{\mathbf{u}}^N(0), \\ \mathbf{u}^{\varepsilon,N} = \int_0^t \dot{\mathbf{u}}^{\varepsilon,N}(s)\,\mathrm{d}s + \mathbf{u}^N(0) \end{cases}$$

and if we define $\mathbf{v}(t) := (\mathbf{v}_1(t),\mathbf{v}_2(t),\dots,\mathbf{v}_N(t))$, where $\mathbf{v}_i(t) = (\mathbf{a}_i(t),\mathbf{b}_i(t),\dot{\mathbf{b}}_i(t))$, $i = 1,2,\dots,N$, we can write the last system in the form

$$\mathbf{v}(t) = \int_0^t \mathbf{w}\bigl(t,s,\mathbf{v}(s)\bigr)\,\mathrm{d}s + \mathbf{v}(0).$$

Now, for a strictly positive constant α, we take

$$E_\alpha := \bigl\{\mathbf{x}/\|\mathbf{x}\|_\infty \le \alpha\bigr\},$$

where $\|\mathbf{x}\|_\infty = \sup_{0\le t\le\tau} |\mathbf{x}_i(t)|$, for $\tau > 0$. We consider the mapping T defined on E_α by

$$\forall \mathbf{x} \in E_\alpha, \quad T(\mathbf{x}) = \int_0^t \mathbf{w}\big(t,s,\mathbf{x}(s)\big)\,\mathrm{d}s + \mathbf{v}(0).$$

Then we have, for $\mathbf{x}$ in E_α,

$$\big\|\mathbf{w}(\mathbf{x})\big\|_\infty = \sup_{0\le t\le\tau}\sup_{0\le s\le t}\big|\mathbf{w}\big(t,s,\mathbf{x}(s)\big)\big| \le C\alpha$$

for a positive constant C. So, if we choose (α,τ) such that $\|\mathbf{v}(0)\|_\infty \le \frac{\alpha}{2}$ and $\tau \le \frac{1}{2C}$, we can write

$$\begin{aligned}\big\|T(\mathbf{x})\big\|_\infty &\le C\alpha\tau + \big\|\mathbf{v}(0)\big\|_\infty \\ &\le C\alpha\tau + \frac{\alpha}{2} \\ &\le \alpha,\end{aligned}$$

that implies $T(\mathbf{x}) \in E_\alpha$. Moreover, for $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ in E_α and for all $t \in [0,\tau]$,

$$\begin{aligned}\big|T\big(\mathbf{x}^{(1)}\big)_i - T\big(\mathbf{x}^{(2)}\big)_i\big| &= \left|\int_0^t \big(\mathbf{w}_i^{(1)} - \mathbf{w}_i^{(2)}\big)\,\mathrm{d}s\right| \\ &\le \tau\big\|\nabla_{\mathbf{x}}\mathbf{w}(\mathbf{x})\big\|_\infty\big\|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}\big\|_\infty.\end{aligned}$$

Since $\|\nabla_{\mathbf{x}}\mathbf{w}(\mathbf{x})\|_\infty$ is bounded in E_α and we can choose τ small enough to have $\tau\|\nabla_{\mathbf{x}}\mathbf{w}(\mathbf{x})\|_\infty < 1$, then we obtain

$$\big\|T\big(\mathbf{x}^{(1)}\big) - T\big(\mathbf{x}^{(2)}\big)\big\|_\infty \le C\big\|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}\big\|_\infty$$

with $0 < C < 1$. Hence T is a contraction mapping of the convex compact set E_α into itself. From the fixed point theorem we deduce the local existence of solutions to the problem that we can extend on $[0,T]$ using a priori estimates. For this, we multiply the first equation of (14) by $\dot{\mathbf{m}}^{\varepsilon,N}$ and the second by $\dot{\mathbf{u}}^{\varepsilon,N}$ integrating in Ω, we obtain

$$\begin{cases}\int_\Omega |\dot{\mathbf{m}}^{\varepsilon,N}|^2\,\mathrm{d}x + a\int_\Omega \Lambda^{2\alpha}\mathbf{m}^{\varepsilon,N}\cdot\dot{\mathbf{m}}^{\varepsilon,N}\,\mathrm{d}x \\ \quad + \int_\Omega \boldsymbol{\ell}(\mathbf{m}^{\varepsilon,N},\mathbf{u}^{\varepsilon,N})\cdot\dot{\mathbf{m}}^{\varepsilon,N}\,\mathrm{d}x + \frac{1}{4\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(|\mathbf{m}^{\varepsilon,N}|^2-1)^2\,\mathrm{d}x = 0, \\ \frac{\varrho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\dot{\mathbf{u}}^{\varepsilon,N}|^2\,\mathrm{d}x + \int_\Omega(\mathcal{S}(\mathbf{u}^{\varepsilon,N}) + \frac{1}{2}\mathcal{L}(\mathbf{m}^{\varepsilon,N}))\cdot\nabla\dot{\mathbf{u}}^{\varepsilon,N}\,\mathrm{d}x = 0.\end{cases}$$

Now we use the following result (the proof can be found in [4]).

Lemma 4.1 *If f and g belong to* $H^{2\alpha}_{per}(\Omega) := \{f \in L^2(\Omega)/\Lambda^{2\alpha}f \in L^2(\Omega)\}$, *then*

$$\int_\Omega \Lambda^{2\alpha}f\cdot g\,\mathrm{d}x = \int_\Omega \Lambda^\alpha f\cdot\Lambda^\alpha g\,\mathrm{d}x.$$

We obtain

$$\begin{cases} \int_\Omega |\dot{\mathbf{m}}^{\varepsilon,N}|^2\,\mathrm{d}x + \frac{a}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Lambda^\alpha \mathbf{m}^{\varepsilon,N}|^2\,\mathrm{d}x \\ \quad + \int_\Omega \boldsymbol{\ell}(\mathbf{m}^{\varepsilon,N},\mathbf{u}^{\varepsilon,N})\cdot\dot{\mathbf{m}}^{\varepsilon,N}\,\mathrm{d}x + \frac{1}{4\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (|\mathbf{m}^{\varepsilon,N}|^2-1)^2\,\mathrm{d}x = 0, \\ \frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\dot{\mathbf{u}}^{\varepsilon,N}|^2\,\mathrm{d}x + \int_\Omega (\mathcal{S}(\mathbf{u}^{\varepsilon,N}) + \frac{1}{2}\mathcal{L}(\mathbf{m}^{\varepsilon,N}))\cdot\nabla\dot{\mathbf{u}}^{\varepsilon,N}\,\mathrm{d}x = 0. \end{cases} \tag{15}$$

Note that $\lambda_{ijkl} = \lambda_{jikl}$, hence (using the components of the vector $\boldsymbol{\ell}$)

$$\begin{aligned} \int_\Omega \boldsymbol{\ell}(\mathbf{m}^{\varepsilon,N},\mathbf{u}^{\varepsilon,N})\cdot\dot{\mathbf{m}}^{\varepsilon,N}\,\mathrm{d}x &= \int_\Omega \lambda_{ijkl} m_j^{\varepsilon,N}\dot{m}_i^{\varepsilon,N}\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})\,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega \lambda_{ijkl}(m_j^{\varepsilon,N}\dot{m}_i^{\varepsilon,N} + m_i^{\varepsilon,N}\dot{m}_j^{\varepsilon,N})\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})\,\mathrm{d}x. \end{aligned}$$

From where

$$\begin{aligned} \int_\Omega \boldsymbol{\ell}(\mathbf{m}^{\varepsilon,N},\mathbf{u}^{\varepsilon,N})\cdot\dot{\mathbf{m}}^{\varepsilon,N}\,\mathrm{d}x &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \lambda_{ijkl} m_i^{\varepsilon,N} m_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})\,\mathrm{d}x \\ &\quad - \frac{1}{2}\int_\Omega \lambda_{ijkl} m_i^{\varepsilon,N} m_j^{\varepsilon,N}\epsilon_{kl}(\dot{\mathbf{u}}^{\varepsilon,N})\,\mathrm{d}x, \end{aligned} \tag{16}$$

and since the tensor $\mathcal{L}$ is symmetric, we have

$$\begin{aligned} \frac{1}{2}\int_\Omega \mathcal{L}(\mathbf{m}^{\varepsilon,N})\cdot\nabla\dot{\mathbf{u}}^{\varepsilon,N}\,\mathrm{d}x &= \frac{1}{2}\int_\Omega \mathcal{L}(\mathbf{m}^{\varepsilon,N})\cdot\epsilon(\dot{\mathbf{u}}^{\varepsilon,N})\,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega \lambda_{ijkl} m_i^{\varepsilon,N} m_j^{\varepsilon,N}\epsilon_{kl}(\dot{\mathbf{u}}^{\varepsilon,N})\,\mathrm{d}x. \end{aligned} \tag{17}$$

Furthermore, (omitting superscripts and) following the idea introduced in [2], we have

$$\begin{aligned} \int_\Omega \mathcal{S}(\mathbf{u}(t))\cdot\nabla\dot{\mathbf{u}}(t)\,\mathrm{d}x &= \int_\Omega \mathcal{G}_{ijkl}(0)\epsilon_{ij}(\mathbf{u}(t))\epsilon_{kl}(\dot{\mathbf{u}}(t))\,\mathrm{d}x \\ &\quad + \int_\Omega\int_0^t \dot{\mathcal{G}}_{ijkl}(t-s)\epsilon_{ij}(\mathbf{u}(s))\epsilon_{kl}(\dot{\mathbf{u}}(t))\,\mathrm{d}s\,\mathrm{d}x \\ &= \int_\Omega \mathcal{G}_{ijkl}(0)\epsilon_{ij}(\mathbf{u}(t))\epsilon_{kl}(\dot{\mathbf{u}}(t))\,\mathrm{d}x \\ &\quad + \int_\Omega\int_0^t \dot{\mathcal{G}}_{ijkl}(s)\epsilon_{ij}(\mathbf{u}(t-s))\epsilon_{kl}(\dot{\mathbf{u}}(t))\,\mathrm{d}s\,\mathrm{d}x \\ &= -\int_\Omega (\mathcal{G}_{ijkl}(t)-\mathcal{G}_{ijkl}(0))\epsilon_{ij}(\mathbf{u}(t))\epsilon_{kl}(\dot{\mathbf{u}}(t))\,\mathrm{d}x \\ &\quad + \int_\Omega \mathcal{G}_{ijkl}(t)\epsilon_{ij}(\mathbf{u}(t))\epsilon_{kl}(\dot{\mathbf{u}}(t))\,\mathrm{d}x \\ &\quad + \int_\Omega\int_0^t \dot{\mathcal{G}}_{ijkl}(s)\epsilon_{ij}(\mathbf{u}(t-s))\epsilon_{kl}(\dot{\mathbf{u}}(t))\,\mathrm{d}s\,\mathrm{d}x \\ &= -\int_\Omega\int_0^t \dot{\mathcal{G}}_{ijkl}(s)\epsilon_{ij}(\mathbf{u}(t))\epsilon_{kl}(\dot{\mathbf{u}}(t))\,\mathrm{d}s\,\mathrm{d}x \\ &\quad + \int_\Omega \mathcal{G}_{ijkl}(t)\epsilon_{ij}(\mathbf{u}(t))\epsilon_{kl}(\dot{\mathbf{u}}(t))\,\mathrm{d}x \\ &\quad + \int_\Omega\int_0^t \dot{\mathcal{G}}_{ijkl}(s)\epsilon_{ij}(\mathbf{u}(t-s))\epsilon_{kl}(\dot{\mathbf{u}}(t))\,\mathrm{d}s\,\mathrm{d}x. \end{aligned}$$

Then

$$\int_\Omega \mathcal{S}\big(\mathbf{u}(t)\big)\cdot\nabla\dot{\mathbf{u}}(t)\,\mathrm{d}x = -\int_\Omega\int_0^t \dot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}(t)\big)-\epsilon_{ij}\big(\mathbf{u}(t-s)\big)\big)\epsilon_{kl}\big(\dot{\mathbf{u}}(t)\big)\,\mathrm{d}s\,\mathrm{d}x + \int_\Omega \mathcal{G}_{ijkl}(t)\epsilon_{ij}\big(\mathbf{u}(t)\big)\epsilon_{kl}\big(\dot{\mathbf{u}}(t)\big)\,\mathrm{d}x.$$

This leads to

$$\begin{aligned}\int_\Omega \mathcal{S}\big(\mathbf{u}(t)\big)\cdot\nabla\dot{\mathbf{u}}(t)\,\mathrm{d}x = {}&-\int_\Omega\int_0^t \dot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}(t)\big)-\epsilon_{ij}\big(\mathbf{u}(t-s)\big)\big)\epsilon_{kl}\big(\dot{\mathbf{u}}(t)\big)\,\mathrm{d}s\,\mathrm{d}x\\ &+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{G}_{ijkl}(t)\epsilon_{ij}\big(\mathbf{u}(t)\big)\epsilon_{kl}\big(\mathbf{u}(t)\big)\,\mathrm{d}x\\ &-\frac{1}{2}\int_\Omega \dot{\mathcal{G}}_{ijkl}(t)\epsilon_{ij}\big(\mathbf{u}(t)\big)\epsilon_{kl}\big(\mathbf{u}(t)\big)\,\mathrm{d}x,\end{aligned}\tag{18}$$

by using (H_1). On the other hand,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\int_\Omega \dot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}(t)\big)-\epsilon_{ij}\big(\mathbf{u}(t-s)\big)\big)\big(\epsilon_{kl}\big(\mathbf{u}(t)\big)-\epsilon_{kl}\big(\mathbf{u}(t-s)\big)\big)\,\mathrm{d}x\,\mathrm{d}s\\ &\quad=\int_\Omega \dot{\mathcal{G}}_{ijkl}(t)\big(\epsilon_{ij}\big(\mathbf{u}(t)\big)-\epsilon_{ij}\big(\mathbf{u}(0)\big)\big)\big(\epsilon_{kl}\big(\mathbf{u}(t)\big)-\epsilon_{kl}\big(\mathbf{u}(0)\big)\big)\,\mathrm{d}x\\ &\qquad+2\int_\Omega\int_0^t \dot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}(t)\big)-\epsilon_{ij}\big(\mathbf{u}(t-s)\big)\big)\epsilon_{kl}\big(\dot{\mathbf{u}}(t)\big)\,\mathrm{d}s\,\mathrm{d}x\\ &\qquad-2\int_\Omega\int_0^t \dot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}(t)\big)-\epsilon_{ij}\big(\mathbf{u}(t-s)\big)\big)\epsilon_{kl}\big(\dot{\mathbf{u}}(t-s)\big)\,\mathrm{d}s\,\mathrm{d}x,\end{aligned}\tag{19}$$

and we have

$$\begin{aligned}&-2\int_\Omega\int_0^t \dot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}(t)\big)-\epsilon_{ij}\big(\mathbf{u}(t-s)\big)\big)\epsilon_{kl}\big(\dot{\mathbf{u}}(t-s)\big)\,\mathrm{d}s\,\mathrm{d}x\\ &\quad=-2\int_\Omega\int_0^t \dot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}(t)\big)-\epsilon_{ij}\big(\mathbf{u}(t-s)\big)\big)\frac{\partial}{\partial s}\big(\epsilon_{kl}\big(\mathbf{u}(t)\big)-\epsilon_{kl}\big(\mathbf{u}(t-s)\big)\big)\,\mathrm{d}s\,\mathrm{d}x\\ &\quad=-\int_\Omega \dot{\mathcal{G}}_{ijkl}(t)\big(\epsilon_{ij}\big(\mathbf{u}(t)\big)-\epsilon_{ij}\big(\mathbf{u}(0)\big)\big)\big(\epsilon_{kl}\big(\mathbf{u}(t)\big)-\epsilon_{kl}\big(\mathbf{u}(0)\big)\big)\,\mathrm{d}x\\ &\qquad+\int_\Omega\int_0^t \ddot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}(t)\big)-\epsilon_{ij}\big(\mathbf{u}(t-s)\big)\big)\big(\epsilon_{kl}\big(\mathbf{u}(t)\big)-\epsilon_{kl}\big(\mathbf{u}(t-s)\big)\big)\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}$$

Substituting in (19), we find

$$\begin{aligned}&-\int_\Omega\int_0^t \dot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}(t)\big)-\epsilon_{ij}\big(\mathbf{u}(t-s)\big)\big)\epsilon_{kl}\big(\dot{\mathbf{u}}(t)\big)\,\mathrm{d}s\,\mathrm{d}x\\ &\quad=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\int_\Omega \dot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}(t)\big)-\epsilon_{ij}\big(\mathbf{u}(t-s)\big)\big)\big(\epsilon_{kl}\big(\mathbf{u}(t)\big)-\epsilon_{kl}\big(\mathbf{u}(t-s)\big)\big)\,\mathrm{d}x\,\mathrm{d}s\\ &\qquad+\frac{1}{2}\int_0^t\int_\Omega \ddot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}(t)\big)-\epsilon_{ij}\big(\mathbf{u}(t-s)\big)\big)\big(\epsilon_{kl}\big(\mathbf{u}(t)\big)-\epsilon_{kl}\big(\mathbf{u}(t-s)\big)\big)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Substituting in (18)

$$\begin{aligned}
&\int_\Omega \mathcal{S}\big(\mathbf{u}(t)\big)\cdot\nabla\dot{\mathbf{u}}(t)\,\mathrm{d}x\\
&\quad= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\int_\Omega \dot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}(t)\big)-\epsilon_{ij}\big(\mathbf{u}(t-s)\big)\big)\big(\epsilon_{kl}\big(\mathbf{u}(t)\big)-\epsilon_{kl}\big(\mathbf{u}(t-s)\big)\big)\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad+\frac{1}{2}\int_0^t\int_\Omega \ddot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}(t)\big)-\epsilon_{ij}\big(\mathbf{u}(t-s)\big)\big)\big(\epsilon_{kl}\big(\mathbf{u}(t)\big)-\epsilon_{kl}\big(\mathbf{u}(t-s)\big)\big)\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{G}_{ijkl}(t)\epsilon_{ij}\big(\mathbf{u}(t)\big)\epsilon_{kl}\big(\mathbf{u}(t)\big)\,\mathrm{d}x\\
&\qquad-\frac{1}{2}\int_\Omega \dot{\mathcal{G}}_{ijkl}(t)\epsilon_{ij}\big(\mathbf{u}(t)\big)\epsilon_{kl}\big(\mathbf{u}(t)\big)\,\mathrm{d}x.
\end{aligned}\tag{20}$$

Substituting (16), (17) and (20) in (15), we obtain after summing

$$\begin{aligned}
&\int_\Omega\big|\dot{\mathbf{m}}^{\varepsilon,N}\big|^2\,\mathrm{d}x+\frac{a}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\big|\Lambda^\alpha\mathbf{m}^{\varepsilon,N}\big|^2\,\mathrm{d}x+\frac{1}{4\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\big(\big|\mathbf{m}^{\varepsilon,N}\big|^2-1\big)^2\,\mathrm{d}x\\
&\quad+\frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\big|\dot{\mathbf{u}}^{\varepsilon,N}\big|^2\,\mathrm{d}x+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\lambda_{ijkl}m_i^{\varepsilon,N}m_j^{\varepsilon,N}\epsilon_{kl}\big(\mathbf{u}^{\varepsilon,N}\big)\,\mathrm{d}x\\
&\quad-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\int_\Omega\dot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}^{\varepsilon,N}(t)\big)\\
&\quad-\epsilon_{ij}\big(\mathbf{u}^{\varepsilon,N}(t-s)\big)\big)\big(\epsilon_{kl}\big(\mathbf{u}^{\varepsilon,N}(t)\big)-\epsilon_{kl}\big(\mathbf{u}^{\varepsilon,N}(t-s)\big)\big)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\frac{1}{2}\int_0^t\int_\Omega\ddot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}^{\varepsilon,N}(t)\big)-\epsilon_{ij}\big(\mathbf{u}^{\varepsilon,N}(t-s)\big)\big)\big(\epsilon_{kl}\big(\mathbf{u}^{\varepsilon,N}(t)\big)-\epsilon_{kl}\big(\mathbf{u}^{\varepsilon,N}(t-s)\big)\big)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\mathcal{G}_{ijkl}(t)\epsilon_{ij}\big(\mathbf{u}^{\varepsilon,N}(t)\big)\epsilon_{kl}\big(\mathbf{u}^{\varepsilon,N}(t)\big)\,\mathrm{d}x\\
&\quad-\frac{1}{2}\int_\Omega\dot{\mathcal{G}}_{ijkl}(t)\epsilon_{ij}\big(\mathbf{u}^{\varepsilon,N}(t)\big)\epsilon_{kl}\big(\mathbf{u}^{\varepsilon,N}(t)\big)\,\mathrm{d}x=0.
\end{aligned}$$

Now integrating in time

$$\begin{aligned}
&\int_Q\big|\dot{\mathbf{m}}^{\varepsilon,N}\big|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{a}{2}\int_\Omega\big|\Lambda^\alpha\mathbf{m}^{\varepsilon,N}(T)\big|^2\,\mathrm{d}x+\frac{1}{4\varepsilon}\int_\Omega\big(\big|\mathbf{m}^{\varepsilon,N}(T)\big|^2-1\big)^2\,\mathrm{d}x\\
&\quad+\frac{\rho}{2}\int_\Omega\big|\dot{\mathbf{u}}^{\varepsilon,N}(T)\big|^2\,\mathrm{d}x+\frac{1}{2}\int_\Omega\lambda_{ijkl}m_i^{\varepsilon,N}m_j^{\varepsilon,N}\epsilon_{kl}\big(\mathbf{u}^{\varepsilon,N}\big)(T)\,\mathrm{d}x\\
&\quad-\frac{1}{2}\int_0^T\int_\Omega\dot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}^{\varepsilon,N}(T)\big)-\epsilon_{ij}\big(\mathbf{u}^{\varepsilon,N}(T-s)\big)\big)\big(\epsilon_{kl}\big(\mathbf{u}^{\varepsilon,N}(T)\big)\\
&\quad-\epsilon_{kl}\big(\mathbf{u}^{\varepsilon,N}(T-s)\big)\big)\,\mathrm{d}s\,\mathrm{d}x\\
&\quad+\frac{1}{2}\int_Q\int_0^t\ddot{\mathcal{G}}_{ijkl}(s)\big(\epsilon_{ij}\big(\mathbf{u}^{\varepsilon,N}(t)\big)-\epsilon_{ij}\big(\mathbf{u}^{\varepsilon,N}(t-s)\big)\big)\big(\epsilon_{kl}\big(\mathbf{u}^{\varepsilon,N}(t)\big)\\
&\quad-\epsilon_{kl}\big(\mathbf{u}^{\varepsilon,N}(t-s)\big)\big)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\frac{1}{2}\int_\Omega\mathcal{G}_{ijkl}(T)\epsilon_{ij}\big(\mathbf{u}^{\varepsilon,N}(T)\big)\epsilon_{kl}\big(\mathbf{u}^{\varepsilon,N}(T)\big)\,\mathrm{d}x\\
&\quad-\frac{1}{2}\int_Q\dot{\mathcal{G}}_{ijkl}(t)\epsilon_{ij}\big(\mathbf{u}^{\varepsilon,N}(t)\big)\epsilon_{kl}\big(\mathbf{u}^{\varepsilon,N}(t)\big)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a}{2}\int_\Omega \bigl|\Lambda^\alpha \mathbf{m}^N(0)\bigr|^2\,\mathrm{d}x + \frac{1}{4\varepsilon}\int_\Omega \bigl(\bigl|\mathbf{m}^N(0)\bigr|^2-1\bigr)^2\,\mathrm{d}x + \frac{\rho}{2}\int_\Omega \bigl|\dot{\mathbf{u}}^N(0)\bigr|^2\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega \lambda_{ijkl} m_i^N m_j^N \epsilon_{kl}\bigl(\mathbf{u}^N\bigr)(0)\,\mathrm{d}x + \frac{1}{2}\int_\Omega \mathcal{G}_{ijkl}(0)\epsilon_{ij}\bigl(\mathbf{u}^N(0)\bigr)\epsilon_{kl}\bigl(\mathbf{u}^N(0)\bigr)\,\mathrm{d}x.
\end{aligned}
$$

Taking into account assumptions (H_3) and (H_4)

$$
\begin{aligned}
&\int_Q \bigl|\dot{\mathbf{m}}^{\varepsilon,N}\bigr|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{a}{2}\int_\Omega \bigl|\Lambda^\alpha \mathbf{m}^{\varepsilon,N}(T)\bigr|^2\,\mathrm{d}x + \frac{1}{4\varepsilon}\int_\Omega \bigl(\bigl|\mathbf{m}^{\varepsilon,N}(T)\bigr|^2-1\bigr)^2\,\mathrm{d}x \\
&\quad + \frac{\rho}{2}\int_\Omega \bigl|\dot{\mathbf{u}}^{\varepsilon,N}(T)\bigr|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \lambda_{ijkl} m_i^{\varepsilon,N} m_j^{\varepsilon,N} \epsilon_{kl}\bigl(\mathbf{u}^{\varepsilon,N}\bigr)(T)\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega \mathcal{G}_{ijkl}(T)\epsilon_{ij}\bigl(\mathbf{u}^{\varepsilon,N}(T)\bigr)\epsilon_{kl}\bigl(\mathbf{u}^{\varepsilon,N}(T)\bigr)\,\mathrm{d}x \\
&\le \frac{a}{2}\int_\Omega \bigl|\Lambda^\alpha \mathbf{m}^N(0)\bigr|^2\,\mathrm{d}x + \frac{1}{4\varepsilon}\int_\Omega \bigl(\bigl|\mathbf{m}^N(0)\bigr|^2-1\bigr)^2\,\mathrm{d}x + \frac{\rho}{2}\int_\Omega \bigl|\dot{\mathbf{u}}^N(0)\bigr|^2\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega \lambda_{ijkl} m_i^N m_j^N \epsilon_{kl}\bigl(\mathbf{u}^N\bigr)(0)\,\mathrm{d}x + \frac{1}{2}\int_\Omega \mathcal{G}_{ijkl}(0)\epsilon_{ij}\bigl(\mathbf{u}^N(0)\bigr)\epsilon_{kl}\bigl(\mathbf{u}^N(0)\bigr)\,\mathrm{d}x. \qquad (21)
\end{aligned}
$$

We call $\mathcal{A}^{\varepsilon,N}(T)$ the left-hand side of (21) and $\mathcal{A}^N(0)$ its right-hand side.

Now, for a positive parameter λ such that $\frac{2\lambda}{9} > \sup_{ijkl}|\lambda_{ijkl}|$, we have by Young's inequality, omitting superscripts,

$$
\begin{aligned}
\bigl|\lambda_{ijkl} m_i m_j \epsilon_{kl}(\mathbf{u})\bigr| &\le \frac{2\lambda}{9}|m_i||m_j|\bigl|\epsilon_{kl}(\mathbf{u})\bigr| \\
&\le \frac{2\lambda}{9}\left(\frac{\lambda}{\beta}|m_i|^2|m_j|^2 + \frac{\beta}{4\lambda}\bigl|\epsilon_{kl}(\mathbf{u})\bigr|^2\right),
\end{aligned}
$$

from where

$$
\begin{aligned}
\sum_{ijkl}\bigl|\lambda_{ijkl} m_i m_j \epsilon_{kl}(\mathbf{u})\bigr| &\le \frac{2\lambda}{9}\left(\frac{9\lambda}{\beta}\sum_i |m_i|^2\sum_j |m_j|^2 + \frac{9\beta}{4\lambda}\sum_{kl}\bigl|\epsilon_{kl}(\mathbf{u})\bigr|^2\right) \\
&= 2\lambda\left(\frac{\lambda}{\beta}\left(\sum_i |m_i|^2\right)^2 + \frac{\beta}{4\lambda}\sum_{kl}\bigl|\epsilon_{kl}(\mathbf{u})\bigr|^2\right) \\
&= \frac{2\lambda^2}{\beta}|\mathbf{m}|^4 + \frac{\beta}{2}\sum_{kl}\bigl|\epsilon_{kl}(\mathbf{u})\bigr|^2.
\end{aligned}
$$

Therefore, following the idea introduced in [7], we have

$$
\begin{aligned}
&\frac{1}{2}\left|\int_\Omega \lambda_{ijkl} m_i m_j \epsilon_{kl}(\mathbf{u})\,\mathrm{d}x\right| \\
&\quad = \frac{1}{2}\left|\int_\Omega \sum_{ijkl}\lambda_{ijkl} m_i m_j \epsilon_{kl}(\mathbf{u})\,\mathrm{d}x\right| \\
&\quad \le \frac{1}{2}\int_\Omega \sum_{ijkl}\bigl|\lambda_{ijkl} m_i m_j \epsilon_{kl}(\mathbf{u})\bigr|\,\mathrm{d}x \\
&\quad \le \frac{\lambda^2}{\beta}\int_\Omega |\mathbf{m}|^4\,\mathrm{d}x + \frac{\beta}{4}\int_\Omega \sum_{kl}\bigl|\epsilon_{kl}(\mathbf{u})\bigr|^2\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\lambda^2}{\beta}\int_\Omega \left(|\mathbf{m}|^2 - 1 + 1\right)^2 \mathrm{d}x + \frac{\beta}{4}\int_\Omega \sum_{kl}\left|\epsilon_{kl}(\mathbf{u})\right|^2 \mathrm{d}x \\
&\le \frac{2\lambda^2}{\beta}\int_\Omega \left(|\mathbf{m}|^2 - 1\right)^2 \mathrm{d}x + \frac{2\lambda^2}{\beta}\operatorname{vol}(\Omega) + \frac{\beta}{4}\int_\Omega \sum_{kl}\left|\epsilon_{kl}(\mathbf{u})\right|^2 \mathrm{d}x \\
&\le \frac{2\lambda^2}{\beta}\int_\Omega \left(|\mathbf{m}|^2 - 1\right)^2 \mathrm{d}x + \frac{2\lambda^2}{\beta}\operatorname{vol}(\Omega) + \frac{1}{4}\int_\Omega \mathcal{G}_{ijkl}\epsilon_{ij}(\mathbf{u})\epsilon_{kl}(\mathbf{u})\,\mathrm{d}x
\end{aligned}
$$

by using (H_5). Now, for $\varepsilon < \frac{\beta}{16\lambda^2}$, we have

$$
\frac{1}{2}\left|\int_\Omega \lambda_{ijkl}m_i m_j \epsilon_{kl}(\mathbf{u})\,\mathrm{d}x\right| \le \frac{1}{8\varepsilon}\int_\Omega \left(|\mathbf{m}|^2 - 1\right)^2 \mathrm{d}x + \frac{2\lambda^2}{\beta}\operatorname{vol}(\Omega) + \frac{1}{4}\int_\Omega \mathcal{G}_{ijkl}\epsilon_{ij}(\mathbf{u})\epsilon_{kl}(\mathbf{u})\,\mathrm{d}x,
$$

which implies

$$
\begin{aligned}
\frac{1}{2}\int_\Omega \lambda_{ijkl}m_i^N m_j^N \epsilon_{kl}\left(\mathbf{u}^N\right)(0)\,\mathrm{d}x \le{}& \frac{1}{8\varepsilon}\int_\Omega \left(\left|\mathbf{m}^N(0)\right|^2 - 1\right)^2 \mathrm{d}x \\
&+ \frac{2\lambda^2}{\beta}\operatorname{vol}(\Omega) + \frac{1}{4}\int_\Omega \mathcal{G}_{ijkl}\epsilon_{ij}\left(\mathbf{u}^N\right)\epsilon_{kl}\left(\mathbf{u}^N\right)(0)\,\mathrm{d}x,
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{1}{8\varepsilon}\int_\Omega \left(\left|\mathbf{m}^{\varepsilon,N}(T)\right|^2 - 1\right)^2 \mathrm{d}x - \frac{2\lambda^2}{\beta}\operatorname{vol}(\Omega) \\
&\quad - \frac{1}{4}\int_\Omega \mathcal{G}_{ijkl}\epsilon_{ij}\left(\mathbf{u}^{\varepsilon,N}\right)\epsilon_{kl}\left(\mathbf{u}^{\varepsilon,N}\right)(T)\,\mathrm{d}x \\
&\le \frac{1}{2}\int_\Omega \lambda_{ijkl}m_i^{\varepsilon,N} m_j^{\varepsilon,N}\epsilon_{kl}\left(\mathbf{u}^{\varepsilon,N}\right)(T)\,\mathrm{d}x.
\end{aligned}
$$

According to the definition of $\mathcal{A}^{\varepsilon,N}(T)$ and $\mathcal{A}^N(0)$, we can write

$$
\begin{aligned}
&\int_Q \left|\dot{\mathbf{m}}^{\varepsilon,N}\right|^2 \mathrm{d}x\,\mathrm{d}t + \frac{a}{2}\int_\Omega \left|\Lambda^\alpha \mathbf{m}^{\varepsilon,N}(T)\right|^2 \mathrm{d}x + \frac{1}{8\varepsilon}\int_\Omega \left(\left|\mathbf{m}^{\varepsilon,N}(T)\right|^2 - 1\right)^2 \mathrm{d}x \\
&\quad + \frac{\rho}{2}\int_\Omega \left|\dot{\mathbf{u}}^{\varepsilon,N}(T)\right|^2 \mathrm{d}x + \frac{1}{4}\int_\Omega \mathcal{G}_{ijkl}\epsilon_{ij}\left(\mathbf{u}^{\varepsilon,N}\right)\epsilon_{kl}\left(\mathbf{u}^{\varepsilon,N}\right)(T)\,\mathrm{d}x - \frac{2\lambda^2}{\beta}\operatorname{vol}(\Omega) \le \mathcal{A}^{\varepsilon,N}(T),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{A}^N(0) \le{}& \frac{a}{2}\int_\Omega \left|\Lambda^\alpha \mathbf{m}^N(0)\right|^2 \mathrm{d}x + \frac{3}{8\varepsilon}\int_\Omega \left(\left|\mathbf{m}^N(0)\right|^2 - 1\right)^2 \mathrm{d}x + \frac{\rho}{2}\int_\Omega \left|\dot{\mathbf{u}}^N(0)\right|^2 \mathrm{d}x \\
&+ \frac{3}{4}\int_\Omega \mathcal{G}_{ijkl}\epsilon_{ij}\left(\mathbf{u}^N\right)\epsilon_{kl}\left(\mathbf{u}^N\right)(0)\,\mathrm{d}x + \frac{2\lambda^2}{\beta}\operatorname{vol}(\Omega).
\end{aligned}
$$

Since $\mathcal{A}^{\varepsilon,N}(T) \le \mathcal{A}^N(0)$, we have

$$
\begin{aligned}
&\int_Q \left|\dot{\mathbf{m}}^{\varepsilon,N}\right|^2 \mathrm{d}x\,\mathrm{d}t + \frac{a}{2}\int_\Omega \left|\Lambda^\alpha \mathbf{m}^{\varepsilon,N}(T)\right|^2 \mathrm{d}x + \frac{1}{8\varepsilon}\int_\Omega \left(\left|\mathbf{m}^{\varepsilon,N}(T)\right|^2 - 1\right)^2 \mathrm{d}x \\
&\quad + \frac{\rho}{2}\int_\Omega \left|\dot{\mathbf{u}}^{\varepsilon,N}(T)\right|^2 \mathrm{d}x + \frac{1}{4}\int_\Omega \mathcal{G}_{ijkl}\epsilon_{ij}\left(\mathbf{u}^{\varepsilon,N}\right)\epsilon_{kl}\left(\mathbf{u}^{\varepsilon,N}\right)(T)\,\mathrm{d}x
\end{aligned}
$$

$$\leq \frac{a}{2}\int_\Omega \left|\Lambda^\alpha \mathbf{m}^N(0)\right|^2 \mathrm{d}x + \frac{3}{8\varepsilon}\int_\Omega \left(\left|\mathbf{m}^N(0)\right|^2 - 1\right)^2 \mathrm{d}x + \frac{\rho}{2}\int_\Omega \left|\dot{\mathbf{u}}^N(0)\right|^2 \mathrm{d}x$$
$$+ \frac{3}{4}\int_\Omega \mathcal{G}_{ijkl}\epsilon_{ij}\left(\mathbf{u}^N\right)\epsilon_{kl}\left(\mathbf{u}^N\right)(0)\,\mathrm{d}x + \frac{4\lambda^2}{\beta}\operatorname{vol}(\Omega).$$

Moreover, $\int_\Omega |\nabla \mathbf{u}^{\varepsilon,N}(T)|^2\,\mathrm{d}x \leq \int_\Omega \sum_{kl} |\epsilon_{kl}(\mathbf{u}^{\varepsilon,N}(T))|^2\,\mathrm{d}x$ and under assumptions (H_5) and (H_6) we have

$$\int_Q \left|\dot{\mathbf{m}}^{\varepsilon,N}\right|^2 \mathrm{d}x\,\mathrm{d}t + \frac{a}{2}\int_\Omega \left|\Lambda^\alpha \mathbf{m}^{\varepsilon,N}(T)\right|^2 \mathrm{d}x + \frac{1}{8\varepsilon}\int_\Omega \left(\left|\mathbf{m}^{\varepsilon,N}(T)\right|^2 - 1\right)^2 \mathrm{d}x$$
$$+ \frac{\rho}{2}\int_\Omega \left|\dot{\mathbf{u}}^{\varepsilon,N}(T)\right|^2 \mathrm{d}x + \frac{\beta}{4}\int_\Omega \left|\nabla \mathbf{u}^{\varepsilon,N}(T)\right|^2 \mathrm{d}x$$
$$\leq \frac{a}{2}\int_\Omega \left|\Lambda^\alpha \mathbf{m}^N(0)\right|^2 \mathrm{d}x + \frac{3}{8\varepsilon}\int_\Omega \left(\left|\mathbf{m}^N(0)\right|^2 - 1\right)^2 \mathrm{d}x + \frac{\rho}{2}\int_\Omega \left|\dot{\mathbf{u}}^N(0)\right|^2 \mathrm{d}x$$
$$+ \frac{3\tau}{4}\int_\Omega \left|\nabla \mathbf{u}^N(0)\right|^2 \mathrm{d}x + \frac{4\lambda^2}{\beta}\operatorname{vol}(\Omega). \tag{22}$$

Since $\mathbf{u}_0 \in \mathbf{H}_0^1(\Omega)$, $\mathbf{u}_1 \in \mathbf{L}^2(\Omega)$ and $\mathbf{m}_0 \in \mathbf{H}^\alpha(\Omega)$, which is embedded into $\mathbf{L}^4(\Omega)$ for $1 < \alpha < \frac{3}{2}$, the right-hand side is uniformly bounded. Indeed, for constants C_1, C_2, C_3 and C_4 independent of N,

$$\begin{aligned}\int_\Omega \left(\left|\mathbf{m}^N(0)\right|^2 - 1\right)^2 \mathrm{d}x &= \int_\Omega \left|\mathbf{m}^N(0)\right|^4 \mathrm{d}x - 2\int_\Omega \left|\mathbf{m}^N(0)\right|^2 \mathrm{d}x + \operatorname{vol}(\Omega)\\ &\leq \left\|\mathbf{m}^N(0)\right\|_{\mathbf{L}^4(\Omega)}^4 + \operatorname{vol}(\Omega)\\ &\leq C_1\left\|\mathbf{m}^N(0)\right\|_{\mathbf{H}^\alpha(\Omega)}^4 + C_2\\ &\leq C_3\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \left|\nabla \mathbf{u}^N(0)\right|^2 \mathrm{d}x &= \int_\Omega \left|\nabla \mathbf{u}^N(0) - \nabla \mathbf{u}_0 + \nabla \mathbf{u}_0\right|^2 \mathrm{d}x\\ &\leq 2\int_\Omega \left|\nabla \mathbf{u}^N(0) - \nabla \mathbf{u}_0\right|^2 \mathrm{d}x + 2\int_\Omega |\nabla \mathbf{u}_0|^2\,\mathrm{d}x\\ &\leq 2\left\|\mathbf{u}^N(0) - \mathbf{u}_0\right\|_{\mathbf{H}_0^1(\Omega)}^2 + 2\|\mathbf{u}_0\|_{\mathbf{H}_0^1(\Omega)}^2\\ &\leq C_4,\end{aligned}$$

thanks to the strong convergences $\mathbf{m}^N(\cdot,0) \to \mathbf{m}_0$ in $\mathbf{H}^\alpha(\Omega)$ and $\mathbf{u}^N(\cdot,0) \to \mathbf{u}_0$ in $\mathbf{H}_0^1(\Omega)$. For the other term $(\dot{\mathbf{u}}^N(0))$, the estimate can be carried out in an analogous way using the strong convergence $\dot{\mathbf{u}}^N(\cdot,0) \to \mathbf{u}_1$ in $\mathbf{L}^2(\Omega)$. Moreover, noting that (for a constant C independent of ε and N)

$$\int_\Omega \left|\mathbf{m}^{\varepsilon,N}\right|^2 \mathrm{d}x = \int_\Omega \left(\left|\mathbf{m}^{\varepsilon,N}\right|^2 - 1 + 1\right) \mathrm{d}x$$
$$\leq \frac{1}{2}\int_\Omega \left(\left|\mathbf{m}^{\varepsilon,N}\right|^2 - 1\right)^2 \mathrm{d}x + C.$$

Therefore, for fixed $\varepsilon > 0$, we have

$$\begin{aligned}
&(\mathbf{m}^{\varepsilon,N})_N \text{ is bounded in } L^\infty(0,T;\mathbf{H}^\alpha(\Omega)),\\
&(\dot{\mathbf{m}}^{\varepsilon,N})_N \text{ is bounded in } L^2(0,T;\mathbf{L}^2(\Omega)),\\
&(|\mathbf{m}^{\varepsilon,N}|^2-1)_N \text{ is bounded in } L^\infty(0,T;L^2(\Omega)),\\
&(\mathbf{u}^{\varepsilon,N})_N \text{ is bounded in } L^2(0,T;\mathbf{H}_0^1(\Omega)),\\
&(\dot{\mathbf{u}}^{\varepsilon,N})_N \text{ is bounded in } L^2(0,T;\mathbf{L}^2(\Omega)).
\end{aligned} \tag{23}$$

Note that (23) is due to the Poincaré lemma. Now, from classical compactness results, there exist two subsequences which we still denote by $(\mathbf{m}^{\varepsilon,N})$ and $(\mathbf{u}^{\varepsilon,N})$ such that for fixed $\varepsilon > 0$

$$\begin{aligned}
&\mathbf{m}^{\varepsilon,N} \rightharpoonup \mathbf{m}^\varepsilon \text{ weakly in } L^2(0,T;\mathbf{H}^\alpha(\Omega)),\\
&\dot{\mathbf{m}}^{\varepsilon,N} \rightharpoonup \dot{\mathbf{m}}^\varepsilon \text{ weakly in } \mathbf{L}^2(Q),\\
&\mathbf{m}^{\varepsilon,N} \to \mathbf{m}^\varepsilon \text{ strongly in } L^2(0,T,\mathbf{H}^\beta(\Omega)) \text{ and a.e. for } 0 \le \beta < \alpha,\\
&|\mathbf{m}^{\varepsilon,N}|^2 - 1 \rightharpoonup \zeta \text{ weakly in } L^2(Q),\\
&\mathbf{u}^{\varepsilon,N} \rightharpoonup \mathbf{u}^\varepsilon \text{ weakly in } L^2(0,T;\mathbf{H}_0^1(\Omega)),\\
&\dot{\mathbf{u}}^{\varepsilon,N} \rightharpoonup \dot{\mathbf{u}}^\varepsilon \text{ weakly in } \mathbf{L}^2(Q),\\
&\mathbf{u}^{\varepsilon,N} \to \mathbf{u}^\varepsilon \text{ strongly in } \mathbf{L}^2(Q).
\end{aligned} \tag{24}$$

Convergence (24) is due to the following lemma (the proof can be found in [14], p.57).

Lemma 4.2 *Assume A, B and C are three Banach spaces and satisfy $A \subset B \subset C$ where the injections are continuous with compact embedding $A \hookrightarrow B$ and A, C are reflexive. Denote*

$$D := \left\{ v \,\middle|\, v \in L^{p_0}(0,T;A), \dot{v} = \frac{dv}{dt} \in L^{p_1}(0,T;C) \right\},$$

where T is finite and $1 < p_i < \infty$, $i = 0,1$. Then D, equipped with the norm

$$\|v\|_{L^{p_0}(0,T;A)} + \|\dot{v}\|_{L^{p_1}(0,T;C)},$$

is a Banach space and the embedding $D \hookrightarrow L^{p_0}(0,T;B)$ is compact.

Another lemma (Lemma 4.3) whose proof can be found in [14], p.12 will ensure that $\zeta = |\mathbf{m}^\varepsilon|^2 - 1$.

Lemma 4.3 *Let Θ be a bounded open set of $\mathbb{R}_x^d \times \mathbb{R}_t$, h_n and h in $L^q(\Theta)$, $1 < q < \infty$ such that $\|h_n\|_{L^q(\Theta)} \le C$, $h_n \to h$ a.e. in Θ, then $h_n \rightharpoonup h$ weakly in $L^q(\Theta)$.*

Now, since $1 < \alpha < \frac{3}{2}$ and from the Sobolev embedding $H^\alpha(Q) \hookrightarrow L^4(Q)$, further compactness result follows

$$m_i^{\varepsilon,N} m_j^{\varepsilon,N} \to m_i^\varepsilon m_j^\varepsilon \text{ strongly in } L^2(Q) \tag{25}$$

and

$$m_i^{\varepsilon,N}\phi_j \to m_i^{\varepsilon}\phi_j \text{ strongly in } L^2(Q).$$

The above estimates allow us to pass to the limit as N goes to infinity and to get the desired result. Indeed consider the variational formulation of (14)

$$\begin{cases} \int_Q \dot{\mathbf{m}}^{\varepsilon,N}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t + a\int_Q \Lambda^\alpha\mathbf{m}^{\varepsilon,N}\cdot\Lambda^\alpha\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t \\ \quad + \int_Q \lambda_{ijkl}m_j^{\varepsilon,N}\epsilon_{kl}(\mathbf{u}^{\varepsilon,N})\boldsymbol{\phi}_i\,\mathrm{d}x\,\mathrm{d}t + \int_Q \frac{|\mathbf{m}^{\varepsilon,N}|^2-1}{\varepsilon}\mathbf{m}^{\varepsilon,N}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t = 0, \\ -\rho\int_Q \dot{\mathbf{u}}^{\varepsilon,N}\cdot\dot{\boldsymbol{\psi}}\,\mathrm{d}x\,\mathrm{d}t + \int_Q \mathcal{G}_{ijkl}(0)\epsilon_{ij}(\mathbf{u}^{\varepsilon,N}(t))\epsilon_{kl}(\boldsymbol{\psi}(t))\,\mathrm{d}x\,\mathrm{d}t \\ \quad + \int_Q\int_0^t \mathcal{G}_{ijkl}(t-s)\epsilon_{ij}(\mathbf{u}^{\varepsilon,N}(s))\epsilon_{kl}(\boldsymbol{\psi}(t))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\ \quad + \frac{1}{2}\int_Q \lambda_{ijkl}m_i^{\varepsilon,N}m_j^{\varepsilon,N}\epsilon_{kl}(\boldsymbol{\psi})\,\mathrm{d}x\,\mathrm{d}t = 0 \end{cases} \tag{26}$$

for any $\boldsymbol{\phi}\in L^2(0,T;\mathbf{H}^\alpha(\Omega))$ and $\boldsymbol{\psi}\in\mathbf{H}_0^1(Q)$. Taking $N\to\infty$ in (26), we find

$$\begin{cases} \int_Q \dot{\mathbf{m}}^{\varepsilon}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t + a\int_Q \Lambda^\alpha\mathbf{m}^{\varepsilon}\cdot\Lambda^\alpha\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t \\ \quad + \int_Q \lambda_{ijkl}m_j^{\varepsilon}\epsilon_{kl}(\mathbf{u}^{\varepsilon})\boldsymbol{\phi}_i\,\mathrm{d}x\,\mathrm{d}t + \int_Q \frac{|\mathbf{m}^{\varepsilon}|^2-1}{\varepsilon}\mathbf{m}^{\varepsilon}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t = 0, \\ -\rho\int_Q \dot{\mathbf{u}}^{\varepsilon}\cdot\dot{\boldsymbol{\psi}}\,\mathrm{d}x\,\mathrm{d}t + \int_Q \mathcal{G}_{ijkl}(0)\epsilon_{ij}(\mathbf{u}^{\varepsilon}(t))\epsilon_{kl}(\boldsymbol{\psi}(t))\,\mathrm{d}x\,\mathrm{d}t \\ \quad + \int_Q\int_0^t \mathcal{G}_{ijkl}(t-s)\epsilon_{ij}(\mathbf{u}^{\varepsilon}(s))\epsilon_{kl}(\boldsymbol{\psi}(t))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\ \quad + \frac{1}{2}\int_Q \lambda_{ijkl}m_i^{\varepsilon}m_j^{\varepsilon}\epsilon_{kl}(\boldsymbol{\psi})\,\mathrm{d}x\,\mathrm{d}t = 0 \end{cases} \tag{27}$$

for any $\boldsymbol{\phi}\in L^2(0,T;\mathbf{H}^\alpha(\Omega))$ and $\boldsymbol{\psi}\in\mathbf{H}_0^1(Q)$. We proved the following result.

Proposition 4.1 *Given* $\mathbf{m}_0\in\mathbf{H}^\alpha(\Omega)$ *such that* $|\mathbf{m}_0| = 1$ *a.e.*, $\mathbf{u}_0\in\mathbf{H}_0^1(\Omega)$ *and* $\mathbf{u}_1\in\mathbf{L}^2(\Omega)$. *Then there exists a solution* $(\mathbf{m}^\varepsilon,\mathbf{u}^\varepsilon)$, *for any positive* ε *small enough, to problem* (11) *in the sense of distributions. Moreover, we have the following energy estimate*:

$$\begin{aligned} &\int_Q \left|\dot{\mathbf{m}}^\varepsilon\right|^2\mathrm{d}x\,\mathrm{d}t + \frac{a}{2}\int_\Omega\left|\Lambda^\alpha\mathbf{m}^\varepsilon(T)\right|^2\mathrm{d}x + \frac{1}{8\varepsilon}\int_\Omega\left(\left|\mathbf{m}^\varepsilon(T)\right|^2-1\right)^2\mathrm{d}x \\ &\quad + \frac{\rho}{2}\int_\Omega\left|\dot{\mathbf{u}}^\varepsilon(T)\right|^2\mathrm{d}x + \frac{\beta}{4}\int_\Omega\left|\nabla\mathbf{u}^\varepsilon(T)\right|^2\mathrm{d}x \\ &\le \frac{a}{2}\int_\Omega\left|\Lambda^\alpha\mathbf{m}_0\right|^2\mathrm{d}x + \frac{\rho}{2}\int_\Omega|\mathbf{u}_1|^2\,\mathrm{d}x + \frac{3\tau}{4}\int_\Omega|\nabla\mathbf{u}_0|^2\,\mathrm{d}x + \frac{4\lambda^2}{\beta}\operatorname{vol}(\Omega). \end{aligned} \tag{28}$$

Remark 2 We can deduce (28) by taking the lower semicontinuous limit in (22).

4.2 Convergence of approximate solutions

The limit process as $\varepsilon\to 0$ makes use also of some convergence results. For this, we will use estimate (28), from which we have

$$\begin{aligned} &(\mathbf{m}^\varepsilon)_\varepsilon \text{ is bounded in } L^\infty(0,T;\mathbf{H}^\alpha(\Omega)), \\ &(\dot{\mathbf{m}}^\varepsilon)_\varepsilon \text{ is bounded in } L^2(0,T;\mathbf{L}^2(\Omega)), \\ &(\left|\mathbf{m}^\varepsilon\right|^2-1)_\varepsilon \text{ is bounded in } L^\infty(0,T;L^2(\Omega)), \\ &(\mathbf{u}^\varepsilon)_\varepsilon \text{ is bounded in } L^2(0,T;\mathbf{H}_0^1(\Omega)), \\ &(\dot{\mathbf{u}}^\varepsilon)_\varepsilon \text{ is bounded in } L^2(0,T;\mathbf{L}^2(\Omega)). \end{aligned}$$

Then there exist two subsequences, which we still denote by $(\mathbf{m}^\varepsilon)$ and $(\mathbf{u}^\varepsilon)$, such that

$$\begin{aligned}
&\mathbf{m}^\varepsilon \rightharpoonup \mathbf{m} \text{ weakly in } L^2\big(0,T;\mathbf{H}^\alpha(\Omega)\big),\\
&\dot{\mathbf{m}}^\varepsilon \rightharpoonup \dot{\mathbf{m}} \text{ weakly in } L^2\big(0,T;\mathbf{L}^2(\Omega)\big),\\
&\mathbf{m}^\varepsilon \to \mathbf{m} \text{ strongly in } L^2\big(0,T,\mathbf{H}^\beta(\Omega)\big) \text{ and a.e. for } 0 \le \beta < \alpha,\\
&\big|\mathbf{m}^\varepsilon\big|^2 - 1 \to 0 \text{ strongly in } L^2(Q) \text{ and a.e.,}\\
&\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u} \text{ weakly in } L^2\big(0,T;\mathbf{H}_0^1(\Omega)\big),\\
&\dot{\mathbf{u}}^\varepsilon \rightharpoonup \dot{\mathbf{u}} \text{ weakly in } \mathbf{L}^2(Q),\\
&\mathbf{u}^\varepsilon \to \mathbf{u} \text{ strongly in } \mathbf{L}^2(Q).
\end{aligned} \tag{29}$$

It can be shown from convergence (29) that $|\mathbf{m}| = 1$ a.e.

Now, in order to pass to the limit $\varepsilon \to 0$ in (27), let $\boldsymbol{\phi} = \mathbf{m}^\varepsilon \times \boldsymbol{\varphi}$, where $\boldsymbol{\varphi} \in \mathbf{C}^\infty(\overline{Q})$. As $\boldsymbol{\phi}$ is in $L^2(0,T;\mathbf{H}^\alpha(\Omega))$, there holds

$$\begin{cases}
\int_Q \dot{\mathbf{m}}^\varepsilon \cdot (\mathbf{m}^\varepsilon \times \boldsymbol{\varphi})\,\mathrm{d}x\,\mathrm{d}t + a\int_Q \Lambda^\alpha \mathbf{m}^\varepsilon \cdot \Lambda^\alpha(\mathbf{m}^\varepsilon \times \boldsymbol{\varphi})\,\mathrm{d}x\,\mathrm{d}t\\
\quad + \int_Q \lambda_{ijkl} m_j^\varepsilon \epsilon_{kl}(\mathbf{u}^\varepsilon)(\mathbf{m}^\varepsilon \times \boldsymbol{\varphi})_i\,\mathrm{d}x\,\mathrm{d}t = 0,\\
-\rho\int_Q \dot{\mathbf{u}}^\varepsilon \cdot \dot{\boldsymbol{\psi}}\,\mathrm{d}x\,\mathrm{d}t + \int_Q \mathcal{G}_{ijkl}(0)\epsilon_{ij}(\mathbf{u}^\varepsilon(t))\epsilon_{kl}(\boldsymbol{\psi}(t))\,\mathrm{d}x\,\mathrm{d}t\\
\quad + \int_Q\int_0^t \mathcal{G}_{ijkl}(t-s)\epsilon_{ij}(\mathbf{u}^\varepsilon(s))\epsilon_{kl}(\boldsymbol{\psi}(t))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\
\quad + \frac{1}{2}\int_Q \lambda_{ijkl} m_i^\varepsilon m_j^\varepsilon \epsilon_{kl}(\boldsymbol{\psi})\,\mathrm{d}x\,\mathrm{d}t = 0.
\end{cases} \tag{30}$$

In (30) we can easily pass to the limit $\varepsilon \to 0$ (with the exception of the terms where there is the fractional Laplacian) thanks to recent convergences that we have set and a result like the one in (25).

Now we consider the convergence of the second term of the first equation. This is by no means obvious since we encounter the fractional order derivatives; for this reason, the classical methods are not applied anymore. However, commutator estimates (Lemma 4.4 (see [15–17] for a proof)) provide us with proper tools, to which the success in the following owes a lot.

Lemma 4.4 (Commutator estimates) *Suppose that $s > 0$ and $p \in (1,+\infty)$. If $f,g \in \mathcal{S}$ (the Schwartz class), then*

$$\big\|\Lambda^s(fg) - f\Lambda^s g\big\|_{L^p} \le C\big(\|\nabla f\|_{L^{p_1}}\|g\|_{\dot{W}^{s-1,p_2}} + \|f\|_{\dot{W}^{s,p_3}}\|g\|_{L^{p_4}}\big) \tag{31}$$

and

$$\big\|\Lambda^s(fg)\big\|_{L^p} \le C\big(\|f\|_{L^{p_1}}\|g\|_{\dot{W}^{s,p_2}} + \|f\|_{\dot{W}^{s,p_3}}\|g\|_{L^{p_4}}\big) \tag{32}$$

with $p_2,p_3 \in (1,+\infty)$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

We start firstly by showing that $\Lambda^\alpha(\mathbf{m}^\varepsilon \times \boldsymbol{\varphi}) \in \mathbf{L}^2(Q)$ (then we comment that the second term in (9) makes sense), indeed applying the multiplicative estimates (32) in Lemma 4.4

to $\mathbf{m}^\varepsilon$ and $\boldsymbol{\varphi}$ (for $s = \alpha, p = 2, p_1 = \frac{6}{3-2\alpha}, p_2 = \frac{3}{\alpha}, p_3 = 2$ and $p_4 = \infty$), we find for a constant C independent of ε

$$\begin{aligned}\|\Lambda^\alpha(\mathbf{m}^\varepsilon \times \boldsymbol{\varphi})\|_{\mathbf{L}^2(\Omega)} &\le C\left(\|\mathbf{m}^\varepsilon\|_{\mathbf{L}^{p_1}(\Omega)}\|\boldsymbol{\varphi}\|_{\dot{\mathbf{W}}^{\alpha,p_2}(\Omega)} + \|\mathbf{m}^\varepsilon\|_{\dot{\mathbf{H}}^\alpha(\Omega)}\|\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(\Omega)}\right)\\ &= C\left(\|\mathbf{m}^\varepsilon\|_{\mathbf{L}^{p_1}(\Omega)}\|\Lambda^\alpha\boldsymbol{\varphi}\|_{\mathbf{L}^{p_2}(\Omega)} + \|\Lambda^\alpha\mathbf{m}^\varepsilon\|_{\mathbf{L}^2(\Omega)}\|\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(\Omega)}\right).\end{aligned}$$

Here is another lemma (see [3] for a detailed proof and for more details on fractional calculus).

Lemma 4.5 *Suppose that $p > q > 1$ and $\frac{1}{p} + \frac{s}{d} = \frac{1}{q}$. Assume that $\Lambda^s f \in L^q$, then $f \in L^p$ and there is a constant $C > 0$ such that*

$$\|f\|_{L^p} \le C\|\Lambda^s f\|_{L^q}.$$

In Lemma 4.5, we take $f = \mathbf{m}^\varepsilon, q = 2, s = \alpha, p = p_1$, then $\|\mathbf{m}^\varepsilon\|_{L^{p_1}(\Omega)} \le C_1\|\Lambda^\alpha\mathbf{m}^\varepsilon\|_{L^2(\Omega)}$. Therefore

$$\begin{aligned}\|\Lambda^\alpha(\mathbf{m}^\varepsilon \times \boldsymbol{\varphi})\|_{\mathbf{L}^2(\Omega)} &\le C\left(C_1\|\Lambda^\alpha\mathbf{m}^\varepsilon\|_{\mathbf{L}^2(\Omega)}\|\Lambda^\alpha\boldsymbol{\varphi}\|_{\mathbf{L}^{p_2}(\Omega)} + \|\Lambda^\alpha\mathbf{m}^\varepsilon\|_{\mathbf{L}^2(\Omega)}\|\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(\Omega)}\right)\\ &\le C\|\Lambda^\alpha\mathbf{m}^\varepsilon\|_{\mathbf{L}^2(\Omega)}\left(C_1\|\Lambda^\alpha\boldsymbol{\varphi}\|_{\mathbf{L}^{p_2}(\Omega)} + \|\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(\Omega)}\right)\\ &\le C_2,\end{aligned}$$

where the constants C_1, C_2 and C are independent of ε.

Now, to ensure the convergence for the nonlinear nonlocal term, we introduce the commutator (see [12])

$$\Gamma_{\boldsymbol{\varphi}}(\mathbf{m}) := \Lambda^\alpha(\mathbf{m} \times \boldsymbol{\varphi}) - \boldsymbol{\varphi} \times \Lambda^\alpha\mathbf{m}$$

and we begin by showing that $\Gamma_{\boldsymbol{\varphi}}(\mathbf{m}) \in \mathbf{L}^2(Q)$. Indeed, applying (31) for $p_1 = \infty, p_2 = 2, p_3 = \frac{3}{\beta}$ and $p_4 = \frac{6}{3-2\beta}$ with $\beta = \alpha - 1$ (note that for the choice of p_4 we have $\dot{\mathbf{H}}^\beta(\Omega) \hookrightarrow \mathbf{L}^{p_4}(\Omega)$), we find

$$\begin{aligned}\|\Gamma_{\boldsymbol{\varphi}}(\mathbf{m})\|_{\mathbf{L}^2(\Omega)} &\le C_1\left(\|\nabla\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(\Omega)}\|\mathbf{m}\|_{\dot{\mathbf{H}}^\beta(\Omega)} + \|\boldsymbol{\varphi}\|_{\dot{\mathbf{W}}^{\alpha,p_3}(\Omega)}\|\mathbf{m}\|_{\mathbf{L}^{p_4}(\Omega)}\right)\\ &\le C_1\left(\|\nabla\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(\Omega)}\|\mathbf{m}\|_{\dot{\mathbf{H}}^\beta(\Omega)} + C_2\|\boldsymbol{\varphi}\|_{\dot{\mathbf{W}}^{\alpha,p_3}(\Omega)}\|\mathbf{m}\|_{\dot{\mathbf{H}}^\beta(\Omega)}\right)\\ &\le C\|\mathbf{m}\|_{\mathbf{H}^\beta(\Omega)}\left(\|\nabla\boldsymbol{\varphi}\|_{\mathbf{L}^\infty(\Omega)} + \|\boldsymbol{\varphi}\|_{\dot{\mathbf{W}}^{\alpha,p_3}(\Omega)}\right)\\ &\le C'\|\mathbf{m}\|_{\mathbf{H}^\beta(\Omega)},\end{aligned}$$

where C_1, C_2, C and C' are constants. Then

$$\|\Gamma_{\boldsymbol{\varphi}}(\mathbf{m})\|_{\mathbf{L}^2(Q)} \le C\|\mathbf{m}\|_{L^2(0,T;\mathbf{H}^\beta(\Omega))}.$$

Once again, similarly

$$\|\Gamma_{\boldsymbol{\varphi}}(\mathbf{m}^\varepsilon - \mathbf{m})\|_{\mathbf{L}^2(Q)} \le C\|\mathbf{m}^\varepsilon - \mathbf{m}\|_{L^2(0,T;\mathbf{H}^\beta(\Omega))}.$$

In what follows, we focus on the convergence of the following term:

$$\mathfrak{I}_\varepsilon := \int_Q \Lambda^\alpha \mathbf{m}^\varepsilon \cdot \Lambda^\alpha \big(\mathbf{m}^\varepsilon \times \boldsymbol{\varphi}\big)\,\mathrm{d}x\,\mathrm{d}t.$$

Let $\mathfrak{I} := \int_Q \Lambda^\alpha \mathbf{m} \cdot \Lambda^\alpha (\mathbf{m} \times \boldsymbol{\varphi})\,\mathrm{d}x\,\mathrm{d}t$, since $\Lambda^\alpha \mathbf{m} \cdot (\Lambda^\alpha \mathbf{m} \times \boldsymbol{\varphi}) = 0$, we have

$$\mathfrak{I}_\varepsilon = \int_Q \Lambda^\alpha \mathbf{m}^\varepsilon \cdot \Gamma_{\boldsymbol{\varphi}}\big(\mathbf{m}^\varepsilon\big)\,\mathrm{d}x\,\mathrm{d}t \quad \text{and} \quad \mathfrak{I} = \int_Q \Lambda^\alpha \mathbf{m} \cdot \Gamma_{\boldsymbol{\varphi}}(\mathbf{m})\,\mathrm{d}x\,\mathrm{d}t,$$

and note that these two integrals are well defined since $\Gamma_{\boldsymbol{\varphi}}(\mathbf{m}^\varepsilon)$ and $\Gamma_{\boldsymbol{\varphi}}(\mathbf{m})$ are in $\mathbf{L}^2(Q)$. Now we will show that $\mathfrak{I}_\varepsilon \to \mathfrak{I}$ as $\varepsilon \to 0$.

We have

$$\begin{aligned}
|\mathfrak{I}_\varepsilon - \mathfrak{I}| &= \left| \int_Q \Lambda^\alpha \mathbf{m}^\varepsilon \cdot \Gamma_{\boldsymbol{\varphi}}\big(\mathbf{m}^\varepsilon\big)\,\mathrm{d}x\,\mathrm{d}t - \int_Q \Lambda^\alpha \mathbf{m} \cdot \Gamma_{\boldsymbol{\varphi}}(\mathbf{m})\,\mathrm{d}x\,\mathrm{d}t \right| \\
&= \left| \int_Q \Lambda^\alpha \mathbf{m}^\varepsilon \cdot \Gamma_{\boldsymbol{\varphi}}\big(\mathbf{m}^\varepsilon - \mathbf{m}\big)\,\mathrm{d}x\,\mathrm{d}t + \int_Q \Lambda^\alpha \big(\mathbf{m}^\varepsilon - \mathbf{m}\big) \cdot \Gamma_{\boldsymbol{\varphi}}(\mathbf{m})\,\mathrm{d}x\,\mathrm{d}t \right| \\
&\le \int_Q \big|\Lambda^\alpha \mathbf{m}^\varepsilon \cdot \Gamma_{\boldsymbol{\varphi}}\big(\mathbf{m}^\varepsilon - \mathbf{m}\big)\big|\,\mathrm{d}x\,\mathrm{d}t + \left| \int_Q \Lambda^\alpha \big(\mathbf{m}^\varepsilon - \mathbf{m}\big) \cdot \Gamma_{\boldsymbol{\varphi}}(\mathbf{m})\,\mathrm{d}x\,\mathrm{d}t \right| \\
&\le C \big\| \Gamma_{\boldsymbol{\varphi}}\big(\mathbf{m}^\varepsilon - \mathbf{m}\big) \big\|_{\mathbf{L}^2(Q)} + \left| \int_Q \Lambda^\alpha \big(\mathbf{m}^\varepsilon - \mathbf{m}\big) \cdot \Gamma_{\boldsymbol{\varphi}}(\mathbf{m})\,\mathrm{d}x\,\mathrm{d}t \right| \\
&\le C' \big\| \mathbf{m}^\varepsilon - \mathbf{m} \big\|_{L^2(0,T;\mathbf{H}^\beta(\Omega))} + \left| \int_Q \Lambda^\alpha \big(\mathbf{m}^\varepsilon - \mathbf{m}\big) \cdot \Gamma_{\boldsymbol{\varphi}}(\mathbf{m})\,\mathrm{d}x\,\mathrm{d}t \right| \\
&\to 0
\end{aligned}$$

by the strong convergence for $\mathbf{m}^\varepsilon$ to $\mathbf{m}$ in $L^2(0,T;\mathbf{H}^\beta(\Omega))$ and the weak convergence in $L^2(0,T;\mathbf{H}^\alpha(\Omega))$.

Therefore

$$\begin{cases}
\int_Q \dot{\mathbf{m}} \cdot (\mathbf{m} \times \boldsymbol{\varphi})\,\mathrm{d}x\,\mathrm{d}t + a \int_Q \Lambda^\alpha \mathbf{m} \cdot \Lambda^\alpha (\mathbf{m} \times \boldsymbol{\varphi})\,\mathrm{d}x\,\mathrm{d}t \\
\quad + \int_Q \lambda_{ijkl} m_j \epsilon_{kl}(\mathbf{u})(\mathbf{m} \times \boldsymbol{\varphi})_i\,\mathrm{d}x\,\mathrm{d}t = 0, \\
-\rho \int_Q \dot{\mathbf{u}} \cdot \dot{\boldsymbol{\psi}}\,\mathrm{d}x\,\mathrm{d}t + \int_Q \mathcal{G}_{ijkl}(0)\epsilon_{ij}(\mathbf{u}(t))\epsilon_{kl}(\boldsymbol{\psi}(t))\,\mathrm{d}x\,\mathrm{d}t \\
\quad + \int_Q \int_0^t \mathcal{G}_{ijkl}(t-s)\epsilon_{ij}(\mathbf{u}(s))\epsilon_{kl}(\boldsymbol{\psi}(t))\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_Q \lambda_{ijkl} m_i m_j \epsilon_{kl}(\boldsymbol{\psi})\,\mathrm{d}x\,\mathrm{d}t = 0.
\end{cases}$$

This being true for every $\boldsymbol{\varphi} \in \mathbf{C}^\infty(\overline{Q})$, $\boldsymbol{\psi} \in \mathbf{H}_0^1(Q)$. Note that from (28) one can easily get (10). Hence $(\mathbf{m}, \mathbf{u})$ is a solution of problem (6)-(7)-(8) in the sense of Definition 3.1. The proof of Theorem 3.2 is complete.

5 Concluding remarks

In this paper, we have considered a model described by the fractional Heisenberg equation for the magnetization field and the viscoelastic integro-differential equation for the displacements. Global existence of weak solutions is proved. However, it would be interesting to prove the global existence of weak solutions in the case where $\nu \neq 0$ and to set up a numerical scheme for the system studied.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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