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Stabilized mixed finite element model for the 2D nonlinear incompressible viscoelastic fluid system

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Abstract
In this study, we first establish a stabilized mixed finite element (SMFE) model based on parameter-free and two local Gauss integrals for the two-dimensional (2D) nonlinear incompressible viscoelastic fluid system. And then, we prove the existence, uniqueness, and convergence of the SMFE solutions. Finally, we use a numerical example to verify the correctness of the previous theoretical results.

MSC: 65N30; 35Q10

Keywords: stabilized mixed finite element model; parameter-free and two local Gauss integrals; two-dimensional nonlinear incompressible viscoelastic fluid system; existence and uniqueness as well as convergence

1 Introduction

In this study, we take into account the following two-dimensional (2D) nonlinear incompressible viscoelastic fluid system (see [1]):

Problem I Find $\boldsymbol{F} = (F_{l,m})_{2\times 2}$, $\boldsymbol{u} = (u_x, u_y)^T$, and p such that

$$\begin{cases} \boldsymbol{F}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{F} = \nabla\boldsymbol{u}\boldsymbol{F}, \quad (x,y,t)\in\Omega\times(0,T), \\ \boldsymbol{u}_t - \mu\triangle\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \nabla\cdot(\boldsymbol{F}\boldsymbol{F}^T), \quad (x,y,t)\in\Omega\times(0,T), \\ \nabla\cdot\boldsymbol{u} = 0, \quad (x,y,t)\in\Omega\times(0,T), \end{cases} \tag{1}$$

subject to initial conditions

$$\boldsymbol{F}(x,y,0) = \boldsymbol{\psi}(x,y), \qquad \boldsymbol{u}(x,y,0) = \boldsymbol{\varphi}(x,y), \quad (x,y)\in\Omega \tag{2}$$

and boundary conditions

$$\boldsymbol{F}(x,y,t) = \boldsymbol{F}_0(x,y,t), \qquad \boldsymbol{u}(x,y,t) = \boldsymbol{u}_0(x,y,t), \quad (x,y,t)\in\partial\Omega\times[0,T], \tag{3}$$

where $\Omega\subset R^2$ is the connected and bounded polygonal domain, $\boldsymbol{F} = (F_{l,m})_{2\times 2}$ represents the unknown deformation tensor, $\boldsymbol{u} = (u_x, u_y)^T$ denotes the unknown fluid velocity, and

p represents the unknown pressure, $\mu = 1/Re$ is the known viscosity coefficient and Re is the Reynolds number, and $\boldsymbol{\psi}(x,y)$, $\boldsymbol{\varphi}(x,y)$, $\boldsymbol{F}_0(x,y,t)$, and $\boldsymbol{u}_0(x,y,t)$ are four given functions. For convenience and without loss of universality, we presume that $\boldsymbol{F}_0(x,y,t) = \mathbf{0}$ and $\boldsymbol{u}_0(x,y,t) = \mathbf{0}$ in hereinafter discussion.

The nonlinear incompressible viscoelastic fluid system (1)-(3) is often used to describe some specific physical phenomena, for example, the rheological behaviors of complex fluids and the electromagnetic behaviors (see [2–5]).

Although the existence of the generalized solution for the nonlinear incompressible viscoelastic fluid system (1)-(3) has been provided in [1, 6], because the system not merely contains the pressure and the velocity vector but also includes the deformation tensor matrix, *i.e.*, it includes seven unknown functions, it is a difficult task to obtain its analytical solution. We have to count on numerical solutions.

However, for all we know, up to now, there has not been any report that the stabilized mixed finite element (SMFE) model for the 2D nonlinear incompressible viscoelastic fluid system (1)-(3) is developed. Even though Bellet [7] established a mixed finite element (MFE) model for the purely viscoplastic compressible flow including three-field formulation (velocity, volumetric strain rate, and pressure), the problem discussed was stationary and linear, and the existence and uniqueness as well as the convergence of the numerical solutions was not presented. Sampaio [8, 9] also developed some MFE models for incompressible viscous flows, but the existence and uniqueness as well as the convergence of the numerical solutions had not been given yet. Faria and Karam-Filho [10] also proposed a regularized-stabilized MFE formulation for the steady flow of an incompressible fluid of Bingham type, but their problem is also stationary and linear. Whereas Problem I here is nonlinear, involves seven unknown functions, and its unknown deformation tensor is an unsymmetrical matric; therefore, it is entirely different from those problems in [7–10] and is more complex than those equations in [7–10]. Thus, the theoretical analysis of the existence and uniqueness as well as the convergence of the SMFE solutions to Problem I faces more difficult and greater challenges than those mentioned above, but the problem has some specific applications. Therefore, in Section 2, we first address a time semi-discrete model and deduce the existence, uniqueness, and convergence for the time semi-discrete solutions. And then, in Section 3, we address a fully discretized SMFE model based on parameter-free and two local Gauss integrals for the 2D nonlinear incompressible viscoelastic fluid system and deduce the existence, uniqueness, and convergence of the SMFE solutions. Finally, we give a numerical example to verify the correctness of the previous theoretical results in Section 4. Thus, we would verify the effectiveness of the SMFE model from two aspects of theory and numerical experiments. This signifies that the current work is significant and is development and improvement over the existing results mentioned above.

2 Establishment of time semi-discrete model

The Sobolev spaces and their norms used hereinafter are normative (see [11]). Let $H = [H_0^1(\Omega)]^2$ and $W = [H_0^1(\Omega)]^{2\times 2}$ as well as $M = L_0^2(\Omega) =: \{q \in L^2(\Omega) : \int_\Omega q \,\mathrm{d}x\,\mathrm{d}y = 0\}$. Thus, the mixed variational form for the 2D nonlinear incompressible viscoelastic fluid system can be described in the following.

Problem II Find $(\boldsymbol{u},\boldsymbol{F},p) \in H \times W \times M$ such that, for $0 < t \leq T$,

$$\begin{cases} (\boldsymbol{u}_t,\boldsymbol{v}) + A(\boldsymbol{u},\boldsymbol{v}) + A_1(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) - B(p,\boldsymbol{v}) = (\operatorname{div}(\boldsymbol{F}\boldsymbol{F}^T),\boldsymbol{v}), \quad \forall \boldsymbol{v} \in H, \\ (\boldsymbol{F}_t,\boldsymbol{\tau}) + A_2(\boldsymbol{u},\boldsymbol{F},\boldsymbol{\tau}) = (\nabla \boldsymbol{u}\boldsymbol{F},\boldsymbol{\tau}), \quad \forall \boldsymbol{\tau} \in W, \\ B(q,\boldsymbol{u}) = 0, \quad \forall q \in M, \\ \boldsymbol{F}(x,y,0) = \boldsymbol{\psi}(x,y), \qquad \boldsymbol{u}(x,y,0) = \boldsymbol{\varphi}(x,y), \quad (x,y) \in \Omega, \end{cases} \tag{4}$$

where $(\cdot,\cdot)$ represents the scalar product of $L^2(\Omega)^{2\times 2}$ or $L^2(\Omega)^2$, and

$$\begin{aligned} &A(\boldsymbol{u},\boldsymbol{v}) = \mu \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \,\mathrm{d}x\,\mathrm{d}y, \quad \forall \boldsymbol{u},\boldsymbol{v} \in H, \\ &B(q,\boldsymbol{v}) = \int_\Omega q \operatorname{div} \boldsymbol{v} \,\mathrm{d}x\,\mathrm{d}y, \quad \forall \boldsymbol{v} \in H, q \in M, \\ &A_1(\boldsymbol{v},\boldsymbol{w},\boldsymbol{u}) = \frac{1}{2}\int_\Omega \bigl[\bigl((\boldsymbol{v}\cdot\nabla)\boldsymbol{w}\bigr)\cdot \boldsymbol{u} - \bigl((\boldsymbol{v}\cdot\nabla)\boldsymbol{u}\bigr)\cdot\boldsymbol{w}\bigr] \,\mathrm{d}x\,\mathrm{d}y, \quad \forall \boldsymbol{v},\boldsymbol{w},\boldsymbol{u} \in H, \\ &A_2(\boldsymbol{w},\boldsymbol{F},\boldsymbol{\tau}) = \frac{1}{2}\int_\Omega \bigl[\bigl((\boldsymbol{w}\cdot\nabla)\boldsymbol{F}\bigr)\boldsymbol{\tau} - \bigl((\boldsymbol{w}\cdot\nabla)\boldsymbol{\tau}\bigr)\boldsymbol{F}\bigr] \,\mathrm{d}x\,\mathrm{d}y, \quad \forall \boldsymbol{w} \in H, \forall \boldsymbol{F},\boldsymbol{\tau} \in W. \end{aligned}$$

The aforesaid trilinear functions $A_1(\cdot,\cdot,\cdot)$ and $A_2(\cdot,\cdot,\cdot)$, bilinear functions $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$ satisfy the following (see, *e.g.*, [12–17]):

$$A_1(\boldsymbol{v},\boldsymbol{w},\boldsymbol{u}) = -A_1(\boldsymbol{v},\boldsymbol{u},\boldsymbol{w}), \qquad A_1(\boldsymbol{v},\boldsymbol{w},\boldsymbol{w}) = 0, \quad \forall \boldsymbol{v},\boldsymbol{w},\boldsymbol{u} \in H, \tag{5}$$

$$A_2(\boldsymbol{w},\boldsymbol{F},\boldsymbol{\tau}) = -A_2(\boldsymbol{w},\boldsymbol{\tau},\boldsymbol{F}), \qquad A_2(\boldsymbol{w},\boldsymbol{\tau},\boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{w} \in H, \forall \boldsymbol{F},\boldsymbol{\tau} \in W, \tag{6}$$

$$A(\boldsymbol{v},\boldsymbol{v}) \geq \mu \|\nabla \boldsymbol{v}\|_0^2, \qquad \bigl|A(\boldsymbol{u},\boldsymbol{v})\bigr| \leq \mu \|\nabla \boldsymbol{u}\|_0 \|\nabla \boldsymbol{v}\|_0, \quad \forall \boldsymbol{u},\boldsymbol{v} \in H, \tag{7}$$

$$\sup_{\boldsymbol{v}\in X} \frac{B(q,\boldsymbol{v})}{\|\nabla \boldsymbol{v}\|_0} \geq \beta \|q\|_0, \quad \forall q \in M, \tag{8}$$

here β is the known coefficient. Put

$$N_0 = \sup_{\boldsymbol{v},\boldsymbol{w},\boldsymbol{u}\in X} \frac{A_1(\boldsymbol{v},\boldsymbol{w},\boldsymbol{u})}{\|\nabla \boldsymbol{v}\|_0 \cdot \|\nabla \boldsymbol{w}\|_0 \cdot \|\nabla \boldsymbol{u}\|_0}. \tag{9}$$

By using the same approach as the one in [1, 6], we can obtain the following conclusion for Problem II.

Theorem 1 *If the initial functions* $\boldsymbol{\varphi}(x,y) \in [L^2(\Omega)]^2$ *and* $\boldsymbol{\psi}(x,y) \in [L^2(\Omega)]^{2\times 2}$, *then Problem* II *has one and only one solution* $(\boldsymbol{u},\boldsymbol{F},p) \in H \times W \times M$, *relying only on the initial functions* $\boldsymbol{\psi}(x,y)$ *and* $\boldsymbol{\varphi}(x,y)$.

Let k be the time step, $N = [T/k]$ indicate the integer part for T/k, and $(\boldsymbol{u}^n,\boldsymbol{F}^n,p^n)$ denote the time semi-discrete solutions of $(\boldsymbol{u}(t),\boldsymbol{F},p)$ at $t_n = nk$ $(n = 0,1,\ldots,N)$. If the derivatives $\boldsymbol{u}_t$ and $\boldsymbol{F}_t$ at time $t = t_n$ are replaced with $(\boldsymbol{u}^n - \boldsymbol{u}^{n-1})/k$ and $(\boldsymbol{F}^n - \boldsymbol{F}^{n-1})/k$, separately, then the time semi-discrete model for Problem II is described in the following.

Problem III Find $(\boldsymbol{u}^n, \boldsymbol{F}^n, p^n) \in H \times W \times M$ $(1 \le n \le N)$ such that

$$\begin{cases} (\boldsymbol{u}^n, \boldsymbol{v}) + kA(\boldsymbol{u}^n, \boldsymbol{v}) + kA_1(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{v}) - kB(p^n, \boldsymbol{v}) \\ \quad = k(\operatorname{div}(\boldsymbol{F}^n \boldsymbol{F}^{nT}), \boldsymbol{v}) + (\boldsymbol{u}^{n-1}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in H, \\ (\boldsymbol{F}^n, \boldsymbol{\tau}) + kA_2(\boldsymbol{u}^{n-1}, \boldsymbol{F}^n, \boldsymbol{\tau}) = k(\nabla \boldsymbol{u}^{n-1} \boldsymbol{F}^n, \boldsymbol{\tau}) + (\boldsymbol{F}^{n-1}, \boldsymbol{\tau}), \quad \forall \boldsymbol{\tau} \in W, \\ B(q, \boldsymbol{u}^n) = 0, \quad \forall q \in M, \\ \boldsymbol{F}^0 = \boldsymbol{\psi}(x,y), \qquad \boldsymbol{u}^0 = \boldsymbol{\varphi}(x,y), \quad (x,y) \in \Omega. \end{cases} \tag{10}$$

For Problem III, we have the following main conclusions.

Theorem 2 *Under the conditions of Theorem* 1, *if there exists a constant* $\alpha > 0$ *that satisfies* $\|\nabla \boldsymbol{u}^{n-1}\|_{0,\infty} \le \alpha$ *and* k *is sufficiently small that satisfies* $(1 - 2k\alpha) > 0$, *Problem* III *has one and only one sequence of solutions* $(\boldsymbol{u}^n, \boldsymbol{F}^n, p^n) \in H \times W \times M$ $(n = 1, 2, \dots, N)$ *that satisfies*

$$\|\boldsymbol{F}^n\|_0 \le \frac{1}{(\sqrt{1-2k\alpha})^N} \|\boldsymbol{\psi}\|_0, \tag{11}$$

$$\|\boldsymbol{u}^n\|_0^2 + \|p^n\|_0^2 + \mu k \sum_{i=1}^{n} \|\nabla \boldsymbol{u}^i\|_0^2 \le C\big(\|\boldsymbol{F}^n \boldsymbol{F}^{nT}\|_0^2 + \|\boldsymbol{\varphi}\|_0^2\big), \tag{12}$$

where C *used hereinafter is a constant that does not rely on* k; *however it relies on* $\boldsymbol{\psi}$, $\boldsymbol{\varphi}$, *and* R_e. *Further, when the exact solution to Problem* I *satisfies* $(\boldsymbol{u}, \boldsymbol{F}, p) \in [H_0^1(\Omega) \cap H^2(\Omega)]^2 \times [H_0^1(\Omega) \cap H^2(\Omega)]^{2\times 2} \times [H^1(\Omega) \cap M]$, *there hold the error estimations*

$$\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|_0^2 + k \sum_{i=1}^{n} \big[\|\nabla(\boldsymbol{u}(t_i) - \boldsymbol{u}^i)\|_0^2 + \|p(t_i) - p^i\|_0^2\big] \le Ck^2, \tag{13}$$

$$\|\boldsymbol{F}(t_n) - \boldsymbol{F}^n\|_0 \le Ck, \quad 1 \le n \le N. \tag{14}$$

Proof Because the second equation in Problem III is linear, in order to deduce that the second equation in Problem III has one and only one sequence of solutions $\{\boldsymbol{F}^n\}_{n=1}^N \subset W$, we only need to prove that when $\boldsymbol{\psi}(x,y) = \mathbf{0}$, there holds $\boldsymbol{F}^n = \mathbf{0}$ $(n = 1, 2, \dots, N)$. Thus, we only prove that (11) is correct. To this end, by taking $\boldsymbol{\tau} = \boldsymbol{F}^n$ in the second equation in Problem III, if $\|\nabla \boldsymbol{u}^{n-1}\|_{0,\infty} \le \alpha$, by means of using (6) and Hölder and Cauchy inequalities, we have

$$\|\boldsymbol{F}^n\|_0^2 = k\big(\nabla \boldsymbol{u}^{n-1} \boldsymbol{F}^n, \boldsymbol{F}^n\big) + \big(\boldsymbol{F}^{n-1}, \boldsymbol{F}^n\big) \le k\alpha \|\boldsymbol{F}^n\|_0^2 + \frac{1}{2}\big(\|\boldsymbol{F}^n\|_0^2 + \|\boldsymbol{F}^{n-1}\|_0^2\big). \tag{15}$$

When k is sufficiently small that satisfies $(1 - 2k\alpha) > 0$, we have

$$\|\boldsymbol{F}^n\|_0 \le \frac{\|\boldsymbol{F}^{n-1}\|_0}{\sqrt{1-2k\alpha}} \le \cdots \le \frac{\|\boldsymbol{\psi}\|_0}{(\sqrt{1-2k\alpha})^n} \le \frac{\|\boldsymbol{\psi}\|_0}{(\sqrt{1-2k\alpha})^N}. \tag{16}$$

Thus, if $\boldsymbol{\psi} = \mathbf{0}$, then the second equation in Problem III has only a sequence of zero solutions, which signifies that it has one and only one sequence of solutions $\{\boldsymbol{F}^n\}_{n=1}^N$.

After we have obtained $\{\boldsymbol{F}^n\}_{n=1}^N$ from the second equation in Problem III, the first and third equations in Problem III constitute the weak-form of nonstationary Navier-Stokes equations. Thus, with the theory of the weak-form of nonstationary Navier-Stokes equations (see, *e.g.*, [12, 13, 17, 18]), the first and third equations in Problem III have one and only one sequence of solutions $(\boldsymbol{u}^n, p^n)$ $(n = 1, 2, \ldots, N)$ that satisfies (12).

Finally, with Taylor's expansion and by using the same proof methods as the ones in Theorems 5.6 and 5.7 in [12] or referring to [18], the error formulas (13) and (14) are easily obtained. □

Remark 1 Theorem 2 implies that the sequence of solutions to Problem III is stabilized and achieves the optimal order convergence about time. Moreover, it is known via the regularity for PDEs that, when the initial data $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ are properly smooth, it can be ensured that $\|\nabla \boldsymbol{u}^{n-1}\|_{0,\infty}$ of the sequence of solutions for Problem III are bounded so that the assumptions $\|\nabla \boldsymbol{u}^{n-1}\|_{0,\infty} \leq \alpha$ are reasonable.

3 Establishment of the fully discretized SMFE model

In the section, we firsthand formulate the fully discretized SMFE model via the time semi-discrete form so that we could avoid the semi-discrete MFE method about spatial variables.

Let $\Im_h = \{K\}$ be the quasi-regular triangulation of Ω (see [12, 18]). The MFE subspaces are chosen as

$$W_h = \{\boldsymbol{\tau}_h \in W \cap [C(\overline{\Omega})]^{2\times 2}; \boldsymbol{\tau}_h|_K \in [\mathscr{P}_1(K)]^{2\times 2}, \forall K \in \Im_h\},$$
$$H_h = \{\boldsymbol{v}_h \in H \cap [C(\overline{\Omega})]^2; \boldsymbol{v}_h|_K \in [\mathscr{P}_1(K)]^2, \forall K \in \Im_h\},$$
$$M_h = \{q_h \in M \cap C(\overline{\Omega}); q_h|_K \in \mathscr{P}_1(K), \forall K \in \Im_h\},$$

where $\mathscr{P}_1(K)$ represents the bivariate linear polynomial set on K.

The next lemma is classical and very serviceable (see [12]).

Lemma 3 *Suppose that* $P_h : H \to H_h$ *represents an elliptic operator, i.e.,* $\forall \boldsymbol{u} \in H$, *there is one and only one* $P_h\boldsymbol{u} \in X_h$ *that satisfies*

$$(\nabla(P_h\boldsymbol{u} - \boldsymbol{u}), \nabla \boldsymbol{v}_h) = 0, \quad \forall \boldsymbol{v}_h \in X_h.$$

Then the following error estimates hold:

$$\|P_h\boldsymbol{u} - \boldsymbol{u}\|_s \leq Ch^{2-s}\|\boldsymbol{u}\|_2, \quad s = 0, 1, \forall \boldsymbol{u} \in [H^2(\Omega)]^2,$$

where C used hereinafter represents a general positive constant which is possibly different at different occurrence and does not rely on h and k.

Suppose that $Q_h : M \to M_h$ *represents an* L^2*-operator, i.e.,* $\forall \omega \in M$, *there is one and only one* $Q_h\omega \in M_h$ *that satisfies*

$$(Q_h\omega - \omega, \omega_h) = 0, \quad \forall \omega_h \in M_h.$$

When $\omega \in H^l(\Omega)$, *the following error estimates hold*:

$$\|Q_h\omega - \omega\|_s \leq Ch^{l-s}\|\omega\|_l, \quad s = 0, 1, l = 1, 2.$$

Suppose that $R_h : W \to W_h$ represents also an L^2-operator, i.e., $\forall \boldsymbol{\tau} \in W$, there is one and only one $R_h\boldsymbol{\tau} \in W_h$ that satisfies

$$(R_h\boldsymbol{\tau} - \boldsymbol{\tau}, \boldsymbol{\tau}_h) = 0, \quad \forall \boldsymbol{\tau}_h \in W_h.$$

Then the following error estimates hold:

$$\|R_h\boldsymbol{\tau} - \boldsymbol{\tau}\|_s \le Ch^{2-s}\|\boldsymbol{\tau}\|_2, \quad s = 0,1, \forall \boldsymbol{\tau} \in \bigl[H^2(\Omega)\bigr]^{2\times 2}.$$

Then the fully discrete SMFE model based on parameter-free and two local Gauss integrals is described in the following.

Problem IV Find $(\boldsymbol{u}_h^n, p_h^n, T_h^n) \in X_h \times M_h \times W_h$ $(1 \le n \le N)$ such that

$$\begin{cases} (\bar{\partial}_t \boldsymbol{u}_h^n, \boldsymbol{v}_h) + A(\boldsymbol{u}_h^n, \boldsymbol{v}_h) + A_1(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n, \boldsymbol{v}_h) - B(p_h^n, \boldsymbol{v}_h) \\ \quad = (\operatorname{div}(\boldsymbol{F}_h^n \boldsymbol{F}_h^{nT}), \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in X_h, \\ B(q_h, \boldsymbol{u}_h^n) + D_h(p_h^n, q_h) = 0, \quad \forall q_h \in M_h, \\ (\bar{\partial}_t \boldsymbol{F}_h^n, \boldsymbol{\tau}_h) + A_2(\boldsymbol{u}_h^{n-1}, \boldsymbol{F}_h^n, \boldsymbol{\tau}_h) = (\nabla \boldsymbol{u}_h^{n-1} \boldsymbol{F}_h^n, \boldsymbol{\tau}_h), \quad \forall \boldsymbol{\tau}_h \in W_h, \\ \boldsymbol{u}_h^0 = P_h \boldsymbol{\varphi}, \qquad \boldsymbol{F}_h^0 = R_h \boldsymbol{\psi}(x,y), \quad (x,y) \in \Omega, \end{cases} \tag{17}$$

where

$$D_h\bigl(p_h^n, q_h\bigr) = \varepsilon \sum_{K \in \Im_h} \left\{ \int_{K,2} p_h^n q_h \,\mathrm{d}x\,\mathrm{d}y - \int_{K,1} p_h^n q_h \,\mathrm{d}x\,\mathrm{d}y \right\}, \quad p_h, q_h \in M_h, \tag{18}$$

here $\varepsilon > 0$, called parameter-free, represents a constant, $\int_{K,i} \lambda(x,y)\,\mathrm{d}x\,\mathrm{d}y$ $(i = 1,2)$ are two proper Gauss integrals on K and accurate for ith order multinomials $(i = 1,2)$, and $\lambda(x,y) = q_h p_h$ is the ith order multinomial $(i = 1,2)$.

Hence, if $q_h \in M_h$, then $p_h \in M_h$ is just a piecewise constant as $i = 1$. Suppose that the operator $\varrho_h : L^2(\Omega) \to \hat{M}_h := \{q_h \in L^2(\Omega) : q_h|_K \in \mathscr{P}_0(K) \forall K \in \Im_h\}$ that satisfies, $\forall p \in L^2(\Omega)$,

$$(p, q_h) = (\varrho_h p, q_h), \quad \forall q_h \in \hat{M}_h, \tag{19}$$

where $\mathscr{P}_0(K)$ is the zero degree polynomial set on K. Therefore, the operator ϱ_h has the following properties (see [12, 19]):

$$\|\varrho_h p\|_0 \le C\|p\|_0, \quad \forall p \in L^2(\Omega), \tag{20}$$

$$\|p - \varrho_h p\|_0 \le Ch\|p\|_1, \quad \forall p \in H^1(\Omega). \tag{21}$$

Thus, by using ϱ_h, the bilinear function $D_h(\cdot,\cdot)$ may be indicated into:

$$D_h(q_h, p_h) = \varepsilon(q_h - \varrho_h q_h, p_h) = \varepsilon(q_h - \varrho_h q_h, p_h - \varrho_h p_h). \tag{22}$$

To discuss the existence, uniqueness, and convergence of the SMFE solutions, it is necessary to use the following discrete Gronwall lemma (see [12, 18]).

Lemma 4 (Discrete Gronwall lemma) *Suppose that the positive sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *and the monotone positive sequence* $\{\epsilon_n\}$ *satisfy* $\alpha_n + \beta_n \le \epsilon_n + \bar{\lambda}\sum_{i=0}^{n-1}\alpha_i$ $(\bar{\lambda} > 0)$ *and* $\alpha_0 + \beta_0 \le \epsilon_0$, *then* $\alpha_n + \beta_n \le \epsilon_n \exp(n\bar{\lambda})$ $(n \ge 0)$.

There is the next main conclusion for Problem IV.

Theorem 5 *Under the assumptions of Theorem* 2, *if there exists a constant* $\alpha > 0$ *such that* $\|\nabla u_h^{n-1}\|_{0,\infty} \le \alpha$ *and when k is sufficiently small such that* $(1 - 2k\alpha) > 0$, *Problem* IV *has one and only one sequence of solutions* $(\boldsymbol{u}_h^n, \boldsymbol{F}_h^n, p_h^n) \in H_h \times W_h \times M_h$ $(n = 1, 2, \ldots, N)$ *that satisfies*

$$\|\boldsymbol{F}_h^n\|_0 \le \frac{1}{(\sqrt{1-2k\alpha})^n}\|\boldsymbol{\psi}\|_0, \tag{23}$$

$$\|\boldsymbol{u}_h^n\|_0^2 + \|p_h^n\|_0^2 + \mu k\sum_{i=1}^{n}\|\nabla\boldsymbol{u}_h^i\|_0^2 \le C\big(\|\boldsymbol{F}_h^n\boldsymbol{F}_h^{nT}\|_0^2 + \|\boldsymbol{\varphi}\|_0^2\big), \tag{24}$$

where C used hereinafter represents a constant that does not rely on k and h, but relies on $\boldsymbol{\psi}$, $\boldsymbol{\varphi}$, *and Re. And if the exact solution to Problem* I *satisfies* $(\boldsymbol{u}, \boldsymbol{F}, p) \in [H_0^1(\Omega) \cap H^2(\Omega)]^2 \times [H_0^1(\Omega) \cap H^2(\Omega)]^{2\times 2} \times [H^1(\Omega) \cap M]$ *and* $N_0\mu^{-1}\|\nabla\boldsymbol{u}^n\|_0 \le 1/4$ *and* $N_0\mu^{-1}\|\nabla\boldsymbol{u}_h^n\|_0 \le 1/4$ *as well as* $\|\boldsymbol{u}_h^{n-1}\|_{0,\infty}$ *and* $\|\operatorname{div}\boldsymbol{F}^n\|_{0,\infty}$ *are bounded* $(n = 1, 2, \ldots, N)$, *we have the following error formulas*:

$$\begin{aligned}&\|\boldsymbol{F}(t_n) - \boldsymbol{F}_h^n\|_0^2 + \|\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n\|_0^2 + k\sum_{i=1}^{n}\big[\|\nabla(\boldsymbol{u}(t_i) - \boldsymbol{u}_h^i)\|_0^2 + \|p(t_i) - p_h^i\|_0^2\big]\\&\quad\le C\big(k^2 + h^2\big), \quad 1 \le n \le N.\end{aligned} \tag{25}$$

Proof Because the third equation in Problem IV is linear, in order to deduce that the third equation in Problem IV has one and only one sequence of solutions $\{\boldsymbol{F}_h^n\}_{n=1}^N \subset W_h$, we only need to prove that when $\boldsymbol{\psi}(x,y) = \mathbf{0}$, there holds $\boldsymbol{F}_h^n = \mathbf{0}$ $(n = 1, 2, \ldots, N)$. Thus, we only prove that (23) is correct. To this end, by choosing $\boldsymbol{\tau}_h = \boldsymbol{F}_h^n$ in the third equation in Problem IV, if $\|\nabla u_h^{n-1}\|_0 \le \alpha$, using (6) and Hölder and Cauchy inequalities, we have

$$\|\boldsymbol{F}_h^n\|_0^2 = k\big(\nabla\boldsymbol{u}^{n-1}\boldsymbol{F}_h^n, \boldsymbol{F}_h^n\big) + \big(\boldsymbol{F}_h^{n-1}, \boldsymbol{F}_h^n\big) \le k\alpha\|\boldsymbol{F}_h^n\|_0^2 + \frac{1}{2}\big(\|\boldsymbol{F}_h^n\|_0^2 + \|\boldsymbol{F}_h^{n-1}\|_0^2\big). \tag{26}$$

When k is sufficiently small such that $(1 - 2k\alpha) > 0$, we have

$$\|\boldsymbol{F}_h^n\|_0 \le \frac{\|\boldsymbol{F}_h^{n-1}\|_0}{\sqrt{1-2k\alpha}} \le \cdots \le \frac{\|\boldsymbol{\psi}\|_0}{(\sqrt{1-2k\alpha})^n} \le \frac{\|\boldsymbol{\psi}\|_0}{(\sqrt{1-2k\alpha})^N}. \tag{27}$$

Thus, if $\boldsymbol{\psi} = \mathbf{0}$, then the third equation in Problem IV has only a sequence of zero solutions, therefore, it has one and only one sequence of solutions $\{\boldsymbol{F}_h^n\}_{n=1}^N$.

After we have obtained $\{\boldsymbol{F}_h^n\}_{n=1}^N$ from the third equation in Problem IV, the first and second equations in Problem IV constitute the fully discrete SMFE model for the nonstationary Navier-Stokes equations. Thus, with the SMFE methods for the nonstationary Navier-Stokes equations (see, *e.g.*, [16, 18–20]), it is obtained that the first and second equations in Problem IV have one and only one sequence of solutions $\{(\boldsymbol{u}_h^n, p_n^n)\}_{n=1}^N \subset H_h \times M_h$ that satisfies (24).

If we let Problem III to subtract Problem IV, and then choose $\boldsymbol{v} = \boldsymbol{v}_h$, $q = q_h$, and $\boldsymbol{\tau} = \boldsymbol{\tau}_h$, we gain three error equations:

$$\begin{aligned}
&\left(\boldsymbol{u}^n - \boldsymbol{u}_h^n, \boldsymbol{v}_h\right) + kA\left(\boldsymbol{u}^n - \boldsymbol{u}_h^n, \boldsymbol{v}_h\right) + kA_1\left(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{v}_h\right) - kA_1\left(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n, \boldsymbol{v}_h\right) \\
&\quad = k\left(\operatorname{div}\left(\boldsymbol{F}^n\boldsymbol{F}^{nT} - \boldsymbol{F}_h^n\boldsymbol{F}_h^{nT}\right), \boldsymbol{v}_h\right) + kB\left(p^n - p_h^n, \boldsymbol{v}_h\right) \\
&\qquad + \left(\boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h\right), \quad \forall \boldsymbol{v}_h \in H_h, \qquad (28)\\
&B\left(q_h, \boldsymbol{u}^n - \boldsymbol{u}_h^n\right) - \varepsilon\left(p_h^n - \varrho_h p_h^n, q_h - \varrho_h q_h\right) = 0, \quad \forall q_h \in M_h, \qquad (29)\\
&\left(\boldsymbol{F}^n - \boldsymbol{F}_h^n, \boldsymbol{\tau}_h\right) + kA_2\left(\boldsymbol{u}^{n-1}, \boldsymbol{F}^n, \boldsymbol{\tau}_h\right) - kA_2\left(\boldsymbol{u}_h^{n-1}, \boldsymbol{F}_h^n, \boldsymbol{\tau}_h\right) \\
&\quad = k\left(\nabla \boldsymbol{u}^{n-1}\boldsymbol{F}^n - \nabla \boldsymbol{u}_h^{n-1}\boldsymbol{F}_h^n, \boldsymbol{\tau}_h\right) + \left(\boldsymbol{F}^{n-1} - \boldsymbol{F}_h^{n-1}, \boldsymbol{\tau}_h\right), \quad \forall \boldsymbol{\tau}_h \in W_h, \qquad (30)
\end{aligned}$$

where $n = 1, 2, \ldots, N$.

Let $\varrho_n = \boldsymbol{F}^n - R_h\boldsymbol{F}^n$, $\boldsymbol{E}_n = R_h\boldsymbol{F}^n - \boldsymbol{F}_h^n$, $\boldsymbol{e}^n = P_h\boldsymbol{u}^n - \boldsymbol{u}_h^n$, $\boldsymbol{\rho}^n = \boldsymbol{u}^n - P_h\boldsymbol{u}^n$, $\eta^n = Q_hp^n - p_h^n$, and $\xi^n = p^n - Q_hp^n$.

First, by noting that $\operatorname{div}(\boldsymbol{F}_h^n\boldsymbol{E}_n)$ is the piecewise bivariate linear polynomial and $\|\boldsymbol{u}_h^{n-1}\|_{0,\infty}$ and $\|\operatorname{div}\boldsymbol{F}^n\|_{0,\infty}$ are bounded and by using the error equation (30), Lemma 3, the properties of $A_2(\cdot,\cdot,\cdot)$, Green's formula, and Hölder and Cauchy inequalities, we have

$$\begin{aligned}
\|\boldsymbol{E}_n\|_0^2 &= (\boldsymbol{E}_n, \boldsymbol{E}_n) = -(\varrho_n, \boldsymbol{E}_n) + \left(\boldsymbol{F}^n - \boldsymbol{F}_h^n, \boldsymbol{E}_n\right) \\
&= k\left(\nabla \boldsymbol{u}^{n-1}\boldsymbol{F}^n - \nabla \boldsymbol{u}_h^{n-1}\boldsymbol{F}_h^n, \boldsymbol{E}_n\right) + \left(\boldsymbol{F}^{n-1} - \boldsymbol{F}_h^{n-1}, \boldsymbol{E}_n\right) \\
&\quad + kA_2\left(\boldsymbol{u}_h^{n-1}, \boldsymbol{F}_h^n, \boldsymbol{E}_n\right) - kA_2\left(\boldsymbol{u}^{n-1}, \boldsymbol{F}^n, \boldsymbol{E}_n\right) \\
&= k\left(\nabla \boldsymbol{u}^{n-1}(\varrho_n + \boldsymbol{E}_n), \boldsymbol{E}_n\right) - k\left(\boldsymbol{e}^{n-1}, \operatorname{div}\left(\boldsymbol{F}_h^n\boldsymbol{E}_n\right)\right) + (\boldsymbol{E}_{n-1}, \boldsymbol{E}_n) \\
&\quad + kA_2\left(\boldsymbol{u}_h^{n-1}, \boldsymbol{F}_h^n, \boldsymbol{E}_n\right) - kA_2\left(\boldsymbol{u}^{n-1}, \boldsymbol{F}^n, \boldsymbol{E}_n\right) \\
&= k\left(\nabla \boldsymbol{u}^{n-1}(\varrho_n + \boldsymbol{E}_n), \boldsymbol{E}_n\right) - k\left(\boldsymbol{e}^{n-1}\operatorname{div}\boldsymbol{F}^n, \varrho^n\right) + (\boldsymbol{E}_{n-1}, \boldsymbol{E}_n) \\
&\quad + kA_2\left(\boldsymbol{u}_h^{n-1}, \boldsymbol{F}_h^n, \boldsymbol{E}_n\right) - kA_2\left(\boldsymbol{u}^{n-1}, \boldsymbol{F}^n, \boldsymbol{E}_n\right) \\
&\le Ckh^2 + \alpha k\|\boldsymbol{E}_n\|_0^2 + \frac{k}{4}\left\|\boldsymbol{e}^{n-1}\right\|_0^2 + \frac{1}{2}\|\boldsymbol{E}_{n-1}\|_0^2 + \frac{1}{2}\|\boldsymbol{E}_n\|_0^2. \qquad (31)
\end{aligned}$$

By simplifying (31), we obtain

$$\|\boldsymbol{E}_n\|_0^2 \le Ckh^2 + \alpha k\|\boldsymbol{E}_n\|_0^2 + \frac{k}{2}\left\|\boldsymbol{e}^{n-1}\right\|_0^2 + \|\boldsymbol{E}_{n-1}\|_0^2. \qquad (32)$$

By summing (32) from 1 to n, using Lemma 3, and noting that $\|\boldsymbol{E}_0\|_0 = \|\boldsymbol{\tau}^n - R_h\boldsymbol{\tau}\|_0 \le Ch^2$, we obtain

$$\|E_n\|_0^2 \le Cnkh^2 + \alpha k\sum_{i=1}^{n}\|\boldsymbol{E}_i\|_0^2 + \frac{k}{2}\sum_{i=1}^{n-1}\left\|\boldsymbol{e}^i\right\|_0^2. \qquad (33)$$

When k is sufficiently small so that $\alpha k \le 1/2$, from (33), we obtain

$$\|E_n\|_0^2 \le Cnkh^2 + \alpha k\sum_{i=0}^{n-1}\|\boldsymbol{E}_i\|_0^2 + k\sum_{i=1}^{n-1}\left\|\boldsymbol{e}^i\right\|_0^2. \qquad (34)$$

Thus, applying Lemma 4 (Gronwall lemma) to (34) yields

$$\|E_n\|_0^2 \le \left[Ch^2 + k\sum_{i=1}^{n-1}\|\boldsymbol{e}^i\|_0^2\right]\exp(\alpha nk) \le Ch^2 + Ck\sum_{i=1}^{n-1}\|\boldsymbol{e}^i\|_0^2. \tag{35}$$

Next, by using the error equations (28) and (29), (5), and Lemma 3, Hölder and Cauchy inequalities, we have

$$\begin{aligned}
&\|\boldsymbol{e}^n\|_0^2 + k\mu\|\nabla\boldsymbol{e}^n\|_0^2 \\
&\quad= (P_h\boldsymbol{u}^n - \boldsymbol{u}_h^n, \boldsymbol{e}^n) + ka(P_h\boldsymbol{u}^n - \boldsymbol{u}_h^n, \boldsymbol{e}^n) \\
&\quad= -(\boldsymbol{\rho}^n, \boldsymbol{e}^n) + (\boldsymbol{u}^n - \boldsymbol{u}_h^n, \boldsymbol{e}^n) + kA(\boldsymbol{u}^n - \boldsymbol{u}_h^n, \boldsymbol{e}^n) + kA(P_h\boldsymbol{u}^n - \boldsymbol{u}^n, \boldsymbol{e}^n) \\
&\quad= (\boldsymbol{\rho}^{n-1} - \boldsymbol{\rho}^n, \boldsymbol{e}^n) + kB(p^n - p_h^n, \boldsymbol{e}^n) + k(\operatorname{div}(\boldsymbol{F}^n\boldsymbol{F}^{nT} - \boldsymbol{F}_h^n\boldsymbol{F}_h^{nT}), \boldsymbol{e}^n) \\
&\qquad+ (\boldsymbol{e}^{n-1}, \boldsymbol{e}^n) - kA_1(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{e}^n) + kA_1(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n, \boldsymbol{e}^n) \\
&\quad= (\boldsymbol{\rho}^{n-1} - \boldsymbol{\rho}^n, \boldsymbol{e}^n) - kA_1(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{e}^n) + kA_1(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n, \boldsymbol{e}^n) + (\boldsymbol{e}^{n-1}, \boldsymbol{e}^n) \\
&\qquad+ k(\operatorname{div}(\boldsymbol{F}^n\boldsymbol{F}^{nT} - \boldsymbol{F}_h^n\boldsymbol{F}_h^{nT}), \boldsymbol{e}^n) - k\varepsilon(\eta^n - \varrho_h\eta^n, \eta^n - \varrho_h\eta^n) + kB(\xi^n, \boldsymbol{e}^n) \\
&\qquad- kB(\eta^n, \boldsymbol{\rho}^n) + k\varepsilon(p_h^n - \varrho_h p^n, \eta^n - \varrho_h\eta^n) + k\varepsilon(Q_h p^n - p^n, \eta^n - \varrho_h\eta^n) \\
&\quad\le C(k^{-1}\|\boldsymbol{\rho}^{n-1} - \boldsymbol{\rho}^n\|_{-1}^2) + \frac{k}{8}\|\eta^n\|_0^2 + Ck\|\xi^n\|_0^2 + Ckh\varepsilon\|\eta^n - \varrho_h\eta^n\|_0^2 \\
&\qquad+ Ck\|\nabla\rho^n\|_0^2 + \frac{k\mu}{8}\|\nabla\boldsymbol{e}^n\|_0^2 + \frac{1}{2}\|\boldsymbol{e}^{n-1}\|_0^2 + \frac{1}{2}\|\boldsymbol{e}^n\|_0^2 - k\varepsilon\|\eta^n - \varrho_h\eta^n\|_0^2 \\
&\qquad- kA_1(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{e}^n) + kA_1(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n, \boldsymbol{e}^n) + k(\operatorname{div}(\boldsymbol{F}^n\boldsymbol{F}^{nT} - \boldsymbol{F}_h^n\boldsymbol{F}_h^{nT}), \boldsymbol{e}^n) \\
&\quad\le C(k^{-1}h^4 + kh^2) + \frac{k}{8}\|\eta^n\|_0^2 + Ck\|\xi^n\|_0^2 + \frac{k\mu}{8}\|\nabla\boldsymbol{e}^n\|_0^2 \\
&\qquad+ \frac{1}{2}\|\boldsymbol{e}^{n-1}\|_0^2 + \frac{1}{2}\|\boldsymbol{e}^n\|_0^2 - \frac{k\varepsilon}{4}(\|\eta^n\|_0^2 - \|\varrho_h\eta^n\|_0^2) \\
&\qquad- kA_1(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{e}^n) + kA_1(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n, \boldsymbol{e}^n) + k(\operatorname{div}(\boldsymbol{F}^n\boldsymbol{F}^{nT} - \boldsymbol{F}_h^n\boldsymbol{F}_h^{nT}), \boldsymbol{e}^n).
\end{aligned} \tag{36}$$

If $N_0\mu^{-1}\|\nabla\boldsymbol{u}^i\|_0 \le 1/4$ and $N_0\mu^{-1}\|\nabla\boldsymbol{u}_h^i\|_0 \le 1/4$ $(i = 1, 2, \ldots, N)$, by using the properties of $A_1(\cdot,\cdot,\cdot)$, Hölder and Cauchy inequalities, and Lemma 3, we have

$$kA_1(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n, \boldsymbol{e}^n) - kA_1(\boldsymbol{u}^n, \boldsymbol{u}^n, \boldsymbol{e}^n) \le Ck\|\nabla\boldsymbol{\rho}^n\|_0^2 + \frac{k\mu}{4}\|\nabla\boldsymbol{e}^n\|_0^2. \tag{37}$$

By using Green's formula and Hölder and Cauchy inequalities, we have

$$\begin{aligned}
&k(\operatorname{div}(\boldsymbol{F}^n\boldsymbol{F}^{nT} - \boldsymbol{F}_h^n\boldsymbol{F}_h^{nT}), \boldsymbol{e}^n) \\
&\quad= -k(\boldsymbol{F}^n\boldsymbol{F}^{nT} - {\boldsymbol{F}_h}^n\boldsymbol{F}^{nT} + {\boldsymbol{F}_h}^n\boldsymbol{F}^{nT} - \boldsymbol{F}_h^n\boldsymbol{F}_h^{nT}, \nabla\boldsymbol{e}^n) \\
&\quad\le Ck\|\boldsymbol{F}^n - \boldsymbol{F}_h^n\|_0\|\nabla\boldsymbol{e}^n\|_0 \le Ck\|\boldsymbol{F}^n - \boldsymbol{F}_h^n\|_0^2 + \frac{k\mu}{8}\|\nabla\boldsymbol{e}^n\|_0^2.
\end{aligned} \tag{38}$$

If $\eta^n \neq 0$, it is easily deduced that $\|\eta^n\|_0^2 > \|\varrho_h\eta\|_0^2$ from (22). Therefore, there exists a constant $\delta \in (0,1)$ such that $\delta\|\eta^n\|_0^2 \ge \|\varrho_h\eta\|_0^2$. By choosing $\varepsilon = (1-\delta)^{-1}$, combining (37) with

(38) and (36), and using Lemma 3, we obtain

$$\|\boldsymbol{e}^n\|_0^2 + k\mu\|\nabla\boldsymbol{e}^n\|_0^2 + 2k\|\eta^n\|_0^2 \le Ckh^2 + Ck\|\boldsymbol{F}^n - \boldsymbol{F}_h^n\|_0^2 + \|\boldsymbol{e}^{n-1}\|_0^2. \tag{39}$$

Because $k\|\eta^n\|_0^2 \le 2k\|\eta^n\|_0^2$, by summing (39) from 1 to n, we have

$$\begin{aligned}\|\boldsymbol{e}^n\|_0^2 + k\sum_{i=1}^{n}\left(\|\nabla\boldsymbol{e}^i\|_0^2 + \|\eta^i\|_0^2\right) &\le Cnkh^2 + Ck\sum_{i=1}^{n}\|\boldsymbol{F}^n - \boldsymbol{F}_h^n\|_0^2 \\ &\le Ch^2 + Ck\sum_{i=1}^{n}\|\boldsymbol{E}_i\|_0^2.\end{aligned} \tag{40}$$

When k is adequately small that satisfies $Ck \le 1/2$, from (35) and (40), we obtain

$$\|\boldsymbol{E}_n\|_0^2 + \|\boldsymbol{e}^n\|_0^2 + k\sum_{i=1}^{n}\left(\|\nabla\boldsymbol{e}^i\|_0^2 + \|\eta^i\|_0^2\right) \le Ch^2 + Ck\sum_{i=0}^{n-1}\left(\|\boldsymbol{E}_i\|_0^2 + \|\boldsymbol{e}^i\|_0^2\right). \tag{41}$$

Applying Lemma 4 to (41) gains

$$\|\boldsymbol{E}_n\|_0^2 + \|\boldsymbol{e}^n\|_0^2 + k\sum_{i=1}^{n}\left(\|\nabla\boldsymbol{e}^i\|_0^2 + \|\eta^i\|_0^2\right) \le Ch^2\exp(Ck) \le Ch^2. \tag{42}$$

Combining (42) with Theorem 2 yields (25). If $\eta^n = 0$, (25) is obviously correct, which accomplishes the proof of Theorem 5. □

Remark 2 Theorem 5 implies that the sequence of solutions for Problem IV is stabilized and convergent. This signifies that it is theoretically valid that the SMFE model is used to solve the 2D nonlinear incompressible viscoelastic fluid system. Moreover, it is known from Theorems 2 and 5 and their proofs that when $\|\boldsymbol{\psi}\|_1$ and $\|\boldsymbol{\varphi}\|_1$ are sufficiently small, the assumptions that $N_0\mu^{-1}\|\nabla\boldsymbol{u}^n\|_0 \le 1/4$ and $N_0\mu^{-1}\|\nabla\boldsymbol{u}_h^n\|_0 \le 1/4$ ($n = 1, 2, \ldots, N$) in Theorem 5 are reasonable.

4 A numerical example

In this section, we give a numerical example of the 2D nonlinear incompressible viscoelastic fluid system to verify the validity of the SMFE model.

In the 2D nonlinear incompressible viscoelastic fluid system, we chose the computational domain as $\Omega = \{(x,y) : 0 \le x \le 50, 30 \le y \le 70\} \cup \{(x,y) : 50 \le x \le 100, 50 \le y \le 100\}$, $Re = 1000$, and the initial and boundary values of the fluid velocity $\boldsymbol{u} = (u_x, u_y)$ as $\boldsymbol{\varphi} = \boldsymbol{u}_0 = (u_{x0}, u_{y0}) = (2(y-30)(70-y), 0)$ ($x = 0$, $30 \le y \le 70$) and $\boldsymbol{\varphi} = \boldsymbol{u}_0 = (u_{x0}, u_{y0})$ satisfying $\partial u_{x0}/\partial x = pRe$ and $u_{y0} = 0$ on $\{(x,y) : x = 100, 0 \le y \le 100\}$, but $\boldsymbol{\varphi} = \boldsymbol{u}_0 = (u_{x0}, u_{y0}) = (0,0)$ on other solid boundaries; whereas the initial and boundary values of the deformation tensor $\boldsymbol{F} = (F_{lm})_{2\times 2}$ satisfy $F_{11} = F_{22} = 1$ and $F_{12} = F_{21} = 0$. In addition, we chose $h = k = 0.01$. We solved the SMFE model on a ThinkPad E530 PC to obtain the numerical solutions $\boldsymbol{u}$, p, F_{11}, F_{12}, F_{21}, and F_{22} at $t = 10$, which were still convergent and were drawn in Figures 1 to 6, respectively. These charts show that the SMFE model has very good stability, and the numerical simulation results could also reflect the actual physical phenomena.

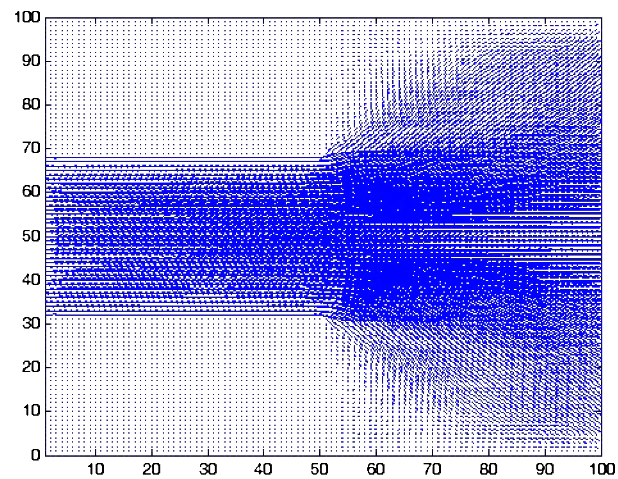

Figure 1 The streamline of the SMFE solution of the fluid velocity $\boldsymbol{u}$ at $t = 10$.

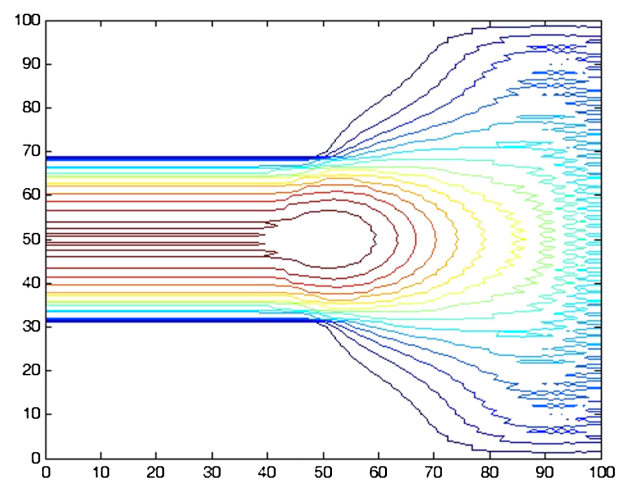

Figure 2 The SMFE solution of the pressure p at $t = 10$.

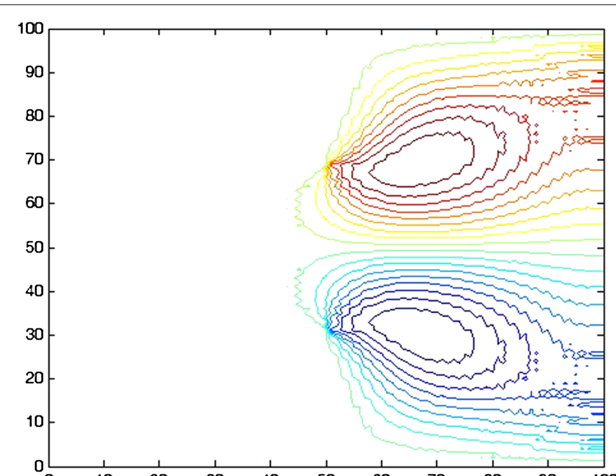

Figure 3 The component F_{11} of the SMFE solution of the deformation tensor $\boldsymbol{F}$ at $t = 10$.

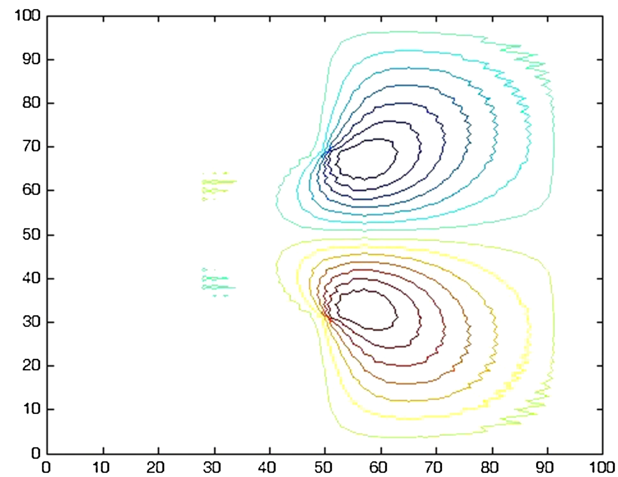

Figure 4 The component F_{12} of the SMFE solution of the deformation tensor *F* at *t* = 10.

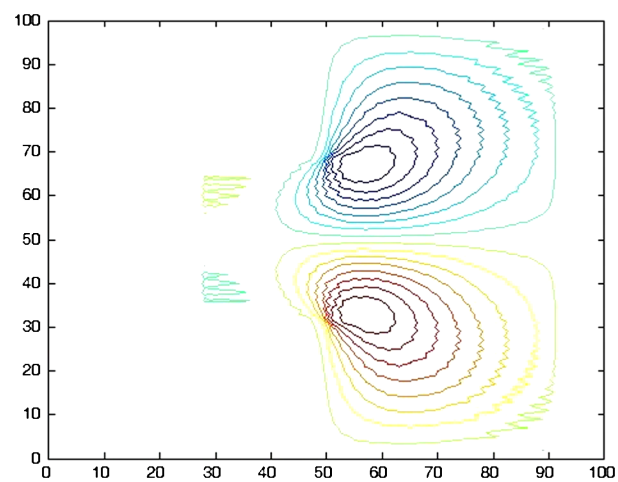

Figure 5 The component F_{21} of the SMFE solution of the deformation tensor *F* at *t* = 10.

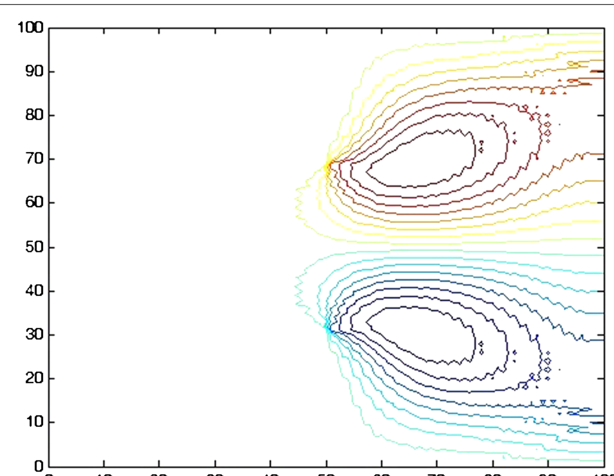

Figure 6 The component F_{22} of the SMFE solution of the deformation tensor *F* at *t* = 10.

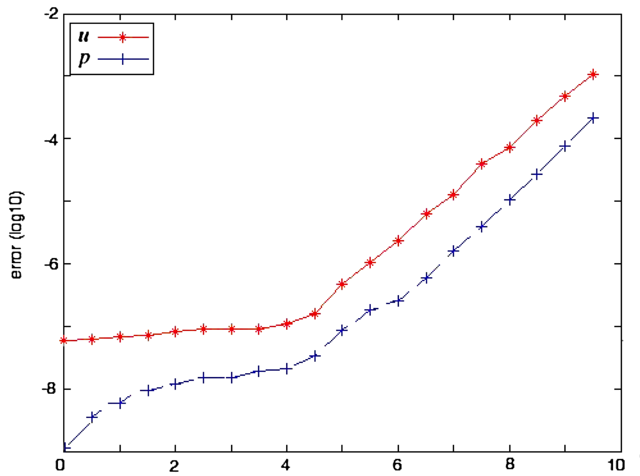

Figure 7 The relative error trending charts of the SMFE solutions of the fluid velocity *u* and the pressure *p* on $0 \le t \le 10$.

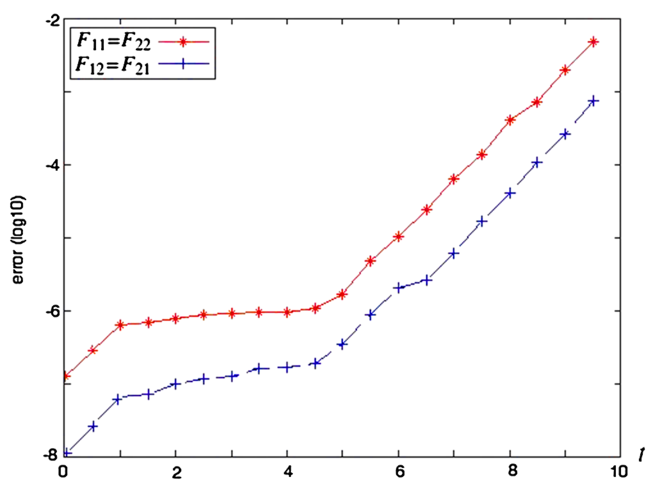

Figure 8 The relative error trending charts of the SMFE solutions of the deformation tensor $\boldsymbol{F} = (F_{lm})_{2\times 2}$ on $0 \le t \le 10$.

Figures 7 and 8 exhibit the relative errors of the SMFE solutions of the fluid velocity $\boldsymbol{u}$ and the pressure p as well as the deformation tensor $\boldsymbol{F} = (F_{lm})_{2\times 2}$ on $0 \le t \le 10$, respectively. The variation tendencies of the error curves are reasonable because the relative errors of the SMFE solutions gradually increase as the truncation errors are accumulated in the computational procedure. These relative error curves also express that the numerical experiment errors are consistent with the theoretical ones because the theoretical and numerical errors all do not exceed 10^{-2}. This signifies that the SMFE model is very effective for solving the 2D nonlinear incompressible viscoelastic fluid system.

5 Conclusions and discussions

In this article, we have developed the SMFE model based on parameter-free and two local Gauss integrals directly from the time semi-discrete model for the 2D nonlinear incompressible viscoelastic fluid system. Thus, we not only could circumvent the semi-discrete

SMFE method about spatial variables, but could also eliminate the restriction of Brezzi-Babuška (B-B) condition so that the theoretical analysis here is far simpler than the subsistent other approaches that have been used in the nonstationary Navier-Stokes equations (see, *e.g.*, [12, 16, 18, 19]). We have also offered the existence, uniqueness, stability, and error estimations of the SMFE solutions and given a numerical example to verify the validity of the SMFE model. This signifies that the approach here is the development of the existing results (see, *e.g.*, [7–10, 12, 16, 18, 19]).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors wrote, read, and approved the final manuscript.

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