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General decay for a viscoelastic wave equation with strong time-dependent delay

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Abstract
A viscoelastic wave equation with strong damping and strong time-dependent delay in the internal feedback is considered. Under the assumption $|\mu_2| < \sqrt{1-d}\mu_1$, we establish the general decay of energy of the problem by using the energy perturbation method.

MSC: 35B35; 35B40; 93D15

Keywords: general decay; wave equation; memory; delay feedbacks

1 Introduction

In this paper, we consider the following viscoelastic wave equation with a strong damping and a strong time-dependent delay in the internal feedback:

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(s)\,ds - \mu_1 \Delta u_t(x,t) - \mu_2 \Delta u_t\big(x,t-\tau(t)\big) = 0, \tag{1.1}$$

where $x \in \Omega$, and $\Omega \subseteq \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. The function $g(t)$ is the relaxation function. μ_1, μ_2 are constants and $\tau(t) > 0$ denotes the time-dependent delay.

We consider the following initial conditions:

$$\begin{cases} u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u_t(x,t) = f_0(x,t), \quad x \in \Omega, t \in [-\tau(0), 0), \end{cases} \tag{1.2}$$

and the following boundary conditions:

$$u = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+. \tag{1.3}$$

Recently the control of wave equations with time delay effects has become an active area of research. The delay effects often appear in many practical problems and may turn a well-behaved system into a wild one. The presence of delay can be a source of instability. Here we mention the work of Nicaise and Pignotti [1]. In this work the authors considered a wave equation with time delay of the form

$$u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0.$$

Under the assumption $0 < \mu_2 < \mu_1$, they proved exponential stability of the system. For viscoelastic wave equation with time delay, Kirane and Said-Houari [2] studied the following equation:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0,$$

and obtained the global well-posedness of solutions and established the energy decay under the assumption $0 < \mu_2 \leq \mu_1$. Liu [3] considered a wave with time-dependent delay

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t + \mu_2 u_t\bigl(t-\tau(t)\bigr) = 0,$$

and established the general decay of the system under $|\mu_1| < \sqrt{1-d}\mu_1$. Dai and Yang [4] considered the same equation as in [2] and improved the above results under weaker conditions. In this work, the authors obtained the global well-posedness without any restrictions on μ_1, μ_2, and established an exponential decay result of energy in the case $\mu_1 = 0$. Benaissa *et al.* [5] investigated a wave equation with nonlinear time delay of the form

$$u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)\,ds + \mu_1 g_1\bigl(u_t(t)\bigr) + \mu_2 g_2\bigl(u_t(t-\tau)\bigr) = 0.$$

They proved the global existence of solution under assumption of a relation between the weight of the delay term in the feedback and the weight of the term without delay. In addition, they obtained the general decay of energy. Liu and Zhang [6] considered a wave equation with past history and time delay in internal feedback

$$u_{tt} - \alpha\Delta u + \int_{-\infty}^t \mu(t-s)\Delta u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau) + f(u) = h.$$

They proved the global well-posedness without any restrictions on μ_1, μ_2. Furthermore, they also proved the exponential decay of energy with $0 < |\mu_2| < \mu_1$. Kafini *et al.* [7] consider a nonlinear damped second-order evolution equation with delay

$$u_{tt} + Au + G(u_t) + \mu G\bigl(u_t(t-\tau)\bigr) = F(u),$$

and they proved that the energy of the solutions blows up in finite time under some suitable assumptions. Recently, Alabau-Boussouira *et al.* [8] studied a wave equation with past history and time delay

$$u_{tt} - \Delta u + \int_0^\infty \mu(s)\Delta u(t-s)\,ds + ku_t(t-\tau) = 0.$$

They showed that the system is exponentially stable if the coefficient of delay k is small enough. They also established the stability in the case $\tau = 0$ and $k < 0$. For some more results concerning the wave equation with a weak time delay term under an appropriate assumption between μ_1 and μ_2, one can refer to Benaissa *et al.* [9], Datko *et al.* [10], Liu [11], Nicaise and Pignotti [12–14], Nicaise *et al.* [15], Nicaise and Valein [16], Xu *et al.* [17], and the references therein. With respect to waves with strong time delay there is just little published work. The only one we found is duo to Messaoudi *et al.* [18]. In this work, they investigated the following equation:

$$u_{tt} - \Delta u - \mu_1 \Delta u_t - \mu_2 \Delta u_t(t-\tau) = 0,$$

and proved the well-posedness under the assumption $|\mu_2| \le \mu_1$ and established exponential decay of energy under the assumption $|\mu_2| < \mu_1$. Moreover, they also studied a wave equation with distributed delay.

In the absence of time delay, problems similar to (1.1) have been extensively studied and there are many results in the literature, most of which are mainly concerned with global well-posedness, asymptotic behavior and blow-up. See, for example, Berrimi and Messaoudi [19], Cavalcanti *et al.* [20, 21], Han and Wang [22, 23], Liu [24], Messaoudi *et al.* [25–31], Tatar [32].

Motivated by [18], we study in the present work the asymptotic behavior for system (1.1)-(1.3). Since the delay is dependent on time, this makes the work different from [18]. The main objective of the present work is to establish a general decay result from which the exponential decay and polynomial decay are only special cases.

The plan of this paper is as follows. In Section 2, we give some assumptions and our main results. In Section 3, we establish the general decay result of the energy by using the energy perturbation method.

2 Assumptions and main results

$L^q(\Omega)$ $(1 \le q \le \infty)$ and $H^1(\Omega)$ denote the Lebesgue integral and Sobolev spaces. $\|\cdot\|_B$ is the norm in the space B, we write $\|\cdot\|$ instead of $\|\cdot\|_2$ for $q = 2$.

For the relaxation function g, we assume $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing C^1 function satisfying

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)\,ds = l > 0, \tag{2.1}$$

and there exists a nonincreasing differentiable function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying, for $t \ge 0$,

$$\zeta(t) > 0, \qquad g'(t) \le -\zeta(t)g(t) \tag{2.2}$$

and

$$\int_0^\infty \zeta(t)\,dt = \infty.$$

Concerning the delay $\tau(t)$, we assume

$$0 < \tau_0 \le \tau(t) \le \tau_1, \quad \forall t > 0, \tag{2.3}$$

where the constants τ_0 and τ_1 are two positive constants. We assume further that

$$\tau(t) \in W^{2,\infty}(0,T) \quad \text{and} \quad \tau'(t) \le d < 1, \quad \forall T, t > 0. \tag{2.4}$$

The weak solutions of (1.1)-(1.3) are defined as follows: for given initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, we call a function $U = (u, u_t) \in C(\mathbb{R}^+, H_0^1(\Omega) \times L^2(\Omega))$ a weak solution to the problem (1.1)-(1.3) if $U(0) = (u_0, u_1)$ and

$$\begin{aligned}&(u_{tt}, \omega) + (\nabla u, \nabla \omega) + \int_0^t g(t-s)\big(\Delta u(s), \omega\big)\, ds \\ &\quad + \mu_1(\nabla u_t, \nabla \omega) + \mu_2\big(\nabla u_t\big(t-\tau(t)\big), \nabla \omega\big) = 0,\end{aligned}$$

for all $\omega \in H_0^1(\Omega)$.

The global well-posedness of problem (1.1)-(1.3) will be given in the following theorem.

Theorem 2.1 *Let $\mu_2 \leq \mu_1$, and assume the assumptions* (2.1)-(2.4) *hold. If the initial data* $(u_0, u_1) \in (H_0^1(\Omega) \times L^2(\Omega))$, $f_0 \in H^1(\Omega \times (-\tau(0), 0))$, *then problem* (1.1)-(1.3) *has a unique weak solution* $(u, u_t) \in C(0, T; H_0^1(\Omega) \times L^2(\Omega))$ *such that, for any* $T > 0$,

$$u \in L^\infty\big(0, T; H_0^1(\Omega)\big), \qquad u_t \in L^\infty\big(0, T; L^2(\Omega)\big).$$

Remark 2.1 By using the classical Faedo-Galerkin method, see, *e.g.*, [2], we can prove the theorem and we omit the proof here.

The energy functional of problem (1.1)-(1.3) is defined by

$$\begin{aligned}E(t) &= \frac{1}{2}\big\|u_t(t)\big\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\, ds\right)\big\|\Delta u(t)\big\|^2 + \frac{1}{2}(g \circ \Delta u) \\ &\quad + \frac{\xi}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\big\|\nabla u_t(s)\big\|^2\, ds,\end{aligned} \tag{2.5}$$

where $\xi > 0$ is a constant to be determined later, the constant $\lambda > 0$, as below, has been introduced in [13],

$$\lambda < \frac{1}{\tau_1}\left|\log \frac{|\mu_2|}{\sqrt{1-d}}\right|,$$

and we have

$$(g \circ v)(t) = \int_0^t g(t-s)\big\|v(t) - v(s)\big\|^2\, ds.$$

Our main stability result is the following theorem.

Theorem 2.2 *Assume the assumptions* (2.1)-(2.4) *hold. Let* $|\mu_2| < \sqrt{1-d}\,\mu_1$. *Let* (u, u_t) *be the weak solutions of problem* (1.1)-(1.3) *with the initial data* $(u_0, u_1) \in (H_0^1(\Omega) \times L^2(\Omega))$, $f_0 \in H^1(\Omega \times (-\tau(0), 0))$. *Then there exist two constants* $\beta > 0$ *and* $\gamma > 0$ *such that the energy* $E(t)$ *satisfies*

$$E(t) \leq \beta \exp\left(-\gamma \int_0^t \zeta(s)\, ds\right), \quad \textit{for all } t \geq 0. \tag{2.6}$$

3 Proof of Theorem 2.2

In this section, we shall study the general decay of energy to problem (1.1)-(1.3) to prove Theorem 2.2. For this purpose, we need the following technical lemmas.

Lemma 3.1 *Under the assumptions of Theorem* 2.2, *the energy functional* $E(t)$ *satisfies, for any* $t \geq 0$,

$$\begin{aligned} E'(t) \leq{}& \frac{1}{2}\bigl(g' \circ \nabla u\bigr)(t) - \frac{1}{2}g(t)\bigl\|\nabla u(t)\bigr\|^2 + \left(\frac{\xi}{2} - \mu_1 + \frac{|\mu_2|}{2\sqrt{1-d}}\right)\bigl\|\nabla u_t(t)\bigr\|^2 \\ &+ \left[\frac{|\mu_2|}{2}\sqrt{1-d} - \frac{\xi}{2}(1-d)e^{-\lambda\tau_1}\right]\bigl\|\nabla u_t\bigl(t-\tau(t)\bigr)\bigr\|^2 \\ &- \frac{\lambda\xi}{2}\int_{t-\tau(t)}^{t} e^{-\lambda(t-s)}\bigl\|\nabla u_t(s)\bigr\|^2\,ds. \end{aligned} \tag{3.1}$$

Proof Differentiating (2.5) and using (1.1), (2.3)-(2.4) and integration by parts, we have

$$\begin{aligned} E(t) ={}& \int_\Omega u_t u_{tt}\,dx - \frac{1}{2}g(t)\|\nabla u\|^2 + \left(1 - \int_0^t g(s)\,ds\right)\int_\Omega \nabla u\cdot\nabla u_t\,dx + \frac{1}{2}\bigl(g'\circ\nabla u\bigr) \\ &+ \int_0^t g(t-s)\,ds\cdot\int_\Omega \nabla u\cdot\nabla u_t\,dx - \int_\Omega \nabla u_t(t)\cdot\int_0^t g(t-s)\nabla u(s)\,ds\,dx \\ &+ \frac{\xi}{2}\|\nabla u_t\|^2 - \frac{\xi}{2}e^{-\lambda\tau(t)}\bigl(1-\tau'(t)\bigr)\bigl\|\nabla u_t\bigl(t-\tau(t)\bigr)\bigr\|^2 \\ &- \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\bigl\|\nabla u_t(s)\bigr\|^2\,ds \\ \leq{}& -\mu_1\|\nabla u_t\|^2 - \frac{1}{2}g(t)\|\nabla u\|^2 + \frac{1}{2}\bigl(g'\circ\nabla u\bigr) + \frac{\xi}{2}\|\nabla u_t\|^2 \\ &- \mu_2\int_\Omega \nabla u_t\cdot\nabla u_t\bigl(t-\tau(t)\bigr)\,dx - \frac{\xi}{2}(1-d)e^{-\lambda\tau_1}\bigl\|\nabla u_t\bigl(t-\tau(t)\bigr)\bigr\|^2 \\ &- \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\bigl\|\nabla u_t(s)\bigr\|^2\,ds. \end{aligned} \tag{3.2}$$

It follows from Young's inequality that

$$-\mu_2\int_\Omega \nabla u_t\cdot\nabla u_t\bigl(t-\tau(t)\bigr)\,dx \leq \frac{|\mu_2|}{2\sqrt{1-d}}\bigl\|\nabla u_t(t)\bigr\|^2 + \frac{|\mu_2|}{2}\sqrt{1-d}\bigl\|\nabla u_t\bigl(t-\tau(t)\bigr)\bigr\|^2,$$

which, together with (3.2) gives us (3.1). The proof is done. □

Lemma 3.2 *Under the assumptions of Theorem* 2.2, *then the functional* $\phi(t)$ *defined as*

$$\phi(t) = \int_\Omega u(t)u_t(t)\,dx, \tag{3.3}$$

satisfies the requirement that there exist positive constants c_1, c_2 *and* c_3 *such that, for any* $t \geq 0$,

$$\phi'(t) \leq -\frac{l}{2}\bigl\|\nabla u(t)\bigr\|^2 + c_1\bigl\|\nabla u_t(t)\bigr\|^2 + c_2\bigl\|\nabla u_t\bigl(t-\tau(t)\bigr)\bigr\|^2 + c_3(g\circ\nabla u)(t). \tag{3.4}$$

Proof By using (1.1), we can get

$$
\begin{aligned}
\phi'(t) &= \int_\Omega u_{tt} u\,dx + \|u_t\|^2 \\
&= \|u_t\|^2 + \int_\Omega u(t) \cdot \left(\Delta u - \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 \Delta u_t + \mu_2 \Delta u_t\bigl(t-\tau(t)\bigr) \right) dx \\
&= \|u_t\|^2 - \|\nabla u\|^2 + \int_\Omega \nabla u(t) \cdot \int_0^t g(t-s)\bigl(\nabla u(s) - \nabla u(t)\bigr)\,ds\,dx \\
&\quad + \int_0^t g(s)\,ds \cdot \|\nabla u\|^2 + \mu_1 \int_\Omega u \cdot \Delta u_t\,dx - \mu_2 \int_\Omega \nabla u \cdot \nabla u_t\bigl(t-\tau(t)\bigr)\,dx \\
&\le \|u_t\|^2 - l\|\nabla u\|^2 + \underbrace{\int_\Omega \nabla u(t) \cdot \int_0^t g(t-s)\bigl(\nabla u(s) - \nabla u(t)\bigr)\,ds\,dx}_{:=I_1} \\
&\quad + \underbrace{\mu_1 \int_\Omega u \cdot \Delta u_t\,dx}_{:=I_2} - \underbrace{\mu_2 \int_\Omega \nabla u \cdot \nabla u_t\bigl(t-\tau(t)\bigr)\,dx}_{:=I_3}. \qquad (3.5)
\end{aligned}
$$

By using Young's inequality and Hölder's inequality, we shall see below, for any $\delta > 0$,

$$
\begin{aligned}
I_1 &\le \delta\|\nabla u\|^2 + \frac{1}{4\delta} \int_\Omega \left(\int_0^t g(t-s)\bigl(\nabla u(s) - \nabla u(t)\bigr)\,ds \right)^2 dx \\
&\le \delta\|\nabla u\|^2 + \frac{1}{4\delta} \int_0^t g(s)\,ds (g \circ \nabla u)(t) \\
&\le \delta\|\nabla u\|^2 + \frac{1-l}{4\delta}(g \circ \nabla u)(t), \qquad (3.6) \\
I_2 &\le \delta\|\nabla u\|^2 + \frac{\mu_1^2}{4\delta}\|\nabla u_t\|^2, \qquad (3.7) \\
I_3 &\le \delta\|\nabla u\|^2 + \frac{\mu_2^2}{4\delta}\bigl\|\nabla u_t\bigl(t-\tau(t)\bigr)\bigr\|^2,
\end{aligned}
$$

which, together with (3.5)-(3.7), implies, for any $\delta > 0$,

$$
\phi'(t) \le \|u_t\|^2 - (l - 3\delta)\|\nabla u\|^2 + \frac{\mu_1^2}{4\delta}\|\nabla u_t\|^2 + \frac{\mu_2^2}{4\delta}\bigl\|\nabla u_t\bigl(t-\tau(t)\bigr)\bigr\|^2 + \frac{1-l}{4\delta}(g \circ \nabla u)(t). \qquad (3.8)
$$

Now taking $\delta > 0$ small enough such that

$$
l - 3\delta > \frac{l}{2},
$$

we can get the desired estimate (3.4) with

$$
c_1 = \frac{1}{\lambda_1} + \frac{\mu_1^2}{4\delta}, \qquad c_2 = \frac{\mu_2^2}{4\delta}, \qquad c_3 = \frac{1-l}{4\delta},
$$

hereafter the positive constant λ_1 represents the Poincaré's constant, *i.e.*, $\lambda_1\|u\|^2 \le \|\nabla u\|$ for $u \in H_0^1(\Omega)$. The proof is hence complete. □

Lemma 3.3 *We define the functional* $\psi(t)$ *as*

$$\psi(t) = -\int_\Omega u_t(t) \cdot \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx. \tag{3.9}$$

Under the assumptions of Theorem 2.2, *the functional* $\psi(t)$ *satisfies, for any* $\delta > 0$,

$$\begin{aligned}\psi'(t) \le{}& -\left(\int_0^t g(s)\,ds - \delta\right)\|u_t\|^2 + \delta\big\|\nabla u_t(t)\big\|^2 + \delta\big\|\nabla u_t\big(t-\tau(t)\big)\big\|^2 \\ &+ c_4(g\circ\Delta u)(t) - c_5\big(g'\circ\Delta u\big)(t),\end{aligned} \tag{3.10}$$

where c_4 *and* c_5 *are positive constants.*

Proof It follows from (1.1) and integration by parts that

$$\begin{aligned}\psi'(t) ={}& -\int_\Omega u_{tt}\cdot\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\ &- \int_\Omega u_t\left[u_t\int_0^t g(t-s)\,ds + \int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\right]dx \\ ={}& \int_\Omega\left(-\Delta u + \int_0^t g(t-s)\Delta u(s)\,ds - \nabla u_t - \mu_2\Delta u_t\big(t-\tau(t)\big)\right) \\ &\times\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx - \int_0^t g(s)\,ds\|u_t\|^2 \\ &- \int_\Omega u_t\int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\ ={}& \left(1-\int_0^t g(s)\,ds\right)\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx \\ &+ \int_\Omega\left(\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)\,ds\right)^2 dx - \int_0^t g(s)\,ds\|u_t\|^2 \\ &- \int_\Omega u_t\int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\ &+ \mu_1\int_\Omega \nabla u_t(t)\cdot\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx \\ &+ \mu_2\int_\Omega \nabla u_t\big(t-\tau(t)\big)\cdot\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx.\end{aligned} \tag{3.11}$$

By using Hölder's inequality, Young's inequality and the Poincaré inequality, we can obtain, for any $\delta > 0$,

$$\mu_2\int_\Omega \nabla u\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx \le \delta\|\nabla u_t\|^2 + \frac{\mu_1^2(1-l)}{4\delta}(g\circ\nabla u)(t), \tag{3.12}$$

$$\begin{aligned}&\mu_2\int_\Omega \nabla u_t\big(t-\tau(t)\big)\cdot\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx \\ &\quad\le \delta\big\|\nabla u_t\big(t-\tau(t)\big)\big\|^2 + \frac{\mu_2^2(1-l)}{4\delta}(g\circ\nabla u)(t),\end{aligned} \tag{3.13}$$

$$
\begin{aligned}
&-\int_\Omega u_t \int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&\quad \le \delta\|u_t\|^2 + \frac{1}{4\delta}\left(\int_0^t \big(-g'(t-s)\big)\big\|u(t)-u(s)\big\|\,ds\right)^2 \\
&\quad \le \delta\|u_t\|^2 - \frac{Cg(0)}{4\delta\lambda_1}\big(g'\circ\Delta u\big)(t),
\end{aligned}
\tag{3.14}
$$

and

$$
\int_\Omega\left(\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)\,ds\right)^2 dx \le \frac{1-l}{\lambda_1}(g\circ\nabla u)(t),
$$

which, combined (3.12)-(3.14) with (3.11), gives us (3.10). The proof is therefore complete. □

Now we define the Lyapunov functional $F(t)$ by

$$
F(t) := E(t) + \varepsilon_1\phi(t) + \varepsilon_2\psi(t), \tag{3.15}
$$

where ε_1 and ε_2 are positive constants to be taken later. First we know that, for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ small enough, there exist two positive constants β_1 and β_2 such that, for any $t > 0$,

$$
\beta_1 E(t) \le F(t) \le \beta_2 E(t). \tag{3.16}
$$

Proof of Theorem 2.2 For any $t_0 > 0$, we get, for any $t \ge t_0$,

$$
\int_0^t g(s)\,ds \ge \int_0^{t_0} g(s)\,ds := g_0.
$$

It follows from (3.1), (3.4) and (3.10) that, for any $t \ge t_0$,

$$
\begin{aligned}
F'(t) \le{}& -\left(\int_0^t g(s)\,ds - \delta\right)\varepsilon_2\big\|u_t(t)\big\|^2 - \left(\frac{l}{2}\varepsilon_1 - \delta\varepsilon_2\right)\big\|\nabla u(t)\big\|^2 \\
&+ \left(\frac{\xi}{2} - \mu_1 + \frac{|\mu_2|}{2\sqrt{1-d}} + c_1\varepsilon_1 + \delta\varepsilon_2\right)\big\|\nabla u_t(t)\big\|^2 \\
&+ \left[\frac{|\mu_2|}{2}\sqrt{1-d} - \frac{\xi}{2}(1-d)e^{-\lambda\tau_1} + c_2\varepsilon_1 + \delta\varepsilon_2\right]\big\|\nabla u_t\big(t-\tau(t)\big)\big\|^2 \\
&+ \left(\frac{1}{2} - c_5\varepsilon_2\right)\big(g'\circ\nabla u\big)(t) + (c_3\varepsilon_1 + c_4\varepsilon_2)(g\circ\nabla u)(t) \\
&- \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\big\|\nabla u_t(s)\big\|^2\,ds.
\end{aligned}
\tag{3.17}
$$

Obviously, $e^{\lambda\tau_1} \to 1$ as $\lambda \to 0$. Because of the continuity of the set of real numbers, we pick $\lambda > 0$ small enough such that there exists a positive constant ξ such that

$$
\frac{e^{\lambda\tau_1}|\mu_2|}{\sqrt{1-d}} < \xi < \mu_1, \tag{3.18}
$$

which implies

$$\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2} < 0 \quad \text{and} \quad \frac{|\mu_2|}{2}\sqrt{1-d} - \frac{\xi}{2e^{\lambda\tau_1}}(1-d) < 0. \tag{3.19}$$

It follows from (3.18) and (3.19) that the energy functional (2.5) is nonincreasing.

Now we take $\delta > 0$ small enough such that, for $t \geq t_0$,

$$\int_0^t g(s)\,ds - \delta \geq \frac{1}{2}g_0.$$

At this point, for any fixed $\delta > 0$, we choose $\varepsilon_2 > 0$ so small that (3.16) holds, and further

$$\varepsilon_2 < \min\left\{\frac{1}{4c_5}, \frac{\mu_1}{2\delta} - \frac{\xi}{4\delta} - \frac{|\mu_2|}{4\delta\sqrt{1-d}}, \frac{\xi}{4\delta}(1-d)e^{-\lambda\tau_1} - \frac{|\mu_2|}{4\delta}\sqrt{1-d}\right\},$$

which yields

$$\frac{1}{2} - c_5\varepsilon_2 > \frac{1}{4}, \qquad \frac{\xi}{2} - \mu_1 + \frac{|\mu_2|}{2\sqrt{1-d}} + \delta\varepsilon_2 < \frac{\xi}{4} - \frac{\mu_1}{2} + \frac{|\mu_2|}{4\sqrt{1-d}}$$

and

$$\frac{|\mu_2|}{2}\sqrt{1-d} - \frac{\xi}{2}(1-d)e^{-\lambda\tau_1} + \delta\varepsilon_2 < \frac{|\mu_2|}{4}\sqrt{1-d} - \frac{\xi}{4}(1-d)e^{-\lambda\tau_1}.$$

In the sequel, for any fixed $\delta > 0$ and $\varepsilon_2 > 0$, we take $\varepsilon_1 > 0$ small so that (3.16) holds, and further

$$\frac{4\delta\varepsilon_2}{l} < \varepsilon_1 < \min\left\{\frac{\mu_1}{4c_1} - \frac{\xi}{8c_1} - \frac{|\mu_2|}{8c_1\sqrt{1-d}}, \frac{\xi}{8c_2}(1-d)e^{-\lambda\tau_1} - \frac{|\mu_2|}{8c_2}\sqrt{1-d}\right\},$$

which gives us

$$\frac{l}{2}\varepsilon_1 - \delta\varepsilon_2 > \frac{l}{4}\varepsilon_2, \qquad \frac{\xi}{4} - \frac{\mu_1}{2} + \frac{|\mu_2|}{4\sqrt{1-d}} + c_1\varepsilon_1 < \frac{\xi}{8} - \frac{\mu_1}{4} + \frac{|\mu_2|}{8\sqrt{1-d}}$$

and

$$\frac{|\mu_2|}{4}\sqrt{1-d} - \frac{\xi}{4}(1-d)e^{-\lambda\tau_1} + c_2\varepsilon_1 < \frac{|\mu_2|}{8}\sqrt{1-d} - \frac{\xi}{8}(1-d)e^{-\lambda\tau_1}.$$

From the above we know that, for positive constants α_1 and α_2,

$$F'(t) \leq -\alpha_1 E(t) + \alpha_2 (g \circ \nabla u)(t), \quad \forall t \geq t_0. \tag{3.20}$$

Multiplying (3.20) by $\zeta(t)$ and using (2.2), we can get, for any $t \geq t_0$,

$$\begin{aligned}
\zeta(t)F'(t) &\leq -\alpha_1\zeta(t)E(t) + \alpha_2\zeta(t)(g \circ \Delta u)(t) \\
&\leq -\alpha_1\zeta(t)E(t) - \alpha_2\big(g' \circ \Delta u\big)(t) \\
&\leq -\alpha_1\zeta(t)E(t) - \alpha_3 E'(t),
\end{aligned} \tag{3.21}$$

where $\alpha_3 > 0$. Denote $\mathcal{E}(t) = \zeta(t)F(t) + \alpha_3 E(t)$, then it is easy to see that $\mathcal{E}(t)$ is equivalent to the energy $E(t)$, *i.e.*, there exist two positive constants β_3 and β_4 such that

$$\beta_3 E(t) \le \mathcal{E}(t) \le \beta_4 E(t). \tag{3.22}$$

Thus we can infer that, for any $t \ge t_0$,

$$\mathcal{E}'(t) \le -\frac{\alpha_1}{\beta_4}\zeta(t)\mathcal{E}(t),$$

which, integrating over (t_0, t) with respect to t, yields, for any $t \ge t_0$,

$$\mathcal{E}(t) \le \mathcal{E}(t_0)\exp\left(-\frac{\alpha_1}{\beta_4}\int_{t_0}^{t}\zeta(s)\,ds\right). \tag{3.23}$$

Therefore (2.6) follows from (3.23) by renaming the constants, and by the continuity and boundedness of $E(t)$ and $\zeta(t)$. The proof is hence complete. □

Remark 3.1 If taking $\zeta(t) = \gamma$ and $\zeta(t) = \gamma(1 + t)^{-1}$, and γ a positive constant, we can obtain the exponential decay and polynomial decay of problem (1.1)-(1.3), respectively. Thus the exponential decay and polynomial decay is a particular case of (2.6). We also find some other examples to illustrate several rates of energy decay; see, for example, [25, 32].

Competing interests
The author declares that he has no competing interests.

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