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Existence and asymptotic expansion of the weak solution for a wave equation with nonlinear source containing nonlocal term

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Abstract
In this paper, we consider the Robin problem for a wave equation with nonlinear source containing nonlocal term. Using the Faedo-Galerkin method and the linearization method for nonlinear term, the existence and uniqueness of a weak solution are proved. An asymptotic expansion of high order in a small parameter of a weak solution is also discussed.

MSC: 35L20; 35L70; 35Q72

Keywords: Faedo-Galerkin method; linear recurrent sequence; Robin conditions; asymptotic expansion in a small parameter

1 Introduction

In this paper, we consider the Robin problem for a wave equation as follows:

$$u_{tt} - u_{xx} = f\big(x, t, u(x,t), u(\eta_1, t), \ldots, u(\eta_q, t), u_t(x,t)\big), \quad 0 < x < 1, 0 < t < T, \tag{1.1}$$

$$u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = \tilde{u}_0(x), \qquad u_t(x,0) = \tilde{u}_1(x), \tag{1.3}$$

where f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and $h_0, h_1 \geq 0$, $\eta_1, \eta_2, \ldots, \eta_q$ are given constants with $h_0 + h_1 > 0$, $0 \leq \eta_1 < \eta_2 < \cdots < \eta_q \leq 1$.

In some special cases, when the nonlinear term has various forms, the following nonlinear wave equation

$$u_{tt} - \Delta u = F(x, t, u, u_t), \tag{1.4}$$

where Δ is a Laplace operator, has been extensively studied by many authors, for example, we refer to [1–10] and the references given therein. In these works, many interesting results about existence, nonexistence, uniqueness, nonuniqueness, regularity, asymptotic behavior, asymptotic expansion, and decay of solutions were obtained.

In [2], Bergounioux considered Prob. (1.3)-(1.4) with the following boundary conditions:

$$u_x(0,t) = P(t), \qquad u_x(1,t) + K_1 u(1,t) + \lambda_1 u_t(1,t) = 0, \tag{1.5}$$

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where f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions, K_1, λ_1 are given constants and the unknown $u(x,t)$ and the unknown boundary value $P(t)$ satisfy the following Cauchy problem for an ordinary differential equation:

$$\begin{cases} P''(t) + \omega^2 P(t) = h u_{tt}(0,t), \quad 0 < t < T, \\ P(0) = P_0, \qquad P'(0) = P_1, \end{cases} \tag{1.6}$$

where $\omega > 0$, $h \geq 0$, P_0, P_1 are given constants and K, λ are given nonnegative constants.

Prob. (1.4)-(1.6), with $F(x,t,u,u_t) = f(x,t) - Ku - \lambda u_t$, describes the shock between a solid body and a linear viscoelastic bar resting on a viscoelastic base with linear elastic constraints at the side, constraints associated with a viscous frictional resistance.

With $F(x,t,u,u_t) = f(x,t) - h(u_t)$, Jokhadze, in [4], considered existence, uniqueness, and nonuniqueness, and nonexistence of a global classical solution for wave equations with nonlinear damping term.

In [5], the authors established the unique existence, stability, and asymptotic expansion of Prob. (1.3)-(1.4) with the nonlocal boundary conditions

$$\begin{cases} u_x(0,t) = g_0(t) + \int_0^t k_0(t-s)u(0,s)\,ds, \\ -u_x(1,t) = g_1(t) + \int_0^t k_1(t-s)u(1,s)\,ds, \end{cases} \tag{1.7}$$

where $F(x,t,u,u_t) = -\lambda u_t - f(u)$, with λ is a given constant and f, g_0, g_1, k_0, k_1 are given functions. The existence and exponential decay for a nonlinear wave equation with a nonlocal boundary condition were also proved in [9].

Beilin, see [1], investigated the existence and uniqueness of a generalized solution for the following wave equation with an integral nonlocal condition:

$$\begin{cases} u_{tt} - \Delta u + c(x,t)u = f(x,t), \quad (x,t) \in \Omega \times (0,T), \\ \frac{\partial u}{\partial \eta} + \int_0^t \int_\Omega k(x,\xi,\tau)u(\xi,\tau)\,d\xi\,d\tau = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = \tilde{u}_0(x), \qquad u_t(x,0) = \tilde{u}_1(x), \quad x \in \Omega, \end{cases} \tag{1.8}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary, η is the unit outward normal on $\partial\Omega$, f, $\tilde{u}_0$, $\tilde{u}_1$, $k(x,\xi,\tau)$ are given functions. Nonlocal conditions come up when values of the function on the boundary are connected to values inside the domain. There are various types of nonlocal boundary conditions of integral form for hyperbolic, parabolic or elliptic equations, the ones were introduced in [1].

In recent years, some close forms of Eq. (1.4), with power-type nonlinearities containing integer power-type, fractional power-type or variable exponent, have been paid attention to by many researchers [3, 11–13]. Benaissa and Messaoudi, in [3], considered the following problem:

$$\begin{cases} u_{tt} - \Delta u + g(u_t) + f(u) = 0, \quad (x,t) \in \Omega \times (0,T), \\ u = 0, \quad x \in \partial\Omega, t \geq 0, \\ u(x,0) = \tilde{u}_0(x), \qquad u_t(x,0) = \tilde{u}_1(x), \quad x \in \Omega, \end{cases} \tag{1.9}$$

where $f(u) = -b|u|^{p-2}u$, $g(u_t) = a(1 + |u_t|^{m-2}u_t)$, $a, b > 0$, $m, p > 2$ and Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. The authors showed that for suitably chosen initial data, (1.9) possesses a global weak solution, which decays exponentially even if $m > 2$. The proof of global existence is based on the use of the potential well theory. In [11], Bhattarai proved the existence and stability of solitary-wave solutions of a system of 2-coupled nonlinear Schrödinger equations with power-type nonlinearities. By using variational methods, Repovš, in [13], established several existence results for Schrödinger-type equations containing Laplace-type operators with variable exponent. Moreover, by using the fractional homotopy analysis transform method, Kumar [12] proposed a modified and simple algorithm for fractional modeling arising in unidirectional propagation of long wave in dispersive media.

In [14], the authors considered a one-dimensional nonlocal nonlinear strongly damped wave equation with dynamical boundary conditions. In other words, they looked to the following problem:

$$\begin{cases} u_{tt} - u_{xx} - \alpha u_{txx} + \varepsilon f(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}) = 0, \\ u(0,t) = 0, \\ u_{tt}(1,t) = -\varepsilon[u_x(1,t) + \alpha u_{tx}(1,t) + r u_t(1,t)] - \varepsilon f(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}}), \end{cases} \tag{1.10}$$

with $0 < x < 1$, $t > 0$, $\alpha, r > 0$, and $\varepsilon > 0$. Prob. (1.10) models a spring-mass-damper system, where the term $\varepsilon f(u(1,t), \frac{u_t(1,t)}{\sqrt{\varepsilon}})$ represents a control acceleration at $x = 1$. By using the invariant manifold theory, the authors proved that for small values of the parameter ε, the solution of (1.10) attracted to a two-dimensional invariant manifold.

In [6], Long and Diem studied Prob. (1.3)-(1.4) with the nonlinear term of the form

$$f(x,t,u,u_x,u_t) + \varepsilon g(x,t,u,u_x,u_t), \tag{1.11}$$

associated with the mixed homogeneous boundary conditions

$$u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0. \tag{1.12}$$

In the case of $f \in C^2([0,1] \times [0,\infty) \times \mathbb{R}^3)$ and $g \in C^1([0,1] \times [0,\infty) \times \mathbb{R}^3)$, an asymptotic expansion of order 2 in ε is obtained for ε sufficiently small.

We consider the following wave equation with the source containing nonlocal term:

$$u_{tt} - u_{xx} = F\left(x,t,u(x,t), \int_0^1 g\big(u(y,t)\big)\,dy\right), \quad 0 < x < 1, 0 < t < T, \tag{1.13}$$

where F, g are given continuous functions. Then, if the function $u(x,t)$ is continuous in x, the integral $\int_0^1 g(u(y,t))\,dy$ can be approximated by its Riemann sum

$$\int_0^1 g\big(u(y,t)\big)\,dy \approx \sum_{i=1}^q \frac{1}{q} g\big(u(\eta_i,t)\big), \tag{1.14}$$

with q is large enough and $\eta_i = i/q$, $i = 1,2,\ldots,q$.

Therefore, the nonlinear term in (1.1) can be considered as an approximation of the one that appeared in (1.13) as follows:

$$F\left(x,t,u(x,t), \int_0^1 g\big(u(y,t)\big)\,dy\right) \approx F\left(x,t,u(x,t), \sum_{i=1}^q \frac{1}{q} g\big(u(\eta_i,t)\big)\right). \tag{1.15}$$

The approximation given in (1.15) and the aforementioned works lead to the ideas to study the existence and asymptotic expansion for the Robin problem for a wave equation with nonlinear source containing nonlocal term (1.1)-(1.3). The paper consists of four sections. In Section 2, we present some preliminaries. In Section 3, we associate with Prob. (1.1)-(1.3) a linear recurrent sequence which is bounded in a suitable space of functions. The existence of a local weak solution and the uniqueness are proved by using the Faedo-Galerkin method and the weak compact method. In Section 4, we establish an asymptotic expansion of a weak solution $u_\varepsilon(x,t)$ of order $N+1$ in a small parameter ε for the equation

$$\begin{aligned} u_{tt} - u_{xx} &= f\big(x,t,u(x,t),u(\eta_1,t),\dots,u(\eta_q,t),u_t(x,t)\big) \\ &\quad + \varepsilon f_1\big(x,t,u(x,t),u(\eta_1,t),\dots,u(\eta_q,t),u_t(x,t)\big), \end{aligned} \tag{1.16}$$

$0<x<1$, $0<t<T$, associated with (1.2), (1.3). The results obtained here may be considered as a relative generalization of the results obtained in [2, 4–6, 9], and [10].

2 Preliminaries

Put $\Omega = (0,1)$. We will omit the definitions of the usual function spaces and denote them by $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle\cdot,\cdot\rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2, and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote $L^p(0,T;X)$, $1\le p\le\infty$ the Banach space of real functions $u:(0,T)\to X$ measurable, such that $\|u\|_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} (\int_0^T \|u(t)\|_X^p\,dt)^{1/p}, & \text{if } 1\le p<\infty, \\ \operatorname{ess\,sup}_{0<t<T}\|u(t)\|_X, & \text{if } p=\infty. \end{cases}$$

Let $u(t)$, $u'(t)=u_t(t)=\dot{u}(t)$, $u''(t)=u_{tt}(t)=\ddot{u}(t)$, $u_x(t)=\nabla u(t)$, $u_{xx}(t)=\Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

With $f\in C^k([0,1]\times\mathbb{R}_+\times\mathbb{R}^{q+2})$, $f=f(x,t,y_1,\dots,y_{q+2})$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_{i+2} f = \frac{\partial f}{\partial y_i}$ with $i=1,\dots,q+2$, and $D^\alpha f = D_1^{\alpha_1}\cdots D_{q+4}^{\alpha_{q+4}} f$, $\alpha=(\alpha_1,\dots,\alpha_{q+4})\in\mathbb{Z}_+^{q+4}$, $|\alpha|=\alpha_1+\cdots+\alpha_{q+4}=k$, $D^{(0,\dots,0)}f=f$.

On H^1, we shall use the following norm:

$$\|v\|_{H^1} = \big(\|v\|^2 + \|v_x\|^2\big)^{1/2}.$$

We put

$$a(u,v) = \int_0^1 u_x(x)v_x(x)\,dx + h_0 u(0)v(0) + h_1 u(1)v(1), \quad u,v\in H^1. \tag{2.1}$$

We have the following lemmas, the proofs of which are straightforward, hence we omit the details.

Lemma 2.1 ([15], Theorem 8.8, pp.212-213) *The imbedding $H^1\hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \le \sqrt{2}\|v\|_{H^1} \quad \textit{for all } v\in H^1. \tag{2.2}$$

Lemma 2.2 *Let $h_0, h_1 \geq 0$, with $h_0 + h_1 > 0$. Then the symmetric bilinear form $a(\cdot,\cdot)$ defined by* (2.1) *is continuous on $H^1 \times H^1$ and coercive on H^1, i.e.,*

$$\begin{aligned} &\text{(i)} \quad \left|a(u,v)\right| \leq a_1 \|u\|_{H^1} \|v\|_{H^1}, \\ &\text{(ii)} \quad a(v,v) \geq a_0 \|v\|_{H^1}^2, \end{aligned} \tag{2.3}$$

for all $u, v \in H^1$, where $a_1 = 1 + 2h_0 + 2h_1$ and

$$a_0 = \frac{1}{4} \min\bigl\{1, \max\{h_0, h_1\}\bigr\}. \tag{2.4}$$

Remark 2.1 It follows from (2.3) that on H^1, $v \longmapsto \|v\|_{H^1}$, $v \longmapsto \|v\|_a = \sqrt{a(v,v)}$ are two equivalent norms satisfying

$$\sqrt{a_0}\|v\|_{H^1} \leq \|v\|_a \leq \sqrt{a_1}\|v\|_{H^1}, \quad \forall v \in H^1. \tag{2.5}$$

Lemma 2.3 *Let $h_0 \geq 0$. Then there exists the Hilbert orthonormal base $\{\widetilde{w}_j\}$ of L^2 consisting of the eigenfunctions $\widetilde{w}_j$ corresponding to the eigenvalue λ_j such that*

$$\begin{cases} 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \qquad \lim_{j\to+\infty} \lambda_j = +\infty, \\ a(\widetilde{w}_j, v) = \lambda_j \langle \widetilde{w}_j, v\rangle \quad \textit{for all } v \in H^1, j = 1,2,\ldots. \end{cases}$$

Furthermore, the sequence $\{\widetilde{w}_j/\sqrt{\lambda_j}\}$ is also a Hilbert orthonormal base of H^1 with respect to the scalar product $a(\cdot,\cdot)$.

On the other hand, we also have $\widetilde{w}_j$ satisfying the following boundary value problem:

$$\begin{cases} -\Delta \widetilde{w}_j = \lambda_j \widetilde{w}_j, \quad \textit{in } (0,1), \\ \widetilde{w}_{jx}(0) - h_0 \widetilde{w}_j(0) = \widetilde{w}_{jx}(1) + h_1 \widetilde{w}_j(1) = 0, \quad \widetilde{w}_j \in C^\infty(\overline{\Omega}). \end{cases}$$

The proof of Lemma 2.3 can be found in ([16], p.87, Theorem 7.7), with $H = L^2$ and $V = H^1$, $a(\cdot,\cdot)$ as defined by (2.1).

Remark 2.2 The weak formulation of the initial-boundary value problem (1.1)-(1.3) can be given in the following manner: Find $u \in \widetilde{W} = \{u \in L^\infty(0,T;H^2) : u_t \in L^\infty(0,T;H^1), u_{tt} \in L^\infty(0,T;L^2)\}$ such that u satisfies the following variational equation:

$$\bigl\langle u_{tt}(t), w\bigr\rangle + a\bigl(u(t), w\bigr) = \bigl\langle f\bigl(\cdot, t, u(t), u(\eta_1, t), \ldots, u(\eta_q, t), u_t(t)\bigr), w\bigr\rangle \tag{2.6}$$

for all $w \in H^1$, a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0, \qquad u_t(0) = \tilde{u}_1. \tag{2.7}$$

3 The existence and uniqueness

We make the following assumptions:

(H_1) $(\tilde{u}_0, \tilde{u}_1) \in H^2 \times H^1$, $\tilde{u}_{0x}(0) - h_0\tilde{u}_0(0) = \tilde{u}_{0x}(1) + h_1\tilde{u}_0(1) = 0$;
(H_2) $f \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R}^{q+2})$.

Fix $T^* > 0$. For each $M > 0$ given, we set the constant $K_M(f)$ as follows:

$$K_M(f) = \sum_{i=1}^{q+4} K_0(M, D_i f),$$

where

$$\begin{cases} K_0(M,f) = \sup_{(x,t,y_1,\ldots,y_{q+2})\in A_1(M)} |f(x,t,y_1,\ldots,y_{q+2})|, \\ A_1(M) = [0,1] \times [0,T^*] \times [-\sqrt{2}M, \sqrt{2}M]^{q+2}. \end{cases}$$

For every $T \in (0, T^*]$ and $M > 0$, we put

$$\begin{cases} W(M,T) = \{v \in L^\infty(0,T;H^2) : v_t \in L^\infty(0,T;H^1), v_{tt} \in L^2(Q_T), \\ \qquad \text{with } \max\{\|v\|_{L^\infty(0,T;H^2)}, \|v_t\|_{L^\infty(0,T;H^1)}, \|v_{tt}\|_{L^2(Q_T)}\} \le M\}, \\ W_1(M,T) = \{v \in W(M,T) : v_{tt} \in L^\infty(0,T;L^2)\}, \end{cases}$$

in which $Q_T = \Omega \times (0,T)$.

Now, we establish the recurrent sequence $\{u_m\}$. The first term is chosen as $u_0 \equiv \tilde{u}_0$, suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.1}$$

we associate Prob. (1.1)-(1.3) with the following problem.

Find $u_m \in W_1(M,T)$ $(m \ge 1)$ satisfying the linear variational problem

$$\begin{cases} \langle u_m''(t), w\rangle + a(u_m(t), w) = \langle F_m(t), w\rangle, \quad \forall w \in H^1, \\ u_m(0) = \tilde{u}_0, \qquad u_m'(0) = \tilde{u}_1, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} F_m(x,t) &= f[u_{m-1}](x,t) \\ &= f\big(x,t,u_{m-1}(x,t),u_{m-1}(\eta_1,t),\ldots,u_{m-1}(\eta_q,t),u_{m-1}'(x,t)\big). \end{aligned} \tag{3.3}$$

Then we have the following theorem.

Theorem 3.1 *Let* (H_1), (H_2) *hold. Then there exist positive constants* $M, T > 0$ *such that, for* $u_0 \equiv \tilde{u}_0$, *there exists a recurrent sequence* $\{u_m\} \subset W_1(M,T)$ *defined by* (3.1)-(3.3).

Proof The proof consists of several steps.

Step 1. The Faedo-Galerkin approximation (introduced by Lions [17]). Consider the basis $\{w_j\}$ for H^1 as in Lemma 2.3. Put

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \tag{3.4}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of linear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + a(u_m^{(k)}(t), w_j) = \langle F_m(t), w_j\rangle, \quad 1 \le j \le k, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \qquad \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{3.5}$$

where $F_m(x,t)$ is defined as in (3.3) and

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \to \tilde{u}_0 \quad \text{strongly in } H^2, \\ \tilde{u}_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \to \tilde{u}_1 \quad \text{strongly in } H^1. \end{cases} \tag{3.6}$$

The system of equations (3.5) can be rewritten in the form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \lambda_j c_{mj}^{(k)}(t) = \langle F_m(t), w_j\rangle, \\ c_m^{(k)}(0) = \alpha_j^{(k)}, \qquad \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)}, \quad 1 \le j \le k. \end{cases} \tag{3.7}$$

It is not difficult to show that (3.7) has a unique solution $c_{mj}^{(k)}(t)$ in $[0,T]$ as follows:

$$\begin{aligned} c_{mj}^{(k)}(t) &= \alpha_j^{(k)} \cos(\sqrt{\lambda_j} t) + \beta_j^{(k)} \frac{\sin(\sqrt{\lambda_j} t)}{\sqrt{\lambda_j}} \\ &\quad + \int_0^t \frac{\sin(\sqrt{\lambda_j}(t-s))}{\sqrt{\lambda_j}} \langle F_m(s), w_j\rangle \, ds, \\ &\quad 0 \le t \le T, 1 \le j \le k. \end{aligned} \tag{3.8}$$

Therefore, (3.5) has a unique solution $u_m^{(k)}(t)$ in $[0,T]$.

Step 2. A priori estimates. First, for all $j = 1,\dots,k$, multiplying $(3.5)_1$ by $\dot{c}_{mj}^{(k)}(t)$, summing on j, and integrating with respect to the time variable from 0 to t, we have

$$X_m^{(k)}(t) = X_m^{(k)}(0) + 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s)\rangle \, ds, \tag{3.9}$$

where

$$X_m^{(k)}(t) = \left\| \dot{u}_m^{(k)}(t) \right\|^2 + \left\| u_m^{(k)}(t) \right\|_a^2. \tag{3.10}$$

Next, by replacing w_j in $(3.5)_1$ by $-\frac{1}{\lambda_j}\Delta w_j$, we obtain that

$$a(\ddot{u}_m^{(k)}(t), w_j) + \langle \Delta u_m^{(k)}(t), \Delta w_j\rangle = \langle F_m(t), -\Delta w_j\rangle, \quad 1 \le j \le k,$$

similar to $(3.5)_1$, it gives

$$\begin{aligned} Y_m^{(k)}(t) &= Y_m^{(k)}(0) + 2\langle F_m(0), \Delta \tilde{u}_{0k}\rangle - 2\langle F_m(t), \triangle u_m^{(k)}(t)\rangle \\ &\quad + 2\int_0^t \langle F'_m(s), \triangle u_m^{(k)}(s)\rangle \, ds, \end{aligned} \tag{3.11}$$

where

$$Y_m^{(k)}(t) = \left\| \dot{u}_m^{(k)}(t) \right\|_a^2 + \left\| \Delta u_m^{(k)}(t) \right\|^2. \tag{3.12}$$

Put

$$\begin{aligned} S_m^{(k)}(t) &= X_m^{(k)}(t) + Y_m^{(k)}(t) + \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|^2 ds \\ &= \left\| \dot{u}_m^{(k)}(t) \right\|^2 + \left\| \dot{u}_m^{(k)}(t) \right\|_a^2 + \left\| u_m^{(k)}(t) \right\|_a^2 + \left\| \Delta u_m^{(k)}(t) \right\|^2 + \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|^2 ds, \end{aligned} \tag{3.13}$$

then we deduce from (3.9), (3.11), and (3.13) that

$$\begin{aligned} S_m^{(k)}(t) &= S_m^{(k)}(0) + 2\langle F_m(0), \triangle \tilde{u}_{0k} \rangle + 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s) \rangle ds \\ &\quad - 2\langle F_m(t), \triangle u_m^{(k)}(t) \rangle + 2\int_0^t \langle F_m'(s), \triangle u_m^{(k)}(s) \rangle ds + \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|^2 ds \\ &\equiv S_m^{(k)}(0) + 2\langle F_m(0), \Delta \tilde{u}_{0k} \rangle + \sum_{j=1}^4 I_j. \end{aligned} \tag{3.14}$$

We estimate all terms on the right-hand side of (3.14) as follows:

$$I_1 = 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s) \rangle ds \le TK_M^2(f) + \int_0^t S_m^{(k)}(s)\, ds; \tag{3.15}$$

$$\begin{aligned} I_2 &= -2\langle F_m(t), \triangle u_m^{(k)}(t) \rangle \\ &\le 4\left\| F_m(0) \right\|^2 + 4T\left(\sqrt{T}\left[1 + (q+1)\sqrt{2}M\right]^2 + M\right)^2 K_M^2(f) + \frac{1}{2} S_m^{(k)}(t); \end{aligned} \tag{3.16}$$

$$\begin{aligned} I_3 &= 2\int_0^t \langle F_m'(s), \triangle u_m^{(k)}(s) \rangle ds \\ &\le 2T\left(\left[1 + (q+1)\sqrt{2}M\right]^2 + M^2\right) K_M^2(f) + \int_0^t S_m^{(k)}(s)\, ds. \end{aligned} \tag{3.17}$$

We note that Eq. $(3.5)_1$ can be written as follows:

$$\langle \ddot{u}_m^{(k)}(t), w_j \rangle - \langle \Delta u_m^{(k)}(t), w_j \rangle = \langle F_m(t), w_j \rangle, \quad 1 \le j \le k. \tag{3.18}$$

Hence, after replacing w_j with $\ddot{u}_m^{(k)}(t)$, we obtain

$$\begin{aligned} \left\| \ddot{u}_m^{(k)}(t) \right\|^2 &= \langle \Delta u_m^{(k)}(t), \ddot{u}_m^{(k)}(t) \rangle + \langle F_m(t), \ddot{u}_m^{(k)}(t) \rangle \\ &\le \left[\left\| \Delta u_m^{(k)}(t) \right\| + \left\| F_m(t) \right\| \right] \left\| \ddot{u}_m^{(k)}(t) \right\| \\ &\le \left[\left\| \Delta u_m^{(k)}(t) \right\| + \left\| F_m(t) \right\| \right]^2, \end{aligned}$$

so

$$\begin{aligned} I_4 &= \int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|^2 ds \le 2\int_0^t \left\| \Delta u_m^{(k)}(s) \right\|^2 ds + 2\int_0^t \left\| F_m(s) \right\|^2 ds \\ &\le 2\int_0^t S_m^{(k)}(s)\, ds + 2TK_M^2(f). \end{aligned} \tag{3.19}$$

It follows from (3.14)-(3.17), (3.19) that

$$S_m^{(k)}(t) \le D_0^{(k)}(f, \tilde{u}_0, \tilde{u}_1, \tilde{u}_{0k}, \tilde{u}_{1k}) + D_1(M,T) + 8\int_0^t S_m^{(k)}(s)\,ds, \tag{3.20}$$

where

$$\begin{cases} D_0^{(k)}(f, \tilde{u}_0, \tilde{u}_1, \tilde{u}_{0k}, \tilde{u}_{1k}) = 2S_m^{(k)}(0) + 4\langle F_m(0), \Delta\tilde{u}_{0k}\rangle + 8\|F_m(0)\|^2, \\ D_1(M,T) = 2[3 + 4(\sqrt{T}[1 + (q+1)\sqrt{2}M]^2 + M)^2 \\ \qquad\qquad + 2([1 + (q+1)\sqrt{2}M]^2 + M^2)]TK_M^2(f). \end{cases} \tag{3.21}$$

By means of the convergence in (3.6), we can deduce the existence of a constant $M > 0$ independent of k and m such that

$$D_0^{(k)}(f, \tilde{u}_0, \tilde{u}_1, \tilde{u}_{0k}, \tilde{u}_{1k}) \le \frac{1}{2}M^2 \quad \text{for all } m, k \in \mathbb{N}. \tag{3.22}$$

We choose $T \in (0, T^*]$ such that

$$\left(\frac{1}{2}M^2 + D_1(M,T)\right)e^{8T} \le M^2 \tag{3.23}$$

and

$$k_T = \left(1 + \frac{1}{\sqrt{a_0}}\right)\sqrt{2Te^T}(q+1)K_M(f) < 1. \tag{3.24}$$

Finally, it follows from (3.20), (3.22), and (3.23) that

$$S_m^{(k)}(t) \le M^2 e^{-8T} + 8\int_0^t S_m^{(k)}(s)\,ds. \tag{3.25}$$

By using Gronwall's lemma, we deduce from (3.25) that

$$S_m^{(k)}(t) \le M^2 e^{-8T} e^{8t} \le M^2 \tag{3.26}$$

for all $t \in [0, T]$, for all m and k. Therefore, we have

$$u_m^{(k)} \in W(M,T) \quad \text{for all } m \text{ and } k. \tag{3.27}$$

Step 3. Limiting process. From (3.26), we deduce the existence of a subsequence of $\{u_m^{(k)}\}$, still so denoted, such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in } L^\infty(0,T;H^2) \text{ weak}^*, \\ \dot{u}_m^{(k)} \to u_m' & \text{in } L^\infty(0,T;H^1) \text{ weak}^*, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in } L^2(Q_T) \text{ weak}, \\ u_m \in W(M,T). \end{cases} \tag{3.28}$$

Passing to limit in (3.5), we have u_m satisfying (3.2), (3.3) in $L^2(0,T)$. On the other hand, it follows from $(3.2)_1$ and $(3.28)_4$ that $u_m'' = \Delta u_m + F_m \in L^\infty(0,T;L^2)$, hence $u_m \in W_1(M,T)$ and the proof of Theorem 3.1 is complete. □

We use the result given in Theorem 3.1 and the compact imbedding theorems to prove the existence and uniqueness of a weak solution of Prob. (1.1)-(1.3). Hence, we get the main result in this section as follows.

Theorem 3.2 *Let* (H_1), (H_2) *hold. Then*

(i) *Prob.* (1.1)-(1.3) *has a unique weak solution* $u \in W_1(M,T)$, *where the constants* $M > 0$ *and* $T > 0$ *are chosen as in Theorem* 3.1.

(ii) *The recurrent sequence* $\{u_m\}$ *defined by* (3.1)-(3.3) *converges to the solution u of Prob.* (1.1)-(1.3) *strongly in the space*

$$W_1(T) = \{v \in L^\infty(0,T;H^1) : v' \in L^\infty(0,T;L^2)\}. \tag{3.29}$$

Furthermore, we also have the estimation

$$\|u_m - u\|_{W_1(T)} \le C_T k_T^m \quad \textit{for all } m \in \mathbb{N}, \tag{3.30}$$

where the constant $k_T \in [0,1)$ *is defined as in* (3.24) *and* C_T *is a constant depending only on* T, h_0, h_1, f, $\tilde{u}_0$, $\tilde{u}_1$, *and* k_T.

Proof (a) *Existence of the solution*. First, we note that $W_1(T)$ is a Banach space with respect to the norm (see Lions [17]).

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;H^1)} + \|v'\|_{L^\infty(0,T;L^2)}. \tag{3.31}$$

We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\begin{cases} \langle w_m''(t), w\rangle + a(w_m(t), w) = \langle F_{m+1}(t) - F_m(t), w\rangle, & \forall w \in H^1, \\ w_m(0) = w_m'(0) = 0. \end{cases} \tag{3.32}$$

Taking $w = w_m'$ in $(3.32)_1$, after integrating in t, we get

$$Z_m(t) = 2\int_0^t \langle F_{m+1}(s) - F_m(s), w_m'(s)\rangle\, ds, \tag{3.33}$$

where

$$Z_m(t) = \|w_m'(t)\|^2 + \|w_m(t)\|_a^2. \tag{3.34}$$

By (H_2) it is clear that

$$\begin{aligned} \|F_{m+1}(t) - F_m(t)\| &\le K_M(f)\big[\sqrt{2}(q+1)\|w_{m-1}(t)\|_{H^1} + \|w_{m-1}'(t)\|\big] \\ &\le \sqrt{2}(q+1)K_M(f)\|w_{m-1}\|_{W_1(T)}. \end{aligned} \tag{3.35}$$

Hence

$$Z_m(t) \leq 2T(q+1)^2 K_M^2(f)\|w_{m-1}\|_{W_1(T)}^2 + \int_0^t Z_m(s)\,ds. \tag{3.36}$$

Using Gronwall's lemma, we deduce from (3.36) that

$$\|w_m\|_{W_1(T)} \leq k_T \|w_{m-1}\|_{W_1(T)} \quad \forall m \in \mathbb{N}, \tag{3.37}$$

where $k_T \in (0,1)$ is defined as in (3.24), which implies that

$$\|u_m - u_{m+p}\|_{W_1(T)} \leq \|u_0 - u_1\|_{W_1(T)}(1-k_T)^{-1}k_T^m \quad \forall m,p \in \mathbb{N}. \tag{3.38}$$

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \to u \quad \text{strongly in } W_1(T). \tag{3.39}$$

Note that $u_m \in W_1(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$such that

$$\begin{cases} u_{m_j} \to u & \text{in } L^\infty(0,T;H^2) \text{ weak}^*, \\ u'_{m_j} \to u' & \text{in } L^\infty(0,T;H^1) \text{ weak}^*, \\ u''_{m_j} \to u'' & \text{in } L^2(Q_T) \text{ weak}, \\ u \in W(M,T). \end{cases} \tag{3.40}$$

We also note that

$$\begin{aligned} &\big\|F_m - f\big(\cdot,t,u(t),u(\eta_1,t),\ldots,u(\eta_q,t),u'(t)\big)\big\|_{L^\infty(0,T;L^2)} \\ &\quad \leq \sqrt{2}(q+1)K_M(f)\|u_{m-1} - u\|_{W_1(T)}. \end{aligned} \tag{3.41}$$

Hence, from (3.39) and (3.41), we obtain

$$F_m \to f\big(\cdot,t,u(t),u(\eta_1,t),\ldots,u(\eta_q,t),u'(t)\big) \quad \text{strongly in } L^\infty\big(0,T;L^2\big). \tag{3.42}$$

Finally, passing to limit in (3.2)-(3.3) as $m = m_j \to \infty$, it implies from (3.39), $(3.40)_{1,3}$, and (3.42) that there exists $u \in W(M,T)$ satisfying (2.6), (2.7).

On the other hand, from assumption (H_2) we obtain from (2.6), $(3.40)_4$, and (3.42) that

$$u'' = u_{xx} + f\big(\cdot,t,u(t),u(\eta_1,t),\ldots,u(\eta_q,t),u'(t)\big) \in L^\infty\big(0,T;L^2\big), \tag{3.43}$$

thus we have $u \in W_1(M,T)$. The existence proof is completed.

(b) *Uniqueness of the solution*. Let $u_1, u_2 \in W_1(M,T)$ be two different weak solutions of Prob. (1.1)-(1.3). Then $u = u_1 - u_2$ satisfies the variational problem

$$\begin{cases} \langle u''(t),w\rangle + a(u(t),w) = \langle F_1(t) - F_2(t),w\rangle, & \forall w \in H^1, \\ u(0) = u'(0) = 0, \end{cases} \tag{3.44}$$

where $F_i(x,t) = f(x,t,u_i(x,t),u_i(\eta_1,t),\ldots,u_i(\eta_q,t),u_i'(x,t))$, $i = 1,2$.

We take $w = u'$ in $(3.44)_1$ and integrate in t to get

$$\left\|u'(t)\right\|^2 + \left\|u(t)\right\|_a^2 \leq \sqrt{\frac{2}{a_0}}(q+1)K_M(f)\int_0^t \left(\left\|u'(s)\right\|^2 + \left\|u(s)\right\|_a^2\right) ds. \tag{3.45}$$

Using Gronwall's lemma, it follows that $\|u'(t)\|^2 + \|u(t)\|_a^2 \equiv 0$, i.e., $u_1 \equiv u_2$.

So (i) is proved and (ii) follows. Theorem 3.2 is proved completely. □

4 Asymptotic expansion of the solution with respect to a small parameter

In this section, let (H_1), (H_2) hold. We make the following additional assumption:

(H_2') $f_1 \in C^1([0,1] \times \mathbb{R}_+ \times \mathbb{R}^{q+2})$.

We consider the following perturbed problem, where ε is a small parameter such that, $|\varepsilon| \leq 1$:

$$(P_\varepsilon) \quad \begin{cases} u_{tt} - u_{xx} = F_\varepsilon[u](x,t), \quad 0 < x < 1, 0 < t < T, \\ u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \qquad u_t(x,0) = \tilde{u}_1(x), \end{cases}$$

where

$$\begin{cases} F_\varepsilon[u](x,t) = f[u](x,t) + \varepsilon f_1[u](x,t), \\ f[u](x,t) = f(x,t,u(x,t),u(\eta_1,t),\dots,u(\eta_q,t),u'(t)), \\ f_1[u](x,t) = f_1(x,t,u(x,t),u(\eta_1,t),\dots,u(\eta_q,t),u'(t)). \end{cases}$$

First, we note that if the functions f, f_1 satisfy (H_2), (H_2'), then the a priori estimates of the Galerkin approximation sequence $\{u_m^{(k)}\}$ in the proof of Theorem 3.1 for Prob. (1.1)-(1.3) corresponding to $f = F_\varepsilon[u]$, $|\varepsilon| \leq 1$, satisfy $u_m^{(k)} \in W_1(M,T)$, where M, T are constants independent of ε. We also note that the positive constants M and T are chosen as in (3.22)-(3.23) with $|f(\cdot,0,\tilde{u}_0,\tilde{u}_0(\eta_1),\dots,\tilde{u}_0(\eta_q),\tilde{u}_1)|$, $K_M(f)$, stand for

$$\begin{aligned} &\left|f\left(\cdot,0,\tilde{u}_0,\tilde{u}_0(\eta_1),\dots,\tilde{u}_0(\eta_q),\tilde{u}_1\right)\right| + \left|f_1\left(\cdot,0,\tilde{u}_0,\tilde{u}_0(\eta_1),\dots,\tilde{u}_0(\eta_q),\tilde{u}_1\right)\right|, \\ &K_M(f) + K_M(f_1), \end{aligned}$$

respectively.

Hence, the limit u_ε in suitable function spaces of the sequence $\{u_m^{(k)}\}$ as $k \to +\infty$, after $m \to +\infty$, is a unique weak solution of the problem (P_ε) satisfying $u_\varepsilon \in W_1(M,T)$.

Then we can prove, in a manner similar to the proof of Theorem 3.2, that the limit u_0 in suitable function spaces of the family $\{u_\varepsilon\}$ as $\varepsilon \to 0$ is a unique weak solution of the problem (P_0) (corresponding to $\varepsilon = 0$) satisfying $u_0 \in W_1(M,T)$.

We shall study the asymptotic expansion of the solution of the problem (P_ε) with respect to a small parameter ε.

We use the following notations. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{Z}_+^N$, and $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, we put

$$\begin{cases} |\alpha| = \alpha_1 + \cdots + \alpha_N, \qquad \alpha! = \alpha_1! \cdots \alpha_N!, \\ \alpha, \beta \in \mathbb{Z}_+^N, \quad \alpha \leq \beta \quad \Longleftrightarrow \quad \alpha_i \leq \beta_i \quad \forall i = 1, \dots, N, \\ x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}. \end{cases}$$

Next, we need the following lemma.

Lemma 4.1 *Let $m, N \in \mathbb{N}$ and $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, $\varepsilon \in \mathbb{R}$. Then*

$$\left(\sum_{i=1}^{N} x_i \varepsilon^i \right)^m = \sum_{k=m}^{mN} P_k^{(m)}[N, x] \varepsilon^k, \tag{4.1}$$

where the coefficients $P_k^{(m)}[N, x]$, $m \leq k \leq mN$ depending on $x = (x_1, \dots, x_N)$ are defined by the formulas

$$P_k^{(m)}[N, x] = \begin{cases} x_k, & 1 \leq k \leq N, m = 1, \\ \sum_{\alpha \in A_k^{(m)}(N)} \frac{m!}{\alpha!} x^\alpha, & m \leq k \leq mN, m \geq 2, \end{cases} \tag{4.2}$$

where $A_k^{(m)}(N) = \{\alpha \in \mathbb{Z}_+^N : |\alpha| = m, \sum_{i=1}^N i\alpha_i = k\}$.

The proof of Lemma 4.1 is easy, hence we omit the details.

Now, we assume that

$(H_2^{(N)})$ $f \in C^{N+1}([0,1] \times \mathbb{R}_+ \times \mathbb{R}^{q+2}), f_1 \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R}^{q+2})$.

Let u_0 be a unique weak solution of the problem (P_0) corresponding to $\varepsilon = 0$, *i.e.*,

$$(P_0) \qquad \begin{cases} u_0'' - \Delta u_0 = f[u_0] \equiv F_0, \quad 0 < x < 1, 0 < t < T, \\ u_{0x}(0,t) - h_0 u_0(0,t) = u_{0x}(1,t) + h_1 u_0(1,t) = 0, \\ u_0(x,0) = \tilde{u}_0(x), \qquad u_0'(x,0) = \tilde{u}_1(x), \\ u_0 \in W_1(M,T). \end{cases}$$

Let us consider the sequence of weak solutions u_k, $1 \leq k \leq N$, defined by the following problems:

$$(\tilde{P}_k) \qquad \begin{cases} u_k'' - \Delta u_k = F_k, \quad 0 < x < 1, 0 < t < T, \\ u_{kx}(0,t) - h_0 u_k(0,t) = u_{kx}(1,t) + h_1 u_k(1,t) = 0, \\ u_k(x,0) = u_k'(x,0) = 0, \\ u_k \in W_1(M,T), \end{cases}$$

where F_k, $1 \leq k \leq N$, are defined by the formulas

$$F_k = \begin{cases} \bar{\Phi}_1[N, f] + f_1[u_0], & k = 1, \\ \bar{\Phi}_k[N, f] + \bar{\Phi}_{k-1}[N-1, f_1], & 2 \leq k \leq N, \end{cases} \tag{4.3}$$

with $\bar{\Phi}_k[N,f] = \bar{\Phi}_k[N,f,u_0,u_0',\vec{u},\vec{u}']$, $0 \le k \le N$, are defined by the formulas

$$\bar{\Phi}_k[N,f] = \begin{cases} f[u_0], & k = 0, \\ \sum_{1\le|\gamma|\le k} \frac{1}{\gamma!} D^\gamma f[u_0]\Psi_k[\gamma,N,\vec{u},\vec{u}'], & 1 \le k \le N, \end{cases} \tag{4.4}$$

where

$$\begin{aligned} &\Psi_k\big[\gamma,N,\vec{u},\vec{u}'\big] \\ &\quad = \sum_{\substack{(\beta_1,\ldots,\beta_{q+2})\in\widetilde{A}(\gamma,N), \\ \beta_1+\cdots+\beta_{q+2}=k}} P_{\beta_1}^{(\gamma_1)}\big[N,\vec{u}(x,t)\big]P_{\beta_2}^{(\gamma_2)}\big[N,\vec{u}(\eta_1,t)\big]\cdots \\ &\qquad \times P_{\beta_{q+1}}^{(\gamma_{q+1})}\big[N,\vec{u}(\eta_q,t)\big]P_{\beta_{q+2}}^{(\gamma_{q+2})}\big[N,\vec{u}'(x,t)\big], \end{aligned} \tag{4.5}$$

with

$$\begin{cases} \widetilde{A}(\gamma,N) = \{(\beta_1,\ldots,\beta_{q+2}) \in \mathbb{Z}_+^{q+2} : \gamma_i \le \beta_i \le N\gamma_i, 1 \le i \le q+2\}, \\ \gamma = (\gamma_1,\ldots,\gamma_{q+2}) \in \mathbb{Z}_+^{q+2}, \quad 1 \le |\gamma| \le N, \end{cases} \tag{4.6}$$

and $\vec{u}(x,t) = (u_1(x,t),\ldots,u_N(x,t))$, $\vec{u}'(x,t) = (\dot{u}_1(x,t),\ldots,\dot{u}_N(x,t))$.

Then, we have the following theorem.

Theorem 4.2 *Let* (H_1) *and* $(H_2^{(N)})$ *hold. Then there exist constants* $M > 0$ *and* $T > 0$ *such that, for every* $\varepsilon \in [-1,1]$, *the problem* (P_ε) *has a unique weak solution* $u_\varepsilon \in W_1(M,T)$ *satisfying the asymptotic estimation up to order* $N+1$ *as follows*:

$$\left\| u_\varepsilon - \sum_{k=0}^{N} u_k\varepsilon^k \right\|_{W_1(T)} \le C_T|\varepsilon|^{N+1}, \tag{4.7}$$

where the functions u_k, $0 \le k \le N$, *are the weak solutions of the problems* (P_0), $(\tilde{P}_k)$, $1 \le k \le N$, *respectively, and* C_T *is a constant depending only on* N, T, f, f_1, u_k, $0 \le k \le N$.

Remark 4.1 By the fact that it is very difficult to find u_ε of the problem (P_ε), we try to search the weak solutions u_k, $0 \le k \le N$, of the problems (P_0), $(\tilde{P}_k)$. Clearly, they are found much more easily than u_ε and u_ε can be approximated by u_k via (4.7).

In order to prove Theorem 4.2, we need the following lemmas.

Lemma 4.3 *Let* $\bar{\Phi}_k[N,f]$, $0 \le k \le N$, *be the functions defined by formulas* (4.4)-(4.6). *Put* $h = \sum_{k=0}^N u_k\varepsilon^k$, *then we have*

$$f[h] = \sum_{k=0}^{N} \bar{\Phi}_k[N,f]\varepsilon^k + |\varepsilon|^{N+1}\hat{R}_N\big[f,u_0,\vec{u},\vec{u}',\varepsilon\big], \tag{4.8}$$

with $\|\hat{R}_N[f,u_0,\vec{u},\vec{u}',\varepsilon]\|_{L^\infty(0,T;L^2)} \le C$, *where* C *is a constant depending only on* N, T, f, u_k, $\dot{u}_k$, $0 \le k \le N$.

Proof of Lemma 4.3 (i) In the case of $N = 1$, the proof of (4.8) is easy, hence we omit the details, which we only prove with $N \geq 2$. Put $h = u_0 + \sum_{k=1}^{N} u_k \varepsilon^k \equiv u_0 + h_1$, we rewrite as follows:

$$\begin{aligned} & f[h](x,t) \\ & \quad = f\big(x,t,h(x,t),h(\eta_1,t),\ldots,h(\eta_q,t),\dot{h}(x,t)\big) \\ & \quad = f\big(x,t,u_0(x,t)+h_1(x,t),u_0(\eta_1,t)+h_1(\eta_1,t),\ldots, \\ & \qquad u_0(\eta_q,t)+h_1(\eta_q,t),\dot{u}_0(x,t)+\dot{h}_1(x,t)\big). \end{aligned} \tag{4.9}$$

By using Taylor's expansion of the function $f[h]$ around the point

$$[u_0] \equiv \big(x,t,u_0(x,t),u_0(\eta_1,t),\ldots,u_0(\eta_q,t),\dot{u}_0(x,t)\big)$$

up to order $N + 1$, we obtain

$$\begin{aligned} f[h] = {} & f[u_0] + \sum_{1 \leq |\gamma| \leq N} \frac{1}{\gamma!} D^{\gamma} f[u_0] h_1^{\gamma_1}(x,t) h_1^{\gamma_2}(\eta_1,t) \cdots \\ & \times h_1^{\gamma_{q+1}}(\eta_q,t) \dot{h}_1^{\gamma_{q+2}}(x,t) + R_N[f,u_0,h_1], \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} & R_N[f,u_0,h_1] \\ & \quad = \sum_{|\gamma|=N+1} \frac{N+1}{\gamma!} \int_0^1 (1-\theta)^N D^{\gamma} f[u_0 + \theta h_1] h_1^{\gamma_1}(x,t) h_1^{\gamma_2}(\eta_1,t) \cdots \\ & \qquad \times h_1^{\gamma_{q+1}}(\eta_q,t) \dot{h}_1^{\gamma_{q+2}}(x,t)\, d\theta \\ & \quad = |\varepsilon|^{N+1} R_N^{(1)}[f,u_0,h_1,\varepsilon], \end{aligned} \tag{4.11}$$

$$\begin{aligned} & \gamma = (\gamma_1,\ldots,\gamma_{q+2}) \in \mathbb{Z}_+^{q+2}, \\ & |\gamma| = \gamma_1 + \cdots + \gamma_{q+2}, \\ & \gamma! = \gamma_1! \cdots \gamma_{q+2}!, \\ & D^{\gamma} f = D_3^{\gamma_1} D_4^{\gamma_2} \cdots D_{q+4}^{\gamma_{q+2}} f, \\ & D^{\gamma} f[u_0] = D^{\gamma} f\big(x,t,u_0(x,t),u_0(\eta_1,t),\ldots,u_0(\eta_q,t),\dot{u}_0(x,t)\big). \end{aligned}$$

By formula (4.1), we get

$$\begin{aligned} & h_1^{\gamma_1}(x,t) = \left(\sum_{k=1}^{N} u_k(x,t)\varepsilon^k\right)^{\gamma_1} = \sum_{k=\gamma_1}^{N\gamma_1} P_k^{(\gamma_1)}\big[N,\vec{u}(x,t)\big]\varepsilon^k, \\ & h_1^{\gamma_2}(\eta_1,t) = \left(\sum_{k=1}^{N} u_k(\eta_1,t)\varepsilon^k\right)^{\gamma_2} = \sum_{k=\gamma_2}^{N\gamma_2} P_k^{(\gamma_2)}\big[N,\vec{u}(\eta_1,t)\big]\varepsilon^k, \end{aligned}$$

$$\begin{aligned}
&\vdots \\
&h_1^{\gamma_{q+1}}(\eta_q,t) = \left(\sum_{k=1}^{N} u_k(\eta_q,t)\varepsilon^k\right)^{\gamma_{q+1}} = \sum_{k=\gamma_{q+1}}^{N\gamma_{q+1}} P_k^{(\gamma_{q+1})}\big[N,\vec{u}(\eta_q,t)\big]\varepsilon^k, \\
&\dot{h}_1^{\gamma_{q+2}}(x,t) = \left(\sum_{k=1}^{N} \dot{u}_k(x,t)\varepsilon^k\right)^{\gamma_{q+2}} = \sum_{k=\gamma_{q+2}}^{N\gamma_{q+2}} P_k^{(\gamma_{q+2})}\big[N,\vec{u}'(x,t)\big]\varepsilon^k,
\end{aligned} \tag{4.12}$$

where $\vec{u}(x,t) = (u_1(x,t),\dots,u_N(x,t))$, $\vec{u}'(x,t) = (\dot{u}_1(x,t),\dots,\dot{u}_N(x,t))$.

Hence, we deduce from (4.12), that

$$\begin{aligned}
&h_1^{\gamma_1}(x,t)h_1^{\gamma_2}(\eta_1,t)\cdots h_1^{\gamma_{q+1}}(\eta_q,t)\dot{h}_1^{\gamma_{q+2}}(x,t) \\
&\quad = \sum_{k=|\gamma|}^{N} \Psi_k\big[\gamma,N,\vec{u},\vec{u}'\big]\varepsilon^k + \sum_{k=N+1}^{|\gamma|N} \Psi_k\big[\gamma,N,\vec{u},\vec{u}'\big]\varepsilon^k,
\end{aligned} \tag{4.13}$$

where

$$\begin{cases}
\Psi_k[\gamma,N,\vec{u},\vec{u}'] \\
\quad = \sum_{(\beta_1,\dots,\beta_{q+2})\in\widetilde{A}(\gamma,N),\beta_1+\cdots+\beta_{q+2}=k} P_{\beta_1}^{(\gamma_1)}[N,\vec{u}(x,t)]P_{\beta_2}^{(\gamma_2)}[N,\vec{u}(\eta_1,t)]\cdots \\
\qquad \times P_{\beta_{q+1}}^{(\gamma_{q+1})}[N,\vec{u}(\eta_q,t)]P_{\beta_{q+2}}^{(\gamma_{q+2})}[N,\vec{u}'(x,t)], \\
\widetilde{A}(\gamma,N) = \{(\beta_1,\dots,\beta_{q+2})\in\mathbb{Z}_+^{q+2} : \gamma_i \le \beta_i \le N\gamma_i, 1\le i\le q+2\}.
\end{cases} \tag{4.14}$$

We deduce from (4.10), (4.13) that

$$\begin{aligned}
f[h] &= f[u_0] + \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!}D^\gamma f[u_0] \sum_{k=|\gamma|}^{N} \Psi_k\big[\gamma,N,\vec{u},\vec{u}'\big]\varepsilon^k + |\varepsilon|^{N+1}\hat{R}_N\big[f,u_0,\vec{u},\vec{u}',\varepsilon\big] \\
&= f[u_0] + \sum_{k=1}^{N}\left(\sum_{1\le|\gamma|\le k} \frac{1}{\gamma!}D^\gamma f[u_0]\Psi_k\big[\gamma,N,\vec{u},\vec{u}'\big]\right)\varepsilon^k + |\varepsilon|^{N+1}\hat{R}_N\big[f,u_0,\vec{u},\vec{u}',\varepsilon\big] \\
&= f[u_0] + \sum_{k=1}^{N} \bar{\Phi}_k[N,f]\varepsilon^k + |\varepsilon|^{N+1}\hat{R}_N\big[f,u_0,\vec{u},\vec{u}',\varepsilon\big],
\end{aligned} \tag{4.15}$$

where $\bar{\Phi}_k[N,f]$, $1\le k\le N$, are defined by (4.4)-(4.6) and

$$\begin{aligned}
&|\varepsilon|^{N+1}\hat{R}_N\big[f,u_0,\vec{u},\vec{u}',\varepsilon\big] \\
&\quad = \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!}D^\gamma f[u_0] \sum_{k=N+1}^{|\gamma|N} \Psi_k\big[\gamma,N,\vec{u},\vec{u}'\big]\varepsilon^k + |\varepsilon|^{N+1}R_N^{(1)}[f,u_0,h_1,\varepsilon].
\end{aligned} \tag{4.16}$$

By the boundedness of the functions u_k, $\dot{u}_k$, $1\le k\le N$, in the function space $L^\infty(0,T;H^1)$, we obtain from (4.11), (4.14), and (4.16) that $\|\hat{R}_N[f,u_0,\vec{u},\vec{u}',\varepsilon]\|_{L^\infty(0,T;L^2)} \le C$, where C is a constant depending only on N, T, f, u_k, $\dot{u}_k$, $1\le k\le N$. Thus, Lemma 4.3 is proved. □

Remark 4.2 Lemma 4.3 is a generalization of the formula contained in ([7], p.262, formula (4.38)) and then Lemma 4.4 follows. These lemmas are the key to establishing the asymptotic expansion of the weak solution u_ε of order $N+1$ in a small parameter ε as below.

Let $u = u_\varepsilon \in W_1(M,T)$ be the unique weak solution of the problem (P_ε). Then $v = u_\varepsilon - \sum_{k=0}^{N} u_k\varepsilon^k \equiv u_\varepsilon - h$ satisfies the problem

$$\begin{cases} v'' - \Delta v = f[v+h] - f[h] + \varepsilon(f_1[v+h] - f_1[h]) \\ \qquad\qquad + E_\varepsilon(x,t), \quad 0 < x < 1, 0 < t < T, \\ v_x(0,t) - h_0 v(0,t) = v_x(1,t) + h_1 v(1,t) = 0, \\ v(x,0) = v'(x,0) = 0, \end{cases} \tag{4.17}$$

where

$$E_\varepsilon(x,t) = f[h] - f[u_0] + \varepsilon f_1[h] - \sum_{k=1}^{N} F_k\varepsilon^k, \tag{4.18}$$

and F_k, $1 \le k \le N$, are defined by formulas (4.3).

Then we have the following lemma.

Lemma 4.4 *Let* (H_1) *and* $(H_2^{(N)})$ *hold. Then there exists a constant* C_* *such that*

$$\|E_\varepsilon\|_{L^\infty(0,T;L^2)} \le C_*|\varepsilon|^{N+1}, \tag{4.19}$$

where C_* *is a constant depending only on* N, T, f, f_1, u_k, $0 \le k \le N$.

Proof of Lemma 4.4 In the case of $N = 1$, the proof of Lemma 4.4 is easy, hence we omit the details, which we only prove with $N \ge 2$.

By using formula (4.8) for the function $f_1[h]$, we obtain

$$f_1[h] = f_1[u_0] + \sum_{k=1}^{N-1} \bar{\Phi}_k[N-1,f_1]\varepsilon^k + |\varepsilon|^N \hat{R}_{N-1}\big[f_1,u_0,\vec{u},\vec{u}',\varepsilon\big], \tag{4.20}$$

where $\|\hat{R}_{N-1}[f_1,u_0,\vec{u},\vec{u}',\varepsilon]\|_{L^\infty(0,T;L^2)} \le C$, with C is a constant depending only on N, T, f_1, u_k, $0 \le k \le N$.

By (4.20), we rewrite $\varepsilon f_1[h]$ as follows:

$$\varepsilon f_1[h] = \varepsilon f_1[u_0] + \sum_{k=2}^{N} \bar{\Phi}_{k-1}[N-1,f_1]\varepsilon^k + \varepsilon|\varepsilon|^N \hat{R}_{N-1}\big[f_1,u_0,\vec{u},\vec{u}',\varepsilon\big]. \tag{4.21}$$

Hence, we deduce from (4.8) and (4.21) that

$$\begin{aligned} &f[h] - f[u_0] + \varepsilon f_1[h] \\ &\quad = \big(f_1[u_0] + \bar{\Phi}_1[N,f]\big)\varepsilon + \sum_{k=2}^{N}\big(\bar{\Phi}_k[N,f] + \bar{\Phi}_{k-1}[N-1,f_1]\big)\varepsilon^k \\ &\qquad + |\varepsilon|^{N+1}\tilde{R}_N\big[f,f_1,u_0,\vec{u},\vec{u}',\varepsilon\big], \end{aligned} \tag{4.22}$$

where

$$|\varepsilon|^{N+1}\tilde{R}_N\big[f,f_1,u_0,\vec{u},\vec{u}',\varepsilon\big] = |\varepsilon|^{N+1}\hat{R}_N\big[f,u_0,\vec{u},\vec{u}',\varepsilon\big] + \varepsilon|\varepsilon|^N\hat{R}_{N-1}\big[f_1,u_0,\vec{u},\vec{u}',\varepsilon\big]. \tag{4.23}$$

Combining (4.3), (4.18), and (4.22) leads to

$$E_\varepsilon(x,t) = |\varepsilon|^{N+1}\tilde{R}_N\big[f,f_1,u_0,\vec{u},\vec{u}',\varepsilon\big]. \tag{4.24}$$

By the boundedness of the functions u_k, u_k', $0 \le k \le N$, in the function space $L^\infty(0,T;H^1)$, we obtain from (4.8), (4.20), (4.23), and (4.24) that

$$\|E_\varepsilon\|_{L^\infty(0,T;L^2)} \le C_*|\varepsilon|^{N+1}, \tag{4.25}$$

where C_* is a constant depending only on N, T, f, f_1, u_k, u_k', $0 \le k \le N$.
The proof of Lemma 4.4 is complete. □

Proof of Theorem 4.2 Consider the sequence $\{v_m\}$ defined by

$$\begin{cases} v_0 \equiv 0, \\ v_m'' - \Delta v_m = f[v_{m-1}+h] - f[h] + \varepsilon(f_1[v_{m-1}+h] - f_1[h]) \\ \qquad\qquad + E_\varepsilon(x,t), \quad 0 < x < 1, 0 < t < T, \\ v_{mx}(0,t) - h_0 v_m(0,t) = v_{mx}(1,t) + h_1 v_m(1,t) = 0, \\ v_m(x,0) = v_m'(x,0) = 0, \quad m \ge 1. \end{cases} \tag{4.26}$$

By multiplying two sides of (4.26) with v_m' and after integration in t, we have

$$\begin{aligned} Z_m(t) &= 2\int_0^t \big\langle E_\varepsilon(s), v_m'(s)\big\rangle\,ds + 2\int_0^t \big\langle f[v_{m-1}+h] - f[h], v_m'(s)\big\rangle\,ds \\ &\quad + 2\varepsilon\int_0^t \big\langle f_1[v_{m-1}+h] - f_1[h], v_m'(s)\big\rangle\,ds \\ &= \bar{J}_1 + \bar{J}_2 + \bar{J}_3, \end{aligned} \tag{4.27}$$

where $Z_m(t) = \|v_m'(t)\|^2 + \|v_m(t)\|_a^2$.
We estimate the integrals on the right-hand side of (4.27) as follows.
Estimating $\bar{J}_1$. By using Lemma 4.4, we deduce that

$$\bar{J}_1 = 2\int_0^t \big\langle E_\varepsilon(s), v_m'(s)\big\rangle\,ds \le TC_*^2|\varepsilon|^{2N+2} + \int_0^t Z_m(s)\,ds. \tag{4.28}$$

Estimating $\bar{J}_2$. We note that

$$\big\|f[v_{m-1}+h] - f[h]\big\| \le \sqrt{2}(q+1)K_{M_*}(f)\|v_{m-1}\|_{W_1(T)}, \tag{4.29}$$

with $M_* = (N+2)M$.

It follows from (4.29) that

$$\begin{aligned}\bar{J}_2 &\leq 2\int_0^t \big\| f[v_{m-1}+h]-f[h]\big\| \, \big\| v_m'(s)\big\| \, ds \\ &\leq 2T(q+1)^2 K_{M_*}^2(f)\|v_{m-1}\|_{W_1(T)}^2 + \int_0^t Z_m(s)\, ds. \end{aligned} \tag{4.30}$$

Estimating $\bar{J}_3$. Similarly,

$$\begin{aligned}\bar{J}_3 &\leq 2\int_0^t \big\| f_1[v_{m-1}+h]-f_1[h]\big\| \, \big\| v_m'(s)\big\| \, ds \\ &\leq 2T(q+1)^2 K_{M_*}^2(f_1)\|v_{m-1}\|_{W_1(T)}^2 + \int_0^t Z_m(s)\, ds. \end{aligned} \tag{4.31}$$

Combining (4.27), (4.28), (4.30), and (4.31) leads to

$$\begin{aligned} Z_m(t) &\leq 2T(q+1)^2\big[K_{M_*}^2(f)+K_{M_*}^2(f_1)\big]\|v_{m-1}\|_{W_1(T)}^2 \\ &\quad + TC_*^2|\varepsilon|^{2N+2} + 3\int_0^t Z_m(s)\, ds. \end{aligned} \tag{4.32}$$

By using Gronwall's lemma, we deduce from (4.32) that

$$\|v_m\|_{W_1(T)} \leq \sigma_T \|v_{m-1}\|_{W_1(T)} + \delta_T(\varepsilon), \quad \text{for all } m \geq 1, \tag{4.33}$$

where

$$\begin{aligned} \sigma_T &= \sqrt{2}(q+1)\left(1+\frac{1}{\sqrt{a_0}}\right)\sqrt{K_{M_*}^2(f)+K_{M_*}^2(f_1)}\sqrt{Te^{3T}}, \\ \delta_T(\varepsilon) &= C_*\left(1+\frac{1}{\sqrt{a_0}}\right)\sqrt{Te^{3T}}|\varepsilon|^{N+1}. \end{aligned}$$

We can assume that

$$\sigma_T < 1, \quad \text{with the suitable constant } T > 0. \tag{4.34}$$

We require the following lemma whose proof is immediate.

Lemma 4.5 *Let the sequence* $\{\gamma_m\}$ *satisfy*

$$\gamma_m \leq \sigma\gamma_{m-1} + \delta \quad \textit{for all } m \geq 1, \qquad \gamma_0 = 0, \tag{4.35}$$

where $0 \leq \sigma < 1$, $\delta \geq 0$ *are the given constants. Then*

$$\gamma_m \leq \delta/(1-\sigma) \quad \textit{for all } m \geq 1. \tag{4.36}$$

Applying Lemma 4.5 with $\gamma_m = \|v_m\|_{W_1(T)}$, $\sigma = \sigma_T < 1$, $\delta = \delta_T(\varepsilon)$, it follows from (4.36) that

$$\|v_m\|_{W_1(T)} \leq \frac{\delta_T(\varepsilon)}{1-\sigma_T} = C_T|\varepsilon|^{N+1}, \tag{4.37}$$

where C_T is a constant depending only on T.

On the other hand, the linear recurrent sequence $\{v_m\}$ defined by (4.26) converges strongly in the space $W_1(T)$ to the solution v of problem (4.17). Hence, letting $m \to +\infty$ in (4.37), we get

$$\|v\|_{W_1(T)} \leq C_T|\varepsilon|^{N+1}. \tag{4.38}$$

This implies (4.7). The proof of Theorem 4.2 is complete. □

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