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Solvability of a third-order differential equation with functional boundary conditions at resonance

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Abstract
By using the coincidence degree theory due to Mawhin and constructing suitable operators, we study the existence of solutions for a third-order functional boundary value problem at resonance with $\dim \operatorname{Ker} L = 1$.

MSC: 34B15

Keywords: coincidence degree theory; functional boundary condition; three-order differential equation; resonance; Fredholm operator

1 Introduction

A boundary value problem is said to be at resonance if the corresponding homogeneous boundary value problem has a non-trivial solution. Boundary value problems at resonance have been studied by many authors. We refer the readers to [1–9] and the references cited therein. In [10], the authors discussed the second-order differential equation

$$x''(t) = f\big(t, x(t), x'(t)\big), \quad t \in (0,1),$$

with functional boundary conditions

$$\Gamma_1(x) = 0, \qquad \Gamma_2(x) = 0,$$

where Γ_1, Γ_2 are linear functionals on $C^1[0,1]$ satisfying the general resonance condition $\Gamma_1(t)\Gamma_2(1) = \Gamma_1(1)\Gamma_2(t)$. (The authors also studied the non-resonant scenario under condition (A_1): $\Gamma_1(t)\Gamma_2(1) \neq \Gamma_1(1)\Gamma_2(t)$.) To be specific, the following resonant cases received attention:

(A_2) $\Gamma_1(t), \Gamma_1(1), \Gamma_2(1) = 0$, $\Gamma_2(t) \neq 0$;
(A_3) $\Gamma_1(t), \Gamma_1(1), \Gamma_2(t) = 0$, $\Gamma_2(1) \neq 0$;
(A_4) $\Gamma_1(1), \Gamma_2(t), \Gamma_2(1) = 0$, $\Gamma_1(t) \neq 0$;
(A_5) $\Gamma_1(t), \Gamma_2(1), \Gamma_2(t) = 0$, $\Gamma_1(1) \neq 0$;
(A_6) $\Gamma_1(1), \Gamma_1(t), \Gamma_2(1), \Gamma_2(t) = 0$.

The cases (A_2) and (A_4) result in $\ker L = \{c : c \in \mathbb{R}\}$, and (A_3) and (A_5) correspond to $\ker L = \{ct : c \in \mathbb{R}\}$. The case (A_6) describes a resonance with $\ker L = \{c_1 t + c_2 : c_1, c_2 \in \mathbb{R}\}$. In [6],

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the authors extended the results of [10] as well as [3, 9] in several respects including the study of the case $\ker L = \{c(at+b) : c \in \mathbb{R}\}$, where $a, b \neq 0$.

This paper is a study of third-order functional boundary value problems (FBVPs) at resonance. It improves and generalizes the results of [1, 7] and the results of [2] applicable to third-order problems. We consider

$$\begin{cases} x'''(t) = f(t, x(t), x'(t), x''(t)), & t \in (0,1), \\ \varphi_1(x) = \varphi_2(x) = \varphi_3(x) = 0, \end{cases} \tag{1.1}$$

where $\varphi_i : C^2[0,1] \to \mathbb{R}$, $i = 1,2,3$, are bounded linear functionals. To the best of our knowledge, this is the first paper devoted to a third-order FBVP at resonance. We present several generalizations to the existing results and improvements to the method based on Mawhin's coincidence degree theory.

The framework of this paper is as follows. In Section 2, we present some notations and the fundamentals of coincidence degree theory. In Section 3, we study problem (1.1) under the conditions

$$\varphi_i(t^j) = 0, \quad i = 1,2,3, j \in \{0,1,2\}, \tag{1.2}$$

respectively. In Section 4, we show the existence of a solution for problem (1.1) under the condition

$$\frac{\varphi_1(t^2)}{\varphi_2(t^2)} = \frac{\varphi_1(t)}{\varphi_2(t)} = \frac{\varphi_1(1)}{\varphi_2(1)}. \tag{1.3}$$

(Here, if $\varphi_2(t^j) = 0$ for some $j \in \{0,1,2\}$, then also $\varphi_1(t^j) = 0$.)

2 Preliminaries

For convenience, we denote

$$\Delta = \begin{vmatrix} \varphi_1(t^2) & \varphi_1(t) & \varphi_1(1) \\ \varphi_2(t^2) & \varphi_2(t) & \varphi_2(1) \\ \varphi_3(t^2) & \varphi_3(t) & \varphi_3(1) \end{vmatrix}, \qquad \Delta_1(y) = \begin{vmatrix} \varphi_1(\int_0^t (t-s)^2 y(s)\,ds) & \varphi_1(t) & \varphi_1(1) \\ \varphi_2(\int_0^t (t-s)^2 y(s)\,ds) & \varphi_2(t) & \varphi_2(1) \\ \varphi_3(\int_0^t (t-s)^2 y(s)\,ds) & \varphi_3(t) & \varphi_3(1) \end{vmatrix},$$

$$\Delta_2(y) = \begin{vmatrix} \varphi_1(t^2) & \varphi_1(\int_0^t (t-s)^2 y(s)\,ds) & \varphi_1(1) \\ \varphi_2(t^2) & \varphi_2(\int_0^t (t-s)^2 y(s)\,ds) & \varphi_2(1) \\ \varphi_3(t^2) & \varphi_3(\int_0^t (t-s)^2 y(s)\,ds) & \varphi_3(1) \end{vmatrix},$$

$$\Delta_3(y) = \begin{vmatrix} \varphi_1(t^2) & \varphi_1(t) & \varphi_1(\int_0^t (t-s)^2 y(s)\,ds) \\ \varphi_2(t^2) & \varphi_2(t) & \varphi_2(\int_0^t (t-s)^2 y(s)\,ds) \\ \varphi_3(t^2) & \varphi_3(t) & \varphi_3(\int_0^t (t-s)^2 y(s)\,ds) \end{vmatrix}.$$

From the last three determinants we can define and derive the following three relations:

$$\Delta_1(Lx) = \begin{vmatrix} \varphi_1(-x''(0)t^2 - 2x'(0)t - 2x(0)) & \varphi_1(t) & \varphi_1(1) \\ \varphi_2(-x''(0)t^2 - 2x'(0)t - 2x(0)) & \varphi_2(t) & \varphi_2(1) \\ \varphi_3(-x''(0)t^2 - 2x'(0)t - 2x(0)) & \varphi_3(t) & \varphi_3(1) \end{vmatrix} = -x''(0)\triangle, \tag{2.1}$$

$$\Delta_2(Lx) = \begin{vmatrix} \varphi_1(t^2) & \varphi_1(-x''(0)t^2 - 2x'(0)t - 2x(0)) & \varphi_1(1) \\ \varphi_2(t^2) & \varphi_2(-x''(0)t^2 - 2x'(0)t - 2x(0)) & \varphi_2(1) \\ \varphi_3(t^2) & \varphi_3(-x''(0)t^2 - 2x'(0)t - 2x(0)) & \varphi_3(1) \end{vmatrix} = -2x'(0)\Delta, \tag{2.2}$$

and $\Delta_3(Lx) = -2x(0)\Delta$. Also, Δ_{ij}, $i,j = 1,2,3$, $\Delta_k(y)_{ij}$, $i,k = 1,2,3$, $j \in \{1,2,3\} \setminus \{k\}$, are the cofactors of $\varphi_i(t^{3-j})$ in Δ, $\Delta_k(y)$, $k = 1,2,3$, respectively.

We introduce some notations and a theorem. For more details, see [11].

Let X and Y be real Banach spaces and $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero, $P : X \to X$, $Q : Y \to Y$ be projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L, \qquad X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q. \tag{2.3}$$

It follows that

$$L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible. We denote the inverse by K_P.

If Ω is an open bounded subset of X, $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ is called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. We rely on Mawhin's theorem for coincidences [8].

Theorem 2.1 *Let* $L : \operatorname{dom} L \subset X \to Y$ *be a Fredholm operator of index zero and* $N : X \to Y$ *be* L*-compact on* $\overline{\Omega}$. *Assume that the following conditions are satisfied*:

(1) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;

(2) $Nx \notin \operatorname{Im} L$ *for every* $x \in \operatorname{Ker} L \cap \partial\Omega$;

(3) $\deg(JQN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, *where* $Q : Y \to Y$ *is a projection such that* $\operatorname{Im} L = \operatorname{Ker} Q$, *and* $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *is an isomorphism.*

Then the equation $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

We work in $X = C^2[0,1]$ with the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty\}$, where $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. We define $Y = L^1[0,1]$ with the norm $\|y\|_1 = \int_0^1 |y(t)|\,dt$.

In this paper, we always suppose that the following condition holds:

(C) There exist constants $k_i > 0$, $i = 1,2,3$, such that $|\varphi_i(x)| \leq k_i\|x\|$, $x \in X$ and the function $f(t,u,v,w)$ satisfies the Carathéodory conditions, that is, $f(\cdot,u,v,w)$ is measurable for each fixed $(u,v,w) \in \mathbb{R}^3$, $f(t,\cdot,\cdot,\cdot)$ is continuous for a.e. $t \in [0,1]$.

3 Solvability of (1.1) with condition (1.2)

Case I. $\varphi_i(1) = 0$, $i = 1,2,3$.

Clearly, $\Delta = 0$. In this case, we assume that there exists $j \in \{1,2,3\}$ such that $\Delta_{j3} \neq 0$. In what follows, we choose and fix such j.

Lemma 3.1 *There exists a function* $g_3 \in Y$ *such that* $\Delta_3(g_3) = 1$.

Proof Suppose the contrary. Then

$$\Delta_3\big(t^n\big) = \begin{vmatrix} \varphi_1(t^2) & \varphi_1(t) & \varphi_1(\int_0^t (t-s)^2 s^n\,ds) \\ \varphi_2(t^2) & \varphi_2(t) & \varphi_2(\int_0^t (t-s)^2 s^n\,ds) \\ \varphi_3(t^2) & \varphi_3(t) & \varphi_3(\int_0^t (t-s)^2 s^n\,ds) \end{vmatrix} = 0, \quad n = 0,1,\ldots.$$

Hence

$$\begin{vmatrix} \varphi_1(t^2) & \varphi_1(t) & \varphi_1(t^{n+3}) \\ \varphi_2(t^2) & \varphi_2(t) & \varphi_2(t^{n+3}) \\ \varphi_3(t^2) & \varphi_3(t) & \varphi_3(t^{n+3}) \end{vmatrix} = 0, \quad n = 0,1,\ldots.$$

It follows from $\Delta_{j3} \neq 0$ and $\varphi_i(1) = 0$, $i = 1,2,3$, that there exist constants a and b such that

$$\varphi_j(t^i) = a\varphi_k(t^i) + b\varphi_l(t^i) = (a\varphi_k + b\varphi_l)(t^i), \quad i = 0,1,2,\ldots,$$

where $k,l \in \{1,2,3\}, k,l \neq j, k \neq l$. Hence $\varphi_j(x) = (a\varphi_k + b\varphi_l)(x), x \in X$. This is a contradiction because $\varphi_1, \varphi_2, \varphi_3$ are linearly independent on X. Hence, there exists a function $h \in Y$ with $\Delta_3(h) \neq 0$ and, as a result, $g_3 = \frac{1}{\Delta_3(h)} h \in Y$ with $\Delta_3(g_3) = 1$. □

Define operators $L : \operatorname{dom} L \subset X \to Y$, $N : X \to Y$ as follows:

$$Lx(t) = x'''(t), \qquad Nx(t) = f\bigl(t, x(t), x'(t), x''(t)\bigr),$$

where $\operatorname{dom} L = \{x \in X : x''' \in Y, \varphi_i(x) = 0, i = 1,2,3\}$.

If $x \in \operatorname{dom} L$ with $Lx = 0$, then $x = at^2 + bt + c$, $a,b,c \in \mathbb{R}$ and $\varphi_i(x) = 0$, $i = 1,2,3$, that is,

$$\begin{aligned} &a\varphi_1(t^2) + b\varphi_1(t) = 0, \\ &a\varphi_2(t^2) + b\varphi_2(t) = 0, \\ &a\varphi_3(t^2) + b\varphi_3(t) = 0. \end{aligned}$$

Since $\Delta_{j3} \neq 0$, we have $a = b = 0$. So, $x \equiv c$, that is, $\operatorname{Ker} L = \{c : c \in \mathbb{R}\}$.

Lemma 3.2 $\operatorname{Im} L = \{y \in Y : \Delta_3(y) = 0\}$.

Proof If $x \in \operatorname{dom} L$, $Lx = y$, then there exist constants a,b,c such that the following equalities hold:

$$\begin{aligned} &x(t) = \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + at^2 + bt + c, \\ &\varphi_1(x) = \frac{1}{2}\varphi_1\biggl(\int_0^t (t-s)^2 y(s)\,ds\biggr) + a\varphi_1(t^2) + b\varphi_1(t) = 0, \\ &\varphi_2(x) = \frac{1}{2}\varphi_2\biggl(\int_0^t (t-s)^2 y(s)\,ds\biggr) + a\varphi_2(t^2) + b\varphi_2(t) = 0, \\ &\varphi_3(x) = \frac{1}{2}\varphi_3\biggl(\int_0^t (t-s)^2 y(s)\,ds\biggr) + a\varphi_3(t^2) + b\varphi_3(t) = 0. \end{aligned}$$

So, y satisfies $\Delta_3(y) = 0$.

Inversely, if $y \in Y$ with $\Delta_3(y) = 0$, we let

$$x(t) = \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds - \frac{\Delta_1(y)_{j3}}{2\Delta_{j3}} t^2 - \frac{\Delta_2(y)_{j3}}{2\Delta_{j3}} t.$$

Obviously, $x'''(t) = y(t)$. Considering $\Delta_1(y)_{j3} = -\Delta_3(y)_{j1}$, $\Delta_2(y)_{j3} = \Delta_3(y)_{j2}$, $\Delta_{j3} = \Delta_3(y)_{j3}$ and

$$\varphi_j(x) = \frac{1}{2}\varphi_j\left(\int_0^t (t-s)^2 y(s)\,ds\right) - \frac{\Delta_1(y)_{j3}}{2\Delta_{j3}}\varphi_j(t^2) - \frac{\Delta_2(y)_{j3}}{2\Delta_{j3}}\varphi_j(t),$$

we have

$$\begin{aligned}\varphi_j(x) &= \frac{1}{2\Delta_{j3}}\left[\varphi_j(t^2)\Delta_3(y)_{j1} - \varphi_j(t)\Delta_3(y)_{j2} + \varphi_j\left(\int_0^t (t-s)^2 y(s)\,ds\right)\Delta_3(y)_{j3}\right] \\ &= \frac{1}{2\Delta_{j3}}\Delta_3(y) = 0.\end{aligned}$$

Clearly, $\varphi_i(x) = 0$, $i \neq j$, $i \in \{1,2,3\}$, which implies that $x \in \operatorname{dom} L$ and, consequently, $y \in \operatorname{Im} L$. □

Define the operators $P_3 : X \to X$, $Q_3 : Y \to Y$ by

$$P_3 x = x(0), \qquad Q_3 y = \Delta_3(y) g_3.$$

Clearly, P_3, Q_3 are projectors such that (2.3) hold.

Define the operator $K_{P_3} : Y \to X$ by

$$K_{P_3} y = \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds - \frac{\Delta_1(y)_{j3}}{2\Delta_{j3}}t^2 - \frac{\Delta_2(y)_{j3}}{2\Delta_{j3}}t.$$

Lemma 3.3 $K_{P_3} = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P_3})^{-1}$.

Proof Let $x \in \operatorname{dom} L \cap \operatorname{Ker} P_3$. Then $\varphi_i(x) = 0$, $i = 1,2,3$, and $x(0) = 0$. So, we get

$$\begin{aligned}K_{P_3} Lx(t) &= \frac{1}{2}\int_0^t (t-s)^2 Lx(s)\,ds - \frac{\Delta_1(Lx)_{j3}}{2\Delta_{j3}}t^2 - \frac{\Delta_2(Lx)_{j3}}{2\Delta_{j3}}t \\ &= x(t) - \frac{x''(0)}{2}t^2 - x'(0)t - \frac{\Delta_1(Lx)_{j3}}{2\Delta_{j3}}t^2 - \frac{\Delta_2(Lx)_{j3}}{2\Delta_{j3}}t.\end{aligned}$$

It follows from (2.1), (2.2) that $\Delta_1(Lx)_{j3} = -x''(0)\Delta_{j3}$, $\Delta_2(Lx)_{j3} = -2x'(0)\Delta_{j3}$. So, $K_{P_3}Lx = x$.

Inversely, $y \in \operatorname{Im} L$ results in $\Delta_3(y) = 0$. As the proof of Lemma 3.2, $\varphi_i(K_{P_3}y) = 0$, $i = 1,2,3$. Clearly, $(K_{P_3}y)''' = y$. Thus, $K_{P_3}y \in \operatorname{dom} L$ and $LK_{P_3}y = y$, $y \in \operatorname{Im} L$. □

We introduce the constants $l_3 = k_1|\Delta_{13}| + k_2|\Delta_{23}| + k_3|\Delta_{33}|$ and

$$l = \max\{k_1k_2, k_1k_3, k_2k_3\}. \tag{3.1}$$

The latter is frequently used in the remainder of the paper.

The next assumption is fulfilled in the main results by virtue of appropriate assumptions on $f(t,\cdot,\cdot,\cdot)$:

(H_1) For any $r > 0$, there exists a function $h_r \in Y$ such that $|f(t,x(t),x'(t),x''(t))| \leq h_r(t)$, $x \in X$, $\|x\| \leq r$.

Lemma 3.4 *If (H_1) holds and $\Omega \subset X$ is bounded, then N is L-compact on $\overline{\Omega}$.*

Proof Take $r \in \mathbb{R}$ large enough such that $\|x\| \leq r$, $x \in \overline{\Omega}$. Then

$$\left|\Delta_3(Nx)\right| \leq \left(k_1|\Delta_{13}| + k_2|\Delta_{23}| + k_3|\Delta_{33}|\right)\left\|\int_0^t (t-s)^2 Nx(s)\,ds\right\| \leq l_3\|h_r\|_1.$$

So, $\|Q_3Nx\|_1 \leq l_3\|h_r\|_1\|g_3\|_1$, which shows that $Q_3N(\overline{\Omega})$ is bounded. For $y \in Y$, we have

$$\|K_{P_3}y\| \leq \|y\|_1 + \frac{2l}{|\Delta_{j3}|}2\|y\|_1 + \frac{4l}{2|\Delta_{j3}|}2\|y\|_1 = \left(1 + \frac{8l}{|\Delta_{j3}|}\right)\|y\|_1,$$

where, for convenience, we define, using (3.1), the constant

$$A_{P_3} = 1 + \frac{8l}{|\Delta_{j3}|}. \tag{3.2}$$

Then

$$\left\|K_{P_3}(I-Q_3)Nx\right\| \leq A_{P_3}\left\|(I-Q_3)Nx\right\|_1 \leq A_{P_3}\left(1 + l_3\|g_3\|_1\right)\|h_r\|_1.$$

Thus, $K_{P_3}(I-Q_3)N(\overline{\Omega})$ is bounded.

For $0 \leq t_1 < t_2 \leq 1$, $x \in \overline{\Omega}$, we have

$$\begin{aligned}\left|\left(K_{P_3}(I-Q_3)Nx\right)''(t_2) - \left(K_{P_3}(I-Q_3)Nx\right)''(t_1)\right| &= \left|\int_{t_1}^{t_2}(I-Q_3)Nx(s)\,ds\right| \\ &\leq \int_{t_1}^{t_2} h_r(s)\,ds + l_3\|h_r\|_1\int_{t_1}^{t_2}\left|g_3(s)\right|ds,\end{aligned}$$

that is, $(K_{P_3}(I-Q_3)N)''(\overline{\Omega})$ is equicontinuous on $[0,1]$ as well as $(K_{P_3}(I-Q_3)N)'(\overline{\Omega})$ and $(K_{P_3}(I-Q_3)N)(\overline{\Omega})$ by the mean value theorem. Therefore, by the Arzela-Ascoli theorem, $K_{P_3}(I-Q_3)N(\overline{\Omega})$ is compact. □

In order to obtain the main results, we impose the following conditions:

(H_2) There exist nonnegative functions $a, b, c, d \in Y$ such that $|f(t,u,v,w)| \leq a(t) + b(t)|u| + c(t)|v| + d(t)|w|$, $t \in [0,1]$, $u, v, w \in \mathbb{R}$;

(H_3) There exists a constant $M_{03} > 0$ such that $\Delta_3(Nx) \neq 0$ if $|x(t)| > M_{03}$, $t \in [0,1]$;

(H_4) There exists a constant $M_{13} > 0$ such that if $|c| > M_{13}$, then one of the following two inequalities holds:

$$c\Delta_3(Nc) > 0, \tag{3.3}$$

or

$$c\Delta_3(Nc) < 0. \tag{3.4}$$

(Here $Nc = f(t,c,0,0)$, $c \in \mathbb{R}$.)

Lemma 3.5 *Assume that* (H_2), (H_3) *hold and let*

$$A_{P_3}\big(\|b\|_1 + \|c\|_1 + \|d\|_1\big) < \frac{1}{2}, \tag{3.5}$$

where A_{P_3} *satisfies* (3.2). *Then* $\Omega_{13} = \{x \in \operatorname{dom} L \setminus \operatorname{Ker} L : Lx = \lambda Nx, \lambda \in (0,1)\}$ *is bounded.*

Proof Since $x \in \Omega_{13}$, then $\Delta_3(Nx) = 0$. By (H_3), there exists $t_0 \in [0,1]$ such that $|x(t_0)| \leq M_{03}$. Now,

$$\big\|(I - P_3)x\big\| = \big\|K_{P_3}L(I - P_3)x\big\| = \|K_{P_3}Lx\| \leq A_{P_3}\|Lx\|_1$$

and

$$\big|P_3x(t_0)\big| = \big|x(t_0) - (I - P_3)x(t_0)\big| \leq M_{03} + A_{P_3}\|Lx\|_1.$$

Thus, $\|P_3x\| = |P_3x(t_0)| \leq M_{03} + A_{P_3}\|Lx\|_1$. It follows from $x = P_3x + (I - P_3)x$ and (H_2) that

$$\begin{aligned}\|x\| \leq M_{03} + 2A_{P_3}\|Lx\|_1 < M_{03} + 2A_{P_3}\|Nx\|_1 \\ \leq M_{03} + 2A_{P_3}\big(\|a\|_1 + \big(\|b\|_1 + \|c\|_1 + \|d\|_1\big)\|x\|\big).\end{aligned}$$

So,

$$\|x\| \leq \frac{M_{03} + 2A_{P_3}\|a\|_1}{1 - 2A_{P_3}(\|b\|_1 + \|c\|_1 + \|d\|_1)}.$$

Therefore, Ω_{13} is bounded by (3.5). □

Lemma 3.6 *Assume that* (H_4) *holds. Then* $\Omega_{23} = \{x \in \operatorname{Ker} L : Nx \in \operatorname{Im} L\}$ *is bounded.*

Proof If $x \in \Omega_{23}$, then $x \equiv c$ and $Q_3(Nc) = 0$, that is, $\Delta_3(Nc) = 0$. By (H_4), it follows that $|c| \leq M_{13}$. Thus, Ω_{23} is bounded. □

Lemma 3.7 *Assume that* (H_4) *holds. Then*

$$\Omega_{33} = \big\{x : \rho\lambda x + (1 - \lambda)\Delta_3(Nx) = 0, x \in \operatorname{Ker} L, \lambda \in [0,1]\big\}$$

is bounded, where $\rho = \begin{cases} 1, & \text{if (3.3) holds,} \\ -1, & \text{if (3.4) holds.} \end{cases}$

Proof Let $x \in \Omega_{33}$. Then $x \equiv c \in \mathbb{R}$ and $\rho\lambda c + (1 - \lambda)\Delta_3(Nc) = 0$. If $\lambda = 0$, then $\Delta_3(Nc) = 0$. By (H_4), $|c| \leq M_{13}$. If $\lambda = 1$, then $c = 0$. If $\lambda \in (0,1)$, then $c = -\frac{1-\lambda}{\lambda\rho}\Delta_3(Nc)$. Hence, $c^2 = -\frac{1-\lambda}{\lambda\rho}c\Delta_3(Nc)$. If $|c| > M_{13}$, by (H_4), we obtain

$$c^2 = -\frac{1 - \lambda}{\lambda\rho}c\Delta_3(Nc) < 0,$$

which is a contradiction. Therefore, $|c| \leq M_{13}$ and Ω_{33} is bounded. □

Theorem 3.8 *Assume that* (H_2)-(H_4) *and* (3.5) *hold. Then problem* (1.1) *has at least one solution.*

Proof Let $\Omega \supset \overline{\Omega}_{13} \cup \overline{\Omega}_{23} \cup \overline{\Omega}_{33}$ be bounded. It follows from Lemmas 3.5 and 3.6 that $Lx \neq \lambda Nx$, $x \in (\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega$, $\lambda \in (0,1)$ and $Nx \notin \operatorname{Im} L$, $x \in \operatorname{Ker} L \cap \partial\Omega$. Let

$$H(x,\lambda) = \lambda\rho x + (1-\lambda)J_3 Q_3 Nx,$$

where $J_3 : \operatorname{Im} Q_3 \to \operatorname{Ker} L$ is an isomorphism defined by $J_3(cg_3) = c$, $c \in \mathbb{R}$. By Lemma 3.7, we know $H(x,\lambda) \neq 0$, $x \in \partial\Omega \cap \operatorname{Ker} L$, $\lambda \in [0,1]$. Since the degree is invariant under a homotopy,

$$\begin{aligned} \deg(J_3 Q_3 N|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= \deg\bigl(H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0\bigr) = \deg\bigl(H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0\bigr) \\ &= \deg(\rho\, \mathrm{I}, \Omega \cap \operatorname{Ker} L, 0) \neq 0. \end{aligned}$$

By Theorem 2.1, $Lx = Nx$ has a solution in $\operatorname{dom} L \cap \overline{\Omega}$. □

Case II. $\varphi_i(t) = 0$, $i = 1,2,3$.

In this case, assume there exists $j \in \{1,2,3\}$ such that $\Delta_{j2} \neq 0$. With an adjustment of the method of Lemma 3.1, we can assert the existence of a function $g_2 \in Y$ such that $\Delta_2(g_2) = 1$.

Clearly, $\Delta = 0$ and $\operatorname{Ker} L = \{ct : c \in \mathbb{R}\}$. Similar to the proof of Lemma 3.2, we can show that $\operatorname{Im} L = \{y \in Y : \Delta_2(y) = 0\}$.

Define the operators $P_2 : X \to X$, $Q_2 : Y \to Y$ by

$$P_2 x = x'(0)t, \qquad Q_2 y = \Delta_2(y) g_2.$$

Obviously, P_2 and Q_2 are continuous linear projectors satisfying (2.3).

Define the operator $K_{P_2} : Y \to X$ as

$$K_{P_2} y = \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds - \frac{\Delta_1(y)_{j2}}{2\Delta_{j2}} t^2 - \frac{\Delta_3(y)_{j2}}{2\Delta_{j2}}.$$

As above, we can obtain that $K_{P_2} = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P_2})^{-1}$ and $\|K_{P_2} y\| \le A_{P_2} \|y\|_1$, where

$$A_{P_2} = 1 + \frac{8l}{|\Delta_{j2}|}. \tag{3.6}$$

Suppose that the following conditions hold:

(H_5) There exists $M_{02} > 0$ such that $\Delta_2(Nx) \neq 0$, if $|x'(t)| > M_{02}$, $t \in [0,1]$;

(H_6) There exists $M_{12} > 0$ such that if $|c| > M_{12}$, then either

$$c\Delta_2\bigl(N(ct)\bigr) > 0, \tag{3.7}$$

or

$$c\Delta_2\bigl(N(ct)\bigr) < 0. \tag{3.8}$$

Lemma 3.9 *Assume that conditions* (H_2), (H_5) *hold and let*

$$A_{P_2}\big(\|b\|_1+\|c\|_1+\|d\|_1\big)<\frac{1}{2}, \tag{3.9}$$

where A_{P_2} *satisfies* (3.6). *Then the set*

$$\Omega_{12}=\big\{x\in \operatorname{dom}L\setminus \operatorname{Ker}L: Lx=\lambda Nx, \lambda\in(0,1)\big\}$$

is bounded.

Proof If $x\in\Omega_{12}$, then $\Delta_2(Nx)=0$. By (H_5), there exists a constant $t_1\in[0,1]$ such that $|x'(t_1)|\le M_{02}$. Since $x(t)=P_2x(t)+(I-P_2)x(t)$, $x'(t_1)=x'(0)+((I-P_2)x)'(t_1)$ and

$$\big\|(I-P_2)x\big\|=\big\|K_{P_2}L(I-P_2)x\big\|=\|K_{P_2}Lx\|\le A_{P_2}\|Lx\|_1<A_{P_2}\|Nx\|_1,$$

we have

$$\big|x'(0)\big|\le M_{02}+\big\|(I-P_2)x\big\|\le M_{02}+A_{P_2}\|Nx\|_1.$$

So,

$$\begin{aligned}\|x\|\le\|P_2x\|+\big\|(I-P_2)x\big\|&\le M_{02}+2A_{P_2}\|Nx\|_1\\&\le M_{02}+2A_{P_2}\big(\|a\|_1+\big(\|b\|_1+\|c\|_1+\|d\|_1\big)\|x\|\big).\end{aligned}$$

Thus,

$$\|x\|\le\frac{M_{02}+2A_{P_2}\|a\|_1}{1-2A_{P_2}(\|b\|_1+\|c\|_1+\|d\|_1)},$$

which proves that Ω_{12} is bounded. □

Lemma 3.10 *Assume that* (H_6) *holds. Then the set*

$$\Omega_{22}=\{x\in\operatorname{Ker}L: Nx\in\operatorname{Im}L\}$$

is bounded.

Proof Since $x\in\Omega_{22}$, $x=ct$, $c\in\mathbb{R}$ and $\Delta_2(N(ct))=0$. By (H_6), we have $|c|\le M_{12}$. So, $\|x\|=|c|\le M_{12}$, that is, Ω_{22} is bounded. □

Lemma 3.11 *Assume that* (H_6) *holds. Then the set*

$$\Omega_{32}=\big\{x\in\operatorname{Ker}L:\rho\lambda x+(1-\lambda)J_2Q_2Nx=0,\lambda\in[0,1]\big\}$$

is bounded, where $J_2:\operatorname{Im}Q_2\to\operatorname{Ker}L$, $J_2(cg_2)(t)=ct$, $c\in\mathbb{R}$, *and* $\rho=\begin{cases}1, & \text{if (3.7) holds,}\\ -1, & \text{if (3.8) holds.}\end{cases}$

Proof If $x \in \Omega_{32}$, then $x = ct$, $c \in \mathbb{R}$ and $\lambda\rho c + (1-\lambda)J_2Q_2(N(ct)) = 0$. So,

$$\lambda\rho c + (1-\lambda)\Delta_2\big(N(ct)\big) = 0.$$

If $\lambda = 0$, then $\Delta_2(N(ct)) = 0$. By (H_6), $|c| \le M_{12}$. If $\lambda = 1$, then $c = 0$. If $\lambda \in (0,1)$, $c = -\frac{1-\lambda}{\lambda\rho}\Delta_2(N(ct))$. So,

$$c^2 = -\frac{1-\lambda}{\lambda\rho}c\Delta_2\big(N(ct)\big).$$

If $|c| > M_{12}$, by (H_6), we obtain $c^2 < 0$, a contradiction. So, $|c| \le M_{12}$, that is, Ω_{32} is bounded. □

Under assumption (H_1), N is L-compact on a bounded set $\overline{\Omega}$ as in the proof of Lemma 3.4.

Theorem 3.12 *Assume that* (H_2), (H_5), (H_6) *and* (3.9) *hold. Then FBVP* (1.1) *has at least one solution.*

The proof is similar to that of Theorem 3.8.

Case III. $\varphi_i(t^2) = 0$, $i = 1,2,3$.

In this case, assume that there exists $j \in \{1,2,3\}$ such that $\Delta_{j1} \neq 0$.

Similarly, there exists a function $g_1 \in Y$ such that $\Delta_1(g_1) = 1$.

Obviously, $\Delta = 0$ and $\operatorname{Ker} L = \{ct^2 : c \in \mathbb{R}\}$. Similar to the proof of Lemma 3.2, we can obtain $\operatorname{Im} L = \{y \in Y : \Delta_1(y) = 0\}$.

Define the operators $P_1 : X \to X$, $Q_1 : Y \to Y$ as

$$P_1x = \frac{1}{2}x''(0)t^2, \qquad Q_1y = \Delta_1(y)g_1.$$

Clearly, P_1 and Q_1 are continuous linear projectors. Introduce the operator $K_{P_1} : Y \to X$ by

$$K_{P_1}y = \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds - \frac{\Delta_2(y)_{j1}}{2\Delta_{j1}}t - \frac{\Delta_3(y)_{j1}}{2\Delta_{j1}}.$$

As above, it is easy to show that $K_{P_1} = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P_1})^{-1}$ and $\|K_{P_1}y\| \le A_{P_1}\|y\|_1$, where

$$A_{P_1} = 1 + \frac{4l}{|\Delta_{j1}|}. \tag{3.10}$$

By the same method we used in Lemma 3.4, we can show that N is L-compact on $\overline{\Omega}$.

To prove the main result, we need the following hypotheses:

(H_7) There exists $M_{01} > 0$ such that $\Delta_1(Nx) \neq 0$ if $|x''(t)| > M_{01}$, $t \in [0,1]$;

(H_8) There exists M_{11} such that if $|c| > M_{11}$, then either $c\Delta_1(N(ct^2)) > 0$ or $c\Delta_1(N(ct^2)) < 0$.

Lemma 3.13 *Assume that* (H_2), (H_7) *hold. In addition, assume that*

$$A_{P_1}\big(\|b\|_1 + \|c\|_1 + \|d\|_1\big) < \frac{1}{2}, \tag{3.11}$$

where A_{P_1} *is given by* (3.10). *Then the set*

$$\Omega_{11} = \{x \in \operatorname{dom} L \setminus \operatorname{Ker} L : Lx = \lambda Nx, \lambda \in (0,1)\}$$

is bounded.

Proof For $x \in \Omega_{11}$, we have $\Delta_1(Nx) = 0$. By (H_7), there exists $t_2 \in [0,1]$ such that $|x''(t_2)| \le M_{01}$. Since $x = P_1x + (I - P_1)x$, $\|(I - P_1)x\| \le A_{P_1}\|Lx\|_1 < A_{P_1}\|Nx\|_1$,

$$\left|(P_1x)''(t_2)\right| = \left|x''(t_2) - \left((I - P_1)x\right)''(t_2)\right| \le M_{01} + \left\|(I - P_1)x\right\|$$

and $(P_1x)''(t_2) = x''(0)$, we get

$$\left|x''(0)\right| = \left|(P_1x)''(t_2)\right| \le M_{01} + \left\|(I - P_1)x\right\| \le M_{01} + A_{P_1}\|Lx\|_1.$$

Combining the inequalities above, we get

$$\|x\| < M_{01} + 2A_{P_1}\left(\|a\|_1 + \left(\|b\|_1 + \|c\|_1 + \|d\|_1\right)\|x\|\right).$$

Thus,

$$\|x\| \le \frac{M_{01} + 2A_{P_1}\|a\|_1}{1 - 2A_{P_1}(\|b\|_1 + \|c\|_1 + \|d\|_1)}.$$

In view of (3.11), Ω_{11} is bounded. □

Similarly, if (H_7) and (H_8) hold, we can prove that $\Omega_{21} = \{x \in \operatorname{Ker} L : Nx \in \operatorname{Im} L\}$ and $\Omega_{31} = \{x \in \operatorname{Ker} L : \rho\lambda x + (1 - \lambda)J_1Q_1Nx = 0, \lambda \in [0,1]\}$, with an isomorphism $J_1 : \operatorname{Im} Q \to \operatorname{Ker} L$, $J_1(cg_1)(t) = ct^2$, $c \in \mathbb{R}$, are bounded.

Theorem 3.14 *Assume that* (H_2), (H_7), (H_8) *and* (3.11) *hold. Then FBVP* (1.1) *has at least one solution.*

4 Solvability of (1.1) with condition (1.3)

We define, for convenience,

$$\frac{\varphi_1(t^2)}{\varphi_2(t^2)} = \frac{\varphi_1(t)}{\varphi_2(t)} = \frac{\varphi_1(1)}{\varphi_2(1)} = k, \qquad \Delta_{1j} \neq 0, j \in \{1,2,3\}. \tag{4.1}$$

By the same method as we used in the proof of Lemma 3.1 (see also [6]), there exists $g \in Y$ such that

$$(\varphi_1 - k\varphi_2)\left(\int_0^t (t-s)^2 g(s)\,ds\right) = 1.$$

It is easy to see that

$$\operatorname{Ker} L = \left\{c\left(\Delta_{11}t^2 - \Delta_{12}t + \Delta_{13}\right) : c \in \mathbb{R}\right\}.$$

Lemma 4.1

$$\operatorname{Im} L = \left\{ y \in Y : \varphi_1\left(\int_0^t (t-s)^2 y(s)\,ds\right) = k\varphi_2\left(\int_0^t (t-s)^2 y(s)\,ds\right)\right\}. \tag{4.2}$$

Proof In fact, if $x \in \operatorname{dom} L$, $Lx = y$, then

$$x(t) = \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + at^2 + bt + c$$

and $\varphi_i(x) = 0, i = 1, 2, 3$. So, we have

$$\begin{aligned}
&\frac{1}{2}\varphi_1\left(\int_0^t (t-s)^2 y(s)\,ds\right) + a\varphi_1\left(t^2\right) + b\varphi_1(t) + c\varphi_1(1) = 0,\\
&\frac{1}{2}\varphi_2\left(\int_0^t (t-s)^2 y(s)\,ds\right) + a\varphi_2\left(t^2\right) + b\varphi_2(t) + c\varphi_2(1) = 0.
\end{aligned}$$

In view of (4.1),

$$\varphi_1\left(\int_0^t (t-s)^2 y(s)\,ds\right) = k\varphi_2\left(\int_0^t (t-s)^2 y(s)\,ds\right).$$

On the other hand, if $y \in Y$ satisfies the identity on the right-hand side of (4.2), we choose

$$\begin{aligned}
x(t) &= \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds - \frac{\Delta_2(y)_{11}}{2\Delta_{11}}t - \frac{\Delta_3(y)_{11}}{2\Delta_{11}}, \quad \text{if } \Delta_{11} \neq 0,\\
x(t) &= \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds - \frac{\Delta_1(y)_{12}}{2\Delta_{12}}t^2 - \frac{\Delta_3(y)_{12}}{2\Delta_{12}}, \quad \text{if } \Delta_{11} = 0, \Delta_{12} \neq 0,\\
x(t) &= \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds - \frac{\Delta_1(y)_{13}}{2\Delta_{13}}t^2 - \frac{\Delta_2(y)_{13}}{2\Delta_{13}}t, \quad \text{if } \Delta_{11} = \Delta_{12} = 0, \Delta_{13} \neq 0.
\end{aligned}$$

Obviously, $Lx = y$. If $\Delta_{11} \neq 0$, then

$$\varphi_1(x) = \frac{1}{2}\varphi_1\left(\int_0^t (t-s)^2 y(s)\,ds\right) - \frac{\Delta_2(y)_{11}}{2\Delta_{11}}\varphi_1(t) - \frac{\Delta_3(y)_{11}}{2\Delta_{11}}\varphi_1(1).$$

Considering $\Delta_{11} = \Delta_1(y)_{11}$, $\Delta_2(y)_{11} = \Delta_1(y)_{12}$, $\Delta_3(y)_{11} = -\Delta_1(y)_{13}$ and $\Delta_1(y) = 0$, we get

$$\begin{aligned}
\varphi_1(x) &= \frac{1}{2\Delta_{11}}\left[\varphi_1\left(\int_0^t (t-s)^2 y(s)\,ds\right)\Delta_1(y)_{11} - \varphi_1(t)\Delta_1(y)_{12} + \varphi_1(1)\Delta_1(y)_{13}\right]\\
&= \frac{1}{2\Delta_{11}}\Delta_1(y) = 0.
\end{aligned}$$

Similarly, $\varphi_1(x) = -\frac{1}{2\Delta_{12}}\Delta_2(y) = 0$, if $\Delta_{12} \neq 0$ and $\varphi_1(x) = \frac{1}{2\Delta_{13}}\Delta_3(y) = 0$, if $\Delta_{13} \neq 0$. It is easy to check $\varphi_2(x) = \varphi_3(x) = 0$.

Thus, $x \in \operatorname{dom} L$, that is, $y \in \operatorname{Im} L$. So, (4.2) holds. □

Define operators $P : X \to X$, $Q : Y \to Y$ by

$$Px(t) = \frac{\Delta_{11}x''(0) - \Delta_{12}x'(0) + \Delta_{13}x(0)}{2\Delta_{11}^2 + \Delta_{12}^2 + \Delta_{13}^2}\left(\Delta_{11}t^2 - \Delta_{12}t + \Delta_{13}\right),$$

$$Qy(t) = (\varphi_1 - k\varphi_2)\left(\int_0^t (t-s)^2 y(s)\,ds\right)g(t),$$

where g is introduced at the beginning of the section. Moreover, $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is defined by

$$J(cg)(t) = c\bigl(\Delta_{11}t^2 - \Delta_{12}t + \Delta_{13}\bigr). \tag{4.3}$$

We define $K_P : Y \to X$ as follows:

$$\begin{aligned}
K_P y(t) &= \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds - \frac{\Delta_2(y)_{11}}{2\Delta_{11}}t - \frac{\Delta_3(y)_{11}}{2\Delta_{11}} \\
&\quad + \frac{\Delta_{13}\Delta_3(y)_{11} - \Delta_{12}\Delta_2(y)_{11}}{2\Delta_{11}(2\Delta_{11}^2 + \Delta_{12}^2 + \Delta_{13}^2)}\bigl(\Delta_{11}t^2 - \Delta_{12}t + \Delta_{13}\bigr), \quad \text{if } \Delta_{11} \neq 0, \\
K_P y(t) &= \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds - \frac{\Delta_1(y)_{12}}{2\Delta_{12}}t^2 - \frac{\Delta_3(y)_{12}}{2\Delta_{12}} \\
&\quad + \frac{\Delta_{13}\Delta_3(y)_{12}}{2\Delta_{12}(\Delta_{12}^2 + \Delta_{13}^2)}(-\Delta_{12}t + \Delta_{13}), \quad \text{if } \Delta_{11} = 0, \Delta_{12} \neq 0, \\
K_P y(t) &= \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds - \frac{\Delta_1(y)_{13}}{2\Delta_{13}}t^2 - \frac{\Delta_2(y)_{13}}{2\Delta_{13}}t, \quad \text{if } \Delta_{11} = \Delta_{12} = 0, \Delta_{13} \neq 0.
\end{aligned}$$

Lemma 4.2 *$K_P = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$ and*

$$\|K_P y\| \le A_P\|y\|_1, \qquad A_P = 1 + \frac{16l}{|\Delta_{1j}|}, \quad \Delta_{1j} \neq 0, j \in \{1,2,3\}. \tag{4.4}$$

Proof If $\Delta_{11} \neq 0$, for $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, considering $\Delta_2(Lx)_{11} = -x''(0)\Delta_{12} - 2x'(0)\Delta_{11}$, $\Delta_3(Lx)_{11} = x''(0)\Delta_{13} - 2x(0)\Delta_{11}$, $\Delta_{11}x''(0) - \Delta_{12}x'(0) + \Delta_{13}x(0) = 0$, we have

$$\begin{aligned}
K_P Lx(t) &= \frac{1}{2}\int_0^t (t-s)^2 Lx(s)\,ds - \frac{\Delta_2(Lx)_{11}}{2\Delta_{11}}t - \frac{\Delta_3(Lx)_{11}}{2\Delta_{11}} \\
&\quad + \frac{\Delta_{13}\Delta_3(Lx)_{11} - \Delta_{12}\Delta_2(Lx)_{11}}{2\Delta_{11}(2\Delta_{11}^2 + \Delta_{12}^2 + \Delta_{13}^2)}\bigl(\Delta_{11}t^2 - \Delta_{12}t + \Delta_{13}\bigr) \\
&= x(t).
\end{aligned}$$

Inversely, if $y \in \operatorname{Im} L$, then $\Delta_i(y) = 0$, $i = 1,2,3$. This, together with $\Delta_2(y)_{11} = \Delta_1(y)_{12}$, $\Delta_3(y)_{11} = -\Delta_1(y)_{13}$, $\Delta_{11} = \Delta_1(y)_{11}$, $\Delta = 0$, implies that

$$\begin{aligned}
\varphi_1(K_P y) &= \frac{1}{2}\varphi_1\left(\int_0^t (t-s)^2 y(s)\,ds\right) - \frac{\Delta_2(y)_{11}}{2\Delta_{11}}\varphi_1(t) - \frac{\Delta_3(y)_{11}}{2\Delta_{11}}\varphi_1(1) \\
&\quad + \frac{\Delta_{13}\Delta_3(y)_{11} - \Delta_{12}\Delta_2(y)_{11}}{2\Delta_{11}(2\Delta_{11}^2 + \Delta_{12}^2 + \Delta_{13}^2)}\bigl[\Delta_{11}\varphi_1\bigl(t^2\bigr) - \Delta_{12}\varphi_1(t) + \Delta_{13}\varphi_1(1)\bigr] \\
&= \frac{1}{2\Delta_{11}}\left[\varphi_1\left(\int_0^t (t-s)^2 y(s)\,ds\right)\Delta_1(y)_{11} - \varphi_1(t)\Delta_1(y)_{12} + \varphi_1(1)\Delta_1(y)_{13}\right] \\
&\quad + \frac{\Delta_{13}\Delta_3(y)_{11} - \Delta_{12}\Delta_2(y)_{11}}{2\Delta_{11}(2\Delta_{11}^2 + \Delta_{12}^2 + \Delta_{13}^2)}\Delta \\
&= \frac{1}{2\Delta_{11}}\Delta_1(y) + \frac{\Delta_{13}\Delta_3(y)_{11} - \Delta_{12}\Delta_2(y)_{11}}{2\Delta_{11}(2\Delta_{11}^2 + \Delta_{12}^2 + \Delta_{13}^2)}\Delta = 0.
\end{aligned}$$

Obviously, $\varphi_2(K_P y) = \varphi_3(K_P y) = 0$. So, $K_P y \in \operatorname{dom} L$. By a simple calculation, we can obtain $K_P y \in \operatorname{Ker} P$ and $LK_P y = y$. Therefore, $K_P = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$. If $\Delta_{11} = 0$, $\Delta_{12} \neq 0$ or $\Delta_{11} = \Delta_{12} = 0$, $\Delta_{13} \neq 0$, we can similarly get the result.

If $\Delta_{11} \neq 0$, then

$$\begin{aligned} \|K_P y\| &\le \|y\|_1 + \frac{2l}{2|\Delta_{11}|} \cdot 2\|y\|_1 + \frac{2l}{2|\Delta_{11}|} \cdot 2\|y\|_1 \\ &\quad + \frac{(2|\Delta_{11}| + |\Delta_{12}| + |\Delta_{13}|)(|\Delta_{13}| + |\Delta_{12}|)}{2|\Delta_{11}|(2\Delta_{11}^2 + \Delta_{12}^2 + \Delta_{13}^2)} 2l \cdot 2\|y\|_1 \\ &\le \left(1 + \frac{16l}{|\Delta_{11}|}\right)\|y\|_1. \end{aligned}$$

If $\Delta_{11} = 0$, $\Delta_{12} \neq 0$, then

$$\begin{aligned} \|K_P y\| &\le \|y\|_1 + \frac{2 \cdot 2l}{2|\Delta_{12}|} \cdot 2\|y\|_1 + \frac{2 \cdot 2l}{2|\Delta_{12}|} \cdot 2\|y\|_1 + \frac{(|\Delta_{12}| + |\Delta_{13}|)|\Delta_{13}|}{|\Delta_{12}|(\Delta_{12}^2 + \Delta_{13}^2)} 4l \cdot \|y\|_1 \\ &\le \left(1 + \frac{8l}{|\Delta_{12}|} + \frac{8l}{|\Delta_{12}|}\right)\|y\|_1 = \left(1 + \frac{16l}{|\Delta_{12}|}\right)\|y\|_1. \end{aligned}$$

Similarly, if $\Delta_{11} = \Delta_{12} = 0$, $\Delta_{13} \neq 0$, then

$$\|K_P y\| \le \left(1 + \frac{16l}{|\Delta_{13}|}\right)\|y\|_1.$$

□

Lemma 4.3 *Assume that (H_1) holds and $\Omega \subset X$ is bounded. Then N is L-compact on Ω.*

Proof For $x \in \overline{\Omega}$, there exists a constant $r > 0$ such that $\|x\| \le r$. By (H_1), we get

$$\begin{aligned} \|QNx\|_1 &= \left|(\varphi_1 - k\varphi_2)\left(\int_0^t (t-s)^2 f\big(s, x(s), x'(s), x''(s)\big)\, ds\right)\right| \|g\|_1 \\ &\le (k_1 + kk_2)\left\|\int_0^t (t-s)^2 f\big(s, x(s), x'(s), x''(s)\big)\, ds\right\| \|g\|_1 \\ &\le (k_1 + kk_2)\|h_r\|_1 \|g\|_1, \end{aligned}$$

that is, $QN(\overline{\Omega})$ is bounded. Hence,

$$\big\|K_P(I-Q)Nx\big\| \le A_P\big(\|Nx\|_1 + \|QNx\|_1\big) \le A_P\big(1 + (k_1 + kk_2)\|g\|_1\big)\|h_r\|_1.$$

So, $K_P(I-Q)N(\overline{\Omega})$ is bounded.

By the same method as used in the proof of Lemma 3.3, we can demonstrate that $K_P(I - Q)N(\overline{\Omega})$ is compact. Thus N is L-compact. □

When $\Delta_{11} \neq 0$, we assume that the following conditions hold:

(H_9) There exists a constant $M_0 > 0$ such that if $|x''(t)| > M_0$, then

$$(\varphi_1 - k\varphi_2)\left(\int_0^t (t-s)^2 Nx(s)\, ds\right) \neq 0;$$

(H_{10}) There exists a constant $M_1 > 0$ such that if $|c| > M_1$, either

$$c(\varphi_1 - k\varphi_2)\left(\int_0^t (t-s)^2 N\big(c\big(\Delta_{11}s^2 - \Delta_{12}s + \Delta_{13}\big)\big)\,ds\right) > 0, \tag{4.5}$$

or

$$c(\varphi_1 - k\varphi_2)\left(\int_0^t (t-s)^2 N\big(c\big(\Delta_{11}s^2 - \Delta_{12}s + \Delta_{13}\big)\big)\,ds\right) < 0. \tag{4.6}$$

Lemma 4.4 *Assume that* $\Delta_{11} \neq 0$, (H_2), (H_9) *and*

$$\|b\|_1 + \|c\|_1 + \|d\|_1 < \frac{2|\Delta_{11}|}{A_P(4|\Delta_{11}| + |\Delta_{12}| + |\Delta_{13}|)} \tag{4.7}$$

hold. Then the set

$$\Omega_1 = \big\{x \in \operatorname{dom} L \setminus \operatorname{Ker} L, Lx = \lambda Nx, \lambda \in (0,1)\big\}$$

is bounded.

Proof Since $x \in \Omega_1$, then $QNx = 0$. By (H_9), there exists $t_0 \in [0,1]$ such that $|x''(t_0)| \le M_0$. Since $x = Px + (I-P)x$,

$$\big\|(I-P)x\big\| < A_P\|Nx\|_1, \tag{4.8}$$

and $x''(t) = (Px)''(t) + ((I-P)x)''(t)$, it follows that

$$\big|(Px)''(t_0)\big| = \big|x''(t_0) - \big((I-P)x\big)''(t_0)\big| < M_0 + A_P\|Nx\|_1.$$

Considering

$$(Px)''(t_0) = \frac{\Delta_{11}x''(0) - \Delta_{12}x'(0) + \Delta_{13}x(0)}{2\Delta_{11}^2 + \Delta_{12}^2 + \Delta_{13}^2} \cdot 2\Delta_{11},$$

we have

$$\left|\frac{\Delta_{11}x''(0) - \Delta_{12}x'(0) + \Delta_{13}x(0)}{2\Delta_{11}^2 + \Delta_{12}^2 + \Delta_{13}^2}\right| \le \frac{1}{2\Delta_{11}}\big(M_0 + A_P\|Nx\|_1\big).$$

This, together with (4.8) and (H_2), means

$$\begin{aligned}
\|x\| &\le \|Px\| + \big\|(I-P)x\big\| \\
&\le \frac{M_0 + A_P\|Nx\|_1}{2|\Delta_{11}|}\big(2|\Delta_{11}| + |\Delta_{12}| + |\Delta_{13}|\big) + A_P\|Nx\|_1 \\
&= \frac{M_0}{2|\Delta_{11}|}\big(2|\Delta_{11}| + |\Delta_{12}| + |\Delta_{13}|\big) + \left(\frac{2|\Delta_{11}| + |\Delta_{12}| + |\Delta_{13}|}{2|\Delta_{11}|} + 1\right)A_P\|Nx\|_1 \\
&\le \frac{M_0}{2|\Delta_{11}|}\big(2|\Delta_{11}| + |\Delta_{12}| + |\Delta_{13}|\big) \\
&\quad + \left(\frac{(4|\Delta_{11}| + |\Delta_{12}| + |\Delta_{13}|)}{2|\Delta_{11}|}\right)A_P\big[\|a\|_1 + \big(\|b\|_1 + \|c\|_1 + \|d\|_1\big)\|x\|\big].
\end{aligned}$$

So,

$$\|x\| \le \frac{M_0(2|\Delta_{11}| + |\Delta_{12}| + |\Delta_{13}|) + A_P\|a\|_1(4|\Delta_{11}| + |\Delta_{12}| + |\Delta_{13}|)}{2|\Delta_{11}| - A_P(4|\Delta_{11}| + |\Delta_{12}| + |\Delta_{13}|)(\|b\|_1 + \|c\|_1 + \|d\|_1)}.$$

Therefore, $\overline{\Omega}_1$ is bounded due to (4.7). □

Lemma 4.5 *Assume that* $\Delta_{11} \neq 0$ *and* (H_{10}) *holds. Then the set* $\Omega_2 = \{x \in \operatorname{Ker} L : Nx \in \operatorname{Im} L\}$ *is bounded.*

Proof For $x \in \Omega_2$, we have $x = c(\Delta_{11}t^2 - \Delta_{12}t + \Delta_{13})$ and $QNx = 0$. By (H_{10}), we get that $|c| \le M_1$. So, Ω_2 is bounded. □

Let $\rho = \begin{cases} 1, & \text{if (4.5) holds,} \\ -1, & \text{if (4.6) holds.} \end{cases}$

Lemma 4.6 *Assume that* $\Delta_{11} \neq 0$ *and* (H_{10}) *holds. The set*

$$\Omega_3 = \bigl\{x \in \operatorname{Ker} L : \lambda\rho x + (1-\lambda)JQNx = 0, \lambda \in [0,1]\bigr\}$$

is bounded, where $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *is defined by* (4.3).

Proof If $x \in \Omega_3$, then $\lambda\rho x + (1-\lambda)JQNx = 0$, $x = c(\Delta_{11}t^2 - \Delta_{12}t + \Delta_{13})$.

If $\lambda = 0$, then $QNx = 0$. It follows from (H_{10}) that $\|x\| \le M_1(2|\Delta_{11}| + |\Delta_{12}| + |\Delta_{13}|)$. If $\lambda = 1$, then $x \equiv 0$. For $\lambda \in (0,1)$, we have

$$\begin{aligned} &c\bigl(\Delta_{11}t^2 - \Delta_{12}t + \Delta_{13}\bigr) \\ &\quad = -\frac{1-\lambda}{\lambda\rho}(\varphi_1 - k\varphi_2)\biggl(\int_0^t (t-s)^2 N\bigl(c\bigl(\Delta_{11}s^2 - \Delta_{12}s + \Delta_{13}\bigr)\bigr)\,ds\biggr)\bigl(\Delta_{11}t^2 - \Delta_{12}t + \Delta_{13}\bigr). \end{aligned}$$

That is,

$$c^2 = -\frac{1-\lambda}{\lambda\rho}(\varphi_1 - k\varphi_2)\biggl(\int_0^t (t-s)^2 N\bigl(c\bigl(\Delta_{11}s^2 - \Delta_{12}s + \Delta_{13}\bigr)\bigr)\,ds\biggr).$$

By (H_{10}), we know that $|c| \le M_1$. So, Ω_3 is bounded. □

Theorem 4.7 *Assume that* $\Delta_{11} \neq 0$ *and* (H_2), (H_9), (H_{10}) *hold and*

$$\|b\|_1 + \|c\|_1 + \|d\|_1 < \frac{2|\Delta_{11}|}{A_P(4|\Delta_{11}| + |\Delta_{12}| + |\Delta_{13}|)},$$

where A_p *is given by* (4.4). *Then FBVP* (1.1) *has at least one solution.*

The proof is similar to that of Theorem 3.8.

The example below illustrates Theorem 4.7.

Consider

$$x'''(t) = \frac{1}{\sqrt{t}} + A\sin x(t) + A\sin x'(t) + Ax''(t), \quad t \in (0,1),$$

where $A = 1/113{,}414$, along with the functional conditions

$$\phi_1(x) = 7x(0) + 4u'(0) + 2x''(0) + 12\int_0^1 x(s)\,ds = 0,$$

$$\phi_2(x) = x(0) + x'(0) + x''(0) + 18\int_0^1 x(s)\,ds = 0,$$

$$\phi_3(x) = -4x(0) - 2x'(0) - \frac{1}{2}x''(0) + 6\int_0^1 x(s)\,ds = 0.$$

In this case $\phi_1(t^2) = \phi_2(t^2) = 8$, $\phi_3(t^2) = 1$, $\phi_1(t) = \phi_2(t) = 10$, $\phi_3(t) = 1$, and $\phi_1(1) = \phi_2(1) = 19$, $\phi_3(1) = 2$, so that $k = 1$. Also, $\triangle = 0$ and $\triangle_{11} = 1$, $\triangle_{12} = -3$, $\triangle_{13} = -2$, and

$$\operatorname{Ker} L = \{c(\triangle_{11}t^2 - \triangle_{12}t + \triangle_{13}) : c \in \mathbb{R}\} = \{c(t^2 + 3t - 2) : c \in \mathbb{R}\}.$$

Subsequently,

$$\phi_1(x) - k\phi_2(x) = 6x(0) + 3x'(0) + x''(0) - 6\int_0^1 x(s)\,ds$$

and

$$\begin{aligned}
&\phi_1\left(\int_0^t (t-s)^2 Nx(s)\,ds\right) - k\phi_2\left(\int_0^t (t-s)^2 Nx(s)\,ds\right)\\
&\quad = -6\int_0^1 \int_0^s (s-\tau)^2 Nx(\tau)\,d\tau\,ds\\
&\quad = -2\int_0^1 (1-s)^3 f\bigl(s, x(s), x'(s), x''(s)\bigr)\,ds.
\end{aligned}$$

We can easily check $(\phi_1 - k\phi_2)(-t^5) = 1$.

The following estimates hold for $x''(t) < -M_0$:

$$f\bigl(t, x(t), x'(t), x''(t)\bigr) = \frac{1}{\sqrt{t}} + A\sin x(t) + A\sin x'(t) + Ax''(t) < \frac{1}{\sqrt{t}} + 2A - AM_0,$$

and hence

$$\begin{aligned}
&\phi_1\left(\int_0^t (t-s)^2 Nx(s)\,ds\right) - k\phi_2\left(\int_0^t (t-s)^2 Nx(s)\,ds\right)\\
&\quad = -2\int_0^1 (1-s)^3 f\bigl(s, x(s), x'(s), x''(s)\bigr)\,ds\\
&\quad > -2\int_0^1 (1-s)^3\left(\frac{1}{\sqrt{s}} + 2A - AM_0\right)ds\\
&\quad = -\frac{64}{35} - A + \frac{1}{2}AM_0\\
&\quad > 0
\end{aligned}$$

provided $M_0 > 2 + \frac{128}{35A}$.

If $x''(t) > M_0$, then

$$\begin{aligned} f\bigl(t,x(t),x'(t),x''(t)\bigr) &= \frac{1}{\sqrt{t}} + A\sin x(t) + A\sin x'(t) + Ax''(t) \\ &> \frac{1}{\sqrt{t}} - 2A + AM_0, \end{aligned}$$

and hence

$$\begin{aligned} &\phi_1\biggl(\int_0^t (t-s)^2 Nx(s)\,ds\biggr) - k\phi_2\biggl(\int_0^t (t-s)^2 Nx(s)\,ds\biggr) \\ &\quad = -2\int_0^1 (1-s)^3 f\bigl(s,x(s),x'(s),x''(s)\bigr)\,ds \\ &\quad < -2\int_0^1 (1-s)^3\biggl(\frac{1}{\sqrt{s}} - 2A + AM_0\biggr)\,ds \\ &\quad = -\frac{64}{35} + A - \frac{1}{2}AM_0 \\ &\quad < 0 \end{aligned}$$

provided $M_0 > 2$.

Therefore, if we choose $M_0 > 2 + \frac{64}{35A}$, then (H_9) holds.

Let now $c \in \mathbb{R}$ and $x_c(t) = c(t^2 + 3t - 2)$. Then

$$Nx_c(t) = \frac{1}{\sqrt{t}} + A\sin\bigl(x_c(t)\bigr) + A\sin\bigl(x_c'(t)\bigr) + 2Ac$$

and

$$(\phi_1 - k\phi_2)\biggl(\int_0^t (t-s)^2 Nx_c(s)\,ds\biggr) = -\frac{64}{35} - 2A\int_0^1 (1-s)^3\bigl(\sin\bigl(x_c(s)\bigr) + \sin\bigl(x_c'(s)\bigr) + 2c\bigr)\,ds.$$

Then, repeating the computation leading to the choice of M_0, we obtain that $|c| > M_1 = M_0/2$ results in

$$\begin{aligned} &c(\phi_1 - k\phi_2)\biggl(\int_0^t (t-s)^2 Nx_c(s)\,ds\biggr) \\ &\quad = c\biggl(-\frac{64}{35} - 2A\int_0^1 (1-s)^3\bigl(\sin\bigl(x_c(s)\bigr) + \sin\bigl(x_c'(s)\bigr) + 2c\bigr)\,ds\biggr) < 0, \end{aligned}$$

that is, (H_{10}) holds.

Finally, note that $k_1 = 25$, $k_2 = 21$, $k_3 = 25/2$, so that $l = 525$. Hence $A_P = 1 + 16l/|\triangle_{11}| = 8{,}401$. Then

$$\|b\|_1 + \|c\|_1 + \|d\|_1 = 3A < 2/75{,}609 = \frac{2|\Delta_{11}|}{A_P(4|\Delta_{11}| + |\Delta_{12}| + |\Delta_{13}|)}.$$

All the conditions of Theorem 4.7 are verified.

The following corollaries are stated without proof.

Corollary 4.8 *Let $\Delta_{11} = 0$, $\Delta_{12} \neq 0$ and assume that (H_2) and the following conditions hold*:

$$\|b\|_1 + \|c\|_1 + \|d\|_1 < \frac{|\Delta_{12}|}{A_P(2|\Delta_{12}| + |\Delta_{13}|)};$$

(H_{11}) *There exists a constant $M_0' > 0$ such that if $|x'(t)| \geq M_0'$, then*

$$(\varphi_1 - k\varphi_2)\left(\int_0^t (t-s)^2 Nx(s)\,ds\right) \neq 0;$$

(H_{12}) *There exists a constant $M_1' > 0$ such that if $|c| > M_1'$, then either*

$$c(\varphi_1 - k\varphi_2)\left(\int_0^t (t-s)^2 N\bigl(c(-\triangle_{12}s + \triangle_{13})\bigr)\,ds\right) > 0,$$

or

$$c(\varphi_1 - k\varphi_2)\left(\int_0^t (t-s)^2 N\bigl(c(-\triangle_{12}s + \triangle_{13})\bigr)\,ds\right) < 0.$$

Then FBVP (1.1) *has at least one solution.*

Corollary 4.9 *Let $\Delta_{11} = \Delta_{12} = 0$, $\Delta_{13} \neq 0$ and assume that (H_2) and the following conditions hold*:

$$\|b\|_1 + \|c\|_1 + \|d\|_1 < \frac{2}{A_P};$$

(H_{13}) *There exists a constant $M_0'' > 0$ such that if $|x(t)| \geq M_0''$, then*

$$(\varphi_1 - k\varphi_2)\left(\int_0^t (t-s)^2 Nx(s)\,ds\right) \neq 0;$$

(H_{14}) *There exists a constant $M_1'' > 0$ such that if $|c| > M_1''$, then either*

$$c(\varphi_1 - k\varphi_2)\left(\int_0^t (t-s)^2 N(c\triangle_{13})\,ds\right) > 0,$$

or

$$c(\varphi_1 - k\varphi_2)\left(\int_0^t (t-s)^2 N(c\triangle_{13})\,ds\right) < 0.$$

Then FBVP (1.1) *has at least one solution.*

5 Conclusion

This paper is a study of third-order functional boundary value problems at resonance; it improves and generalizes some of the existent results. We present several generalizations to the existing results and improvements to the method based on Mawhin's coincidence degree theory.

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