Hu and Zhang Boundary Value Problems (2017) 2017:88
DOI 10.1186/s13661-017-0819-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence results for a coupled system of fractional differential equations with p-Laplacian operator and infinite-point boundary conditions

Lei Hu[1*] and Shuqin Zhang[2]

*Correspondence: huleimath@163.com
[1]School of Science, Shandong Jiaotong University, Jinan, 250357, China
Full list of author information is available at the end of the article

Abstract

By means of coincidence degree theory, we present the existence of solutions of a coupled system of fractional differential equations with p-Laplacian operator and infinite-point boundary conditions. This paper enriches and extends some existing literature. Finally, an example is given to support our results.

MSC: 26A33; 34B15

Keywords: fractional differential equation; infinite-point boundary value conditions; p-Laplacian; resonance

1 Introduction

In this paper, we study the existence of solutions for higher-order nonlinear fractional differential equations with p-Laplacian operator:

$$\begin{cases} D_{0+}^{\beta_1}\phi_p(D_{0+}^{\alpha_1}u(t)) = f(t, v(t), D_{0+}^{\alpha_2-1}v(t), \dots, D_{0+}^{\alpha_2-(n-1)}v(t)), & t \in (0,1), \\ D_{0+}^{\beta_2}\phi_p(D_{0+}^{\alpha_2}v(t)) = g(t, u(t), D_{0+}^{\alpha_1-1}u(t), \dots, D_{0+}^{\alpha_1-(n-1)}u(t)), & t \in (0,1), \\ u'(0) = \cdots = u^{(n-1)}(0) = D_{0+}^{\alpha_1}u(0) = 0, & u(0) = \sum_{i=1}^{\infty} a_i u(\xi_i), \\ v'(0) = \cdots = v^{(n-1)}(0) = D_{0+}^{\alpha_2}v(0) = 0, & v(0) = \sum_{i=1}^{\infty} b_i v(\eta_i), \end{cases} \tag{1.1}$$

where the p-Laplacian operator is defined as $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_q(s) = \phi_p^{-1}(s)$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \beta_1, \beta_2 < 1$, $n - 1 < \alpha_1, \alpha_2 < n$, $0 < \xi_1 < \xi_2 < \cdots < \xi_i < \cdots < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_i < \cdots < 1$, $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} b_i = 1$, $\sum_{i=1}^{\infty} |a_i| < \infty$, $\sum_{i=1}^{\infty} |b_i| < \infty$, $D_{0+}^{\alpha_1}$, $D_{0+}^{\beta_1}$, $D_{0+}^{\alpha_2}$, $D_{0+}^{\beta_2}$ denote the Caputo fractional derivatives and $f, g : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ are continuous.

The theory of fractional differential equations is a branch of differential equation theory, which occurs more frequently in different research areas and engineering, such as fluid mechanics, control system, viscoelasticity, chemistry, electromagnetic, etc. (see [1–5]). In the last few decades, many authors devoted their attention to the study of resonant boundary value problems for nonlinear fractional differential equations, see [6–19]. Meanwhile, some important results relative to the existence of solutions for a coupled system of fractional differential equations with p-Laplacian operator at resonance have been obtained, see [11–16].

In [15], Hu *et al.* considered the two-point boundary value problem for nonlinear fractional differential equations with p-Laplacian operator at resonance:

$$\begin{cases} D_{0^+}^{\beta}\phi_p(D_{0^+}^{\alpha}u(t)) = f(t, v(t), D_{0^+}^{\delta}u(t)), & t \in (0,1), \\ D_{0^+}^{\gamma}\phi_p(D_{0^+}^{\delta}v(t)) = g(t, u(t), D_{0^+}^{\alpha}u(t)), & t \in (0,1), \\ D_{0^+}^{\alpha}u(0) = D_{0^+}^{\alpha}u(1) = D_{0^+}^{\delta}v(0) = D_{0^+}^{\delta}v(1) = 0, \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$ is the p-Laplacian operator, $0 < \alpha, \beta < 1$, $1 < \alpha + \beta < 2$, $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ $D_{0^+}^{\gamma}$ $D_{0^+}^{\delta}$ denote the Caputo fractional derivatives and $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous.

In [16], Cheng *et al.* considered the two-point boundary value problem for nonlinear fractional p-Laplacian differential equations with $\operatorname{Ker} L = n \geq 2$:

$$\begin{cases} D_{0^+}^{\gamma}\phi_p(D_{0^+}^{\alpha}u(t)) = f(t, v(t)), & t \in (0,1), \\ D_{0^+}^{\gamma}\phi_p(D_{0^+}^{\beta}v(t)) = g(t, u(t)), & t \in (0,1), \\ D_{0^+}^{\alpha}u(0) = D_{0^+}^{\alpha}u(1) = D_{0^+}^{\beta}v(0) = D_{0^+}^{\beta}v(1) = 0, \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$ is the p-Laplacian operator, $0 < \gamma < 1$, $n - 1 < \alpha, \beta < n$, $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$, $D_{0^+}^{\gamma}$ denote the Caputo fractional derivatives and $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous.

In recent years, the subject of infinite-point boundary value problems of fractional differential equations which can extend many previous results have attracted more attention. Most of the results are mainly at nonresonance. For the resonance case, the existing results of fractional differential equations with infinite-point boundary value problems are few. We refer the reader to [20–23] and the references cited therein.

From the above work, we see that recent study on a coupled system of fractional p-Laplacian differential equations is mainly at two-point boundary value problem. The theory for fractional p-Laplacian differential equations with multi-point and even infinite-point at resonance has yet been sufficiently developed. To the best of our knowledge, this is the first paper to study higher order fractional differential equations with p-Laplacian and infinite-point boundary value conditions at resonance. Motivated by the works above, we consider the existence of solutions for BVP (1.1).

The rest of this paper is organized as follows. In Section 2, we give some necessary notations, definitions and lemmas. In Section 3, we study the existence of solutions of (1.1) by the coincidence degree theory due to Mawhin [24]. Finally, an example is given to illustrate our results in Section 4.

2 Preliminaries

We present the necessary definitions and lemmas from fractional calculus theory that will be used to prove our main theorems.

Definition 2.1 ([1]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 ([1]) The Caputo fractional derivative of order $\alpha > 0$ of a function $f \in AC^{n-1}[0,1]$ is given by

$$D^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}\,ds,$$

where $n-1 < \alpha \leq n$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.1 ([1]) *Let $n-1 < \alpha \leq n$, $u \in AC^{n-1}[0,1]$, then*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,\ldots,n-1$.

Lemma 2.2 ([1]) *If $\beta > 0$, $\alpha + \beta > 0$, then the equation*

$$I^{\alpha}_{0+}I^{\beta}_{0+}f(x) = I^{\alpha+\beta}_{0+}f(x)$$

is satisfied for an integrable function f.

Lemma 2.3 ([23]) *For any $u, v \geq 0$, then*

$$\phi_p(u+v) \leq \phi_p(u) + \phi_p(v) \quad \text{if } p < 2;$$
$$\phi_p(u+v) \leq 2^{p-1}\bigl(\phi_p(u) + \phi_p(v)\bigr) \quad \text{if } p \geq 2.$$

Firstly, we briefly recall some definitions on the coincidence degree theory. For more details, see [14].

Let Y, Z be real Banach spaces, $L : \operatorname{dom} L \subset Y \to Z$ be a Fredholm map of index zero and $P : Y \to Y$, $Q : Z \to Z$ be continuous projectors such that

$$\operatorname{Ker} L = \operatorname{Im} P, \qquad \operatorname{Im} L = \operatorname{Ker} Q, \qquad Y = \operatorname{Ker} L \oplus \operatorname{Ker} P, \qquad Z = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \quad \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible. We denote the inverse of this map by K_P.

If Ω is an open bounded subset of Y, the map N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_{P,Q}N = K_P(I-Q)N : \overline{\Omega} \to Y$ is compact.

Theorem 2.1 *Let L be a Fredholm operator of index zero and N be L-compact on $\overline{\Omega}$. Suppose that the following conditions are satisfied:*

(1) *$Lx \neq \lambda Nx$ for each $(x,\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;*
(2) *$Nx \notin \operatorname{Im} L$ for each $x \in \operatorname{Ker} L \cap \partial\Omega$;*
(3) *$\deg(JQN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, where $Q : Z \to Z$ is a continuous projection as above with $\operatorname{Im} L = \operatorname{Ker} Q$ and $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is any isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

3 Main results

In this section, we begin to prove the existence of solutions to problem (1.1). Consider the functions $\phi_1(z) = \sum_{i=1}^{\infty} a_i\xi_i^z$, $\phi_2(z) = \sum_{i=1}^{\infty} b_i\eta_i^z$, $z \in [0,\infty)$. According to $\sum_{i=1}^{\infty} |a_i| < \infty$, $\sum_{i=1}^{\infty} |b_i| < \infty$, one has the series are (uniformly) convergent and thus ϕ_1, ϕ_2 are continuous on $[0,\infty)$.

The following assumption will be used in our main results:

(H_0) There exist z_0, $\tilde{z}_0$ with $z_0 \geq \alpha_1$, $\tilde{z}_0 \geq \alpha_2$ such that $\phi_1(z_0) \cdot \phi_2(\tilde{z}_0) \neq 0$.

The following lemma is fundamental in the proofs of our main results.

Lemma 3.1 *Problem* (1.1) *is equivalent to the following equation*:

$$\begin{cases} D_{0+}^{\alpha_1}u(t) = \phi_q[I_{0+}^{\beta_1}f(t,v(t),D_{0+}^{\alpha_2-1}v(t),\dots,D_{0+}^{\alpha_2-(n-1)}v(t))], & t\in(0,1),\\ D_{0+}^{\alpha_2}v(t) = \phi_q[I_{0+}^{\beta_2}g(t,u(t),D_{0+}^{\alpha_1-1}u(t),\dots,D_{0+}^{\alpha_1-(n-1)}u(t))], & t\in(0,1),\\ u'(0)=\cdots=u^{(n-1)}(0)=0, \qquad u(0)=\sum_{i=1}^{\infty}a_iu(\xi_i),\\ v'(0)=\cdots=v^{(n-1)}(0)=0, \qquad v(0)=\sum_{i=1}^{\infty}b_iv(\eta_i). \end{cases} \tag{3.1}$$

Proof By Lemma 2.1, $D_{0+}^{\beta_1}\phi_p(D_{0+}^{\alpha_1}u(t)) = f(t,v(t),D_{0+}^{\alpha_2-1}v(t),\dots,D_{0+}^{\alpha_2-(n-1)}v(t))$ has the following solution:

$$\phi_p\big(D_{0+}^{\alpha_1}u(t)\big) = I_{0+}^{\beta_1}f\big(t,v(t),D_{0+}^{\alpha_2-1}v(t),\dots,D_{0+}^{\alpha_2-(n-1)}v(t)\big) + c, \quad c\in\mathbb{R}.$$

Substituting $t = 0$ into the above formula, by $D_{0^+}^{\alpha_1}u(0) = 0$, we obtain $c = 0$. Then we have

$$\phi_p\big(D_{0+}^{\alpha_1}u(t)\big) = I_{0+}^{\beta_1}f\big(t,v(t),D_{0+}^{\alpha_2-1}v(t),\dots,D_{0+}^{\alpha_2-(n-1)}v(t)\big). \tag{3.2}$$

Applying the operator ϕ_q to the both sides of (3.2) respectively, we have

$$D_{0+}^{\alpha_1}u(t) = \phi_q\big[I_{0+}^{\beta_1}f\big(t,v(t),D_{0+}^{\alpha_2-1}v(t),\dots,D_{0+}^{\alpha_2-(n-1)}v(t)\big)\big].$$

By a similar argument, we have

$$D_{0+}^{\beta_2}\phi_p\big(D_{0+}^{\alpha_2}v(t)\big) = g\big(t,u(t),D_{0+}^{\alpha_1-1}u(t),\dots,D_{0^+}^{\alpha_1-(n-1)}u(t)\big)$$

is equivalent to

$$D_{0+}^{\alpha_2}v(t) = \phi_q\big[I_{0+}^{\beta_2}g\big(t,u(t),D_{0^+}^{\alpha_1-1}u(t),\dots,D_{0+}^{\alpha_1-(n-1)}u(t)\big)\big].$$

Therefore, BVP (1.1) is rewritten by (3.1)

It is easy to verify that equation (1.1) has a solution (u,v) if and only if (u,v) solves equation (3.1). □

Let $E = C[0,1]$ with the norm $\|x\|_\infty = \max_{0\leq t\leq 1}|x(t)|$. Now, we set $X_1 = \{u(t) : u(t), D_{0+}^{\alpha_1-i}u(t) \in E, i = 1,2,\dots,n-1\}$ with the norm

$$\|u\|_{X_1} = \max\big\{\|u\|_\infty, \big\|D_{0+}^{\alpha_1-1}u\big\|_\infty,\dots,\big\|D_{0+}^{\alpha_1-(n-1)}u\big\|_\infty\big\}$$

and $X_2 = \{v(t) : v(t), D_{0+}^{\alpha_2 - i} v(t) \in E, i = 1, 2, \ldots, n-1\}$ with the norm

$$\|v\|_{X_2} = \max\left\{\|v\|_\infty, \left\|D_{0+}^{\alpha_2-1} v\right\|_\infty, \ldots, \left\|D_{0+}^{\alpha_2-(n-1)} v\right\|_\infty\right\}.$$

Let $Y = X_1 \times X_2$ with the norm $\|(u, v)\|_Y = \max\{\|u\|_{X_1}, \|v\|_{X_2}\}$ and $Z = E \times E$ with the norm $\|(x, y)\|_Z = \max\{\|x\|_\infty, \|y\|_\infty\}$.

Clearly, X and Y are Banach spaces.

Define the linear operator $L_1 : \operatorname{dom} L_1 \to E$ by setting

$$\operatorname{dom} L_1 = \left\{u \in X_1 \middle| u'(0) = \cdots = u^{(n-1)}(0) = 0, u(0) = \sum_{i=1}^{\infty} a_i u(\xi_i)\right\}$$

and

$$L_1 u = D_{0+}^{\alpha_1} u, \quad u \in \operatorname{dom} L_1.$$

Define the linear operator L_2 from $\operatorname{dom} L_2 \to E$ by setting

$$\operatorname{dom} L_2 = \left\{v \in X_2 \middle| v'(0) = \cdots = v^{(n-1)}(0)) = 0, v(0) = \sum_{i=1}^{\infty} b_i v(\eta_i)\right\}$$

and

$$L_2 v = D_{0+}^{\alpha_2} v, \quad v \in \operatorname{dom} L_2.$$

Define the operator $L : \operatorname{dom} L \to Z$ with

$$\operatorname{dom} L = \left\{(u, v) \in Y | u \in \operatorname{dom} L_1, v \in \operatorname{dom} L_2\right\}$$

and

$$L(u, v) = (L_1 u, L_2 v).$$

Let $N : Y \to Z$ be the Nemytskii operator

$$N(u, v) = (N_1 v, N_2 u),$$

where $N_1 : X \to E$ is defined by

$$N_1 v(t) = \phi_q\left[I_{0+}^{\beta_1} f\left(t, v(t), D_{0+}^{\alpha_2-1} v(t), \ldots, D_{0+}^{\alpha_2-(n-1)} v(t)\right)\right]$$

and $N_2 : X \to E$ is defined by

$$N_2 u(t) = \phi_q\left[I_{0+}^{\beta_2} g\left(t, u(t), D_{0+}^{\alpha_1-1} u(t), \ldots, D_{0+}^{\alpha_1-(n-1)} u(t)\right)\right].$$

Then BVP (3.1) can be written as $L(u, v) = N(u, v)$.

Lemma 3.2 *L is defined as above, then*

$$\operatorname{Ker} L = \{(u,v) \in X : (u,v) = (c_0, d_0), c_0, d_0 \in \mathbb{R}\}, \tag{3.3}$$

$$\operatorname{Im} L = \left\{(x,y) \in Z : \sum_{i=1}^{\infty} a_i I_{0+}^{\alpha_1} x(\xi_i) = 0; \sum_{i=1}^{\infty} b_i I_{0+}^{\alpha_2} y(\eta_i) = 0\right\}. \tag{3.4}$$

Proof For $(u,v) \in \operatorname{Ker} L$, then $L_1 u = L_2 v = 0$. By Lemma 2.1, the equation $D_{0+}^{\alpha_1} u(t) = 0$ has solution

$$u(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}.$$

In view of $u^{(i)}(0) = 0$, $i = 1,2,\ldots,n-1$, we get $c_i = 0$, $i = 1,2,\ldots,n-1$. Then $u(t) = c_0$. Similarly, for $v \in \operatorname{Ker} L_2$, we have $v(t) = d_0 \in \mathbb{R}$. Thus, we obtain (3.3).

Next we prove that (3.4) holds. Let $(x,y) \in \operatorname{Im} L$, so there exists $(u,v) \in \operatorname{dom} L$ such that $x(t) = D_{0+}^{\alpha_1} u(t)$, $y(t) = D_{0+}^{\alpha_2} v(t)$. By Lemma 2.1, we have

$$u(t) = I_{0+}^{\alpha_1} x(t) + \sum_{i=0}^{n-1} c_i t^i, \qquad v(t) = I_{0+}^{\alpha_2} y(t) + \sum_{i=0}^{n-1} d_i t^i, \quad c_i, d_i \in \mathbb{R}.$$

In view of $u^{(i)}(0) = v^{(i)}(0) = 0$, $i = 1,2,\ldots,n-1$, we get $c_i = d_i = 0$, $i = 1,2,\ldots,n-1$. Hence,

$$u(t) = I_{0+}^{\alpha_1} x(t) + c_0, \qquad v(t) = I_{0+}^{\alpha_2} y(t) + d_0.$$

According to $u(0) = \sum_{i=1}^{\infty} a_i u(\xi_i)$ and $v(0) = \sum_{i=1}^{\infty} b_i v(\eta_i)$, we have

$$u(0) = I_{0+}^{\alpha_1} x(0) + c_0 = \sum_{i=1}^{\infty} a_i u(\xi_i) = \sum_{i=1}^{\infty} a_i \bigl(I_{0+}^{\alpha_1} x(\xi_i) + c_0\bigr) = \sum_{i=1}^{\infty} a_i I_{0+}^{\alpha_1} x(\xi_i) + c_0,$$

$$v(0) = I_{0+}^{\alpha_2} y(0) + d_0 = \sum_{i=1}^{\infty} b_i v(\xi_i) = \sum_{i=1}^{\infty} b_i \bigl(I_{0+}^{\alpha_2} y(\eta_i) + c_0\bigr) = \sum_{i=1}^{\infty} b_i I_{0+}^{\alpha_2} y(\eta_i) + d_0,$$

that is,

$$\sum_{i=1}^{\infty} a_i I_{0+}^{\alpha_1} x(\xi_i) = 0, \qquad \sum_{i=1}^{\infty} b_i I_{0+}^{\alpha_2} y(\eta_i) = 0.$$

On the other hand, suppose that (x,y) satisfies the above equations. Let $u(t) = I_{0+}^{\alpha_1} x(t)$ and $v(t) = I_{0+}^{\alpha_2} y(t)$, we can prove $(u,v) \in \operatorname{dom} L$ and $L(u,v) = (x,y)$. Then (3.4) holds. □

Lemma 3.3 *The mapping $L : \operatorname{dom} L \subset Y \to Z$ is a Fredholm operator of index zero.*

Proof The linear continuous projector operator $P(u,v) = (P_1 u, P_2 v)$ can be defined as

$$P_1 u = u(0), \qquad P_2 v = v(0).$$

Obviously, $P_1^2 = P_1$ and $P_2^2 = P_2$.

It is clear that

$$\operatorname{Ker} P = \{(u,v) : u(0) = 0, v(0) = 0\}.$$

It follows from $(u,v) = (u,v) - P(u,v) + P(u,v)$ that $Y = \operatorname{Ker} P + \operatorname{Ker} L$. For $(u,u) \in \operatorname{Ker} L \cap \operatorname{Ker} P$, then $u = c_0$, $v = d_0$, $c_0, d_0 \in \mathbb{R}$. Furthermore, by the definition of $\operatorname{Ker} P$, we have $c_0 = d_0 = 0$. Thus, we get

$$Y = \operatorname{Ker} L \oplus \operatorname{Ker} P.$$

By (H_0), the linear operator $Q(x,y) = (Q_1x, Q_2y)$ can be defined as

$$Q_1x(t) = t^{\theta_1} \cdot \frac{\sum_{i=1}^{\infty} a_i I_{0^+}^{\alpha_1} x(\xi_i)}{\sum_{i=1}^{\infty} a_i (I_{0^+}^{\alpha_1} t^{\theta_1})(\xi_i)} = t^{\theta_1} \cdot \frac{\Gamma(1+\alpha_1+\theta_1)}{\phi_1(z_0)\Gamma(1+\theta_1)} \sum_{i=1}^{\infty} a_i I_{0^+}^{\alpha_1} x(\xi_i),$$

$$Q_2y(t) = t^{\theta_2} \cdot \frac{\sum_{i=1}^{\infty} b_i I_{0^+}^{\alpha_2} y(\eta_i)}{\sum_{i=1}^{\infty} b_i (I_{0^+}^{\alpha_2} t^{\theta_2})(\eta_i)} = t^{\theta_2} \cdot \frac{\Gamma(1+\alpha_2+\theta_2)}{\phi_2(\tilde{z}_0)\Gamma(1+\theta_2)} \sum_{i=1}^{\infty} b_i I_{0^+}^{\alpha_2} y(\eta_i),$$

where $\theta_1 = z_0 - \alpha_1$, $\theta_2 = \tilde{z}_0 - \alpha_2$.

Obviously, $Q(x,y) = (Q_1x(t), Q_2y(t)) \cong \mathbb{R}^2$.

For $x(t) \in E$, we have

$$\begin{aligned}
Q_1\bigl(Q_1x(t)\bigr) &= \frac{\Gamma(1+\alpha_1+\theta_1)}{\phi_1(z_0)\Gamma(1+\theta_1)} \sum_{i=1}^{\infty} a_i I_{0^+}^{\alpha_1} x(\xi_i) \cdot Q_1\bigl(t^{\theta_1}\bigr) \\
&= \frac{\Gamma(1+\alpha_1+\theta_1)}{\phi_1(z_0)\Gamma(1+\theta_1)} \sum_{i=1}^{\infty} a_i I_{0^+}^{\alpha_1} x(\xi_i) \cdot t^{\theta_1} \cdot \frac{\Gamma(1+\alpha_1+\theta_1)}{\phi_1(z_0)\Gamma(1+\theta_1)} \cdot \sum_{i=1}^{\infty} a_i \bigl(I_{0^+}^{\alpha_1} t^{\theta_1}\bigr)(\xi_i) \\
&= \frac{\Gamma(1+\alpha_1+\theta_1)}{\phi_1(z_0)\Gamma(1+\theta_1)} \sum_{i=1}^{\infty} a_i I_{0^+}^{\alpha_1} x(\xi_i) \cdot t^{\theta_1} \\
&\quad \cdot \frac{\Gamma(1+\alpha_1+\theta_1)}{\phi_1(z_0)\Gamma(1+\theta_1)} \cdot \sum_{i=1}^{\infty} a_i \frac{\Gamma(1+\theta_1)\xi_i^{\alpha_1+\theta_1}}{\Gamma(1+\alpha_1+\theta_1)} \\
&= \frac{\Gamma(1+\alpha_1+\theta_1)}{\phi_1(z_0)\Gamma(1+\theta_1)} \sum_{i=1}^{\infty} a_i I_{0^+}^{\alpha_1} x(\xi_i) \cdot t^{\theta_1} \cdot \frac{\Gamma(1+\alpha_1+\theta_1)}{\phi_1(z_0)\Gamma(1+\theta_1)} \cdot \sum_{i=1}^{\infty} a_i \frac{\Gamma(1+\theta_1)\xi_i^{z_0}}{\Gamma(1+\alpha_1+\theta_1)} \\
&= \frac{\Gamma(1+\alpha_1+\theta_1)}{\phi_1(z_0)\Gamma(1+\theta_1)} \sum_{i=1}^{\infty} a_i I_{0^+}^{\alpha_1} x(\xi_i) \cdot t^{\theta_1} \cdot \frac{\Gamma(1+\alpha_1+\theta_1)}{\phi_1(z_0)\Gamma(1+\theta_1)} \cdot \frac{\phi_1(z_0)\Gamma(1+\theta_1)}{\Gamma(1+\alpha_1+\theta_1)} \\
&= t^{\theta_1} \cdot \frac{\Gamma(1+\alpha_1+\theta_1)}{\phi_1(z_0)\Gamma(1+\theta_1)} \sum_{i=1}^{\infty} a_i I_{0^+}^{\alpha_1} x(\xi_i) \\
&= Q_1x(t).
\end{aligned}$$

Similarly, $Q_2^2 = Q_2$, that is to say, the operator Q is idempotent. It follows from $(x,y) = (x,y) - Q(x,y) + Q(x,y)$ that $Z = \operatorname{Im} L + \operatorname{Im} Q$. Moreover, by $\operatorname{Ker} Q = \operatorname{Im} L$ and $Q_2^2 = Q_2$, we get $\operatorname{Im} L \cap \operatorname{Im} Q = \{(0,0)\}$. Hence,

$$Z = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

Now, $\operatorname{Ind} L = \dim \operatorname{Ker} L - \operatorname{codim} \operatorname{Im} L = 0$, so L is a Fredholm mapping of index zero. □

For every $(u,v) \in Y$,

$$\|P(u,v)\|_Y = \max\{\|P_1u\|_{X_1}; \|P_2v\|_{X_2}\} = \max\{|u(0)|; |v(0)|\}. \tag{3.5}$$

Furthermore, the operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ can be defined

$$K_P(x,y) = \left(I_{0+}^{\alpha_1}x, I_{0+}^{\alpha_2}y\right).$$

For $(x,y) \in \operatorname{Im} L$, we have

$$LK_P(x,y) = L\left(I_{0+}^{\alpha_1}x, I_{0+}^{\alpha_2}y\right) = \left(D_{0+}^{\alpha_1}I_{0+}^{\alpha_1}x, D_{0+}^{\alpha_2}I_{0+}^{\alpha_2}y\right) = (x,y). \tag{3.6}$$

On the other hand, for $(u,v) \in \operatorname{dom} L \cap \operatorname{Ker} P$, according to Lemma 2.1, we have

$$I_{0+}^{\alpha_1}L_1u(t) = I_{0+}^{\alpha_1}D_{0+}^{\alpha_1}u(t) = u(t) + c_0 + c_1t + \cdots + c_{n-1}t^{n-1},$$
$$I_{0+}^{\alpha_2}L_2v(t) = I_{0+}^{\alpha_2}D_{0+}^{\alpha_2}v(t) = v(t) + d_0 + d_1t + \cdots + d_{n-1}t^{n-1}.$$

By the definitions of $\operatorname{dom} L$ and $\operatorname{Ker} P$, one has $u^{(i)}(0) = v^{(i)}(0)$, $i = 0,1,\dots,n-1$, which implies that $c_i = d_i$, $i = 0,1,\dots,n-1$. Thus, we obtain

$$K_pL(x,y) = \left(I_{0+}^{\alpha_1}D_{0+}^{\alpha_1}x, I_{0+}^{\alpha_2}D_{0+}^{\alpha_2}y\right) = (x,y). \tag{3.7}$$

Combining (3.6) and (3.7), we get $K_P = (L_{\operatorname{dom} L\cap\operatorname{Ker} P})^{-1}$.

For $(x,y) \in \operatorname{Im} L$, we have

$$\begin{aligned}\|K_P(x,y)\|_Y &= \left\|\left(I_{0+}^{\alpha_1}x, I_{0+}^{\alpha_2}y\right)\right\|_Y = \max\left\{\left\|I_{0+}^{\alpha_1}x\right\|_{X_1}; \left\|I_{0+}^{\alpha_2}y\right\|_{X_2}\right\}\\ &\le \max\left\{\max\left\{\left\|I_{0+}^{\alpha_1}x\right\|_\infty, \left\|D_{0+}^{\alpha_1-1}I_{0+}^{\alpha_1}x\right\|_\infty, \dots, \left\|D_{0+}^{\alpha_1-(n-1)}I_{0+}^{\alpha_1}x\right\|_\infty\right\};\right.\\ &\quad \left.\max\left\{\left\|I_{0+}^{\alpha_2}y\right\|_\infty, \left\|D_{0+}^{\alpha_2-1}I_{0+}^{\alpha_2}y\right\|_\infty, \dots, \left\|D_{0+}^{\alpha_2-(n-1)}I_{0+}^{\alpha_2}y\right\|_\infty\right\}\right\}\\ &= \max\{\|x\|_\infty; \|y\|_\infty\}.\end{aligned} \tag{3.8}$$

Again, for $(u,v) \in \Omega_1$, $(u,v) \in \operatorname{dom}(L) \setminus \operatorname{Ker}(L)$, then $(I-P)(u,v) \in \operatorname{dom} L \cap \operatorname{Ker} P$ and $LP(u,v) = (0,0)$, thus from (3.8) we have

$$\begin{aligned}\|(I-P)(u,v)\|_Y &= \|K_PL(I-P)(u,v)\|_Y = \|K_P(L_1u, L_2v)\|_Y\\ &\le \max\{\|N_1v\|_\infty; \|N_2u\|_\infty\}.\end{aligned} \tag{3.9}$$

By similar arguments as in [11] or [12], we have the following lemma. We omit the proof of it.

Lemma 3.4 *$K_P(I-Q)N : Y \to Y$ is completely continuous.*

For simplicity of notation, we set

$$a = \frac{1}{\Gamma(\alpha_1+1)}; \qquad b = \left[\frac{1}{\Gamma(\beta_1+1)}\right]^{q-1}; \qquad \tilde{a} = \frac{1}{\Gamma(\alpha_2+1)}; \qquad \tilde{b} = \left[\frac{1}{\Gamma(\beta_2+1)}\right]^{q-1}.$$

Theorem 3.1 *Assume that* (H_0) *and the following conditions hold.*

(H1) *There exist nonnegative functions* $\psi(t), \tilde{\psi}(t), \varphi_i(t), \tilde{\varphi}_i(t) \in E$, $i = 1, 2, \ldots, n-1$, *such that for* $t \in [0,1]$, $(u_1, u_2, \ldots, u_n), (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, *one has*

$$\begin{aligned} \left|f(t, u_1, u_2, \ldots, u_n)\right| &\leq \psi(t) + \varphi_1(t)|u_1|^{p-1} + \cdots + \varphi_{n-1}(t)|u_n|^{p-1}, \\ \left|g(t, v_1, v_2, \ldots, v_n)\right| &\leq \tilde{\psi}(t) + \tilde{\varphi}_1(t)|v_1|^{p-1} + \cdots + \tilde{\varphi}_{n-1}(t)|v_n|^{p-1}. \end{aligned}$$

(H2) *There exists* $A > 0$ *such that if* $|u| > A$ *or* $|v| > A$, $\forall t \in [0,1]$, *one has*

$$\begin{aligned} &u \cdot \left[\sum_{i=1}^{\infty} a_i \phi_q\left[I_{0+}^{\beta_1} f\left(t, v(t), D_{0+}^{\alpha_2-1} v(t), \ldots, D_{0^+}^{\alpha_2-(n-1)} v(t)\right)\right]\Big|_{t=\xi_i}\right] > 0, \\ &v \cdot \left[\sum_{i=1}^{\infty} b_i \phi_q\left[I_{0+}^{\beta_2} g\left(t, u(t), D_{0+}^{\alpha_1-1} u(t), \ldots, D_{0^+}^{\alpha_1-(n-1)} u(t)\right)\right]\Big|_{t=\eta_i}\right] > 0, \end{aligned}$$

or

$$\begin{aligned} &u \cdot \left[\sum_{i=1}^{\infty} a_i \phi_q\left[I_{0+}^{\beta_1} f\left(t, v(t), D_{0+}^{\alpha_2-1} v(t), \ldots, D_{0^+}^{\alpha_2-(n-1)} v(t)\right)\right]\Big|_{t=\xi_i}\right] < 0, \\ &v \cdot \left[\sum_{i=1}^{\infty} b_i \phi_q\left[I_{0+}^{\beta_2} g\left(t, u(t), D_{0+}^{\alpha_1-1} u(t), \ldots, D_{0^+}^{\alpha_1-(n-1)} u(t)\right)\right]\Big|_{t=\eta_i}\right] < 0. \end{aligned}$$

Then BVP (3.1) *has at least a solution in X provided that*

$$\begin{aligned} &\max\left\{2^{q-1}\tilde{a}\tilde{b}\tilde{c} + 2^{q-1}bc, 2^{q-1}abc + 2^{q-1}\tilde{b}\tilde{c}, \right. \\ &\quad \left. 2^{q-1}abc + 2^{q-1}bc, 2^{q-1}\tilde{a}\tilde{b}\tilde{c} + 2^{q-1}\tilde{b}\tilde{c}\right\} < 1 \quad \text{for } p < 2, \qquad (3.10) \\ &\max\{\tilde{a}\tilde{b}\tilde{c} + bc, abc + \tilde{b}\tilde{c}, abc + bc, \tilde{a}\tilde{b}\tilde{c} + \tilde{b}\tilde{c}\} < 1 \quad \text{for } p \geq 2, \qquad (3.11) \end{aligned}$$

where $c = (\sum_{i=1}^{n-1} \|\varphi_i(t)\|_\infty)^{q-1}$ *and* $\tilde{c} = (\sum_{i=1}^{n-1} \|\tilde{\varphi}_i(t)\|_\infty)^{q-1}$.

Proof According to the definitions of N_1 and N_2, we have the following inequalities.

For $1 < p \leq 2$, one has

$$\begin{aligned} \|N_1 v\|_\infty &= \left\|\phi_q\left[I_{0+}^{\beta_1} f\left(t, v(t), D_{0+}^{\alpha_2-1} v(t), \ldots, D_{0+}^{\alpha_2-(n-1)} v(t)\right)\right]\right\|_\infty \\ &= \max\left|I_{0+}^{\beta_1} f\left(s, v(s), D_{0+}^{\alpha_2-1} v(s), \ldots, D_{0+}^{\alpha_2-(n-1)} v(s)\right)\right|^{q-1} \\ &\leq \left|\frac{1}{\Gamma(\beta_1+1)}\left[\|\psi\|\infty + \|v\|_{X_2}^{p-1} \cdot \sum_{i=1}^{n-1}\left\|\varphi_i(t)\right\|_\infty\right]\right|^{q-1} \\ &\leq 2^{q-1} b\|\psi\|_\infty^{q-1} + 2^{q-1} b\left(\sum_{i=1}^{n-1}\left\|\varphi_i(t)\right\|_\infty\right)^{q-1} \cdot \|v\|_{X_2} \\ &= 2^{q-1} b\|\psi\|_\infty^{q-1} + 2^{q-1} bc \cdot \|v\|_{X_2} \qquad (3.12) \end{aligned}$$

and

$$\begin{aligned}\|N_2u\|_\infty &= \left\|\phi_q\left[I_{0+}^{\beta_2}g\left(t,u(t),D_{0+}^{\alpha_1-1}u(t),\dots,D_{0+}^{\alpha_1-(n-1)}u(t)\right)\right]\right\|_\infty \\ &= \max\left|I_{0+}^{\beta_2}g\left(t,u(t),D_{0+}^{\alpha_1-1}u(t),\dots,D_{0+}^{\alpha_1-(n-1)}u(t)\right)\right|^{q-1} \\ &\le \left|\frac{1}{\Gamma(\beta_2+1)}\left[\|\tilde{\psi}\|_\infty + \|u\|_{X_1}^{p-1}\cdot\sum_{i=1}^{n-1}\left\|\varphi_i(t)\right\|_\infty\right]\right|^{q-1} \\ &\le 2^{q-1}\tilde{b}\|\tilde{\psi}\|_\infty^{q-1} + 2^{q-1}\tilde{b}\left(\sum_{i=1}^{n-1}\left\|\tilde{\varphi}_i(t)\right\|_\infty\right)^{q-1}\cdot\|u\|_{X_1} \\ &= 2^{q-1}\tilde{b}\|\tilde{\psi}\|_\infty^{q-1} + 2^{q-1}\tilde{b}\tilde{c}\cdot\|u\|_{X_1}. \end{aligned} \tag{3.13}$$

By the similar proof of (3.12) and (3.13), one has

$$\|N_1v\|_\infty \le b\|\psi\|_\infty^{q-1} + bc\cdot\|v\|_{X_2} \quad \text{for } p\ge 2, \tag{3.14}$$

$$\|N_2u\|_\infty \le \tilde{b}\|\tilde{\psi}\|_\infty^{q-1} + \tilde{b}\tilde{c}\cdot\|u\|_{X_1} \quad \text{for } p\ge 2. \tag{3.15}$$

Let

$$\Omega_1 = \left\{(u,v)\in \operatorname{dom}L\setminus\operatorname{Ker}L : L(u,v)=\lambda N(u,v), \lambda\in(0,1)\right\}.$$

First, we give a proof that for $1 < p \le 2$, Ω_1 is bounded.

Let $L(u,v) = \lambda N(u,v) \in \operatorname{Im}L = \operatorname{Ker}Q$, that is, $L_1u = \lambda N_1v \in \operatorname{Ker}Q_1$ and $L_2v = \lambda N_2u \in \operatorname{Ker}Q_2$. By the definition of $\operatorname{Ker}Q_1$ and $\operatorname{Ker}Q_2$, we have

$$\sum_{i=1}^{\infty} a_i\cdot\lambda\phi_q\left[I_{0+}^{\beta_1}f\left(t,v(t),D_{0+}^{\alpha_2-1}v(t),\dots,D_{0^+}^{\alpha_2-(n-1)}v(t)\right)\right]_{t=\xi_i} = 0,$$

$$\sum_{i=1}^{\infty} b_i\cdot\lambda\phi_q\left[I_{0+}^{\beta_2}g\left(t,u(t),D_{0+}^{\alpha_1-1}u(t),\dots,D_{0^+}^{\alpha_1-(n-1)}u(t)\right)\right]_{t=\eta_i} = 0.$$

According to (H2), there exist $t_0, t_1 \in (0,1)$ such that $|u(t_0)| \le A$ and $|v(t_1)| \le A$. Again, $L_1u = \lambda N_1v$, $u \in \operatorname{dom}L_1\setminus\operatorname{Ker}L_1$, that is, $D_{0^+}^{\alpha_1}u = \lambda N_1v$, we have

$$u(t) = \frac{\lambda}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\phi_q\left[I_{0+}^{\beta_1}f\left(s,v(s),D_{0+}^{\alpha_2-1}v(s),\dots,D_{0^+}^{\alpha_2-(n-1)}v(s)\right)\right]ds + c_0.$$

Substituting $t = t_0$ into the above equation, we get

$$u(t_0) = \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{t_0} (t_0-s)^{\alpha_1-1}\phi_q\left[I_{0+}^{\beta_1}f\left(s,v(s),D_{0+}^{\alpha_2-1}v(s),\dots,D_{0^+}^{\alpha_2-(n-1)}v(s)\right)\right]ds + c_0.$$

So, we obtain

$$\begin{aligned} u(t)-u(t_0) &= \frac{\lambda}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\phi_q\left[I_{0+}^{\beta_1}f\left(s,v(s),D_{0+}^{\alpha_2-1}v(s),\dots,D_{0^+}^{\alpha_2-(n-1)}v(s)\right)\right]ds \\ &\quad - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{t_0} (t_0-s)^{\alpha_1-1}\phi_q\left[I_{0+}^{\beta_1}f\left(s,v(s),D_{0+}^{\alpha_2-1}v(s),\dots,D_{0^+}^{\alpha_2-(n-1)}v(s)\right)\right]ds. \end{aligned}$$

Together with $|u(t_0)| \le A$ and (3.12), we have

$$\begin{aligned}
|u(0)| &\le |u(t_0)| + \left| \frac{\lambda}{\Gamma(\alpha_1)} \int_0^{t_0} (t_0 - s)^{\alpha_1 - 1} \phi_q\big[I_{0+}^{\beta_1} f\big(s, v(s), D_{0+}^{\alpha_2 - 1} v(s), \ldots, D_{0^+}^{\alpha_2 - (n-1)} v(s)\big)\big]\, ds \right| \\
&\le A + \frac{1}{\Gamma(\alpha_1)} \int_0^{t_0} (t_0 - s)^{\alpha_1 - 1} \big|\phi_q\big[I_{0+}^{\beta_1} f\big(s, v(s), D_{0+}^{\alpha_2 - 1} v(s), \ldots, D_{0^+}^{\alpha_2 - (n-1)} v(s)\big)\big]\big|\, ds \\
&= A + \frac{1}{\Gamma(\alpha_1)} \cdot \big(2^{q-1} b \|\psi\|_\infty^{q-1} + 2^{q-1} bc \cdot \|v\|_{X_2}\big) \cdot \int_0^{t_0} (t_0 - s)^{\alpha_1 - 1}\, ds \\
&\le A + 2^{q-1} ab \|\psi\|_\infty^{q-1} + 2^{q-1} abc \cdot \|v\|_{X_2}.
\end{aligned} \tag{3.16}$$

Similarly, by (3.13), we obtain

$$\begin{aligned}
|v(0)| &\le |v(t_0)| \\
&\quad + \left| \frac{\lambda}{\Gamma(\alpha_2)} \int_0^{t_0} (t_0 - s)^{\alpha_2 - 1} \phi_q\big[I_{0+}^{\beta_2} g\big(s, u(s), D_{0+}^{\alpha_1 - 1} u(s), \ldots, D_{0^+}^{\alpha_1 - (n-1)} u(s)\big)\big]\, ds \right| \\
&\le A + \frac{1}{\Gamma(\alpha_2)} \int_0^{t_0} (t_0 - s)^{\alpha_2 - 1} \big|\phi_q\big[I_{0+}^{\beta_2} g\big(s, u(s), D_{0+}^{\alpha_1 - 1} u(s), \ldots, D_{0^+}^{\alpha_1 - (n-1)} u(s)\big)\big]\big|\, ds \\
&= A + \frac{1}{\Gamma(\alpha_2)} \cdot \big(2^{q-1} \tilde{b} \|\tilde{\psi}\|_\infty^{q-1} + 2^{q-1} \tilde{b}\tilde{c} \cdot \|u\|_{X_1}\big) \cdot \int_0^{t_0} (t_0 - s)^{\alpha_2 - 1}\, ds \\
&\le A + 2^{q-1} \tilde{a}\tilde{b} \|\tilde{\psi}\|_\infty^{q-1} + 2^{q-1} \tilde{a}\tilde{b}\tilde{c} \cdot \|u\|_{X_1}.
\end{aligned} \tag{3.17}$$

For $(u, v) \in \Omega_1$, by (3.5) and (3.9), we have

$$\begin{aligned}
\|(u, v)\|_Y &= \|P(u, v) + (I - P)(u, v)\|_Y \le \|P(u, v)\|_Y + \|(I - P)(u, v)\|_Y \\
&\le \max\big\{|u(0)| + \|N_1 v\|_\infty; |u(0)| + \|N_2 u\|_\infty; \\
&\qquad |v(0)| + \|N_1 v\|_\infty; |v(0)| + \|N_2 u\|_\infty\big\}.
\end{aligned}$$

The following proof is divided into four cases.

Case 1. $\|(u, v)\|_Y \le |u(0)| + \|N_1 v\|_\infty$.

By (3.12) and (3.16), we have

$$\begin{aligned}
\|v\|_{X_2} &\le \|(u, v)\|_Y \le |u(0)| + \|N_1 v\|_\infty \\
&\le A + 2^{q-1} ab \|\psi\|_\infty^{q-1} + 2^{q-1} abc \cdot \|v\|_{X_2} + 2^{q-1} b \|\psi\|_\infty^{q-1} + 2^{q-1} bc \cdot \|v\|_{X_2} \\
&= A + 2^{q-1} ab \|\psi\|_\infty^{q-1} + 2^{q-1} b \|\psi\|_\infty^{q-1} + \big(2^{q-1} abc + 2^{q-1} bc\big) \cdot \|v\|_{X_2}.
\end{aligned}$$

According to (3.10), we can derive

$$\|v\|_{X_2} \le \frac{A + 2^{q-1} ab \|\psi\|_\infty^{q-1} + 2^{q-1} b \|\psi\|_\infty^{q-1}}{1 - (2^{q-1} abc + 2^{q-1} bc)} := M_1.$$

Thus, Ω_1 is bounded.

Case 2. $\|(u, v)\|_Y \le |u(0)| + \|N_2 u\|_\infty$.

By (3.13) and (3.16), we have

$$\begin{aligned}\|(u,v)\|_Y &\le |u(0)| + \|N_2 u\|_\infty \\ &\le A + 2^{q-1}ab\|\psi\|_\infty^{q-1} + 2^{q-1}abc\cdot\|v\|_{X_2} + 2^{q-1}\tilde{b}\|\tilde{\psi}\|_\infty^{q-1} + 2^{q-1}\tilde{b}\tilde{c}\cdot\|u\|_{X_1} \\ &= A + 2^{q-1}ab\|\psi\|_\infty^{q-1} + 2^{q-1}\tilde{b}\|\tilde{\psi}\|_\infty^{q-1} + 2^{q-1}abc\cdot\|v\|_{X_2} + 2^{q-1}\tilde{b}\tilde{c}\cdot\|u\|_{X_1} \\ &\le A + 2^{q-1}ab\|\psi\|_\infty^{q-1} + 2^{q-1}\tilde{b}\|\tilde{\psi}\|_\infty^{q-1} + \left(2^{q-1}abc + 2^{q-1}\tilde{b}\tilde{c}\right)\cdot\|(u,v)\|_Y.\end{aligned}$$

By (3.10), we can derive

$$\|(u,v)\|_Y \le \frac{A + 2^{q-1}ab\|\psi\|_\infty^{q-1} + 2^{q-1}\tilde{b}\|\tilde{\psi}\|_\infty^{q-1}}{1 - 2^{q-1}abc - 2^{q-1}\tilde{b}\tilde{c}} := M_2.$$

Then Ω_1 is bounded.

Case 3. $\|(u,v)\|_Y \le |v(0)| + \|N_1 v\|_\infty$.

According to (3.12) and (3.17), we have

$$\begin{aligned}\|(u,v)\|_Y &\le |v(0)| + \|N_1 v\|_\infty \\ &\le A + 2^{q-1}\tilde{a}\tilde{b}\|\tilde{\psi}\|_\infty^{q-1} + 2^{q-1}\tilde{a}\tilde{b}\tilde{c}\cdot\|u\|_{X_1} + 2^{q-1}b\|\psi\|_\infty^{q-1} + 2^{q-1}bc\cdot\|v\|_{X_2} \\ &= A + 2^{q-1}\tilde{a}\tilde{b}\|\tilde{\psi}\|_\infty^{q-1} + 2^{q-1}b\|\psi\|_\infty^{q-1} + 2^{q-1}\tilde{a}\tilde{b}\tilde{c}\cdot\|u\|_{X_1} + 2^{q-1}bc\cdot\|v\|_{X_2} \\ &\le A + 2^{q-1}\tilde{a}\tilde{b}\|\tilde{\psi}\|_\infty^{q-1} + 2^{q-1}b\|\psi\|_\infty^{q-1} + \left(2^{q-1}\tilde{a}\tilde{b}\tilde{c} + 2^{q-1}bc\right)\cdot\|(u,v)\|_Y.\end{aligned}$$

By (3.10), we have

$$\|(u,v)\|_Y \le \frac{A + 2^{q-1}\tilde{a}\tilde{b}\|\tilde{\psi}\|_\infty^{q-1} + 2^{q-1}b\|\psi\|_\infty^{q-1}}{1 - (2^{q-1}\tilde{a}\tilde{b}\tilde{c} + 2^{q-1}bc)} := M_3.$$

Then Ω_1 is bounded.

Case 4. $\|(u,v)\|_Y \le |v(0)| + \|N_2 u\|_\infty$.

According to (3.13) and (3.17), we have

$$\begin{aligned}\|u\|_{X_1} \le \|(u,v)\|_Y &\le |v(0)| + \|N_2 u\|_\infty \\ &\le A + 2^{q-1}\tilde{a}\tilde{b}\|\tilde{\psi}\|_\infty^{q-1} + 2^{q-1}\tilde{a}\tilde{b}\tilde{c}\cdot\|u\|_{X_1} + 2^{q-1}\tilde{b}\|\tilde{\psi}\|_\infty^{q-1} + 2^{q-1}\tilde{b}\tilde{c}\cdot\|u\|_{X_1} \\ &= A + 2^{q-1}\tilde{a}\tilde{b}\|\tilde{\psi}\|_\infty^{q-1} + 2^{q-1}\tilde{b}\|\tilde{\psi}\|_\infty^{q-1} + \left(2^{q-1}\tilde{a}\tilde{b}\tilde{c} + 2^{q-1}\tilde{b}\tilde{c}\right)\cdot\|u\|_{X_1}.\end{aligned}$$

By (3.10), we get

$$\|u\|_{X_1} \le \frac{A + 2^{q-1}\tilde{a}\tilde{b}\|\tilde{\psi}\|_\infty^{q-1} + 2^{q-1}\tilde{b}\|\tilde{\psi}\|_\infty^{q-1}}{1 - (2^{q-1}\tilde{a}\tilde{b}\tilde{c} + 2^{q-1}\tilde{b}\tilde{c})} := M_4.$$

Then Ω_1 is bounded.

Therefore, we have proved that Ω_1 is bounded for $1 < p \le 2$. By similar arguments as the above proof, according to (3.11), (3.14) and (3.15), we can prove that Ω_1 is also bounded for $p > 2$. We omit the proof of it.

Let

$$\Omega_2 = \{(u,v) \in \operatorname{Ker} L : N(u,v) \in \operatorname{Im} L\}.$$

Let $(u,v) \in \operatorname{Ker} L$, so we have $u = c_0$, $v = d_0$. In view of $N(u,v) = (N_1 v, N_2 u) \in \operatorname{Im} L = \operatorname{Ker} Q$, we have $Q_1(N_1 v) = 0$, $Q_2(N_2 u) = 0$, that is,

$$\sum_{i=1}^{\infty} a_i \phi_q \bigl[I_{0+}^{\beta_1} f\bigl(t, v(t), D_{0+}^{\alpha_2 - 1} v(t), \ldots, D_{0^+}^{\alpha_2-(n-1)} v(t)\bigr)\bigr]\big|_{t=\xi_i} = 0,$$
$$\sum_{i=1}^{\infty} b_i \phi_q \bigl[I_{0+}^{\beta_2} g\bigl(t, u(t), D_{0+}^{\alpha_1 - 1} u(t), \ldots, D_{0^+}^{\alpha_1-(n-1)} u(t)\bigr)\bigr]\big|_{t=\eta_i} = 0.$$

By (H2), there exist constants $t_0, t_1 \in [0,1]$ such that

$$\bigl|u(t_0)\bigr| = |c_0| \le A, \qquad \bigl|v(t_1)\bigr| = |d_0| \le A.$$

Therefore, Ω_2 is bounded.

Let

$$\Omega_3 = \{(u,v) \in \operatorname{Ker} L : \lambda(u,v) + (1-\lambda)QN(u,v) = (0,0), \lambda \in [0,1]\}.$$

For $(u,v) \in \operatorname{Ker} L$, so we have $u = c_0$ and $v = d_0$. By the definition of the set Ω_3, we have

$$\lambda c_0 + (1-\lambda)Q_1 N_1(d_0) = 0, \qquad \lambda d_0 + (1-\lambda)Q_2 N_2(c_0) = 0. \tag{3.18}$$

If $\lambda = 0$, similar to the proof of the boundedness of Ω_2, we have $|c_0| \le A$ and $|d_0| \le A$. If $\lambda = 1$, we have $c_0 = d_0 = 0$. If $\lambda \in (0,1)$, we also have $|c_0| \le A$ and $|d_0| \le A$. Otherwise, if $|c_0| > A$ or $|d_0| > A$, in view of the first part of (H2), we obtain

$$\lambda c_0^2 + (1-\lambda)c_0 \cdot Q_1 N_1(d_0) > 0, \qquad \lambda d_0^2 + (1-\lambda)d_0 \cdot Q_2 N_2(c_0) > 0,$$

which contradict (3.18). Thus, Ω_3 is bounded.

If the second part of (H2) holds, then we can prove that the set

$$\Omega_3' = \{(u,v) \in \operatorname{Ker} L : -\lambda(u,v) + (1-\lambda)QN(u,v) = (0,0), \lambda \in [0,1]\}$$

is bounded.

Finally, let Ω to be a bounded open set of Y such that $\bigcup_{i=1}^{3} \overline{\Omega_i} \subset \Omega$. By Lemma 3.4, N is L-compact on Ω. Then, by the above arguments, we get

(1) $L(u,v) \neq \lambda N(u,v)$, for every $(u,v) \in [(\operatorname{dom} L \setminus KerL) \cap \partial\Omega] \times (0,1)$;
(2) $N(u,v) \notin \operatorname{Im} L$ for every $(u,v) \in \operatorname{Ker} L \cap \partial\Omega$;
(3) Let $H((u,v),\lambda) = \pm\lambda I(u,v) + (1-\lambda)JQN(u,v)$, where I is the identical operator. Via the homotopy property of degree, we obtain that

$$\begin{aligned}\deg(JQN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= \deg\bigl(H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0\bigr) \\ &= \deg\bigl(H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0\bigr)\end{aligned}$$

$$= \deg(I, \Omega \cap \operatorname{Ker} L, 0)$$
$$= 1 \neq 0.$$

Applying Theorem 2.1, we conclude that $Lu = Nu$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$. □

4 Example

Let us consider the following fractional differential equations with p-Laplacian operator at resonance:

$$\begin{cases} D_{0+}^{0.6}\phi_3(D_{0+}^{2.6}u(t)) = f(t, v(t), D_{0+}^{1.8}v(t), D_{0+}^{0.8}v(t)), & t \in (0,1), \\ D_{0+}^{0.7}\phi_3(D_{0+}^{2.8}u(t)) = f(t, u(t), D_{0+}^{1.6}u(t), D_{0+}^{0.6}u(t)), & t \in (0,1), \\ u'(0) = u''(0) = D_{0+}^{2.6}u(0) = 0, \qquad u(0) = \sum_{i=1}^{\infty} \frac{1}{2^i} u(\frac{1}{2i}), \\ v'(0) = v''(0) = D_{0+}^{2.8}v(0) = 0, \qquad v(0) = \sum_{i=1}^{\infty} \frac{2}{3^i} u(\frac{1}{3i}), \end{cases} \tag{4.1}$$

where

$$f(t, x_1, x_2, x_3) = \frac{t}{10} + \frac{1}{10}x_1^2 + \frac{|\sin x_2|}{20} + \frac{|\arctan x_3|}{10\pi},$$
$$g(t, y_1, y_2, y_3) = \frac{t^2}{20} + \frac{1}{20}y_1^2 + \frac{\cos^2 y_2}{20} + \frac{e^{-|y_3|}}{40}.$$

Corresponding to BVP (1.1), we have that $\alpha_1 = 2.6$, $\beta_1 = 0.6$, $\alpha_2 = 2.8$, $\beta_2 = 0.7$, $n = 3$, $p = 3$, $q = 1.5$, $a = (\Gamma(\alpha_1 + 1))^{-1} = (\Gamma(3.6))^{-1} \approx 0.269$, $b = (\Gamma(\beta_1 + 1))^{1-q} = (\Gamma(1.6))^{-0.5} \approx 1.058$, $\tilde{a} = (\Gamma(\alpha_2 + 1))^{-1} = (\Gamma(3.8))^{-1} \approx 0.213$, $\tilde{b} = (\Gamma(\beta_2 + 1))^{1-q} = (\Gamma(1.7))^{-0.5} \approx 1.049$, $a_i = \frac{1}{2^i}$, $\xi_i = \frac{1}{2i}$, $b_i = \frac{2}{3^i}$, $\eta_i = \frac{1}{3i}$, $i = 1, 2, \ldots$. Then we have $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} |a_i| = \sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} |b_i| = 1$. Taking $z_0 = \tilde{z}_0 = 3$, we have

$$\phi_1(z_0)\phi_2(\tilde{z}_0) = \sum_{i=1}^{\infty} a_i \xi_i^{z_0} \cdot \sum_{i=1}^{\infty} b_i \eta_i^{\tilde{z}_0} = \sum_{i=1}^{\infty} \frac{1}{2^i}\left(\frac{1}{2i}\right)^3 \cdot \sum_{i=1}^{\infty} \frac{2}{3^i}\left(\frac{1}{3i}\right)^3 \neq 0,$$

which implies that (H_0) holds.

By a simple proof, we have

$$\left|f(t, x_1, x_2, x_3)\right| = \left|\frac{t}{10} + \frac{1}{10}x_1^2 + \frac{|\sin x_2|}{20} + \frac{|\arctan x_3|}{10\pi}\right| \leq \frac{1}{5} + \frac{1}{10}x_1^2,$$
$$\left|g(t, y_1, y_2, y_3)\right| = \left|\frac{t^2}{20} + \frac{1}{20}y_1^2 + \frac{\cos^2 y_2}{20} + \frac{e^{-|y_3|}}{40}\right| \leq \frac{1}{8} + \frac{1}{20}x_1^2.$$

Choose $\psi(t) = \frac{1}{5}$, $\varphi_1(t) = \frac{1}{10}$, $\varphi_2 = \varphi_3 = 0$, $\tilde{\psi}(t) = \frac{1}{8}$, $\tilde{\varphi}_1(t) = \frac{1}{20}$, $\tilde{\varphi}_2 = \tilde{\varphi}_3 = 0$, then we have (H1) of Theorem 3.1 is satisfied.

By a simple computation, we have $c = (\sum_{i=1}^{n-1} \|\varphi_i(t)\|_\infty)^{q-1} = (\varphi_1)^{q-1} = \sqrt{0.1} \approx 0.316$, $\tilde{c} = (\sum_{i=1}^{n-1} \|\tilde{\varphi}_i(t)\|_\infty)^{q-1} = (\tilde{\varphi}_1)^{q-1} = \sqrt{0.05} \approx 0.224$, $\tilde{a}\tilde{b}\tilde{c} + bc \approx 0.287$, $abc + \tilde{b}\tilde{c} \approx 0.298$, $abc + bc \approx 0.301$, $\tilde{a}\tilde{b}\tilde{c} + \tilde{b}\tilde{c} \approx 0.240$. So, (3.11) holds.

In addition, by choosing $A = 1$, we have if $u > 1$, or $v > 1$, then f, g are positive functions. So, the first inequality of (H2) is satisfied.

Thus, all the conditions of Theorem 3.1 are satisfied; consequently, its conclusion implies that problem (4.1) has a solution on $[0, 1]$.

5 Conclusion

In this paper, we have obtained the existence of solutions for a coupled system of fractional differential equations with p-Laplacian operator and infinite-point boundary conditions at resonance. We base our analysis on the known coincidence degree theory. The issue on the existence of solutions of infinite-point boundary value problems is interesting. As applications, an example is presented to illustrate the main results. In the future, we will consider the positive solutions for the fractional infinite-point boundary value problems at resonance.

Acknowledgements
We are grateful to the referees for their careful reading of the manuscript. Their comments and suggestions were of great importance in helping to improve the quality of the paper. This work is supported by the National Natural Science Foundation of China (Grant No. 11371364).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this article. They read and approved the final manuscript.

Author details
[1]School of Science, Shandong Jiaotong University, Jinan, 250357, China. [2]School of Science, China University of Mining and Technology, Beijing, 100083, China.

Publisher's Note

Received: 5 December 2016 Accepted: 30 May 2017 Published online: 07 June 2017

References

1. Kilbas, AA, Srivastava, HH, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
2. Oldham, KB, Spanier, J: The Fractional Calculus. Academic Press, New York (1974)
3. Gaul, L, Klein, P, Kempfle, S: Damping description involving fractional operators. Mech. Syst. Signal Process. **5**, 81-88 (1991)
4. Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
5. Ahmad, B, Agarwal, PR, Alsaedi, A: Fractional differential equations and inclusions with semiperiodic and three-point boundary conditions. Bound. Value Probl. **2016**, 28 (2016)
6. Liu, R, Kou, C, Xie, X: Existence results for a coupled system of nonlinear fractional boundary value problems at resonance. Math. Probl. Eng. **2013**, 1-9 (2013)
7. Ahmad, B, Ntouyas, SK, Agarwal, RP, Alsaedi, A: Existence results for sequential fractional integro-differential equations with nonlocal multi-point and strip conditions. Bound. Value Probl. **2016**, 205 (2016)
8. Agarwal, RP, Ntouyas, SK, Ahmad, B, Alzahrani, AK: Hadamard-type fractional functional differential equations and inclusions with retarded and advanced arguments. Adv. Differ. Equ. **2016**, 92 (2016)
9. Jiang, W: Solvability of fractional differential equations with p-Laplacian at resonance. Appl. Math. Comput. **260**, 48-56 (2015)
10. Kosmatov, N: A boundary value problem of fractional order at resonance. Electron. J. Differ. Equ. **2010**, 135 (2010)
11. Hu, L, Zhang, S: On existence results for nonlinear fractional differential equations involving the p-Laplacian at resonance. Mediterr. J. Math. **13**, 955-966 (2016)
12. Hu, L, Zhang, S, Shi, A: Existence result for nonlinear fractional differential equation with p-Laplacian operator at resonance. J. Appl. Math. Comput. **48**, 519-532 (2015)
13. Shen, T, Liu, W, Chen, T, Shen, X: Solvability of fractional multi-point boundary-value problems with p-Laplacian operator at resonance. Electron. J. Differ. Equ. **2014**, 58 (2014)
14. Chen, T, Liu, W, Hu, Z: A boundary value problem for fractional differential equation with p-Laplacian operator at resonance. Nonlinear Anal. **75**, 3210-3217 (2012)
15. Hu, Z, Liu, W, Liu, J: Existence of solutions for a coupled system of fractional p-Laplacian equations at resonance. Adv. Differ. Equ. **2013**, 312 (2013)
16. Cheng, L, Liu, W, Ye, Q: Boundary value problem for a coupled system of fractional differential equations with p-Laplacian operator at resonance. Electron. J. Differ. Equ. **2014**, 60 (2014)
17. Kosmatov, N, Jiang, W: Resonant functional problems of fractional order. Chaos Solitons Fractals **91**, 573-579 (2016)
18. Aljoudi, S, Ahmad, B, Nieto, JJ, Alsaedi, A: A coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions. Chaos Solitons Fractals **91**, 39-46 (2016)
19. Alsaedi, A, Aljoudi, S, Ahmad, B: Existence of solutions for Riemann-Liouville type coupled systems of fractional integro-differential equations and boundary conditions. Electron. J. Differ. Equ. **2016**, 211 (2016)

20. Zhang, X: Positive solutions for a class of singular fractional differential equation with infinite-point boundary value conditions. Appl. Math. Lett. **39**, 22-27 (2015)
21. Gao, H, Han, X: Existence of positive solutions for fractional differential equation with nonlocal boundary condition. Int. J. Differ. Equ. **2011**, 328394 (2011)
22. Zhong, Q, Zhang, X: Positive solution for higher-order singular infinite-point fractional differential equation with p-Laplacian. Adv. Differ. Equ. **2016**, 11 (2016)
23. Ge, F, Zhou, H, Kou, C: Existence of solutions for coupled fractional differential equation with infinitely many points boundary conditions at resonance on an unbounded domain. Differ. Equ. Dyn. Syst. **24**, 1-17 (2016)
24. Mawhin, J: Topological degree and boundary value problems for nonlinear differential equations in topological methods for ordinary differential equations. Lect. Notes Math. **1537**, 74-142 (1993)