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Global well-posedness of the non-isothermal model for incompressible nematic liquid crystals

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Abstract

In this paper, a macromolecular non-isothermal model for the incompressible hydrodynamics flow of nematic liquid crystals on $\mathbb{T}^3$ is considered. By a Galerkin approximation, we prove the local existence of a unique strong solution if the initial data u_0, d_0 and θ_0 satisfy some natural conditions and provided that the viscosity coefficients μ and the heat conductivity κ, h, which are temperature dependent, are properly differentiable and bounded. Moreover, with small initial data, we can extend the local strong solution to be a global one by the argument of contradiction. In this case, the exponential time decay rate is also established.

MSC: 76N10; 35Q30; 35R35

Keywords: incompressible flow; nematic liquid crystals; non-isothermal; a priori estimates; local and global existence and uniqueness

1 Introduction

Liquid crystals, which exist in an intermediate state between isotropic liquid and solid, are materials with rheological properties. There are three phases of liquid crystals (nematic, cholesteric and sematic), and in the nematic phase, the long axis of the constituent molecules tend to align parallel to each other along some commonly preferred direction. The continuum theory for the dynamics of liquid crystal flow was developed by Ericksen [1] and Leslie [2] in the 1960s, and it contains three physical conservation laws: the conservation of *mass*, the conservation of *momentum* and the conservation of *angular momentum*. In the Ericksen-Leslie system, those three conservation laws govern the motions of the fluid by the density ρ, the velocity of the flow u and the mean orientation field d, respectively. For more details, we refer the reader to De Gennes [3]. However, the Ericksen-Leslie system is so complicated that there was no further well-posedness study result until 1989 when Lin [4] derived a simplified version of the Ericksen-Leslie system. Since then, there have been considerable research results on this simplified model.

In 1995, Lin and Liu [5] established the global existence of the weak solutions to the simplified system with a Ginzburg-Landau approximation $f(d) = \frac{1-|d|^2}{\varepsilon^2} d$ instead of $|\nabla d|^2 d$

to relax the constraint $|d| = 1$, that is,

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p - \mu \Delta u = -\lambda \nabla \cdot (\nabla d \odot \nabla d), \\ \nabla \cdot u = 0, \\ d_t + u \cdot \nabla d = \gamma (\Delta d + f(d)), \end{cases} \tag{1.1}$$

which is a semilinear system called the penalized model. When the viscosity coefficient is large enough, the global existence of the unique classical solution was proved in [5]. Subsequently, they obtained the partial regularity of suitable weak solutions to (1.1) (*cf.* [6]) and established the same well-posedness theory as in [5] to the full Ericksen-Leslie system with Ginzburg-Landau approximation [7]. When the density is taken into account, the global existence of weak solutions was established in [8] for the incompressible penalized model and in [9, 10] for the compressible case.

On the other hand, there has been much work concerning the non-penalized model of (1.1) where $f(d)$ in (1.1) is replaced by $|\nabla d|^2 d$ so that the constraint $|d| = 1$ is satisfied. The latter one is much more difficult to tackle since now one has a quasilinear system with a quadratic growth term $|\nabla d|^2 d$ and a non-convex constraint $|d| = 1$. A complete discussion of the convergence $\varepsilon \to 0$ in (1.1) is unsolved. We refer the reader to [11–15] for the existence of global weak solutions in dimension two and to [16–18] for the uniqueness. Nevertheless, in general cases, the regularity and uniqueness of weak solution is still an open problem for the three-dimensional case; see [19, 20]. In addition, the reader can also refer to [21–26] for the global existence of unique strong solution to the incompressible model and [27, 28] for the compressible one. The review of the literature is not intended to be exhaustive but as a rough description. A more complete bibliography on the development of the mathematical theories of liquid crystals has been given by Lin and Wang in [29].

As was shown by Leslie [30] that the mechanical balance equations must be supplemented by an equation to describe the absolute temperature θ, the physical conservation law *the conservation of energy* should also be included. In [31], the authors proposed a non-isothermal penalized model for the nematic liquid crystals with temperature-dependent viscosity, elasticity and thermal conductivities,

$$\begin{cases} \nabla \cdot u = 0, \\ u_t + u \cdot \nabla u + \nabla p = \nabla \cdot (\mu(\theta)(\nabla u + \nabla^T u) - \lambda(\theta) \nabla d \odot \nabla d), \\ d_t + u \cdot \nabla d = \gamma (\Delta d + f(d)), \\ \theta_t + u \cdot \nabla \theta - \nabla \cdot q = \mu(\theta)(\nabla u + \nabla^T u) : \nabla u - \lambda(\theta) \nabla d \odot \nabla d : \nabla u, \end{cases} \tag{1.2}$$

where $q = \kappa(\theta)\nabla\theta + h(\theta)(d \cdot \nabla\theta)d$, and established the existence of the global weak solutions to this model, see [32] for local existence of the unique strong solution. Later, in [33], the authors proposed another non-isothermal model to take the effects of stretching and rotation for large molecules into account; *cf.* [5]. [33] and [31]; also one dealt with the penalized models.

In this paper, a non-isothermal, large molecule model proposed by Li and Xin in [34] is considered, in which the constraint $|d| = 1$ is maintained. Precisely, we study the system

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = \nabla \cdot (\mathbb{S} + \sigma^{nd}), \\ \nabla \cdot u = 0, \\ d_t + u \cdot \nabla d - d \cdot \nabla u + (d \cdot \nabla u \cdot d)d = \gamma(\Delta d + |\nabla d|^2 d), \\ \theta_t + \nabla \cdot (u\theta) + \nabla \cdot q = \mathbb{S} : \nabla u + \lambda\gamma|\Delta d + |\nabla d|^2 d|^2, \end{cases} \tag{1.3}$$

where

$$\mathbb{S} = \mu(\theta)\big(\nabla u + \nabla^T u\big) + \frac{\lambda}{\gamma}(d \cdot \nabla u \cdot d)d \otimes d, \qquad q = -\kappa(\theta)\nabla\theta - h(\theta)(d \cdot \nabla\theta)d,$$
$$\sigma^{nd} = -\lambda\nabla d \odot \nabla d - \lambda\big(\Delta d + |\nabla d|^2 d\big) \otimes d.$$

We impose the initial conditions

$$\begin{aligned} &(u,\theta,d)|_{t=0} = (u_0,\theta_0,d_0), \\ &\nabla \cdot u_0 = 0, \qquad \operatorname*{ess\,inf}_{\Omega} \theta_0 = \underline{\theta}_0 > 0, \qquad |d_0| = 1, \text{ a.e.} \qquad \Omega = [-D,D]^3 \subset \mathbb{R}^3; \end{aligned} \tag{1.4}$$

and the periodic boundary conditions

$$\begin{aligned} &u(x + De_i, t) = u(x - De_i, t), \qquad \theta(x + De_i, t) = \theta(x - De_i, t), \\ &d(x + De_i, t) = d(x - De_i, t), \quad \forall x \in \partial\Omega, i = 1,2,3. \end{aligned} \tag{1.5}$$

Moreover, the temperature-dependent coefficients $\mu(\cdot),\kappa(\cdot),h(\cdot) \in C^2([0,\infty))$ are supposed to satisfy

$$\begin{aligned} &\underline{\mu} \le \mu(\theta) \le \bar{\mu}, \qquad \big|\mu'(\theta)\big| \le \bar{\mu'}, \qquad \big|\mu''(\theta)\big| \le \bar{\mu''} \quad \forall\theta \ge 0, \\ &\underline{\kappa} \le \kappa(\theta) \le \bar{\kappa}, \qquad \big|\kappa'(\theta)\big| \le \bar{\kappa'}, \qquad \big|\kappa''(\theta)\big| \le \bar{\kappa''} \quad \forall\theta \ge 0, \\ &\underline{h} \le h(\theta) \le \bar{h}, \qquad \big|h'(\theta)\big| \le \bar{h'}, \qquad \big|h''(\theta)\big| \le \bar{h''} \quad \forall\theta \ge 0, \end{aligned} \tag{1.6}$$

where $\underline{\kappa},\bar{\kappa},\bar{\kappa'},\bar{\kappa''},\underline{h},\bar{h},\bar{h'},\bar{h''},\underline{\mu},\bar{\mu},\bar{\mu'},\bar{\mu''}$ are given positive constants.

The existence of global weak solutions for (1.3)-(1.5) in dimension two has been established in [34]. In this paper, we aim to prove the local and global existence of the unique strong solution.

Before going on, we would like to have some more words on this system. When the direction field d is a constant vector field, the system (1.3) reduces to a non-isentropic incompressible Navier-Stokes system with temperature-dependent viscosity coefficients, to which the global existence of weak solutions was showed in [35]. Moreover, Cho and Kim in [36] studied a more general non-isentropic incompressible Navier-Stokes system, where the transport coefficients depend on both temperature and mass density. The authors established the local existence and uniqueness of strong solution; see also [37]. For the compressible cases, there are many articles on the global existence of unique classical solution with large initial data for one-dimensional problems; see [38–41] for examples.

The rest of this paper is arranged as follows. In Section 2, we introduce some notations and give the main theorems. The approximate solutions to this problem will be constructed by means of Galerkin method in Section 3. After establishing some suitable *a priori* estimates in Section 4, we will prove the local existence, global existence, and uniqueness in Section 5, 6, 7, respectively.

2 Notations and theorems

We introduce some notations.

- $W = \{u \in H^1(\Omega) \mid \nabla \cdot u = 0, u$ satisfies periodic boundary conditions (1.5)$\}$.
- $\int f := \int_\Omega f \, \mathrm{d}x, L^p := L^p(\Omega), H^k := H^k(\Omega) = W^{k,2}(\Omega), p \geq 2, k \geq 1,$
- $\|\cdot\|_q := \|\cdot\|_{L^q(\Omega)}, \|\cdot\|_{H^k} := \|\cdot\|_{H^k(\Omega)}, q \geq 2, k \geq 1.$
- $A \lesssim B$ denotes $A \leq CB$, where $C > 0$ is an absolute positive constant.
- For 3×3 matrices $A = (a_{ij})_{1 \leq i,j \leq 3}$ and $B = (b_{ij})_{1 \leq i,j \leq 3}$, we denote

$$A : B = \sum_{i,j=1}^{3} a_{ij} b_{ij}.$$

Moreover, the usual summation convention is used.

The main results of this paper are the following theorems.

Theorem 2.1 *Assume that* $(u_0, \theta_0, d_0) \in H^2 \cap W \times H^2 \times H^3$. *Then there exists* $T^* > 0$, *such that, for any positive time* $T \in (0, T^*)$, *there exists a unique solution* (u, θ, d, p) *to the system* (1.3)-(1.5) *on* $[0, T]$ *satisfying*

$$\begin{aligned} & u \in L^\infty(0, T; H^2 \cap W) \cap L^2(0, T; H^3), \qquad u_t \in L^2(0, T; H^1); \\ & d \in L^\infty(0, T; H^3) \cap L^2(0, T; H^4), \qquad d_t \in L^2(0, T; H^2); \\ & \theta \in L^\infty(0, T; H^2) \cap L^2(0, T; H^3), \qquad \theta_t \in L^2(0, T; H^1); \\ & p \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2). \end{aligned} \tag{2.1}$$

With small initial data, one can extend the local solution in Theorem 2.1 to a global one.

Theorem 2.2 *Assume that the initial data* (u_0, d_0, θ_0) *satisfies* $\nabla \cdot u_0 = 0, |d_0| = 1$, *and* $\theta_0 \geq \underline{\theta}_0 > 0$. *Then there exists a unique global-in-time strong solution* (u, d, θ) *to the system* (1.3)-(1.5) *satisfying*

$$\begin{aligned} & \sup_{t \geq 0} \|(u, \nabla d, \theta)(t)\|_{H^2}^2 + \int_0^{+\infty} \|(\nabla u, \Delta d, \nabla \theta)(t)\|_{H^2}^2 \, dt \\ & \quad \leq C \|(u_0, \nabla d_0, \theta_0)\|_{H^2}^2, \end{aligned} \tag{2.2}$$

provided that

$$\|\Delta u_0\|_2^2 + \|\nabla \Delta d_0\|_2^2 + \|\Delta \theta_0\|_2^2 < \delta_0, \tag{2.3}$$

for sufficiently small $\delta_0 > 0$.

Moreover, there exists a constant $\alpha > 0$, *such that the decay estimate*

$$\left\|(\nabla u, \nabla\theta)(t)\right\|_{H^1}^2 + \left\|\nabla d(t)\right\|_{H^2}^2 \le Ce^{-\alpha t} \tag{2.4}$$

is valid for some positive constant C.

3 Galerkin approximation

In this section, we will get an approximate solution for the system (1.3)-(1.5) by means of the Galerkin approximation.

Denote $X = H^2 \cap W$ and its finite-dimensional subspaces

$$X^m = \operatorname{span}\{\Phi^1, \Phi^2, \dots, \Phi^m\}, \quad m = 1, 2, \dots,$$

where $\{\Phi^m\}_{m=1}^\infty$ is a complete normal orthogonal basis of X. This means that $(\Phi^i, \Phi^j) = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$, while $\delta_{ij} = 0$ if $i \ne j$.

For any given m, and $k = 1, 2, \dots, m$, we study the following approximate system:

$$\begin{cases} (u_t^m, \Phi^k) + (u^m \cdot \nabla u^m, \Phi^k) + (\mathbb{S}^m, \nabla\Phi^k) = -(\sigma^{nd,m}, \nabla\Phi^k), \\ \nabla \cdot u^m = 0, \\ d_t^m + u^m \cdot \nabla d^m - d^m \cdot \nabla u^m + (d^m \cdot \nabla u^m \cdot d^m)d^m = \gamma(\Delta d^m + |\nabla d^m|^2 d^m), \\ \theta_t^m + \nabla \cdot (u^m\theta^m) + \nabla \cdot q^m = \mathbb{S}^m : \nabla u^m + \lambda\gamma|\Delta d^m + |\nabla d^m|^2 d^m|^2. \\ (u^m, \theta^m, d^m)|_{t=0} = (u_0^m, \theta_0, d_0), \\ u^m(x + De_i, t) = u^m(x - De_i, t), \qquad d^m(x + De_i, t) = d^m(x - De_i, t), \\ \theta^m(x + De_i, t) = \theta^m(x - De_i, t), \quad \forall x \in \partial\Omega, i = 1, 2, 3, \end{cases} \tag{3.1}$$

where

$$\mathbb{S}^m = \mu(\theta^m)(\nabla u^m + \nabla^T u^m) + \frac{\lambda}{\gamma}(d^m \cdot \nabla u^m \cdot d^m)d^m \otimes d^m,$$

$$\sigma^{nd,m} = -\lambda\nabla d^m \odot \nabla d^m - \lambda(\Delta d^m + |\nabla d^m|^2 d^m) \otimes d^m,$$

$$q^m = -\kappa(\theta^m)\nabla\theta^m - h(\theta^m)(d^m \cdot \nabla\theta^m)d^m,$$

$$u_0^m = \sum_{i=1}^m (u_0, \Phi^i)\Phi^i,$$

and $(f, g) := \int_\Omega f \cdot g \, dx$ is the inner product of $L^2(\Omega; \mathbb{R}^3)$.

By applying the fixed point theorem, we can show that there exist a $T_0 > 0$ depending on u_0, d_0, θ_0, m and Ω such that the system (3.1) possesses a strong solution (u^m, d^m, θ^m), in which the velocity field is in the form $u^m(x,t) = \sum_{i=1}^m \tilde{g}_i^m(t)\Phi^i(x)$ where $\tilde{g}_i^m(t)$ satisfies $\tilde{g}_i^m(0) = (u_0, \Phi^i)$ and $\tilde{g}_i^m(t)$ is continuously differentiable. Since this process is similar to that in [32], we omit the details here. Similar to the system in [32], the following lemma follows from the maximum principle.

Lemma 3.1 *Let* (u^m, d^m, θ^m) *be a solution to system* (3.1) *with initial data* (u_0^m, d_0, θ_0) *satisfy* $|d_0| = 1$ *and* $\sup_\Omega \theta_0 \ge \underline{\theta}_0 > 0$. *Then, for any* $(x, t) \in \Omega \times (0, T_0)$, *it is valid that*

$$\left|d^m(x,t)\right| = 1 \quad \textit{and} \quad \sup_\Omega \theta^m(x,t) \ge \underline{\theta}_0. \tag{3.2}$$

It should be noted that the constant T_0 depends on m. For a uniformly valid $T_0 > 0$ for every $m \in \mathbb{N}^*$, we need the estimates in next section.

4 Uniform *a priori* estimation

It should be noted that by applying a Helmholtz decomposition, $(3.1)_1$ can be rewritten as

$$u_t^m + u^m \cdot \nabla u^m + \nabla p^m = \nabla \cdot \left(\mathbb{S}^m + \sigma^{nd,m}\right), \tag{4.1}$$

where ∇p^m is decided by the property that $\Phi^i, i = 1, 2, \ldots, m$ are all divergence-free. Thus, the system (3.1) is equivalent to

$$\begin{cases} u_t^m + u^m \cdot \nabla u^m + \nabla p^m = \nabla \cdot (\mathbb{S}^m + \sigma^{nd,m}), \\ \nabla \cdot u^m = 0, \qquad |d^m| = 1, \\ d_t^m + u^m \cdot \nabla d^m - d^m \cdot \nabla u^m + (d^m \cdot \nabla u^m \cdot d^m) d^m = \gamma (\Delta d^m + |\nabla d^m|^2 d^m), \\ \theta_t^m + \nabla \cdot (u^m \theta^m) + \nabla \cdot q^m = \mathbb{S}^m : \nabla u^m + \lambda\gamma |\Delta d^m + |\nabla d^m|^2 d^m|^2, \\ (u^m, \theta^m, d^m)|_{t=0} = (u_0^m, \theta_0, d_0), \\ u^m(x + De_i, t) = u^m(x - De_i, t), d^m(x + De_i, t) = d^m(x - De_i, t), \\ \theta^m(x + De_i, t) = \theta^m(x - De_i, t), \quad \forall x \in \partial\Omega, i = 1, 2, 3, \end{cases} \tag{4.2}$$

where

$$\mathbb{S}^m = \mu(\theta^m)(\nabla u^m + \nabla^T u^m) + \frac{\lambda}{\gamma}(d^m \cdot \nabla u^m \cdot d^m) d^m \otimes d^m,$$
$$\sigma^{nd,m} = -\lambda \nabla d^m \odot \nabla d^m - \lambda(\Delta d^m + |\nabla d^m|^2 d^m) \otimes d^m,$$
$$q^m = -\kappa(\theta^m) \nabla \theta^m - h(\theta^m)(d^m \cdot \nabla \theta^m) d^m.$$

To show the solvability of the original system (1.3), in this section, we will omit the superscript m for convenience to establish the uniform estimates, which enable us to take $m \to \infty$ in (4.2). On account of the periodic boundary conditions, it is easy to deduce the equivalence of $\|\nabla^2 f\|_2$ and $\|\Delta f\|_2$ by integration by parts and the fact, by the divergence theorem, that

$$\int_\Omega \nabla^l f \, dx = \int_{\partial\Omega} \nabla^{l-1} f \vec{n} \, dS = 0, \qquad \text{for } l \geq 1,$$

which enables us to apply the Poincaré inequality to $\nabla^l f$, that is,

$$\|\nabla^l f\|_2 \lesssim \|\nabla^{l+1} f\|_2, \quad l \geq 1.$$

In addition, since $|d| = 1$, we have $|\nabla d|^2 = -\Delta d \cdot d \leq |\Delta d|$.

Lemma 4.1 (Basic energy inequality) *For the solution* (u, d, θ) *of system* (4.2)

$$\int \left(|u(t)|^2 + \lambda|\nabla d(t)|^2 + 2\theta(t)\right) = \int \left(|u_0|^2 + \lambda|\nabla d_0|^2 + 2\theta_0\right). \tag{4.3}$$

Proof Multiplying $(4.2)_1$ by u, integrating by parts over Ω and using $(4.2)_2$ give

$$\frac{1}{2}\frac{d}{dt}\int |u|^2 = -\int \mathbb{S}:\nabla u \\ + \lambda\int \nabla d\odot\nabla d:\nabla u + \lambda\int \big(\Delta d + |\nabla d|^2 d\big)\otimes d:\nabla u. \tag{4.4}$$

Multiplying $(4.2)_3$ by $\lambda(\Delta d + |\nabla d|^2 d)$ and integrating by parts over Ω imply that

$$\frac{\lambda}{2}\frac{d}{dt}\int |\nabla d|^2 + \lambda\gamma\int \big|\Delta d + |\nabla d|^2 d\big|^2 \\ = \lambda\int \big(u\cdot\nabla d - d\cdot\nabla u + (d\cdot\nabla u\cdot d)d\big)\cdot\big(\Delta d + |\nabla d|^2 d\big). \tag{4.5}$$

Integrating $(4.2)_4$ over Ω yields

$$\frac{d}{dt}\int \theta = \int \mathbb{S}:\nabla u + \lambda\gamma\int \big|\Delta d + |\nabla d|^2 d\big|^2. \tag{4.6}$$

Putting (4.4)-(4.6) together, one gets

$$\frac{d}{dt}\int\left(\frac{1}{2}|u|^2 + \frac{\lambda}{2}|\nabla d|^2 + \theta\right) \\ = \lambda\int \nabla d\odot\nabla d:\nabla u + \lambda\int \Delta d\otimes d:\nabla u + \lambda\int |\nabla d|^2 d\otimes d:\nabla u \\ + \lambda\int (u\cdot\nabla d)\cdot\Delta d - \lambda\int d\cdot\nabla u\cdot\Delta d - \lambda\int (d\cdot\nabla u)|\nabla d|^2 d \\ + \lambda\int (d\cdot\nabla u\cdot d)(d\cdot\Delta d) + \lambda\int (d\cdot\nabla u\cdot d)|\nabla d|^2. \tag{4.7}$$

Notice that

$$\int \nabla d\odot\nabla d:\nabla u + \int (u\cdot\nabla d)\Delta d = 0,$$
$$\int \Delta d\otimes d:\nabla u - \int d\cdot\nabla u\cdot\Delta d = 0,$$
$$\int |\nabla d|^2 d\otimes d:\nabla u - \int (d\cdot\nabla u)|\nabla d|^2 d = 0,$$
$$\int (d\cdot\nabla u\cdot d)(d\cdot\Delta d) + \int (d\cdot\nabla u\cdot d)|\nabla d|^2 = 0.$$

Therefore, it follows that

$$\frac{d}{dt}\int\left(\frac{1}{2}|u|^2 + \frac{\lambda}{2}|\nabla d|^2 + \theta\right) = 0. \tag{4.8}$$

This lemma follows from integrating (4.8) over $(0,t)$. □

In the next lemma, we give some high-order estimates.

Lemma 4.2 *For any $t > 0$, it is valid that*

$$\frac{d}{dt}\|(\Delta u, \nabla\Delta d, \Delta\theta)\|_2^2 + \|(\nabla\Delta u, \Delta^2 d, \nabla\Delta\theta)\|_2^2 \le C\bigl(1 + \|(\Delta u, \nabla\Delta d, \Delta\theta)\|_2^2\bigr)^3, \tag{4.9}$$

where the constant C depends only on the initial data and known constants.

Proof First of all, applying ∇ to $(4.2)_1$, testing the result by $\nabla\Delta u$ and integrating by parts over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\Delta u|^2 = {}& -\int\Delta(u\cdot\nabla u)\cdot\Delta u \\ & + \lambda\int\Delta\bigl[\nabla d\odot\nabla d + \bigl(\Delta d + |\nabla d|^2 d\bigr)\otimes d\bigr]:\nabla\Delta u \\ & - \int\Delta\Bigl[\mu(\theta)\bigl(\nabla u + \nabla^T u\bigr) + \frac{\lambda}{\gamma}(d\otimes d:\nabla u)d\otimes d\Bigr]:\nabla\Delta u \\ := {}& \sum_{i=1}^{3} I_i,\end{aligned} \tag{4.10}$$

where

$$I_1 \le C\int|\nabla u|\bigl|\nabla^2 u\bigr|^2 \le C\|\nabla u\|_3\|\Delta u\|_3^2, \tag{4.11}$$

$$\begin{aligned} I_2 &\le C\int\bigl(|\nabla\Delta d||\nabla d| + \bigl|\nabla^2 d\bigr|^2\bigr)|\nabla\Delta u| + \lambda\int\Delta^2 d\otimes d:\nabla\Delta u \\ &\le C\bigl(\|\nabla\Delta d\|_2\|\nabla d\|_\infty + \bigl\|\nabla^2 d\bigr\|_4^2\bigr)\|\nabla\Delta u\|_2 + \lambda\int\Delta^2 d\otimes d:\nabla\Delta u,\end{aligned} \tag{4.12}$$

$$\begin{aligned} I_3 \le {}& -\int\frac{\mu(\theta)}{2}\bigl|\nabla\Delta u + \nabla^T\Delta u\bigr|^2 - \int|d\otimes d:\nabla\Delta u|^2 \\ & + C\bigl(\|\nabla\theta\|_6^2 + \|\Delta\theta\|_3 + \|\Delta d\|_3\bigr)\|\nabla u\|_6\|\nabla\Delta u\|_2 \\ & + C\|\nabla\theta\|_6\bigl\|\nabla^2 u\bigr\|_3\|\nabla\Delta u\|_2 + C\|\nabla d\|_\infty\bigl\|\nabla^2 u\bigr\|_2\|\nabla\Delta u\|_2.\end{aligned} \tag{4.13}$$

Substituting (4.11)-(4.13) into (4.10), using the Sobolev embedding inequalities, interpolation inequalities and Young inequalities yield

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int|\Delta u|^2 + \int\mu(\theta)|\nabla\Delta u|^2 + \frac{\lambda}{\gamma}\int|d\otimes d:\nabla\Delta u|^2 \\ & \quad \le \lambda\int\Delta^2 d\otimes d:\nabla\Delta u + C\|\Delta u\|_2^2\|\nabla\Delta u\|_2 + C\|\nabla\Delta d\|_2^2\|\nabla\Delta u\|_2 \\ & \qquad + C\bigl(\|\Delta\theta\|_2^2 + \|\Delta\theta\|_2^{1/2}\|\nabla\Delta\theta\|_2^{1/2} + \|\nabla\Delta d\|_2\bigr)\|\Delta u\|_2\|\nabla\Delta u\|_2 \\ & \qquad + C\|\Delta\theta\|_2\|\Delta u\|_2^{1/2}\|\nabla\Delta u\|_2^{3/2} + C\|\nabla\Delta d\|_2\|\Delta u\|_2\|\nabla\Delta u\|_2 \\ & \quad \le \varepsilon\bigl(\|\nabla\Delta u\|_2^2 + \|\nabla\Delta\theta\|_2^2\bigr) + C\bigl(1 + \|\nabla\Delta d\|_2^2 + \|\Delta u\|_2^2 + \|\Delta\theta\|_2^2\bigr)^3 \\ & \qquad + \lambda\int\Delta^2 d\otimes d:\nabla\Delta u.\end{aligned} \tag{4.14}$$

Then, applying $\nabla\Delta$ to $(4.2)_3$, testing the result by $\lambda\nabla\Delta d$ and integrating by parts over Ω, one sees that

$$\begin{aligned}
&\frac{\lambda}{2}\frac{d}{dt}\int|\nabla\Delta d|^2+\lambda\gamma\int\left|\Delta^2 d\right|^2\\
&\quad=\lambda\int\Delta\big(u\cdot\nabla d-|\nabla d|^2 d\big)\cdot\Delta^2 d+\lambda\int\Delta\big[(d\cdot\nabla u\cdot d)d\big]\cdot\Delta^2 d\\
&\qquad-\lambda\int\Delta(d\cdot\nabla u)\cdot\Delta^2 d\\
&\quad\le\lambda\int\Delta(u\cdot\nabla d)\cdot\Delta^2 d+\lambda\int(d\otimes d:\nabla\Delta u)d\cdot\Delta^2 d\\
&\qquad-\lambda\int\Delta^2 d\otimes d:\nabla\Delta u\\
&\qquad+C\int\big(|\nabla d||\nabla\Delta d|+\left|\nabla^2 d\right|^2+\left|\nabla^2 u\right||\nabla d|+\left|\nabla^2 d\right||\nabla u|\big)\left|\Delta^2 d\right|. \qquad (4.15)
\end{aligned}$$

Notice that

$$\begin{aligned}
\int\Delta(u\cdot\nabla d)\cdot\Delta^2 d&=\int u\cdot\nabla\Delta d\cdot\Delta^2 d+\int\big(2\nabla u:\nabla^2 d+\Delta u\cdot\nabla d\big)\cdot\Delta^2 d\\
&=-\int\nabla u\otimes\nabla\Delta d:\nabla\Delta d+\int\big(2\nabla u:\nabla^2 d+\Delta u\cdot\nabla d\big)\cdot\Delta^2 d\\
d\cdot\Delta^2 d=\Delta(d\cdot\Delta d)-|\Delta d|^2-2\nabla d:\nabla\Delta d&=-\Delta\big(|\nabla d|^2\big)-|\Delta d|^2-2\nabla d:\nabla\Delta d.
\end{aligned}$$

Thus, it follows from (4.15) that

$$\begin{aligned}
&\frac{\lambda}{2}\frac{d}{dt}\int|\nabla\Delta d|^2+\lambda\gamma\int\left|\Delta^2 d\right|^2\\
&\quad\le C\int\big(|\nabla d||\nabla\Delta d|+\left|\nabla^2 d\right|^2+\left|\nabla^2 u\right||\nabla d|+\left|\nabla^2 d\right||\nabla u|\big)\left|\Delta^2 d\right|\\
&\qquad+\int|\nabla u||\nabla\Delta d|^2+C\int\big(\left|\nabla^2 d\right|^2+|\Delta d||\nabla d|\big)|\nabla\Delta u|-\lambda\int\Delta^2 d\otimes d:\nabla\Delta u\\
&\quad\le C\big(\|\nabla d\|_\infty\|\nabla\Delta d\|_2+\left\|\nabla^2 d\right\|_4^2+\left\|\nabla^2 u\right\|_2\|\nabla d\|_\infty+\|\nabla u\|_3\left\|\nabla^2 d\right\|_6\big)\left\|\Delta^2 d\right\|_2\\
&\qquad+\|\nabla u\|_\infty\|\nabla\Delta d\|_2^2+C\big(\left\|\nabla^2 d\right\|_4^2+\|\nabla\Delta d\|_2\|\nabla d\|_\infty\big)\|\nabla\Delta u\|_2\\
&\qquad-\lambda\int\Delta^2 d\otimes d:\nabla\Delta u\\
&\quad\le C\big(\|\Delta u\|_2^2+\|\nabla\Delta d\|_2^2\big)\left\|\Delta^2 d\right\|_2+C\|\nabla\Delta d\|_2^2\|\nabla\Delta u\|_2-\lambda\int\Delta^2 d\otimes d:\nabla\Delta u\\
&\quad\le\varepsilon\big(\|\nabla\Delta u\|_2^2+\left\|\Delta^2 d\right\|_2^2\big)+C\big(1+\|\Delta u\|_2^2+\|\nabla\Delta d\|_2^2\big)^2\\
&\qquad-\lambda\int\Delta^2 d\otimes d:\nabla\Delta u, \qquad (4.16)
\end{aligned}$$

where the Sobolev embedding inequalities, interpolation inequalities and Young inequalities have been used.

Furthermore, applying ∇ to $(4.2)_4$, testing it by $\nabla\Delta\theta$ and integrating on Ω imply that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\Delta\theta|^2 &= -\int\Delta(u\cdot\nabla\theta)\Delta\theta-\int\Delta\big[\kappa(\theta)\nabla\theta+h(\theta)(d\cdot\nabla\theta)d\big]\cdot\nabla\Delta\theta\\&\quad+\int\Delta\Big[\mu(\theta)\big(\nabla u+\nabla^T u\big)+\frac{\lambda}{\gamma}(d\otimes d:\nabla u)d\otimes d\Big]\Delta\theta\\&\quad+\lambda\gamma\int\Delta\big(\Delta d+|\nabla d|^2 d\big)\Delta\theta:=\sum_{i=1}^4 J_i,\end{aligned}\tag{4.17}$$

where

$$\begin{aligned}J_1&\le C\int|\Delta u||\Delta\theta||\nabla\theta|+C\int|\nabla u|\nabla^2\theta|^2\\&\le C\|\Delta u\|_2\|\Delta\theta\|_3\|\nabla\theta\|_6+C\|\nabla u\|_3\big\|\nabla^2\theta\big\|_3^2,\end{aligned}\tag{4.18}$$

$$\begin{aligned}J_2&\le-\int\kappa(\theta)|\nabla\Delta\theta|^2-\int h(\theta)|d\cdot\nabla\Delta\theta|^2+C\int\big(|\nabla\theta|+|\nabla d|\big)|\nabla\theta|^2|\nabla\Delta\theta|\\&\quad+C\int\big(|\nabla\theta|+|\nabla d|\big)\big|\nabla^2\theta\big||\nabla\Delta\theta|+C\int|\Delta d||\nabla\theta||\nabla\Delta\theta|\\&\le-\int\kappa(\theta)|\nabla\Delta\theta|^2-\int h(\theta)|d\cdot\nabla\Delta\theta|^2+C\big(\|\nabla\theta\|_6+\|\nabla d\|_6\big)\|\nabla\theta\|_6^2\|\nabla\Delta\theta\|_2\\&\quad+C\big(\|\nabla\theta\|_6+\|\nabla d\|_6\big)\big\|\nabla^2\theta\big\|_3\|\nabla\Delta\theta\|_2+C\|\Delta d\|_3\|\nabla\theta\|_6\|\nabla\Delta\theta\|_2,\end{aligned}\tag{4.19}$$

$$\begin{aligned}J_3&\le C\int|\Delta\theta|^2|\nabla u|+C\int\big(|\nabla\theta|^2+|\Delta d|\big)|\nabla u||\Delta\theta|\\&\quad+C\int\big(|\nabla\theta|+|\nabla d|\big)\big|\nabla^2 u\big||\Delta\theta|+C\int|\nabla\Delta u||\Delta\theta|\\&\le C\|\Delta\theta\|_3^2\|\nabla u\|_3+C\big(\|\nabla\theta\|_6^2+\|\Delta d\|_3\big)\|\nabla u\|_6\|\Delta\theta\|_2\\&\quad+C\big(\|\nabla\theta\|_6+\|\nabla d\|_6\big)\big\|\nabla^2 u\big\|_2\|\Delta\theta\|_3+C\|\nabla\Delta u\|_2\|\Delta\theta\|_2,\end{aligned}\tag{4.20}$$

$$\begin{aligned}J_4&\le C\int\big(\big|\Delta^2 d\big|+|\nabla d||\nabla\Delta d|+\big|\nabla^2 d\big|^2\big)|\Delta\theta|\\&\le C\big(\big\|\Delta^2 d\big\|_2+\|\nabla d\|_\infty\|\nabla\Delta d\|_2+\big\|\nabla^2 d\big\|_4^2\big)\|\Delta\theta\|_2.\end{aligned}\tag{4.21}$$

Substituting (4.18)-(4.21) into (4.17), using the Sobolev embedding inequalities, interpolation inequalities and Young inequalities give

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|\Delta\theta|^2+\int\kappa(\theta)|\nabla\Delta\theta|^2+\int h(\theta)|d\cdot\nabla\Delta\theta|^2\\&\quad\lesssim\|\Delta u\|_2\|\Delta\theta\|_2^{3/2}\|\nabla\Delta\theta\|_2^{1/2}+\|\Delta u\|_2\|\Delta\theta\|_2\|\nabla\Delta\theta\|_2+\|\Delta\theta\|_2^3\|\nabla\Delta\theta\|_2\\&\qquad+\|\nabla\Delta d\|_2\|\Delta\theta\|_2^2\|\nabla\Delta\theta\|_2+\big(\|\Delta\theta\|_2+\|\nabla\Delta d\|_2\big)\|\Delta\theta\|_2^{1/2}\|\nabla\Delta\theta\|_2^{3/2}\\&\qquad+\|\Delta u\|_2\|\Delta\theta\|_2^3+\|\Delta u\|_2\|\Delta\theta\|_2\|\nabla\Delta d\|_2\\&\qquad+\|\Delta\theta\|_2\big(\|\nabla\Delta u\|_2+\big\|\Delta^2 d\big\|_2+\|\nabla\Delta d\|_2^2\big)\\&\quad\le\big(1+\|\Delta u\|_2^2+\|\nabla\Delta d\|_2^2+\|\Delta\theta\|_2^2\big)^3+\varepsilon\big(\|\nabla\Delta u\|_2^2+\|\nabla\Delta\theta\|_2^2+\big\|\Delta^2 d\big\|_2^2\big).\end{aligned}\tag{4.22}$$

Summing up (4.14), (4.16), and (4.22) with $\varepsilon>0$ small enough, one shows (4.9). □

With these lemmas, we can obtain the local uniform estimates for the approximate solutions.

Lemma 4.3 *There exists a finite time $T^* > 0$ such that, for any $T \in (0, T^*)$, it is valid that*

$$\sup_{0\le t\le T} \|(\nabla u, \nabla\theta, \Delta d)\|_{H^1}^2 + \int_0^T \|(\Delta u, \Delta\theta, \nabla\Delta d)\|_{H^1}^2 \le C, \tag{4.23}$$

where C depends only on initial data and known constants.

Proof

Denote

$$\mathcal{F}(t) := \|(\Delta u, \Delta\theta, \nabla\Delta d)\|_2^2 + 1,$$
$$\mathcal{H}(t) := \|(\nabla\Delta u, \nabla\Delta\theta, \Delta^2 d)\|_2^2.$$

Then it follows from (4.9) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) + \mathcal{H}(t) \le C\mathcal{F}(t)^3. \tag{4.24}$$

In particular, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) \le C\mathcal{F}(t)^3,$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t)^{-2} \ge -2C.$$

Integrating this inequality on $[0, t]$ yields

$$\mathcal{F}(t)^{-2} \ge \mathcal{F}(0)^{-2} - 2Ct.$$

Take $T^* := [2C\mathcal{F}(0)^2]^{-1}$. Then, for any $T \in (0, T^*)$, one has

$$\mathcal{F}(t)^2 \le \left[\mathcal{F}(0)^{-2} - 2CT\right]^{-1}, \quad \forall 0 < t \le T. \tag{4.25}$$

Consequently, it follows from (4.24) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) + \mathcal{H}(t) \le C\left[\mathcal{F}(0)^{-2} - 2CT\right]^{-\frac{3}{2}} \le C, \tag{4.26}$$

Integrating it on $[0, t] \subset [0, T]$ and using the Poincaré inequality, we deduce (4.23). □

For a more complete description to the approximate solution, we need the lemma below.

Lemma 4.4 *For any* $0 < T < T^*$, *it is valid that*

$$\sup_{0\le t\le T}\left\|(\nabla p,\theta,d_t)\right\|_2^2+\int_0^T\left\|(\nabla p,u_t,\theta_t,\nabla d_t)\right\|_{H^1}^2\le C, \tag{4.27}$$

where T^* *is defined in Lemma* 4.3 *and* C *depends only on initial data and known constants.*

Proof First, we give some estimates for ∇p.

Since $\nabla\cdot u = 0$, it yields by taking the divergence to $(4.2)_1$

$$\Delta p=-\nabla\cdot(u\cdot\nabla u)+\nabla^2:\left(\mathbb{S}+\sigma^{nd}\right). \tag{4.28}$$

Then, by elliptic estimates, the Hölder inequality, the Sobolev embedding inequalities, the Young inequality, and (4.3), one has

$$\begin{aligned}
\|\nabla p\|_2^2&\lesssim\|u\cdot\nabla u\|_2^2+\left\|\nabla\cdot\left[\mu(\theta)\left(\nabla u+\nabla^T u\right)+\frac{\lambda}{\gamma}(d\cdot\nabla u\cdot d)d\otimes d\right]\right\|_2^2\\
&\quad+\left\|\nabla\cdot(\nabla d\odot\nabla d)\right\|_2^2+\left\|\nabla\cdot\left[\left(\Delta d+|\nabla d|^2d\right)\otimes d\right]\right\|_2^2\\
&\lesssim\|u\|_\infty^2\|\nabla u\|_2^2+\|\nabla\theta\|_4^2\|\nabla u\|_4^2+\|\Delta u\|_2^2+\|\nabla d\|_\infty^2\|\nabla u\|_2^2\\
&\quad+\|\nabla d\|_\infty^2\|\Delta d\|_2^2+\|\nabla\Delta d\|_2^2+\|\nabla d\|_\infty^2\|\nabla d\|_4^4\\
&\lesssim\left(1+\left\|(\nabla u,\Delta d,\nabla\theta)\right\|_{H^1}^2\right)^2,
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
\left\|\nabla^2 p\right\|_2^2&\lesssim\left\|\nabla\cdot(u\cdot\nabla u)\right\|_2^2+\left\|\nabla^2:\mathbb{S}\right\|_2^2+\left\|\nabla^2:\sigma^{nd}\right\|_2^2\\
&\lesssim\|\nabla u\|_4^4+\left(\|\nabla\theta\|_4^4+\|\Delta\theta\|_2^2+\|\Delta d\|_2^2+\|\nabla d\|_4^4\right)\|\nabla u\|_\infty^2\\
&\quad+\left(\|u\|_\infty^2+\|\nabla d\|_\infty^2+\|\nabla\theta\|_\infty^2\right)\|\Delta u\|_2^2+\|\nabla\Delta u\|_2^2+\|\Delta d\|_4^4\\
&\quad+\left(\|\nabla\Delta d\|_2^2+\|\nabla d\|_\infty^2\|\Delta d\|_2^2+\|\nabla d\|_6^6\right)\|\nabla d\|_\infty^2+\left\|\Delta^2 d\right\|_2^2\\
&\lesssim\left(1+\left\|(\nabla u,\nabla\theta,\Delta d)\right\|_{H^1}^2\right)^3\left(\left\|(\Delta u,\Delta\theta,\nabla\Delta d)\right\|_{H^1}^2+1\right).
\end{aligned} \tag{4.30}$$

Second, we give estimates for ∇u_t.

Applying ∇ to $(4.2)_1$, testing the result by ∇u_t, integrating by parts on Ω, and using the Cauchy inequality, we have

$$\begin{aligned}
\|\nabla u_t\|_2^2&\lesssim\left\|\nabla(u\cdot\nabla u)\right\|_2^2+\left\|\nabla\nabla\cdot\left(\mu(\theta)\left(\nabla u+\nabla^T u\right)\right\|_2^2+\left\|\nabla\nabla\cdot\left[(d\cdot\nabla u\cdot d)d\otimes d\right]\right\|_2^2\\
&\quad+\|\nabla\mathrm{div}\nabla d\odot\nabla d\|_2^2+\left\|\nabla\nabla\cdot\left[\left(\Delta d+|\nabla d|^2d\right)\otimes d\right]\right\|_2^2\\
&\lesssim\|\nabla u\|_4^4+\left(\|u\|_\infty^2+\|\nabla\theta\|_\infty^2\right)\|\Delta u\|_2^2+\left(\|\nabla\theta\|_4^4+\|\Delta\theta\|_2^2\right)\|\nabla u\|_\infty^2\\
&\quad+\|\nabla\Delta u\|_2^2+\left(\|\Delta d\|_2^2+\|\nabla d\|_4^4\right)\|\nabla u\|_\infty^2+\|\nabla d\|_\infty^2\|\Delta u\|_2^2\\
&\quad+\|\nabla\Delta d\|_2^2\|\nabla d\|_\infty^2+\|\Delta d\|_4^4+\left\|\Delta^2 d\right\|_2^2+\left(\|\Delta d\|_2^2+\|\nabla d\|_4^4\right)\|\nabla d\|_\infty^4\\
&\lesssim\left(1+\left\|(\nabla u,\nabla\theta,\Delta d)\right\|_{H^1}^2\right)^3\left(\left\|(\Delta u,\Delta\theta,\nabla\Delta d)\right\|_{H^1}^2+1\right).
\end{aligned} \tag{4.31}$$

Third, we give estimates for Δd_t.

Applying Δ to $(4.2)_3$, it is easy to see that

$$\begin{aligned}
\|\Delta d_t\|_2^2 &\lesssim \left\|\Delta\left[-u\cdot\nabla d + d\cdot\nabla u - (d\cdot\nabla u\cdot d)d + \gamma\left(\Delta d + |\nabla d|^2 d\right)\right]\right\|_2^2 \\
&\lesssim \left(\|\nabla u\|_6^2 + \left\|\nabla^2 d\right\|_6^2\right)\|\nabla d\|_6^4 + \left(\|u\|_\infty^2 + \|\nabla d\|_\infty^2\right)\|\nabla\Delta d\|_2^2 \\
&\quad + \|\nabla u\|_6^2\left\|\nabla^2 d\right\|_3^2 + \|\nabla d\|_\infty^2\|\Delta u\|_2^2 + \|\nabla\Delta u\|_2^2 + \left\|\Delta^2 d\right\|_2^2 + \|\Delta d\|_4^4 \\
&\lesssim \|\nabla\Delta u\|_2^2 + \left\|\Delta^2 d\right\|_2^2 + \left(1 + \left\|(\nabla u,\nabla\theta,\Delta d)\right\|_{H^1}^2\right)^2. \qquad (4.32)
\end{aligned}$$

Fourthly, we give estimates for $\nabla\theta_t$.

Applying ∇ to $(4.2)_4$, we get

$$\begin{aligned}
\|\nabla\theta_t\|_2^2 &\lesssim \left\|\nabla\nabla\cdot(u\theta + q)\right\|_2^2 + \left\|\nabla(\mathbb{S}:\nabla u)\right\|_2^2 + \left\|\nabla\left(\left|\Delta d + |\nabla d|^2 d\right|^2\right)\right\|_2^2 \\
&\lesssim \|\Delta\theta\|_3^2\left(\|u\|_6^2 + \|\nabla\theta\|_6^2\right) + \|\nabla\theta\|_4^2\left(\|\nabla u\|_4^2 + \|\Delta d\|_4^2\right) + \|\nabla\theta\|_6^6 \\
&\quad + \left(\|\nabla\theta\|_6^2 + \|\nabla d\|_6^2\right)\left(\|\nabla\theta\|_6^2\|\nabla d\|_6^2 + \|\nabla u\|_6^4\right) + \|\nabla\Delta\theta\|_2^2 \\
&\quad + \|\nabla u\|_6^2\|\Delta u\|_3^2 + \|\Delta d\|_6^2\left(\|\nabla\Delta d\|_3^2 + \|\nabla d\|_\infty^2\|\nabla d\|_6^4\right) \\
&\lesssim \left(1 + \left\|(\nabla u,\nabla\theta,\Delta d)\right\|_{H^1}^2\right)^3 + \left\|(\Delta u,\Delta\theta,\nabla\Delta d)\right\|_{H^1}^2. \qquad (4.33)
\end{aligned}$$

Finally, it is easy to obtain from the above estimates that

$$\left\|(u_t,\theta,\theta_t,d_t,\nabla d_t)\right\|_2^2 \lesssim \left(1 + \left\|(\nabla u,\nabla\theta,\Delta d)\right\|_{H^1}^2\right)^2. \qquad (4.34)$$

Equation (4.27) follows from (4.29)-(4.34) and (4.23). □

Now we conclude that there exists a $T_0 > 0$ such that, for any given $m \in \mathbb{N}^*$, there exists a strong solution $(u^m, d^m, \theta^m, p^m)$ to system (4.2) satisfying (4.23) and (4.27).

5 Local existence

It is concluded from the previous section that there exists $T^* > 0$, such that, for any $T \in (0, T^*)$, the following estimates are valid:

$$\sup_{0\le t\le T}\left\|u^m\right\|_{H^2}^2 + \int_0^T\left(\left\|\nabla u^m\right\|_{H^2}^2 + \left\|u_t^m\right\|_{H^1}^2\right) \le C, \qquad (5.1)$$

$$\sup_{0\le t\le T}\left\|\nabla d^m\right\|_{H^2}^2 + \int_0^T\left(\left\|\Delta d^m\right\|_{H^2}^2 + \left\|d_t^m\right\|_{H^2}^2\right) \le C, \quad \left|d^m\right| = 1 \text{ a.e.} \quad \Omega\times[0,T], \qquad (5.2)$$

$$\sup_{0\le t\le T}\left\|\theta^m\right\|_{H^2}^2 + \int_0^T\left(\left\|\nabla\theta^m\right\|_{H^2}^2 + \left\|\theta_t^m\right\|_{H^1}^2\right) \le C, \quad \theta^m \ge \underline{\theta}_0 > 0 \text{ a.e.} \quad \Omega\times[0,T], \qquad (5.3)$$

$$\sup_{0\le t\le T}\left\|\nabla p^m\right\|_2^2 + \int_0^T \|\nabla p\|_{H^1}^2 \le C. \qquad (5.4)$$

To prove the convergence of sequences $(u^m, d^m, \theta^m, p^m)$, we need the compactness lemma.

Lemma 5.1 (Simon [42]) *Assume that X, B and Y are three Banach spaces with $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$. Then the following hold true.*

(i) *If F is a bounded subset of $L^p(0,T;X)$, $1 \le p < \infty$, and $\frac{\partial F}{\partial t} = \{\frac{\partial f}{\partial t} \mid f \in F\}$ is bounded in $L^1(0,T;Y)$, then F is relatively compact in $L^p(0,T;B)$.*

(ii) *If F is a bounded subset of $L^\infty(0,T;X)$, and $\frac{\partial F}{\partial t}$ is bounded in $L^r(0,T;Y)$, where $r > 1$, then F is relatively compact in $C([0,T];B)$.*

Since $H^2 \hookrightarrow\hookrightarrow H^1 \hookrightarrow L^2$, it is clear from Lemma 5.1 and (5.1) that $\{u^m\}_{m=1}^\infty \subset C(0,T;H^1)$ is relatively compact, which means the existence of convergent subsequence, still denoted by $\{u^m\}$, satisfying $u^m \to u$ strongly in $C([0,T];H^1)$. Similarly, we have $d^m \to d$ strongly in $C([0,T];H^2)$ and $\theta^m \to \theta$ strongly in $C([0,T];H^1)$.

With these convergence arguments in hand, it is easy to conclude that, as $m \to \infty$, each term of system (4.2) converges to the corresponding term in (1.3). Here we select two terms to show the convergence, and the others can be done in a similar manner.

Lemma 5.2 *Assume that*

$$u^m \rightharpoonup u \quad \text{weakly in } L^2(0,T;H^3), \qquad u^m \to u \quad \text{strongly in } C([0,T];H^1),$$
$$d^m \to d \quad \text{strongly in } C([0,T];H^2), \qquad \theta^m \to \theta \quad \text{strongly in } C([0,T];H^1).$$

Then it is true that

$$\begin{aligned} &\lim_{m\to\infty} \int_0^T \int \nabla \cdot \big[(d^m \cdot \nabla u^m \cdot d^m) d^m \otimes d^m \big] \\ &\quad = \int_0^T \int \nabla \cdot \big[(d \cdot \nabla u \cdot d) d \otimes d \big], \end{aligned} \tag{5.5}$$

$$\begin{aligned} &\lim_{m\to\infty} \int_0^T \int \mu(\theta^m) (\nabla u^m + \nabla^T u^m) : \nabla u^m \\ &\quad = \int_0^T \int \mu(\theta) (\nabla u + \nabla^T u) : \nabla u. \end{aligned} \tag{5.6}$$

Proof Notice that

$$\begin{aligned} &\nabla \cdot \big[(d^m \cdot \nabla u^m \cdot d^m) d^m \otimes d^m \big] - \nabla \cdot \big[(d \cdot \nabla u \cdot d) d \otimes d \big] \\ &\quad = \nabla \cdot \big[\big((d^m - d) \cdot \nabla u^m \cdot d^m \big) d^m \otimes d^m \big] + \nabla \cdot \big[\big(d \cdot \nabla u^m \cdot (d^m - d) \big) d^m \otimes d^m \big] \\ &\qquad + \nabla \cdot \big[(d \cdot \nabla u^m \cdot d) (d^m - d) \otimes d^m \big] + \nabla \cdot \big[(d \cdot \nabla u^m \cdot d) d \otimes (d^m - d) \big] \\ &\qquad + \nabla (d \otimes d) : (\nabla u^m - \nabla u) \cdot (d \otimes d) + (d \otimes d) : (\nabla u^m - \nabla u) \nabla \cdot (d \otimes d) \\ &\qquad + (d \otimes d) : (\nabla^2 u^m - \nabla^2 u) \cdot (d \otimes d) := K_1 + K_2 + \cdots + K_7. \end{aligned}$$

Then, as $m \to \infty$, the uniform estimates and convergence of (u^m, d^m, θ^m) imply that

$$\begin{aligned} \left| \int_0^T \int \sum_{i=1}^6 K_i \right| &\lesssim \int_0^T \int |d^m - d| |\nabla u^m| \big(|\nabla d^m| + |\nabla d| \big) + \int_0^T \int |d^m - d| |\nabla^2 u^m| \\ &\quad + \int_0^T \int |\nabla d^m - \nabla d| |\nabla u^m| + \int_0^T \int |\nabla u^m - \nabla u| |\nabla d| \end{aligned}$$

$$
\begin{aligned}
&\lesssim \int_0^T \|d^m - d\|_2 \left(\|\nabla u^m\|_2 \left(\|\nabla d^m\|_\infty + \|\nabla d\|_{L^\infty}\right) + \|\nabla^2 u^m\|_2\right) \\
&\quad + \int_0^T \|\nabla d^m - \nabla d\|_2 \|\nabla u^m\|_2 + \int_0^T \|\nabla u^m - \nabla u\|_2 \|\nabla d\|_2 \\
&\lesssim T \sup_{0\le t\le T} \|d^m - d\|_{H^1} \sup_{0\le t\le T} \left(1 + \|\nabla d^m\|_{H^2}^2 + \|\nabla d\|_{H^2}^2 + \|\nabla u^m\|_{H^1}^2\right) \\
&\quad + T \sup_{0\le t\le T} \|\nabla u^m - \nabla u\|_2 \sup_{0\le t\le T} \|\nabla d\|_2 \to 0,
\end{aligned}
\tag{5.7}
$$

$$
\int_0^T \int (d\otimes d) : \nabla^2 u^m \cdot (d\otimes d) \to \int_0^T \int (d\otimes d) : \nabla^2 u \cdot (d\otimes d).
\tag{5.8}
$$

Thus, (5.6) follows. Similarly, as $m \to \infty$,

$$
\begin{aligned}
&\left| \int_0^T \int \left[\mu(\theta^m)\left(\nabla u^m + \nabla^T u^m\right) : \nabla u^m - \mu(\theta)\left(\nabla u + \nabla^T u\right) : \nabla u\right] \right| \\
&\quad \le \left| \int_0^T \int \left[\mu(\theta^m) - \mu(\theta)\right]\left(\nabla u^m + \nabla^T u^m\right) : \nabla u^m \right| \\
&\qquad + \left| \int_0^T \int \mu(\theta)\left[\nabla\left(u^m - u\right) + \nabla^T\left(u^m - u\right)\right] : \nabla u^m \right| \\
&\qquad + \left| \int_0^T \int \mu(\theta)\left(\nabla u + \nabla^T u\right) : \nabla\left(u^m - u\right) \right| \\
&\quad \lesssim \int_0^T \int \left|\theta^m - \theta\right| \left|\nabla u^m\right|^2 + \int_0^T \int \left|\nabla u^m - \nabla u\right| \left(\left|\nabla u^m\right| + |\nabla u|\right) \\
&\quad \lesssim \int_0^T \|\theta^m - \theta\|_3 \|\nabla u^m\|_3^2 + \int_0^T \|\nabla u^m - \nabla u\|_2 \left(\|\nabla u^m\|_2 + \|\nabla u\|_2\right) \\
&\quad \lesssim T \sup_{0\le t\le T} \|\theta^m - \theta\|_{H^1} \sup_{0\le t\le T} \|\nabla u^m\|_{H^1}^2 \\
&\qquad + T \sup_{0\le t\le T} \|\nabla u^m - \nabla u\|_2 \sup_{0\le t\le T} \left(\|\nabla u^m\|_2 + \|\nabla u\|_2\right) \to 0.
\end{aligned}
$$

The proof of this lemma is completed. □

In conclusion, $(u(x,t), \theta(x,t), d(x,t), p(x,t))$ solves system (1.3)-(1.5) on $\Omega \times [0,T]$ and satisfies

$$
\begin{aligned}
&u(x,t) \in L^\infty\left(0,T;H^2\right) \cap L^2\left(0,T;H^3\right), \qquad u_t \in L^2\left(0,T;H^1\right); \\
&d(x,t) \in L^\infty\left(0,T;H^3\right) \cap L^2\left(0,T;H^4\right) \cap L^\infty\left(\Omega \times [0,T]\right); \\
&d_t(x,t) \in L^2\left(0,T;H^2\right), \qquad \left|d(x,t)\right| \le 1 \text{ a.e.} \qquad \Omega \times [0,T]; \\
&\theta(x,t) \in L^\infty\left(0,T;H^2\right) \cap L^2\left(0,T;H^3\right); \qquad \theta_t(x,t) \in L^2\left(0,T;H^1\right); \\
&\theta(x,t) \ge \underline{\theta}_0 > 0 \text{ a.e.} \qquad \Omega \times [0,T], \qquad \nabla p(x,t) \in L^\infty\left(0,T;L^2\right) \cap L^2\left(0,T;H^1\right).
\end{aligned}
$$

6 Global existence

In this section, we will prove that the local strong solutions constructed in Section 5 are also global in time provided that the initial data is small enough.

On one hand, we re-estimate (4.14), (4.16), and (4.22), respectively, as follows:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Delta u|^2 + \underline{\mu}\int |\nabla\Delta u|^2 + \frac{\lambda}{\gamma}\int |d\otimes d:\nabla\Delta u|^2 \\
&\quad\le \lambda\int \Delta^2 d\otimes d:\nabla\Delta u + C\|\Delta u\|_2^2\|\nabla\Delta u\|_2 + C\|\nabla\Delta d\|_2^2\|\nabla\Delta u\|_2 \\
&\qquad + C\big(\|\Delta\theta\|_2^2 + \|\Delta\theta\|_2^{1/2}\|\nabla\Delta\theta\|_2^{1/2} + \|\nabla\Delta d\|_2\big)\|\Delta u\|_2\|\nabla\Delta u\|_2 \\
&\qquad + C\|\Delta\theta\|_2\|\Delta u\|_2^{1/2}\|\nabla\Delta u\|_2^{3/2} + C\|\nabla\Delta d\|_2\|\Delta u\|_2\|\nabla\Delta u\|_2 \\
&\quad\le \lambda\int \Delta^2 d\otimes d:\nabla\Delta u + C\|\Delta u\|_2\|\nabla\Delta u\|_2^2 + C\|\nabla\Delta d\|_2\big\|\Delta^2 d\big\|_2\|\nabla\Delta u\|_2 \\
&\qquad + C\big(\|\Delta\theta\|_2^2 + \|\nabla\Delta d\|_2 + \|\Delta\theta\|_2\big)\|\nabla\Delta u\|_2^2 + C\|\Delta u\|_2\|\nabla\Delta u\|_2\|\nabla\Delta\theta\|_2, \qquad (6.1)
\end{aligned}$$

$$\begin{aligned}
&\frac{\lambda}{2}\frac{d}{dt}\int |\nabla\Delta d|^2 + \lambda\gamma\int \big|\Delta^2 d\big|^2 \\
&\quad\le C\big(\|\Delta u\|_2^2 + \|\nabla\Delta d\|_2^2\big)\big\|\Delta^2 d\big\|_2 + C\|\nabla\Delta d\|_2^2\|\nabla\Delta u\|_2 - \lambda\int \Delta^2 d\otimes d:\nabla\Delta u \\
&\quad\le C\|\Delta u\|_2\|\nabla\Delta u\|_2\big\|\Delta^2 d\big\|_2 + C\|\nabla\Delta d\|_2\big\|\Delta^2 d\big\|_2^2 \\
&\qquad + C\|\nabla\Delta d\|_2\|\nabla\Delta u\|_2\big\|\Delta^2 d\big\|_2 - \lambda\int \Delta^2 d\otimes d:\nabla\Delta u, \qquad (6.2)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Delta\theta|^2 + \underline{\kappa}\int |\nabla\Delta\theta|^2 + \int h(\theta)|d\cdot\nabla\Delta\theta|^2 \\
&\quad\lesssim \|\Delta u\|_2\|\Delta\theta\|_2^{3/2}\|\nabla\Delta\theta\|_2^{1/2} + \|\Delta u\|_2\|\Delta\theta\|_2\|\nabla\Delta\theta\|_2 + \|\nabla\Delta d\|_2\|\Delta\theta\|_2^2\|\nabla\Delta\theta\|_2 \\
&\qquad + \|\Delta\theta\|_2^3\|\nabla\Delta\theta\|_2 + \big(\|\Delta\theta\|_2 + \|\nabla\Delta d\|_2\big)\|\Delta\theta\|_2^{1/2}\|\nabla\Delta\theta\|_2^{3/2} + \|\Delta u\|_2\|\Delta\theta\|_2^3 \\
&\qquad + \|\Delta u\|_2\|\Delta\theta\|_2\|\nabla\Delta d\|_2 + \|\Delta\theta\|_2\big(\|\nabla\Delta u\|_2 + \big\|\Delta^2 d\big\|_2 + \|\nabla\Delta d\|_2^2\big) \\
&\quad\le C\big(\|\Delta u\|_2 + \|\nabla\Delta d\|_2 + \|\Delta\theta\|_2\big)\big(1 + \|\Delta\theta\|_2\big)\|\nabla\Delta\theta\|_2^2 + C\|\Delta\theta\|_2\big\|\Delta^2 d\big\|_2^2 \\
&\qquad + C\|\Delta u\|_2\|\nabla\Delta\theta\|_2\big\|\Delta^2 d\big\|_2 + C_1\big(\|\nabla\Delta u\|_2 + \big\|\Delta^2 d\big\|_2^2\big) + \frac{\underline{\kappa}}{2}\|\nabla\Delta\theta\|_2. \qquad (6.3)
\end{aligned}$$

Then, for suitable larger C_2, calculating $((6.1) + (6.2)) \times C_2 + (6.3)$, we have

$$\frac{d}{dt}\mathcal{E}(t) + \mathcal{D}(t) \le C_3\big(\sqrt{\mathcal{E}(t)} + \mathcal{E}(t)\big)\mathcal{D}(t), \qquad (6.4)$$

where we denote

$$\mathcal{E}(t) := \big\|\Delta u(t)\big\|_2^2 + \big\|\nabla\Delta d(t)\big\|_2^2 + \big\|\Delta\theta(t)\big\|_2^2, \qquad (6.5)$$

$$\mathcal{D}(t) := \big\|\nabla\Delta u(t)\big\|_2^2 + \big\|\Delta^2 d(t)\big\|_2^2 + \big\|\Delta\theta(t)\big\|_2^2. \qquad (6.6)$$

Define

$$T_0 := \sup\Big\{t > 0 \;\Big|\; \sup_{0\le s\le t}\mathcal{E}(s) < 2\mathcal{E}(0)\Big\}. \qquad (6.7)$$

Then, for any $t \in (0, T_0)$, (6.4) implies that

$$\frac{d}{dt}\mathcal{E}(t) + \big[1 - C_3\big(\sqrt{2\mathcal{E}(0)} + 2\mathcal{E}(0)\big)\big]\mathcal{D}(t) \le 0. \qquad (6.8)$$

Thus, integrating (6.8) over $(0,s) \subset (0,t)$ gives

$$\sup_{0\le s\le t} \mathcal{E}(s) + \int_0^t \bigl[1 - C_3\bigl(\sqrt{2\mathcal{E}(0)} + 2\mathcal{E}(0)\bigr)\bigr]\mathcal{D}(s)\,ds \le \mathcal{E}(0). \tag{6.9}$$

In particular, we have

$$\sup_{0\le s\le t} \mathcal{E}(s) \le \mathcal{E}(0), \tag{6.10}$$

once $\mathcal{E}(0)$ satisfies $C_3(\sqrt{2\mathcal{E}(0)} + 2\mathcal{E}(0)) < 1$, *i.e.* $\mathcal{E}(0) < \frac{2+C_3}{4C_3} - \frac{1}{4}\sqrt{1+\frac{4}{C_3}}$.

If $T_0 < +\infty$, then it follows from (6.7) and (6.10) that

$$2\mathcal{E}(0) = \sup_{0\le s\le T_0} \mathcal{E}(s) \le \mathcal{E}(0) < 2\mathcal{E}(0),$$

which is a contradiction. Therefore, $T_0 = +\infty$ and

$$\sup_{t\ge 0} \mathcal{E}(t) + \varepsilon_0 \int_0^{+\infty} \mathcal{D}(s)\,ds \le \mathcal{E}(0), \tag{6.11}$$

where $\varepsilon_0 := 1 - C_3(\sqrt{2\mathcal{E}(0)} + 2\mathcal{E}(0))$.

On the other hand, multiplying $(4.2)_4$ by θ and integrating by parts over Ω gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\theta|^2 + \int \kappa(\theta)|\nabla\theta|^2 \le C\bigl(\|\nabla u\|_\infty^2 + \|\Delta d\|_\infty^2\bigr)\int \theta. \tag{6.12}$$

Then it follows from the Sobolev embedding inequality, (4.3), (6.11), and the Hölder inequality that

$$\sup_{t\ge 0}\bigl\|\theta(t)\bigr\|_2^2 + \int_0^{+\infty} \bigl\|\nabla\theta(t)\bigr\|_2^2\,dt \le C\bigl\|(u_0, \nabla d_0, \theta_0)\bigr\|_{H^2}^2. \tag{6.13}$$

Thus, by virtue of (4.3), (6.11) and (6.13), (2.2) is deduced with $\delta_0 = \frac{2+C_3}{4C_3} - \frac{1}{4}\sqrt{1+\frac{4}{C_3}}$.

Moreover, applying the Poincaré inequality to (6.8), we get

$$\frac{d}{dt}\mathcal{E}(t) + C\varepsilon_0\mathcal{E}(t) \le 0. \tag{6.14}$$

Consequently, taking $\alpha = C\varepsilon_0$ and applying the Gronwall inequality to (6.14) we deduce (2.4).

7 Uniqueness

Assume that $(u_i, d_i, \theta_i, \nabla p_i), i = 1, 2$ are two strong solutions to system (1.3)-(1.5) on $\Omega \times [0,T]$ satisfying (5.1)-(5.4) and denote

$$\bar{u} := u_1 - u_2, \qquad \bar{d} := d_1 - d_2, \qquad \bar{\theta} := \theta_1 - \theta_2, \qquad \nabla\bar{p} := \nabla p_1 - \nabla p_2,$$

where $T = T^*$ for the local solutions and $T = +\infty$ for the global ones.

Then $(\bar{u}, \bar{d}, \bar{\theta}, \nabla\bar{p})$ solves the following initial-boundary value problem:

$$\begin{cases} \bar{u}_t + \bar{u}\cdot\nabla u_1 + u_2\cdot\nabla\bar{u} + \nabla\bar{p} = \nabla\cdot(\bar{\mathcal{S}}_1 + \bar{\mathcal{S}}_2 - \bar{\mathcal{O}}_1 - \bar{\mathcal{O}}_2), \\ \nabla\cdot\bar{u} = 0, \\ \bar{d}_t + u_2\cdot\nabla\bar{d} + \bar{u}\cdot\nabla d_1 - \bar{\mathcal{P}} = \gamma[\Delta\bar{d} + (\nabla\bar{d} : (\nabla d_1 + \nabla d_2))d_1 + |\nabla d_2|^2\bar{d}], \\ \bar{\theta}_t + \bar{u}\cdot\nabla\theta_1 + u_2\cdot\nabla\bar{\theta} - \nabla\cdot(\bar{\mathcal{Q}}_1 + \bar{\mathcal{Q}}_2) = \bar{\mathcal{M}}_1 + \bar{\mathcal{M}}_2 + \bar{\mathcal{N}}, \\ (\bar{u}, \bar{\theta}, \bar{d})|_{t=0} = (0,0,0), \\ \bar{u}(x - De_i, t) = \bar{u}(x + De_i, t), \qquad \bar{d}(x - De_i, t) = \bar{d}(x + De_i, t), \\ \bar{\theta}(x - De_i) = \bar{\theta}(x + De_i), \quad \forall x \in \partial\Omega, i = 1,2,3, \end{cases} \tag{7.1}$$

where, for convenience, we denote

$$\begin{aligned}
\bar{\mathcal{S}}_1 &:= \big(\mu(\theta_1) - \mu(\theta_2)\big)\big(\nabla u_1 + \nabla^T u_1\big) + \mu(\theta_2)\big(\nabla\bar{u} + \nabla^T\bar{u}\big) \\
\bar{\mathcal{S}}_2 &:= \frac{\lambda}{\gamma}\big[(\bar{d}\otimes d_1 : \nabla u_1)d_1\otimes d_1 + (d_2\otimes\bar{d} : \nabla u_1)d_1\otimes d_1 + (d_2\otimes d_2 : \nabla u_1)\bar{d}\otimes d_1\big] \\
&\quad + \frac{\lambda}{\gamma}\big[(d_2\times d_2 : \nabla u_1)d_2\otimes\bar{d} + (d_2\otimes d_2 : \nabla\bar{u})d_2\otimes d_2\big], \\
\bar{\mathcal{O}}_1 &:= \lambda\nabla\bar{d}\odot\nabla d_1 + \lambda\nabla d_2\odot\nabla\bar{d}, \\
\bar{\mathcal{O}}_2 &:= \lambda\big(\Delta d_1 + |\nabla d_1|^2 d_1\big)\otimes d_1 - \lambda\big(\Delta d_2 + |\nabla d_2|^2 d_2\big)\otimes d_2, \\
\bar{\mathcal{P}} &:= \big[d_1\cdot\nabla u_1 - (d_1\cdot\nabla u_1\cdot d_1)d_1\big] - \big[d_2\cdot\nabla u_2 - (d_2\cdot\nabla u_2\cdot d_2)d_2\big], \\
\bar{\mathcal{Q}}_1 &:= \big(\kappa(\theta_1) - \kappa(\theta_2)\big)\nabla\theta_2 + \kappa(\theta_1)\nabla\bar{\theta}, \\
\bar{\mathcal{Q}}_2 &:= h(\theta_1)(d_1\cdot\nabla\bar{\theta})d_1 + \big(h(\theta_1) - h(\theta_2)\big)(d_1\cdot\nabla\theta_2)d_1 + h(\theta_2)(\bar{d}\cdot\nabla\theta_2)d_1 \\
&\quad + h(\theta_2)(d_2\cdot\nabla\theta_2)\bar{d}, \\
\bar{\mathcal{M}}_1 &:= \big(\mu(\theta_1) - \mu(\theta_2)\big)\big(\nabla\mu_1 + \nabla^T u_1\big) : \nabla u_1 + \mu(\theta_2)\big(\nabla\bar{u} + \nabla^T\bar{u}\big) : \nabla u_1 \\
&\quad + \mu(\theta_2)\big(\nabla u_2 + \nabla^T u_2\big) : \nabla\bar{u}, \\
\bar{\mathcal{M}}_2 &:= \frac{\lambda}{\gamma}[\bar{d}\otimes d_1 : \nabla u_1 + d_2\otimes\bar{d} : \nabla u_1 + d_2\otimes d_2 : \nabla\bar{u}](d_1\otimes d_1 : \nabla u_1) \\
&\quad + \frac{\lambda}{\gamma}(d_2\otimes d_2 : \nabla u_2)[\bar{d}\otimes d_1 : \nabla u_1 + d_2\otimes\bar{d} : \nabla u_1 + d_2\otimes d_2 : \nabla\bar{u}], \\
\bar{\mathcal{N}} &:= \lambda\gamma\Delta\bar{d}\cdot(\Delta d_1 + \Delta d_2) - \lambda\gamma\big(|\nabla d_1|^2 + |\nabla d_2|^2\big)(\nabla d_1 + \nabla d_2) : \nabla\bar{d}.
\end{aligned}$$

Following the idea given by [43], we indicate that

$$\begin{aligned}
\bar{\mathcal{O}}_2 &= \lambda\big[(d_1\times\Delta d_1)\times d_1\big]\otimes d_1 - \lambda\big[(d_2\times\Delta d_2)\times d_2\big]\otimes d_2 \\
&= \lambda\big[(d_1\times\Delta\bar{d})\times d_1\big]\otimes d_1 + \mathcal{R}_s, |\mathcal{R}_s| \le 3|\Delta d_2||\bar{d}|,
\end{aligned} \tag{7.2}$$

$$\begin{aligned}
\bar{\mathcal{P}} &= \big[d_1\times(d_1\cdot\nabla u_1)\big]\times d_1 - \big[d_2\times(d_2\cdot\nabla u_2)\big]\times d_2 \\
&= \big[d_1\times(d_1\cdot\nabla\bar{u})\big]\times d_1 + \mathcal{R}_q, |\mathcal{R}_q| \le 3|\nabla u_2||\bar{d}|.
\end{aligned} \tag{7.3}$$

In what follows, we will show that $(\bar{u}, \bar{\theta}, \bar{d}, \nabla\bar{p}) = (0,0,0,0)$ a.e. in $\Omega\times[0,T]$.

First of all, multiplying $(7.1)_1$ by $\bar{u}$ and integrating by parts over Ω give

$$\frac{1}{2}\frac{d}{dt}\int|\bar{u}|^2 = -\int \bar{u}\cdot\nabla u_1\cdot\bar{u} - \int(\bar{\mathcal{S}}_1+\bar{\mathcal{S}}_2):\nabla\bar{u} + \int\bar{\mathcal{O}}_1:\nabla\bar{u} + \int\bar{\mathcal{O}}_2:\nabla\bar{u}, \tag{7.4}$$

where

$$\begin{aligned} -\int(\bar{\mathcal{S}}_1+\bar{\mathcal{S}}_2):\nabla\bar{u} &\le -\int\mu(\theta_2)\left|\nabla\bar{u}+\nabla^T\bar{u}\right|^2 - \int\frac{\lambda}{\gamma}|d_2\otimes d_2:\nabla\bar{u}|^2 \\ &\quad + C\int|\nabla u_1|\big(|\bar{\theta}|+|\bar{d}|\big)|\nabla\bar{u}|, \end{aligned} \tag{7.5}$$

$$\int\bar{\mathcal{O}}_1:\nabla\bar{u} - \int\bar{u}\cdot\nabla u_1\cdot\bar{u} \le \int|\bar{u}|^2|\nabla u_1| + \int\big(|\nabla d_1|+|\nabla d_2|\big)|\nabla\bar{d}||\nabla\bar{u}|, \tag{7.6}$$

$$\int\bar{\mathcal{O}}_2:\nabla\bar{u} \le \lambda\int\big[(d_1\times\Delta\bar{d})\times d_1\big]\otimes d_1:\nabla\bar{u} + 3\int|\Delta d_2||\bar{d}||\nabla\bar{u}|. \tag{7.7}$$

Substituting (7.5)-(7.7) into (7.4), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int|\bar{u}|^2 + \int\mu(\theta_2)\left|\nabla\bar{u}+\nabla^T\bar{u}\right|^2 + \int\frac{\lambda}{\gamma}|d_2\otimes d_2:\nabla\bar{u}|^2 \\ &\quad\le C\int|\nabla u_1|\big(|\bar{\theta}|+|\bar{d}|\big)|\nabla\bar{u}| + \int|\bar{u}|^2|\nabla u_1| + \int\big(|\nabla d_1|+|\nabla d_2|\big)|\nabla\bar{d}||\nabla\bar{u}| \\ &\qquad + \int|\Delta d_2||\bar{d}||\nabla\bar{u}| + \lambda\int\big[(d_1\times\Delta\bar{d})\times d_1\big]\otimes d_1:\nabla\bar{u}. \end{aligned} \tag{7.8}$$

Similarly, multiplying $(7.1)_3$ by $\bar{d}$ and integrating by parts over Ω we deduce

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int|\bar{d}|^2 + \gamma\int|\nabla\bar{d}|^2 \\ &\quad= \int\bar{u}\cdot\nabla d_1\cdot\bar{d} + \int\bar{\mathcal{P}}\cdot\bar{d} + \gamma\int|\nabla d_2|^2|\bar{d}|^2 + \gamma\int\big(\nabla\bar{d}:(\nabla d_1+\nabla d_2)\big)d_1\cdot\bar{d} \\ &\quad\le C\int\big(|\nabla d_1||\bar{u}|+\nabla\bar{u}\big)|\bar{d}| \\ &\qquad + C\int\big(|\nabla d_1|+|\nabla d_2|\big)|\nabla\bar{d}||\bar{d}| + \int|\bar{d}|^2\big(|\nabla u_2|+|\nabla d_2|^2\big). \end{aligned} \tag{7.9}$$

Moreover, multiplying $(7.1)_3$ by $\lambda\Delta\bar{d}$ and integrating by parts over Ω yield

$$\begin{aligned} &\frac{\lambda}{2}\frac{d}{dt}\int|\nabla\bar{d}|^2 + \lambda\gamma\int|\Delta\bar{d}|^2 \\ &\quad= -\lambda\int\bar{\mathcal{P}}\cdot\Delta\bar{d} - \lambda\int\big[u_2\cdot\nabla\bar{d} + \bar{u}\cdot\nabla d_1 + \gamma\big(\nabla\bar{d}:(\nabla d_1+\nabla d_2)\big)d_1 + \gamma|\nabla d_2|^2\bar{d}\big]\cdot\Delta\bar{d} \\ &\quad\le C\int|\bar{u}||\Delta\bar{d}||\nabla d_1| + C\int\big(|u_2|+|\nabla d_1|\big)|\nabla\bar{d}||\Delta\bar{d}| + C\int\big(|\nabla u_2|+|\nabla d_2|^2\big)|\bar{d}||\Delta\bar{d}| \\ &\qquad - \lambda\int\big[d_1\times(d_1\cdot\nabla\bar{u})\big]\times d_1\cdot\Delta\bar{d}. \end{aligned} \tag{7.10}$$

In addition, multiplying $(7.1)_4$ by $\bar{\theta}$ and integrating by parts over Ω give

$$\frac{1}{2}\frac{d}{dt}\int|\bar{\theta}|^2+\int(\bar{\mathcal{Q}}_1+\bar{\mathcal{Q}}_2)\cdot\nabla\bar{\theta}=-\int\bar{u}\cdot\nabla\theta_1\bar{\theta}+\int(\bar{\mathcal{M}}_1+\bar{\mathcal{M}}_2+\bar{\mathcal{N}})\bar{\theta},\tag{7.11}$$

where

$$\int(\bar{\mathcal{Q}}_1+\bar{\mathcal{Q}}_2)\cdot\nabla\bar{\theta}\geq\int\big(\kappa(\theta_1)|\nabla\bar{\theta}|^2+h(\theta_1)|d\cdot\nabla\bar{\theta}|^2\big)-C\int|\nabla\theta_2||\nabla\bar{\theta}|\big(|\bar{\theta}|+|\bar{d}|\big),\tag{7.12}$$

$$\begin{aligned}\int(\bar{\mathcal{M}}_1+\bar{\mathcal{M}}_2)\bar{\theta}\leq{}&\int|\nabla u_1|^2|\bar{\theta}|^2+C\int\big(|\nabla u_1|+|\nabla u_2|\big)|\nabla\bar{u}||\bar{\theta}|\\&+C\int\big(|\nabla u_1|^2+|\nabla u_2|^2\big)|\bar{d}||\bar{\theta}|,\end{aligned}\tag{7.13}$$

$$\begin{aligned}\int(\bar{\mathcal{N}}-\bar{u}\cdot\nabla\theta_1)\bar{\theta}\leq{}&\int|\nabla\theta_1||\bar{u}||\bar{\theta}|+\int\big(|\Delta d_1|+|\Delta d_2|\big)|\Delta\bar{d}||\bar{\theta}|\\&+C\int\big(|\nabla d_1|^3+|\nabla d_2|^3\big)|\nabla\bar{d}||\bar{\theta}|.\end{aligned}\tag{7.14}$$

Substituting (7.12)-(7.14) into (7.11), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int|\bar{\theta}|^2+\int\kappa(\theta_1)|\nabla\bar{\theta}|^2+\int h(\theta_1)|d\cdot\nabla\bar{\theta}|^2\\&\quad\leq C\int|\nabla\theta_2||\nabla\bar{\theta}|\big(|\bar{\theta}|+|\bar{d}|\big)+\int|\nabla u_1|^2|\bar{\theta}|^2+C\int\big(|\nabla u_1|+|\nabla u_2|\big)|\nabla\bar{u}||\bar{\theta}|\\&\qquad+C\int\big(|\nabla u_1|^2+|\nabla u_2|^2\big)|\bar{d}||\bar{\theta}|+\int|\nabla\theta_1||\bar{u}||\bar{\theta}|+\int\big(|\Delta d_1|+|\Delta d_2|\big)|\Delta\bar{d}||\bar{\theta}|\\&\qquad+\int\big(|\nabla d_1|^3+|\nabla d_2|^3\big)|\nabla\bar{d}||\bar{\theta}|.\end{aligned}\tag{7.15}$$

Finally, summing up (7.8)-(7.10) and (7.15), we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\big(|\bar{u}|^2+|\bar{d}|^2+|\nabla\bar{d}|^2+|\bar{\theta}|^2\big)+\int\big(\underline{\mu}|\nabla\bar{u}|^2+\gamma|\nabla\bar{d}|^2+\gamma|\Delta\bar{d}|^2+\underline{\kappa}|\nabla\bar{\theta}|^2\big)\\&\quad\lesssim\int\big(|\nabla u_1|+|\Delta d_2|\big)|\bar{d}||\nabla\bar{u}|+\int|\bar{u}|^2|\nabla u_1|+\int\big(|\nabla d_1|+|\nabla d_2|\big)|\nabla\bar{d}||\nabla\bar{u}|\\&\qquad+\int\big(|\nabla d_1||\bar{u}|+\nabla\bar{u}\big)|\bar{d}|+\int\big(|\nabla d_1|+|\nabla d_2|\big)|\nabla\bar{d}||\bar{d}|+\int|\bar{d}|^2\big(|\nabla u_2|+|\nabla d_2|^2\big)\\&\qquad+\int|\bar{u}||\Delta\bar{d}||\nabla d_1|+\int\big(|u_2|+|\nabla d_1|\big)|\nabla\bar{d}||\Delta\bar{d}|+\int\big(|\nabla u_2|+|\nabla d_2|^2\big)|\bar{d}||\Delta\bar{d}|\\&\qquad+\int|\nabla\theta_2||\nabla\bar{\theta}|\big(|\bar{\theta}|+|\bar{d}|\big)+\int|\nabla u_1|^2|\bar{\theta}|^2+\int\big(|\nabla u_1|+|\nabla u_2|\big)|\nabla\bar{u}||\bar{\theta}|\\&\qquad+\int\big(|\nabla u_1|^2+|\nabla u_2|^2\big)|\bar{d}||\bar{\theta}|+\int|\nabla\theta_1||\bar{u}||\bar{\theta}|+\int\big(|\Delta d_1|+|\Delta d_2|\big)|\Delta\bar{d}||\bar{\theta}|\\&\qquad+\int\big(|\nabla d_1|^3+|\nabla d_2|^3\big)|\nabla\bar{d}||\bar{\theta}|:=\sum_{i=1}^{16}M_i,\end{aligned}\tag{7.16}$$

where the assumption (1.6) has been used to get $\underline{\mu},\underline{\kappa}$, and the formula $(a\times b)\cdot c=(b\times c)\cdot a$ has been used to eliminate the last term in (7.8) and in (7.10).

Notice that

$$\begin{aligned}\sum_{i=1}^{3} M_i &\lesssim \big[\big\|(\nabla u_1, \Delta d_2)\big\|_3 \|\bar{d}\|_6 + \big\|(\nabla d_1, \nabla d_2)\big\|_\infty \|\nabla \bar{d}\|_2\big] \|\nabla \bar{u}\|_2 + \|\bar{u}\|_3^2 \|\nabla u_1\|_3 \\ &\le C\big(\|\bar{d}\|_{H^1}^2 + \|\bar{u}\|_2^2\big)\big(1 + \|\nabla u_1\|_{H^1}^2 + \big\|(\nabla d_1, \nabla d_2)\big\|_{H^2}^2\big) + \varepsilon \|\nabla \bar{u}\|_2^2, \end{aligned} \tag{7.17}$$

$$\begin{aligned}\sum_{i=4}^{6} M_i &\lesssim \big(\|\nabla d_1\|_\infty \|\bar{u}\|_2 + \|\nabla \bar{u}\|_2\big)\|\bar{d}\|_2 + \big\|(\nabla d_1, \nabla d_2)\big\|_\infty \|\nabla \bar{d}\|_2 \|\bar{d}\|_2 \\ &\quad + \|\bar{d}\|_3^2\big(\|\nabla u_2\|_3 + \|\nabla d_2\|_6^2\big) \\ &\le C\big(\|\bar{d}\|_{H^1}^2 + \|\bar{u}\|_2^2\big)\big(1 + \|\nabla u_1\|_{H^1}^2 + \big\|(\nabla d_1, \nabla d_2)\big\|_{H^2}^2\big) + \varepsilon \|\nabla \bar{u}\|_2^2, \end{aligned} \tag{7.18}$$

$$\begin{aligned}\sum_{i=7}^{9} M_i &\lesssim \|\bar{u}\|_2 \|\Delta \bar{d}\|_2 \|\nabla d_1\|_\infty + \big\|(u_2, \nabla d_1)\big\|_\infty \|\nabla \bar{d}\|_2 \|\Delta \bar{d}\|_2 \\ &\quad + \big(\|\nabla u_2\|_3 + \|\nabla d_2\|_6^2\big)\|\bar{d}\|_6 \|\Delta \bar{d}\|_2 \\ &\le C\big(\|\bar{d}\|_{H^1}^2 + \|\bar{u}\|_2^2\big)\big(1 + \|\nabla u_1\|_{H^1}^2 + \big\|(\nabla d_1, \nabla d_2)\big\|_{H^2}^4\big) + \varepsilon \|\Delta \bar{d}\|_2^2, \end{aligned} \tag{7.19}$$

$$\begin{aligned}\sum_{i=10}^{12} M_i &\lesssim \|\nabla \theta_2\|_6 \|\nabla \bar{\theta}\|_2 \big\|(\bar{\theta}, \bar{d})\big\|_3 + \|\nabla u_1\|_6^2 \|\bar{\theta}\|_3^2 + \big\|(\nabla u_1, \nabla u_2)\big\|_6 \|\nabla \bar{u}\|_2 \|\bar{\theta}\|_3 \\ &\le C\big(\|\bar{d}\|_{H^1}^2 + \|\bar{\theta}\|_2^2\big)\big(1 + \big\|(\nabla u_1, \nabla u_2, \nabla \theta_2)\big\|_{H^1}^2\big)^2 + \varepsilon \big\|(\nabla \bar{u}, \nabla \bar{\theta})\big\|_2^2, \end{aligned} \tag{7.20}$$

$$\begin{aligned}\sum_{i=13}^{16} &\lesssim \big\|(\nabla u_1, \nabla u_2)\big\|_6^2 \|\bar{d}\|_6 \|\bar{\theta}\|_2 + \|\nabla \theta_1\|_6 \|\bar{u}\|_2 \|\bar{\theta}\|_3 \\ &\quad + \big\|(\Delta d_1, \Delta d_2)\big\|_6 \|\Delta \bar{d}\|_2 \|\bar{\theta}\|_3 + \big\|(\nabla d_1, \nabla d_2)\big\|_\infty^3 \|\nabla \bar{d}\|_2 \|\bar{\theta}\|_2 \\ &\le C\big(\|\bar{d}\|_{H^1}^2 + \big\|(\bar{u}, \bar{\theta})\big\|_2^2\big)\big(1 + \big\|(\nabla d_1, \nabla d_2)\big\|_{H^2}^2 + \big\|(\nabla u_1, \nabla u_2, \nabla \theta_2)\big\|_{H^1}^2\big)^2 \\ &\quad + \varepsilon \big\|(\nabla \bar{\theta}, \Delta \bar{d})\big\|_2^2. \end{aligned} \tag{7.21}$$

Then, substituting (7.17)-(7.21) into (7.16), taking ε small enough and using (5.1)-(5.4) for (u_i, d_i, θ_i), $i = 1, 2$ imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|(\bar{u}, \bar{d}, \nabla \bar{d}, \bar{\theta})\big\|_2^2 + \big\|(\nabla \bar{u}, \nabla \bar{d}, \Delta \bar{d}, \nabla \bar{\theta})\big\|_2^2 \le C\big\|(\bar{u}, \bar{d}, \nabla \bar{d}, \bar{\theta})\big\|_2^2. \tag{7.22}$$

Consequently, it follows from the Gronwall inequality and the initial condition $(\bar{u}, \bar{d}, \bar{\theta})|_{t=0} = (0, 0, 0)$ that

$$\sup_{0 \le t \le T}\big(\|\bar{u}\|_2^2 + \|\bar{d}\|_2^2 + \|\bar{\theta}\|_{H^1}^2\big) + \int_0^T \big(\|\nabla \bar{u}\|_2^2 + \|\nabla \bar{d}\|_{H^1}^2 + \|\nabla \bar{\theta}\|_2^2\big) \le 0, \tag{7.23}$$

from which one infers that $\bar{u} = \bar{d} = \bar{\theta} = 0$ a.e. $\Omega \times [0, T]$. The fact that $\nabla \bar{p} = 0$, a.e. $\Omega \times [0, T]$ can also be got.

The proof of uniqueness of the strong solution to the initial-boundary value problem (1.3)-(1.5) is completed, where p is uniquely determined up to a constant.

8 Conclusion

There is a lot literature focused on the isothermal models for liquid crystals, but still few papers considering the non-isothermal ones. Based on the Galerkin approximation, we first established the existence of the local-in-time strong solutions to system (1.3)-(1.5) by a fixed point theorem and energy methods. Furthermore, the uniqueness of the strong solution is also proved by energy methods. Finally, under the assumption of small initial data, we showed the global-in-time existence of the established unique strong solution and obtained the decay rate of the global strong solution. These results for a bounded domain with periodic boundary condition extend the corresponding known results for the isothermal models and is a part of the well-posedness theories for the non-isothermal model. Based on the results in this paper, we shall consider a general bounded domain case and Cauchy problems in the whole space in forthcoming studies. The author believes that the studies of the non-isothermal model might lead us to a more complete understanding of the behavior of the flow of liquid crystals.

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Competing interests
The author declares that there is no competing interest.

Author's contributions
The author declares that this study is independently finished.

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