Zhan *Boundary Value Problems* (2017) 2017:117
DOI 10.1186/s13661-017-0848-z

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

A new kind of the solutions of a convection-diffusion equation related to the $p(x)$-Laplacian

Huashui Zhan*

*Correspondence:
huashuizhan@163.com
School of Applied Mathematics,
Xiamen University of Technology,
Xiamen, 361024, P.R. China

Abstract

A new kind of the solutions of the convection-diffusion equation related to the $p(x)$-Laplacian is introduced. The equation is degenerate on the boundary, accordingly, the usual boundary value condition cannot be imposed in Dirichlet's way. The test function chosen to verify the uniqueness of the solutions should be independent of the boundary value condition. By the new definition, one can study the stability of the weak solutions without any boundary value condition. The main results of the paper show that the usual homogeneous boundary value condition can be replaced by the degeneracy of the diffusion coefficient and the degeneracy of the convection term on the boundary.

MSC: 35L65; 35L85; 35R35

Keywords: convection-diffusion equation; $p(x)$-Laplacian; stability

1 Introduction and the main results

The initial-boundary value problem of the evolutionary $p(x)$-Laplacian equation

$$u_t = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

has been widely studied [1–6]. It is well-known that the equation arises in many applications in the electrorheological fluids, physics and biology [1–3]. Here, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p(x) \in C^1(\overline{\Omega})$, and we denote $p^+ = \max_{x\in\Omega} p(x), p^- = \min_{x\in\Omega} p(x) > 1$. Let $d(x) = \operatorname{dist}(x, \partial\Omega)$ be the distance function, the constant $\alpha > 0$. The well-posedness of the solutions of the equation

$$u_t = \operatorname{div}\left(d^\alpha(x)|\nabla u|^{p-2}\nabla u\right), \quad (x,t) \in Q_T, \tag{1.2}$$

was first studied by Yin-Wang [7], and later by Zhan-Xie [8] *et al.* A similar equation related to the $p(x)$-Laplacian

$$u_t = \operatorname{div}\left(d^\alpha(x)|\nabla u|^{p(x)-2}\nabla u\right), \quad (x,t) \in Q_T, \tag{1.3}$$

was studied by Zhan-Wen [9, 10] recently. In [9], the stability of the weak solutions is proved in a similar way as that of [7]. But there remained a gap when $p^- - 1 \le \alpha \le p^+ - 1$.

In [10], if $p(x)$ is required to satisfy the logarithmic Hölder continuity condition

$$\bigl|p(x) - p(y)\bigr| \le \omega\bigl(|x - y|\bigr), \quad \forall x, y \in \Omega, |x - y| < \frac{1}{2},$$

with

$$\overline{\lim_{s\to 0^+}}\, \omega(s) \ln\left(\frac{1}{s}\right) = C < \infty,$$

by complicate calculations, the gap had been filled up.

In this paper, we will establish the well-posedness of the solutions of equation

$$u_t = \operatorname{div}\bigl(d^\alpha(x)|\nabla u|^{p(x)-2}\nabla u\bigr) + \frac{\partial b_i(u,x,t)}{\partial x_i}, \quad (x,t) \in Q_T, \tag{1.4}$$

with the initial value

$$u|_{t=0} = u_0(x), \quad x \in \Omega, \tag{1.5}$$

but without any boundary value condition. The initial-boundary value problem of equation (1.4) was first considered by the author in [11], it was shown that the convection term $\frac{\partial b_i(u,x,t)}{\partial x_i}$ may influence the boundary value condition. We conjectured that, to ensure the well-posedness of the solutions, a partial boundary value condition should be imposed on equation (1.4). From then on, I had spent much time to consider the problem, and found that it is difficult to determine which part of the boundary should be imposed the boundary value. Thereupon, in this paper, we turn our attention to a study of the well-posedness of the solutions without any boundary value condition. We will introduce a new kind of the weak solutions matching up with equation (1.4), and try to prove the uniqueness of the new weak solutions only dependent on the initial value.

We denote

$$W_\alpha^{1,p(x)} = \left\{u \in W_{\mathrm{loc}}^{1,p(x)}(\Omega) : \int_\Omega d^\alpha(x)|\nabla u|^{p(x)}\,dx < \infty\right\}. \tag{1.6}$$

Clearly,

$$W_\alpha^{1,p(x)} \subseteq W_{\mathrm{loc}}^{1,p(x)}(\Omega). \tag{1.7}$$

Here $W^{1,p(x)}(\Omega)$ is the variable exponent Sobolev space, one can refer to [12–14] for the details. Some basic properties of the space are quoted in the following lemma.

Lemma 1.1

(i) *The spaces* $(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)})$, $(W^{1,p(x)}(\Omega), \|\cdot\|_{W^{1,p(x)}(\Omega)})$ *and* $W_0^{1,p(x)}(\Omega)$ *are reflexive Banach spaces.*

(ii) $p(x)$*-Hölder's inequality. Let* $q_1(x)$ *and* $q_2(x)$ *be real functions with* $\frac{1}{q_1(x)} + \frac{1}{q_2(x)} = 1$ *and* $q_1(x) > 1$. *Then the conjugate space of* $L^{q_1(x)}(\Omega)$ *is* $L^{q_2(x)}(\Omega)$. *And for any* $u \in L^{q_1(x)}(\Omega)$ *and* $v \in L^{q_2(x)}(\Omega)$, *we have*

$$\left|\int_\Omega uv\,dx\right| \le 2\|u\|_{L^{q_1(x)}(\Omega)}\|v\|_{L^{q_2(x)}(\Omega)}.$$

(iii)

$$\begin{aligned}
&\textit{If } \|u\|_{L^{p(x)}(\Omega)} = 1, \quad \textit{then } \int_\Omega \|u\|^{p(x)}\,dx = 1,\\
&\textit{if } \|u\|_{L^{p(x)}(\Omega)} > 1, \quad \textit{then } \|u\|^{p^-}_{L^{p(x)}(\Omega)} \le \int_\Omega |u|^{p(x)}\,dx \le \|u\|^{p^+}_{L^{p(x)}(\Omega)},\\
&\textit{if } \|u\|_{L^{p(x)}(\Omega)} < 1, \quad \textit{then } \|u\|^{p^+}_{L^{p(x)}(\Omega)} \le \int_\Omega |u|^{p(x)}\,dx \le \|u\|^{p^-}_{L^{p(x)}(\Omega)}.
\end{aligned}$$

The new kind of the weak solutions matching up with equation (1.4) is defined as follows.

Definition 1.2 A function $u(x,t)$ is said to be a solution of equation (1.4) with the initial condition (1.5), if

$$u \in L^\infty(Q_T), \qquad u_t \in L^2(Q_T), \qquad d^\alpha|\nabla u|^{p(x)} \in L^\infty\big(0,T;L^1(\Omega)\big), \tag{1.8}$$

and

$$\iint_{Q_T}\left[u_t(\varphi_1\varphi_2) + d^\alpha(x)|\nabla u|^{p(x)-2}\nabla u\cdot\nabla(\varphi_1\varphi_2) + b_i(u,x,t)\frac{\partial(\varphi_1\varphi_2)}{\partial x_i}\right]dx\,dt = 0, \tag{1.9}$$

where $\varphi_2 \in C_0^1(Q_T)$ as usual, but φ_1 only satisfies, for any given t, $\varphi_1(x,t) \in W_\alpha^{1,p(x)}$, and, for any given x, $|\varphi_1(x,t)| \le c$. The initial condition (1.5) is satisfied in the sense of that

$$\lim_{t\to 0}\int_\Omega \big|u(x,t) - u_0(x)\big|\,dx = 0. \tag{1.10}$$

A basic result of the existence of the solution is the following.

Theorem 1.3 *If* $p^- > 2$ *and* $0 < \alpha < \frac{p^- -2}{2}$, $\beta > 0$, $b_i(s,x,t)$ *and its partial derivatives satisfy the condition*

$$\big|b_i(s,x,t)\big| \le c|s|^{1+\beta}, \qquad \big|b_{is}(s,x,t)\big| \le c|s|^\beta, \tag{1.11}$$

and u_0 *satisfies*

$$u_0 \in L^\infty(\Omega), \quad d^\alpha|\nabla u_0|^{p^+} \in L^1(\Omega), \tag{1.12}$$

then equation (1.4) *with initial value* (1.5) *has a solution.*

We can prove Theorem 1.3 in a similar way to Theorem 1.2 in [11], though Definition 1.2 here is different from that of the weak solution in [11]. We omit the details of the proof here.

In our paper, we will prove another existence result, which seems more interesting.

Theorem 1.4 *Let* $b_i(s,x,t)$ *be a* C^1 *function*, $p(x) \ge 2$. *If* $|s| \le c$,

$$\big|b_{is}(s,x,t)\big| \le cd^{\frac{\alpha}{p(x)}} \tag{1.13}$$

and

$$u_0 \in L^\infty(\Omega), \quad d^\alpha |\nabla u_0|^{p(x)} \in L^\infty\big(0,T;L^1(\Omega)\big), \tag{1.14}$$

then there is a solution of equation (1.4) *with the initial value* (1.5).

One can see that only if $\alpha > 0$ in Theorem 1.4 is required, while $0 < \alpha < \frac{p^- -2}{2}$ in Theorem 1.3 has a stronger restriction. Moreover, there is a difference between the condition (1.11) and the condition (1.13). As we had shown in [11] only if $\alpha < p^- - 1$, the usual Dirichlet boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.15}$$

can be imposed, and by the condition (1.11), $b_i(0,x,t) = 0$. Accordingly, the stability of the weak solutions can be proved. Instead of (1.11), the condition (1.13) has the degeneracy on the boundary independent of the boundary value condition.

The most significant result of our paper is the following stability theorems.

Theorem 1.5 *Let u, v be two solutions of* (1.4) *with the initial values $u_0(x), v_0(x)$, respectively. If $b_i(s,x,t)$ satisfies*

$$\big|b_i(u,x,t) - b_i(v,x,t)\big| \le c d^{\frac{\alpha}{p(x)}} |u-v|^{1+\frac{1}{q(x)}}, \tag{1.16}$$

and the constant α satisfies

$$n \big\| d^{\alpha - 1 + \frac{\alpha}{p(x)}} \big\|_{L^{p(x)}(\Omega \setminus \Omega_{\frac{1}{n}})} \le c, \tag{1.17}$$

then

$$\int_\Omega \big|u(x,t) - v(x,t)\big|\, dx \le \int_\Omega \big|u_0(x) - v_0(x)\big|\, dx. \tag{1.18}$$

Here $\Omega_{\frac{1}{n}} = \{x \in \Omega : d(x) > \frac{1}{n}\}$.

Theorem 1.6 *Let u, v be two solutions of* (1.4) *with the initial values $u_0(x), v_0(x)$, respectively, and*

$$n \left(\int_{\Omega \setminus \Omega_{\frac{1}{n}}} d^\alpha |\nabla u|^{p(x)}\, dx \right)^{\frac{1}{q^+}} \le c, \qquad n \left(\int_{\Omega \setminus \Omega_{\frac{1}{n}}} d^\alpha |\nabla v|^{p(x)}\, dx \right)^{\frac{1}{q^+}} \le c. \tag{1.19}$$

If $b_i(s,x,t)$ satisfies (1.16), *then the global stability* (1.18) *is true. Here,* $q(x) = \frac{p(x)}{p(x)-1}$, $q^+ = \max_{x \in \overline{\Omega}} q(x)$.

One can see that, in Theorem 1.6, the stability is obtained only in the kind of weak solutions which satisfy the condition (1.19). While the restriction in Theorem 1.5 is the condition (1.17), no restrictions are imposed on the solutions themselves. At the end of the introduction, let us give two sufficient conditions of the condition (1.17).

If (1.17) is true, then $\|d^{\alpha-1+\frac{\alpha}{p(x)}}\|_{L^{p(x)}(\Omega\setminus\Omega_{\frac{1}{n}})} < 1$, thus by (iii) of Lemma 1.1,

$$\begin{aligned} n\left\|d^{\alpha-1+\frac{\alpha}{p(x)}}\right\|_{L^{p(x)}(\Omega\setminus\Omega_{\frac{1}{n}})} &\leq n\left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} d^{(\alpha-1+\frac{\alpha}{p(x)})p(x)}\,dx\right)^{\frac{1}{p^+}} \\ &= n\left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} d^{p(x)(\alpha-1)+\alpha}\,dx\right)^{\frac{1}{p^+}} \\ &= n\left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} d^{\alpha[p(x)(1-\frac{1}{\alpha})+1]}\,dx\right)^{\frac{1}{p^+}}. \end{aligned} \tag{1.20}$$

If $\alpha \geq \frac{p^+}{p^-+1} \geq \frac{p(x)}{p(x)+1}$, then $p(x)(1-\frac{1}{\alpha})+1 \geq 0$, and

$$\begin{aligned} n\left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} d^{\alpha[p(x)(1-\frac{1}{\alpha})+1]}\,dx\right)^{\frac{1}{p^+}} &\leq n\left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} d^{\alpha[p^+(1-\frac{1}{\alpha})+1]}\,dx\right)^{\frac{1}{p^+}} \\ &\leq \left(\frac{1}{n}\right)^{[p^+(1-\frac{1}{\alpha})+2]\frac{1}{p^+}-1}, \end{aligned} \tag{1.21}$$

which goes to zero as n goes the infinity, provided that $\alpha \geq \frac{p^+}{2}$, which implies that

$$\left[p^+\left(1-\frac{1}{\alpha}\right)+2\right]\frac{1}{p^+}-1 \geq 0.$$

Thus, the condition

$$\alpha \geq \max\left\{\frac{p^+}{p^-+1}, \frac{p^+}{2}\right\}, \tag{1.22}$$

is a sufficient condition of (1.17).

If

$$\frac{p(x)}{p(x)+2} < \alpha < \frac{p(x)}{p(x)+1}, \tag{1.23}$$

then

$$\begin{aligned} &-1 < p(x)\left(1-\frac{1}{\alpha}\right)+1 < 0, \\ &n\left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} d^{\alpha[p(x)(1-\frac{1}{\alpha})+1]}\,dx\right)^{\frac{1}{p^+}} = n\left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} \left(\frac{1}{d^\alpha}\right)^{-[p(x)(1-\frac{1}{\alpha})+1]}\,dx\right)^{\frac{1}{p^+}} \\ &\qquad \leq n\left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} \left(\frac{1}{d^\alpha}\right)^{p^+(\frac{1}{\alpha}-1)-1}\,dx\right)^{\frac{1}{p^+}} \leq \left(\frac{1}{n}\right)^{\frac{1}{\alpha}-2}. \end{aligned} \tag{1.24}$$

Thus, if

$$\frac{p^+}{p^-+2} < \alpha < \min\left\{\frac{p^-}{p^++1}, \frac{1}{2}\right\}, \tag{1.25}$$

then $\alpha \leq \frac{1}{2}$ and $-1 < p(x)(1-\frac{1}{\alpha})+1 < 0$, so

$$n\left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} d^{\alpha[p(x)(1-\frac{1}{\alpha})+1]}\,dx\right)^{\frac{1}{p^+}} \leq \left(\frac{1}{n}\right)^{\frac{1}{\alpha}-2} \leq c.$$

Consequently, the condition (1.25) is another sufficient condition of (1.17).

The paper is arranged as follows. In the first section, we have introduced the basic background and the main results. In the second section, the existence of the weak solution is proved. In the third section, the stability results are obtained. In the last section, we will give a local stability of the weak solutions, without the restriction (1.17).

2 The proof of existence

Consider the regularized equation

$$u_t = \operatorname{div}\bigl((d_\varepsilon+\varepsilon)^\alpha\bigl(|\nabla u|^2+\varepsilon\bigr)^{\frac{p(x)-2}{2}}\nabla u\bigr) + \frac{\partial b_i(u,x,t)}{\partial x_i}, \quad (x,t)\in Q_T, \tag{2.1}$$

with the initial-boundary conditions

$$u(x,0) = u_{0\varepsilon}(x), \quad x\in\Omega, \tag{2.2}$$

$$u(x,t) = 0, \quad (x,t)\in\partial\Omega\times(0,T). \tag{2.3}$$

Here, $\varepsilon > 0$, $d_\varepsilon = d*\delta_\varepsilon$ is the mollified function of d, δ_ε is the mollifier. For all $\varepsilon > 0$, selecting $u_{\varepsilon,0}$ such that $\|u_{\varepsilon,0}\|_{L^\infty(\Omega)}$ and $\|d_\varepsilon^\alpha|\nabla u_{\varepsilon,0}|^{p(x)}\|_{L^1(\Omega)}$ are uniformly bounded, and $u_{\varepsilon,0}$ converges to u_0 in $W^{1,p(x)}_{\mathrm{loc}}(\Omega)$. It is well known that the problem (2.1)-(2.3) has a unique classical solution [15, 16].

Proof of Theorem 1.4 Multiplying (2.1) by u_ε and integrating it over $Q_t = \Omega\times(0,t)$ for any $t\in[0,T)$, we easily obtain

$$\iint_{Q_t} d^\alpha|\nabla u_\varepsilon|^{p(x)}\,dx\,dt \leq \iint_{Q_t}(d_\varepsilon+\varepsilon)^\alpha\bigl(|\nabla u|^2+\varepsilon\bigr)^{\frac{p(x)-2}{2}}|\nabla u_\varepsilon|^2\,dx\,dt \leq c, \tag{2.4}$$

then

$$\int_0^t\int_{\Omega_1}|\nabla u_\varepsilon|^{p(x)}\,dx\,dt \leq c(\Omega_1,T), \tag{2.5}$$

for any $\overline{\Omega_1}\subseteq\Omega$.

Multiplying (2.1) by $u_{\varepsilon t}$, integrating it over Q_T,

$$\begin{aligned}\iint_{Q_t}(u_{\varepsilon t})^2\,dx\,dt ={}& \iint_{Q_t}\operatorname{div}\bigl(\bigl(d_\varepsilon^\alpha+\varepsilon\bigr)\bigl(|\nabla u|^2+\varepsilon\bigr)^{\frac{p(x)-2}{2}}\nabla u_\varepsilon\bigr)\cdot u_{\varepsilon t}\,dx\,dt\\ &+\iint_{Q_t}u_{\varepsilon t}\frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}\,dx\,dt.\end{aligned} \tag{2.6}$$

Noticing that

$$\bigl(|\nabla u_\varepsilon|^2+\varepsilon\bigr)^{\frac{p(x)-2}{2}}\nabla u_\varepsilon\cdot\nabla u_{\varepsilon t} = \frac{1}{2}\frac{d}{dt}\int_0^{|\nabla u_\varepsilon(x,t)|^2+\varepsilon} s^{\frac{p(x)-2}{2}}\,ds, \tag{2.7}$$

then

$$\begin{aligned}
&\iint_{Q_t} \operatorname{div}\big((d_\varepsilon^\alpha + \varepsilon)\big(|\nabla u_\varepsilon|^2 + \varepsilon\big)^{\frac{p(x)-2}{2}} \nabla u_\varepsilon \cdot u_{\varepsilon t}\big)\,dx\,dt \\
&= -\iint_{Q_t} \big(d_\varepsilon^\alpha + \varepsilon\big)\big(|\nabla u_\varepsilon|^2 + \varepsilon\big)^{\frac{p(x)-2}{2}} \nabla u_\varepsilon \nabla u_{\varepsilon t}\,dx\,dt \\
&= -\frac{1}{2}\iint_{Q_t} \big(d_\varepsilon^\alpha + \varepsilon\big)\frac{d}{dt}\int_0^{|\nabla u_\varepsilon(x,t)|^2+\varepsilon} s^{\frac{p(x)-2}{2}}\,ds\,dx\,dt \\
&= -\frac{1}{2}\int_\Omega \big(d_\varepsilon^\alpha + \varepsilon\big)\int_0^{|\nabla u_\varepsilon(x,t)|^2+\varepsilon} s^{\frac{p(x)-2}{2}}\,ds\,dx \\
&\quad + \frac{1}{2}\int_\Omega \big(d_\varepsilon^\alpha + \varepsilon\big)\int_0^{|\nabla u_\varepsilon(x,0)|^2+\varepsilon} s^{\frac{p(x)-2}{2}}\,ds\,dx. \qquad (2.8)
\end{aligned}$$

Since (1.14), by Young inequality, we have

$$\begin{aligned}
&\iint_{Q_t} u_{\varepsilon t}\frac{\partial b_i(u_\varepsilon, x, t)}{\partial x_i}\,dx\,dt \le \iint_{Q_T} \big|b_{iu_\varepsilon}(u_\varepsilon, x, t)\big||u_{\varepsilon x_i}||u_{\varepsilon t}|\,dx\,dt \\
&\le \varepsilon\iint_{Q_t} |u_{\varepsilon t}|^{q(x)}\,dx\,dt + c(\varepsilon)\iint_{Q_t} d^\alpha|\nabla u_\varepsilon|^{p(x)}\,dx\,dt \\
&\le \varepsilon\iint_{Q_t} (u_{\varepsilon t})^2\,dx\,dt + c(\varepsilon)\iint_{Q_t} d^\alpha|\nabla u_\varepsilon|^{p(x)}\,dx\,dt + c. \qquad (2.9)
\end{aligned}$$

Here, we have used the fact that $p(x) \ge 2$, then $q(x) = \frac{p(x)-1}{p(x)} \le 2$, and by the Young inequality,

$$\iint_{Q_t} |u_{\varepsilon t}|^{q(x)}\,dx\,dt \le \varepsilon\iint_{Q_T} |u_{\varepsilon t}|^2\,dx\,dt + c. \qquad (2.10)$$

Combining (2.4)-(2.10), we have

$$\iint_{Q_t} (u_{\varepsilon t})^2\,dx\,dt + \iint_{Q_t} d_\varepsilon^\alpha\frac{d}{dt}\int_0^{|\nabla u_\varepsilon(x,t)|^2} s^{\frac{p-2}{2}}\,ds\,dx\,dt \le c,$$

by the inequality, we have

$$\iint_{Q_t} (u_{\varepsilon t})^2\,dx\,dt \le c + c\int_\Omega (d_\varepsilon + \varepsilon)^\alpha|\nabla u_{\varepsilon,0}|^{p(x)}\,dx \le c. \qquad (2.11)$$

Thus

$$\int_0^T\int_\Omega |u_{\varepsilon t}|^2\,dx\,dt \le c, \qquad (2.12)$$

and there exist a function u and a n-dimensional vector $\overrightarrow{\zeta} = (\zeta_1, \ldots, \zeta_n)$ satisfying with $\overrightarrow{\zeta} = (\zeta_1, \ldots, \zeta_n)$

$$u \in L^\infty(Q_T), \qquad \frac{\partial u}{\partial t} \in L^2(Q_T), \qquad |\overrightarrow{\zeta}| \in L^\infty\big(0, T; L^{\frac{p(x)}{p(x)-1}}(\Omega)\big), \qquad (2.13)$$

and $u_\varepsilon \to u$ a.e. $\in Q_T$,

$$\begin{aligned}
&u_\varepsilon \rightharpoonup u, \quad \text{weakly star in } L^\infty(Q_T),\\
&u_\varepsilon \to u, \quad \text{in } L^2\bigl(0,T;L^2_{\mathrm{loc}}(\Omega)\bigr),\\
&\frac{\partial u_\varepsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^2(Q_T),\\
&d^\alpha |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \rightharpoonup \overrightarrow{\zeta} \quad \text{weakly star in } L^\infty\bigl(0,T;L^{\frac{p(x)}{p(x)-1}}(\Omega)\bigr).
\end{aligned}$$

In order to prove u is the solution of equation (1.4), we notice that for any function $\varphi \in C_0^1(Q_T)$,

$$\iint_{Q_T} \biggl[u_{\varepsilon t}\varphi + \bigl(d_\varepsilon^\alpha + \varepsilon\bigr)\bigl(|\nabla u_\varepsilon|^2 + \varepsilon\bigr)^{p(x)-2}\nabla u_\varepsilon \cdot \nabla\varphi + \frac{\partial b_i(u,x,t)}{\partial x_i}\varphi_{x_i}\biggr]\,dx\,dt = 0. \tag{2.14}$$

Since $d(x) > 0$ when $x \in \Omega$, then $c > \sup_{\operatorname{supp}\varphi} \frac{|\nabla\varphi|}{d^\alpha} > 0$ due to $\varphi \in C_0^1(Q_T)$, we have

$$\begin{aligned}
&\varepsilon \iint_{Q_T} \bigl|\bigl(|\nabla u_\varepsilon|^2 + \varepsilon\bigr)^{p(x)-2}\nabla u_\varepsilon \cdot \nabla\varphi\bigr|\,dx\,dt\\
&\quad \le \varepsilon \sup_{\operatorname{supp}\varphi} \frac{|\nabla\varphi|}{d^\alpha} \iint_{Q_T} d^\alpha\bigl(|\nabla u_\varepsilon|^{p(x)} + c\bigr)\,dx\,dt \to 0,
\end{aligned} \tag{2.15}$$

as $\varepsilon \to 0$. Similar to the general evolutionary p-Laplacian equation ([15]), by (2.14)-(2.15), we are able to prove that

$$\iint_{Q_T} \bigl[u_t\varphi + \vec{\zeta} \cdot \nabla\varphi + b_i(u,x,t)\varphi_{x_i}\bigr]\,dx\,dt = 0 \tag{2.16}$$

and

$$\iint_{Q_T} d^\alpha |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\,dx\,dt = \iint_{Q_T} \overrightarrow{\zeta} \cdot \nabla\varphi\,dx\,dt, \tag{2.17}$$

for any function $\varphi \in C_0^1(Q_T)$. Then

$$\iint_{Q_T} \bigl[u_t\varphi + d^\alpha|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi + b_i(u,x,t)\varphi_{x_i}\bigr]\,dx\,dt = 0. \tag{2.18}$$

If we denote $\Omega_\varphi = \operatorname{supp}\varphi$, then

$$\int_0^T \int_{\Omega_\varphi} \bigl[u_t\varphi + d^\alpha|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi + b_i(u,x,t)\varphi_{x_i}\bigr]\,dx\,dt = 0. \tag{2.19}$$

Now, for any $\varphi_2 \in C_0^1(Q_T)$, $\varphi_1(x,t) \in W_\alpha^{1,p(x)}$ for any given t, and $|\varphi_1(x,t)| \le c$ for any given x, it is clearly that $\varphi_1 \in W^{1,p^-}(\Omega_{\varphi_2})$. By the fact of that $C_0^\infty(\Omega_{\varphi_2})$ is dense in $W^{1,p^-}(\Omega_{\varphi_2})$, by a limit process, we have

$$\int_0^T \int_{\Omega_{\varphi_2}} \bigl[u_t(\varphi_1\varphi_2) + d^\alpha|\nabla u|^{p(x)-2}\nabla u \cdot \nabla(\varphi_1\varphi_2) + b_i(u,x,t)(\varphi_1\varphi_2)_{x_i}\bigr]\,dx\,dt = 0, \tag{2.20}$$

which implies that

$$\int_0^T \int_\Omega \left[u_t(\varphi_1\varphi_2) + d^\alpha |\nabla u|^{p(x)-2}\nabla u \cdot \nabla(\varphi_1\varphi_2) + b_i(u,x,t)(\varphi_1\varphi_2)_{x_i}\right] dx\,dt = 0. \tag{2.21}$$

At the same time, we can prove (1.5) as in [17], we also omit the details here. Then u satisfies equation (1.4) in the sense of Definition 1.2. Theorem 1.3 is proved. □

Corollary 2.1 *If $b_i \equiv 0$, then the condition $p(x) \geq 2$ in Theorem* 1.3 *can be weakened to $p(x) > 1$.*

Proof We notice that the condition $p(x) \geq 2$ is used only in the proof of (2.9), thus, if $b_i \equiv 0$, only if $p(x) > 1$, there exists a weak solution of equation (1.4) with the initial value (1.2), provided that u_0 satisfies (1.15). □

3 The global stability

Proof of Theorems 1.5 Let u and v be two weak solutions of equation (1.1) with the initial values $u(x,0)$, $v(x,0)$, respectively. For any given positive integer n, let $g_n(s)$ be an odd function, and

$$g_n(s) = \begin{cases} 1, & s > \frac{1}{n}, \\ n^2 s^2 \mathrm{e}^{1-n^2 s^2}, & 0 \leq s \leq \frac{1}{n}. \end{cases}$$

Clearly,

$$\lim_{n\to 0} g_n(s) = \operatorname{sgn}(s), \quad s \in (-\infty, +\infty), \left|g_n'(s)s\right| \leq c. \tag{3.1}$$

Denoting $\Omega_\lambda = \{x \in \Omega : d^\alpha(x) > \lambda\}$, let

$$\phi_n(x) = \begin{cases} 1, & \text{if } x \in \Omega_{\frac{1}{n}}, \\ nd^\alpha(x), & x \in \Omega \setminus \Omega_{\frac{1}{n}}. \end{cases}$$

By a limit process, we can choose $\phi_n g_n(u-v)$ as the test function, then

$$\begin{aligned} &\int_\Omega \phi_n(x) g_n(u-v)\frac{\partial(u-v)}{\partial t}\,dx \\ &\quad + \int_\Omega d^\alpha \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right) \cdot \nabla(u-v) g_n'(u-v)\phi_n(x)\,dx \\ &\quad + \int_\Omega d^\alpha \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right) \cdot (u-v) g_n(u-v)\nabla\phi_n\,dx \\ &\quad + \int_\Omega \left[b_i(u,x,t) - b_i(v,x,t)\right]\phi_{nx_i} g_n(u-v)\,dx \\ &\quad + \int_\Omega \left[b_i(u,x,t) - b_i(v,x,t)\right](u-v)_{x_i}\phi_n g_n'(u-v)\,dx. \end{aligned} \tag{3.2}$$

In the first place,

$$\int_\Omega d^\alpha \big(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\big)\cdot\nabla(u-v)g'_n(u-v)\phi_n(x)\,dx \geq 0, \tag{3.3}$$

and from the proof of that $\frac{\partial u}{\partial t}\in L^2(Q_T)$, we deduce that, for any given $t\in(0,T)$, $\frac{\partial u}{\partial t}\in L^2(\Omega)$,

$$\int_\Omega \left|\frac{\partial u}{\partial t}\right| dx \leq c\left(\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx\right)^{\frac{1}{2}} \leq c,$$

by the Lebesgue dominated theorem,

$$\lim_{n\to\infty}\int_\Omega \phi_n(x)g_n(u-v)\frac{\partial(u-v)}{\partial t}\,dx = \frac{d}{dt}\|u-v\|_{L^1(\Omega)}. \tag{3.4}$$

Since $\nabla\phi_n = n\nabla d^\alpha$ when $x\in\Omega\setminus\Omega_{\frac{1}{n}}$, in the other places, it is identical to zero, and we have

$$\begin{aligned}
&\left|\int_\Omega d^\alpha\big(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\big)\cdot\nabla\phi_n g_n(u-v)\,dx\right| \\
&\quad= \left|\int_{\Omega\setminus\Omega_{\frac{1}{n}}} d^\alpha\big(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\big)\cdot\nabla\phi_n g_n(u-v)\,dx\right| \\
&\quad\leq n\int_{\Omega\setminus\Omega_{\frac{1}{n}}} d^\alpha|\nabla u|^{p(x)-1} + |\nabla v|^{p(x)-1}\big|\nabla d^\alpha g_n(u-v)\big|\,dx \\
&\quad\leq cn\big\|d^{\frac{\alpha}{q(x)}}\big(|\nabla u|^{p(x)-1}+|\nabla v|^{p(x)-1}\big)\big\|_{L^{q(x)}(\Omega\setminus\Omega_{\frac{1}{n}})}\big\|d^{\frac{\alpha}{p(x)}}\nabla d^\alpha g_n(u-v)\big\|_{L^{p(x)}(\Omega\setminus\Omega_{\frac{1}{n}})},
\end{aligned} \tag{3.5}$$

where $q(x) = \frac{p(x)}{p(x)-1}$ as before.

Since $|\nabla d| = 1$, by the assumption (1.17)

$$\begin{aligned}
&n\big\|d^{\frac{\alpha}{p(x)}}\nabla d^\alpha g_n(u-v)\big\|_{L^{p(x)}(\Omega\setminus\Omega_{\frac{1}{n}})} \\
&\quad= \alpha n\big\|d^{\alpha-1+\frac{\alpha}{p(x)}}\nabla d g_n(u-v)\big\|_{L^{p(x)}(\Omega\setminus\Omega_{\frac{1}{n}})} \\
&\quad\leq \alpha n\big\|d^{\alpha-1+\frac{\alpha}{p(x)}}\big\|_{L^{p(x)}(\Omega\setminus\Omega_{\frac{1}{n}})} \leq c.
\end{aligned} \tag{3.6}$$

Then by (3.5)-(3.6), we have

$$\begin{aligned}
&\left|\int_\Omega d^\alpha\big(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\big)\cdot\nabla\phi_n g_n(u-v)\,dx\right| \\
&\quad\leq c\left[\left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} d^\alpha|\nabla u|^{p(x)}\right)^{\frac{1}{q^+}} + \left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} d^\alpha|\nabla v|^{p(x)}\right)^{\frac{1}{q^+}}\right],
\end{aligned} \tag{3.7}$$

which goes to 0 as $n\to 0$.

In the second place, since $b_i(s,x,t)$ satisfies the condition (1.16)

$$\left|b_i(u,x,t)-b_i(v,x,t)\right| \leq c d^{\frac{\alpha}{p(x)}} |u-v|^{1+\frac{1}{q(x)}},$$

we have

$$\begin{aligned}
&\left|\int_\Omega \left[b_i(u,x,t)-b_i(v,x,t)\right](u-v)_{x_i}\phi_n g_n'(u-v)\,dx\right| \\
&\quad \leq c\int_\Omega \left|g_n'(u-v)(u-v)\right| d^{\frac{\alpha}{p(x)}}(u-v)_{x_i}\phi_n\,dx \\
&\quad \leq c\left\| d^{\frac{\alpha}{p(x)}}\left((u-v)_{x_i}\right)\right\|_{L^{p(x)}(\Omega)}\left\|(u-v)^{\frac{1}{q(x)}}\right\|_{L^{q(x)}(\omega)} \\
&\quad \leq c\left\|(u-v)^{\frac{1}{q(x)}}\right\|_{L^{q(x)}(\omega)} \\
&\quad \leq c\left(\int_\Omega |u-v|\,dx\right)^{\frac{1}{q_1}},
\end{aligned} \tag{3.8}$$

where $q_1 = q^+$ or q^- according to $\int_\Omega |u-v|\,dx \geq 1$ or < 1.

Last but not least, by using some techniques from [11], we can prove that

$$\lim_{n\to\infty}\int_\Omega \left[b_i(u,x,t)-b_i(v,x,t)\right](u-v)_{x_i}\phi_n g_n'(u-v)\,dx = 0. \tag{3.9}$$

Now, let $n\to\infty$ in (3.2). Then

$$\frac{d}{dt}\|u-v\|_{L^1(\Omega)} \leq \left(\int_\Omega |u-v|\,dx\right)^{\frac{1}{q_1}}.$$

It implies that

$$\begin{aligned}
&\int_\Omega \left|u(x,t)-v(x,t)\right| dx \\
&\quad \leq \int_\Omega |u_0-v_0|\,dx + c\left(\int_0^t\int_\Omega |u-v|\,dx\,dt\right)^{\frac{1}{q_1}}, \quad \forall t\in[0,T).
\end{aligned} \tag{3.10}$$

By (3.10), we easily get

$$\int_\Omega \left|u(x,t)-v(x,t)\right| dx \leq \int_\Omega |u_0-v_0|\,dx. \qquad \square$$

Proof of Theorem 1.6 Just as the proof of Theorem 1.5, we have (3.1)-(3.5). By the assumption (1.19),

$$\begin{aligned}
&n\left\| d^{\frac{\alpha}{q(x)}}\left(|\nabla u|^{p(x)-1}+|\nabla v|^{p(x)-1}\right)\right\|_{L^{q(x)}(\Omega\setminus\Omega_{\frac{1}{n}})} \\
&\quad \leq n\left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} d^\alpha |\nabla u|^{p(x)}\,dx\right)^{\frac{1}{q^+}} + n\left(\int_{\Omega\setminus\Omega_{\frac{1}{n}}} d^\alpha |\nabla v|^{p(x)}\,dx\right)^{\frac{1}{q^+}} \leq c,
\end{aligned} \tag{3.11}$$

from (3.5), we have

$$\begin{aligned}&\left|\int_\Omega d^\alpha\left(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\right)\cdot\nabla\phi_n g_n(u-v)\,dx\right|\\&\quad\le cn\left\|d^{\frac{\alpha}{q(x)}}\left(|\nabla u|^{p(x)-1}+|\nabla v|^{p(x)-1}\right)\right\|_{L^{q(x)}(\Omega\setminus\Omega_{\frac{1}{n}})}\left\|d^{\frac{\alpha}{p(x)}}\nabla d^\alpha g_n(u-v)\right\|_{L^{p(x)}(\Omega\setminus\Omega_{\frac{1}{n}})}\\&\quad\le c\left\|d^{\alpha-1+\frac{\alpha}{p(x)}}\right\|_{L^{p(x)}(\Omega\setminus\Omega_{\frac{1}{n}})},\end{aligned}$$

which goes to 0 as $n\to 0$.

Now, letting $n\to\infty$ in (3.2), we have the conclusion. □

4 The local stability of the solutions

In what follows, we will give a local stability of the solutions.

Theorem 4.1 *Let $p^->1$, $b_i(s)$ be a Lipschitz function. If u, v are two solutions of equation* (1.4) *with the initial values $u_0(x), v_0(x)$, respectively. Then for any $\overline{\Omega_1}\subset\Omega$,*

$$\int_{\Omega_1}\left|u(x,s)-v(x,s)\right|^2dx\le c(\Omega_1)\int_\Omega|u_0-v_0|^2\,dx \tag{4.1}$$

is true.

If $b_i\equiv 0$, $u_t\in L^2(Q_T)$ and $v_t\in L^2(Q_T)$ are as in the preconditions, the same conclusion had appeared in our previous work [10]. Thanks to the existence of the weak solution, Theorem 1.4, $u_t\in L^2(Q_T)$ and $v_t\in L^2(Q_T)$ are naturally true in our paper. Moreover, $p(x)$ is required to satisfy the logarithmic Hölder continuity condition in [10], but it does not appear in Theorem 4.1. However, the method used in what follows is similar to that in [10], we only need to deal with the convection term carefully.

Proof For any fixed $\tau,s\in[0,T]$, $\chi_{[\tau,s]}$ is the characteristic function on $[\tau,s]$, by a limit process, we may choose $\varphi_1=d^\beta$, $\varphi_2=\chi_{[\tau,s]}(u-v)$, $\varphi=\varphi_1\varphi_2$ as a test function. We choose β is large enough, and let $Q_{\tau s}=\Omega\times[\tau,s]$. Then

$$\begin{aligned}&\iint_{Q_{\tau s}}(u-v)d^\beta\frac{\partial(u-v)}{\partial t}\,dx\,dt\\&\quad=-\iint_{Q_{\tau s}}d^\alpha\left(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\right)\nabla\left[(u-v)d^\beta\right]dx\,dt\\&\qquad-\iint_{Q_{\tau s}}\left[b_i(u,x,t)-b_i(v,x,t)\right]\left[(u-v)d^\beta\right]_{x_i}dx\,dt.\end{aligned}\tag{4.2}$$

We only need to deal with the last term of (4.2)

$$\begin{aligned}&\iint_{Q_{\tau s}}\left[b_i(u,x,t)-b_i(v,x,t)\right]\left[(u-v)d^\beta\right]_{x_i}dx\,dt\\&\quad=\iint_{Q_{\tau s}}\left[b_i(u,x,t)-b_i(v,x,t)\right](u-v)d^\beta_{x_i}\,dx\,dt\\&\qquad+\iint_{Q_s}\left[b_i(u,x,t)-b_i(v,x,t)\right](u-v)_{x_i}d^\beta\,dx\,dt,\end{aligned}\tag{4.3}$$

since $b_i(s,x,t)$ is a Lipschitz function, $u,v\in L^\infty(Q_T)$, when $\beta\geq 2$ we have

$$\begin{aligned}
&\iint_{Q_{\tau s}}\bigl[b_i(u,x,t)-b_i(v,x,t)\bigr](u-v)d^{\beta}_{x_i}\,dx\,dt\\
&\quad=\int_\tau^s\int_\Omega\bigl[b_i(u,x,t)-b_i(v,x,t)\bigr](u-v)\beta d^{\beta-1}|d_{x_i}|\,dx\\
&\quad\leq c\int_\tau^s\int_\Omega|u-v|^2d^{\beta-1}\,dx\\
&\quad\leq c\biggl(\int_\tau^s\int_\Omega\bigl[d^{\frac{\beta}{2}}|u-v|\bigr]^2\,dx\biggr)^{\frac12}\biggl(\int_\tau^s\int_\Omega\bigl[d^{\frac{\beta}{2}-1}|u-v|\bigr]^2\,dx\biggr)^{\frac12}\\
&\quad\leq c\biggl(\int_\tau^s\int_\Omega d^{\beta}|u-v|^2\,dx\biggr)^{\frac12}
\end{aligned}\tag{4.4}$$

and

$$\begin{aligned}
&\iint_{Q_{\tau s}}\bigl[b_i(u,x,t)-b_i(v,x,t)\bigr](u-v)_{x_i}d^{\beta}\,dx\,dt\\
&\quad\leq c\int_\tau^s\bigl\|d^{(\beta-\frac{\alpha}{p(x)})q(x)}\bigl(\bigl|b_i(u,x,t)-b_i(v,x,t)\bigr|\bigr)\bigr\|_{L^{q(x)}(\Omega)}\bigl\|d^{\frac{\alpha}{p(x)}}(u-v)_{x_i}\bigr\|_{L^{p(x)}(\Omega)}\,dt\\
&\quad\leq\int_\tau^s\biggl(\int_\Omega d^{(\beta-\frac{\alpha}{p(x)})q(x)}\bigl(\bigl|b_i(u,x,t)-b_i(v,x,t)\bigr|\bigr)^{q(x)}\,dx\,dt\biggr)^{\frac{1}{q_1}}\\
&\qquad\times\biggl(\int_\Omega d^{\alpha}\bigl(|\nabla u|^{p(x)}+|\nabla v|^{p(x)}\bigr)\,dx\biggr)^{\frac{1}{p_1}}\,dt\\
&\quad\leq c\int_\tau^s\biggl(\int_\Omega d^{(\beta-\frac{\alpha}{p(x)})q(x)}\bigl(\bigl|b_i(u,x,t)-b_i(v,x,t)\bigr|\bigr)^{q(x)}\,dx\,dt\biggr)^{\frac{1}{q_1}}\,dt\\
&\quad\leq c\biggl(\int_\tau^s\int_\Omega d^{(\beta-\frac{\alpha}{p(x)})q(x)}|u-v|^{q(x)}\,dx\,dt\biggr)^{\frac{1}{q_1}}\\
&\quad\leq c\biggl(\int_\tau^s\int_\Omega d^{(\beta-\frac{\alpha}{p(x)})q(x)}|u-v|^{q(x)-1}|u-v|\,dx\,dt\biggr)^{\frac{1}{q_1}}\\
&\quad\leq c\biggl(\int_\tau^s\int_\Omega d^{k}|u-v|\,dx\,dt\biggr)^{\frac{1}{q_1}}\leq c\biggl(\int_\tau^s\int_\Omega d^{\beta}|u-v|^2\,dx\,dt\biggr)^{\frac{1}{2q_1}}.
\end{aligned}\tag{4.5}$$

Here

$$k=\biggl(\beta-\frac{\alpha}{p(x)}+1\biggr)q(x)-1,$$

we have chosen that β is large enough such that $2k\geq\beta$.

By the fact

$$\iint_{Q_{\tau s}}d^{\alpha+\beta}\bigl(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\bigr)\nabla(u-v)\,dx\,dt\geq 0,\tag{4.6}$$

similar to the proof of [10], then we have

$$\int_{\Omega} d^{\beta}\big[u(x,s)-v(x,s)\big]^2\,dx - \int_{\Omega} d^{\beta}\big[u(x,\tau)-v(x,\tau)\big]^2\,dx \\ \leq c\biggl(\int_0^s \int_{\Omega} d^{\beta}\big|u(x,t)-v(x,t)\big|^2\,dx\,dt\biggr)^l, \tag{4.7}$$

where $l < 1$. By (4.7), we easily prove that

$$\int_{\Omega} d^{\beta}\big|u(x,s)-v(x,s)\big|^2\,dx \leq \int_{\Omega} d^{\beta}|u_0-v_0|^2\,dx, \tag{4.8}$$

which implies that for any $\overline{\Omega_1} \subset \Omega$, (4.1) is true. The proof is complete. □

At the end of the paper, we would like to suggest that if $\alpha < p^- - 1$, then the weak solution u of equation (1.4) with the initial value (1.2) belongs to $L^{\infty}(0,T;W^{1,\gamma}(\Omega))$, and so we can impose the usual Dirichlet homogeneous boundary value condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{4.9}$$

However, the stability theorems in our paper show that the condition (4.9) can be replaced by the degeneracy of $a(x)$ and $b_i(s,x,t)$ on the boundary.

5 Conclusion

The equations considered in this paper come from many applied fields. The main character of the equation is its degeneracy on the boundary, since the weak solutions generally lack the regularity to define the trace on the boundary. Thus, if one wants to prove the uniqueness or the stability of the weak solutions, the boundary value condition cannot be used in the usual way. In other words, since the equation is nonlinear, how to quote a suitable boundary value condition matching the equation seems very difficult. The most significant result of this paper lies in that we have found that, if we combine the degeneracy of the diffusion coefficient with the degeneracy of the convection term, by introducing a new kind of the weak solutions, we may prove the stability of the weak solutions without any boundary value condition.

Acknowledgements
The paper is supported by NSF of Fujian Province (Grant No. 2015J1092), supported by SF of Xiamen University of Technology (Grant No. XYK201448), China.

Competing interests
The author declares that he has no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Publisher's Note

Received: 21 April 2017 Accepted: 1 August 2017 Published online: 18 August 2017

References

1. Ruzicka, M: Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Math., vol. 1748. Springer, Berlin (2000)
2. Acerbi, E: Mingione G regularity results for stationary electrorheological fluids. Arch. Ration. Mech. Anal. **164**, 213-259 (2002)
3. Acerbi, E, Mingione, G, Seregin, GA: Regularity results for parabolic systems related to a class of non-Newtonian fluids. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **21**, 25-60 (2004)
4. Antontsev, S, Shmarev, S: Anisotropic parabolic equations with variable nonlinearity. Publ. Mat. **53**, 355-399 (2009)
5. Antontsev, S, Shmarev, S: Parabolic equations with double variable nonlinearities. Math. Comput. Simul. **81**, 2018-2032 (2011)
6. Lian, S, Gao, W, Yuan, H, Cao, C: Existence of solutions to an initial Dirichlet problem of evolutional $p(x)$-Laplace equations. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **29**, 377-399 (2012)
7. Yin, J, Wang, C: Properties of the boundary flux of a singular diffusion process. Chin. Ann. Math., Ser. B **25**(2), 175-182 (2004)
8. Zhan, H, Xie, Q: The boundary degeneracy of a singular diffusion equation. J. Inequal. Appl. **2014**, 284 (2014). doi:10.1186/1029-242x-2014-284
9. Zhan, H, Wen, J: Evolutionary $p(x)$-Laplacian equation free from the limitation of the boundary value. Electron. J. Differ. Equ. **2016**, 143 (2016)
10. Zhan, H, Wen, J: Well-posedness of weak solutions to electrorheological fluid equations with degeneracy on the boundary. Electron. J. Differ. Equ. **2017**, 13 (2017)
11. Zhan, H: The boundary value condition of an evolutionary $p(x)$-Laplacian equation. Bound. Value Probl. **2015**, 112 (2015). doi:10.1186/s13661-015-0377-6
12. Zhikov, VV: On the density of smooth functions in Sobolev-Orlicz spaces. Otdel. Mat. Inst. Steklov. (POMI) **310**, 67-81 (2004), translation in J. Math. Sci. (N.S.) **132**, 285-294 (2006)
13. Fan, XL, Zhao, D: On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}$. J. Math. Anal. Appl. **263**, 424-446 (2001)
14. Kovácik, O, Rákosník, J: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslov. Math. J. **41**, 592-618 (1991)
15. Taylor, ME: Partial Differential Equations III. Springer, Berlin (1999)
16. Wu, Z, Zhao, J, Yin, J, Li, H: Nonlinear Diffusion Equations. World Scientific, Singapore (2001)
17. Zhao, J: Existence and nonexistence of solutions for $u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(\nabla u, u, x, t)$. J. Math. Anal. Appl. **172**(1), 130-146 (1993)