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Infinitely many singularities and denumerably many positive solutions for a second-order impulsive Neumann boundary value problem

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Abstract

Using a fixed point theorem of cone expansion and compression of norm type and a new method to deal with the impulsive term, we prove that the second-order singular impulsive Neumann boundary value problem has denumerably many positive solutions. Noticing that $M > 0$, our main results improve many previous results.

Keywords: denumerably many positive solutions; infinitely many singularities; Neumann impulsive boundary conditions; cone expansion and compression

1 Introduction

We are concerned with the existence of denumerably many positive solutions of the second-order singular impulsive Neumann boundary value problem

$$\begin{cases} -x''(t) + Mx(t) = \omega(t)f(t,x(t)), & t \in J, \\ -\Delta x'|_{t=t_k} = I_k(x(t_k)), & k = 1,2,\dots,m, \\ x'(0) = x'(1) = 0, \end{cases} \tag{1.1}$$

where M is a positive constant, $J = [0,1]$, $t_k \in \mathrm{R}$, $k = 1,\dots,m$, $m \in \mathrm{N}$, satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$, $-\Delta x'|_{t=t_k}$ denotes the jump of $x'(t)$ at $t = t_k$, that is, $-\Delta x'|_{t=t_k} = x'((t_k)^+) - x'((t_k)^-)$, here $x'((t_k)^+)$ and $x'((t_k)^-)$, respectively, represent the right-hand limit and left-hand limit of $x'(t)$ at $t = t_k$.

In addition, ω, f and I_k satisfy the following conditions:

(H_1) $\omega(t) \in L^p[0,1]$ for some $p \in [1,+\infty)$, and there exists $N > 0$ such that $\omega(t) \geq N$ a.e. on J;

(H_2) $f \in C(J \times \mathrm{R}^+, \mathrm{R}^+)$, $I_k \in C(\mathrm{R}^+, \mathrm{R}^+)$, where $\mathrm{R}^+ = [0,+\infty)$;

(H_3) there exists a sequence $\{t_i'\}_{i=1}^{\infty}$ such that $t_1' < \delta$, where $\delta = \min\{t_1, \frac{1}{2}\}$, $t_i' \downarrow t^* \geq 0$ and $\lim_{t\to t_i'} \omega(t) = +\infty$ for all $i = 1,2,\dots$.

For the case $M = 0$ and $I_k = 0$ $(k = 1,2,\dots,m)$, problem (1.1) reduces to the problem studied by Kaufmann and Kosmatov in [1]. By using Krasnosel'skiĭ's fixed point theorem and Hölder's inequality, the authors showed the existence of countably many positive solutions. The other related results can be found in [2–13]. However, there are almost no papers considering second-order impulsive Neumann boundary value problem with infinitely many singularities. To identify a few, we refer the reader to [14–27] and the references therein.

The main reason is that $M \neq 0$ in problem (1.1), which shows that the solution of problem (1.1) has no concave properties. On the other hand, under the case $M \neq 0$ and $\omega(t)$ with infinitely many singularities, the properties of the corresponding Green's function for problem (1.1) are more complicated.

Our plan of the paper is as follows: in Section 2, we collect some well-known results to be used in the subsequent sections. In particular, we also present some new properties of Green's function under the case $M \neq 0$ and $\omega(t)$ with infinitely many singularities. In Section 3, we obtain some new sufficient conditions for the existence of denumerably many positive solutions for problem (1.1). In Section 4, we give an example of a family functions $\omega(t)$ such that (H_3) holds.

2 Preliminaries

In this installment, we list some definitions and lemmas which are needed throughout this paper.

Let $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$ and $E = C[0,1]$. We define $PC^1[0,1]$ in E by

$$PC^1[0,1] = \{x \in E : x'(t) \in C(t_k, t_{k+1}), \exists x'(t_k^-), x'(t_k^+), k = 1, 2, \ldots, m\}. \tag{2.1}$$

Then $PC^1[0,1]$ is a real Banach space with norm

$$\|x\|_{PC^1} = \max\{\|x\|_\infty, \|x'\|_\infty\}, \tag{2.2}$$

where $\|x\|_\infty = \sup_{t\in J} |x(t)|$, $\|x'\|_\infty = \sup_{t\in J} |x'(t)|$.

Suppose that $G(t,s)$ is the Green's function of the boundary value problem

$$-x''(t) + Mu(t) = 0, \qquad x'(0) = x'(1) = 0,$$

then

$$G(t,s) = \frac{1}{\gamma \sinh\gamma} \begin{cases} \cosh\gamma(1-t)\cosh\gamma s, & 0 \le s \le t \le 1, \\ \cosh\gamma(1-s)\cosh\gamma t, & 0 \le t \le s \le 1. \end{cases} \tag{2.3}$$

Lemma 2.1 *By the definition of $G(t,s)$ and the properties of sinhx and coshx, we have the following results.*

(a) *For any $t, s \in J$, there is*

$$A = \frac{1}{\gamma \sinh\gamma} \le G(t,s) \le \frac{\cosh\gamma}{\gamma\sinh\gamma} = B. \tag{2.4}$$

Then it follows from (2.4) *that*

$$A \le G(t,s) \le G(s,s) \le B.$$

(b) *For any* $\tau \in (0,\delta)$,

$$\frac{D'_k}{\gamma \sinh\gamma} \le G(t,s) \le \frac{\cosh\gamma(1-\tau)\cosh\gamma\tau'_k}{\gamma\sinh\gamma}, \quad \forall t \in [\tau,\tau'_k], s \in J, \tag{2.5}$$

where

$$\tau'_k = \max\{1-\tau, 1-t_k\}, \qquad D'_k = \max\{\cosh\gamma\tau, \cosh\gamma(1-\tau'_k)\}, \quad k = 1,2,3,\ldots,m.$$

(c)

$$G'_t(t,s) = \frac{1}{\sinh\gamma}\begin{cases} -\sinh\gamma(1-t)\cosh\gamma s, & 0 \le s \le t \le 1, \\ \sinh\gamma(1-s)\cosh\gamma t, & 0 \le t \le s \le 1, \end{cases} \tag{2.6}$$

and

$$\max_{t,s\in J, t\neq s} |G'_t(t,s)| \le \sinh\gamma. \tag{2.7}$$

Proof We can get equations (2.4)-(2.7) by the definition of $G(t,s)$, so we omit it here. □

To establish the existence of positive solutions to problem (1.1), for a fixed $\tau \in (0,\delta)$, we construct the cone K_τ in $PC^1[0,1]$ by

$$K_\tau = \Big\{x \in PC^1[0,1] : x(t) \ge 0, t \in J, \min_{t\in[\tau,\tau'_k]} x(t) \ge \sigma_k \|x\|_{PC^1}\Big\}, \tag{2.8}$$

where

$$\sigma_k = \frac{D'_k}{\rho\gamma\sinh\gamma}, \quad k = 1,2,\ldots,m, \tag{2.9}$$

$$\rho = \max\{B, \sinh\gamma\}. \tag{2.10}$$

It is easy to see K_τ is a closed convex cone of $PC^1[0,1]$.

Let $\{\tau_i\}_{i=1}^\infty$ be such that $t'_{i+1} < \tau_i < t'_i$, $i = 1,2,\ldots$. Then for any $i \in \mathrm{N}$, we define the cone K_{τ_i} by

$$K_{\tau_i} = \Big\{x(t) \in PC^1[0,1] : x(t) \ge 0, t \in J, \min_{t\in[\tau_i,\tau'_{ik}]} x(t) \ge \sigma_{ik}\|x\|_{PC^1}\Big\}, \tag{2.11}$$

where

$$\tau'_{ik} = \max\{1-\tau_i, 1-t_k\}, \qquad \sigma_{ik} = \frac{D'_{ik}}{\rho\gamma\sinh\gamma}, \tag{2.12}$$

$$D'_{ik} = \max\{\cosh\gamma\tau_i, \cosh\gamma(1-\tau'_k)\}, \quad i = 1,2,\ldots, k = 1,2,\ldots,m. \tag{2.13}$$

It is easy to see K_{τ_i} is a closed convex cone of $PC^1[0,1]$.

Remark 2.1 For any $i = 1,2,\ldots$, $k = 1,2,\ldots,m$, it follows from the definition of σ_k and σ_{ik} that $0 < \sigma_k, \sigma_{ik} < 1$.

Lemma 2.2 *If* (H_1)-(H_3) *hold, then problem* (1.1) *has a unique solution x given by*

$$x(t) = \int_0^1 G(t,s)\omega(s)f\big(s,x(s)\big)\,ds + \sum_{k=1}^{m} G(t,t_k)I_k\big(t_k,x(t_k)\big).$$

Proof The proof is similar to that of Lemma 2.4 in [26]. □

Definition 2.1 A function $x(t)$ is said to be a solution of problem (1.1) on J if:

(i) $x(t)$ is absolutely continuous on each interval $(0,t_1]$ and $(t_k,t_{k+1}]$, $k = 1,2,\ldots,n$;

(ii) for any $k = 1,2,\ldots,m$, $x(t_k^+)$, $x(t_k^-)$ exist;

(iii) $x(t)$ satisfies (1).

Define an operator $T : K_\tau \to PC^1[0,1]$ by

$$(Tx)(t) = \int_0^1 G(t,s)\omega(s)f\big(s,x(s)\big)\,ds + \sum_{k=1}^{m} G(t,t_k)I_k\big(t_k,x(t_k)\big). \tag{2.14}$$

From (2.14), we know that $x(t) \in PC^1[0,1]$ is a solution of problem (1.1) if and only if x is a fixed point of the operator T. Also, for a positive number r, define Ω_r by

$$\Omega_r = \big\{x \in PC^1[0,1] : \|x\|_{PC^1} < r\big\}.$$

Note that $\partial\Omega_r = \{x \in PC^1[0,1] : \|x\|_{PC^1} = r\}$ and $\bar{\Omega}_r = \{x \in PC^1[0,1] : \|x\|_{PC^1} \le r\}$.

Definition 2.2 An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Lemma 2.3 *Assume that* (H_1)-(H_3) *hold. Then* $T(K_\tau) \subset K_\tau$ *and* $T : K_\tau \to K_\tau$ *is a completely continuous.*

Proof For $t \in J$, $x \in K_\tau$, it follows from ((2.5)) and (2.14) that

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s)\omega(s)f\big(s,x(s)\big)\,ds + \sum_{k=1}^{m} G(t,t_k)I_k\big(t_k,x(t_k)\big)\\ &\le B\left[\int_0^1 \omega(s)f\big(s,x(s)\big)\,ds + \sum_{k=1}^{m} I_k\big(t_k,x(t_k)\big)\right].\end{aligned} \tag{2.15}$$

On the other hand, it follows from (2.6), (2.7) and (2.14) that

$$\begin{aligned}\big|(Tx)'(t)\big| &= \left|\int_0^1 G_t'(t,s)\omega(s)f\big(s,x(s)\big)\,ds + \sum_{k=1}^{m} G_t'(t,t_k)I_k\big(t_k,x(t_k)\big)\right|\\ &\le \int_0^1 \big|G_t'(t,s)\big|\omega(s)f\big(s,x(s)\big)\,ds + \sum_{k=1}^{m}\big|G_t'(t,t_k)\big|I_k\big(t_k,x(t_k)\big)\\ &\le \sinh\gamma\left[\int_0^1 \omega(s)f\big(s,x(s)\big)\,ds + \sum_{k=1}^{m} I_k\big(t_k,x(t_k)\big)\right].\end{aligned} \tag{2.16}$$

For any $t \in J$, combined with (2.15) and (2.16), we have

$$\|Tx\|_{PC^1} \le \rho \left[\int_0^1 \omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^m I_k\bigl(t_k, x(t_k)\bigr) \right]. \tag{2.17}$$

Then, by (2.5), (2.8) and (2.17), we have

$$\begin{aligned}
\min_{t\in[\tau,\tau_k]} (Tx)(t) &= \min_{t\in[\tau,\tau_k]} \left[\int_0^1 G(t,s)\omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^m G(t,t_k) I_k\bigl(t_k, x(t_k)\bigr) \right] \\
&\ge \frac{D_k'}{\gamma \sinh \gamma} \left[\int_0^1 \omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^m I_k\bigl(t_k, x(t_k)\bigr) \right] \\
&\ge \frac{D_k'}{\rho\gamma \sinh \gamma} \rho \left[\int_0^1 \omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^m I_k\bigl(t_k, x(t_k)\bigr) \right] \\
&\ge \sigma_k \|Tx\|_{PC^1}.
\end{aligned} \tag{2.18}$$

Evidently, $T(K_\tau) \subset K_\tau$.

Next, we prove that the operator $T : K_\tau \to K_\tau$ is a completely continuous.

It is obvious that T is continuous.

Let $B_d = \{x \in PC^1[0,1] \mid \|x\|_{PC^1} \le d\}$ be bounded set. Then, for all $x \in B_d$, by the definition of $\|Tx\|_\infty$, $\|Tx'\|_\infty$, $\|Tx\|_{PC^1}$, we have

$$\begin{aligned}
\|Tx\|_\infty &= \sup_{t\in J} \bigl| Tx(t) \bigr| \\
&\le B \left[\int_0^1 \omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^m I_k\bigl(t_k, x(t_k)\bigr) \right] \\
&\le B\bigl(\|\omega\|_1 L + mL^*\bigr) \\
&= \Gamma_0, \\
\bigl\| Tx' \bigr\|_\infty &= \sup_{t\in J} \bigl| Tx'(t) \bigr| \\
&\le \sinh \gamma \left[\int_0^1 \omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^m I_k\bigl(t_k, x(t_k)\bigr) \right] \\
&\le \sinh \gamma \bigl(\|\omega\|_1 L + mL^*\bigr) \\
&= \Gamma_1,
\end{aligned}$$

and

$$\|Tx\|_{PC^1} = \max\bigl\{ \|Tx\|_\infty, \bigl\|Tx'\bigr\|_\infty \bigr\} \le \max\{\Gamma_0, \Gamma_1\},$$

where

$$L = \max_{t\in J, x\in K_\tau, \|x\|_{PC^1} \le d} f(t,x), \qquad L^* = \max\{L_k, k = 1, 2, \ldots, m\},$$

$$L_k = \max_{t\in J, x\in K_\tau, \|x\|_{PC^1}\le d} I_k\bigl(t_k, x(t_k)\bigr).$$

Therefore $T(B_d)$ is uniformly bounded.

On the other hand, for all $t_1, t_2 \in J_k$ with $t_1 < t_2$, we have

$$\bigl|(Tx)(t_1) - (Tx)(t_2)\bigr| = \left|\int_{t_1}^{t_2} (Tx)'(t)\,dt\right| \le \Gamma_1 |t_1 - t_2| \to 0 \quad (t_1 \to t_2).$$

Noting (2.7), we know that $G'(t,s)$ is a constant and

$$\begin{aligned}\bigl|(Tx)'(t_1) - (Tx)'(t_2)\bigr| &= \biggl|\int_0^1 \bigl[G_t'(t_1,s) - G_t'(t_2,s)\bigr]\omega(s) f\bigl(s,x(s)\bigr)\,ds \\ &\quad + \sum_{k=1}^n \bigl[G_t'(t_1,t_k) - G_t'(t_2,t_k)\bigr] I_k\bigl(t_k, x(t_k)\bigr)\biggr| \\ &\le \int_0^1 \bigl|G_t'(t_1,s) - G_t'(t_2,s)\bigr|\omega(s) f\bigl(s,x(s)\bigr)\,ds \\ &\quad + \sum_{k=1}^n \bigl|G_t'(t_1,t_k) - G_t'(t_2,t_k)\bigr| I_k\bigl(t_k, x(t_k)\bigr) \to 0 \quad (t_1 \to t_2),\end{aligned}$$

which shows that $T(B_d)$ is equicontinuous. The Arzelà-Ascoli theorem implies that T is completely continuous, and the lemma is proved. □

Lemma 2.4 (Hölder) *Let $e \in L^p[a,b]$ with $p > 1$, $h \in L^q[a,b]$ with $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $eh \in L^1[a,b]$ and*

$$\|eh\|_1 \le \|e\|_p \|h\|_q.$$

Let $e \in L^1[a,b]$, $h \in L^\infty[a,b]$. Then $eh \in L^1[a,b]$ and

$$\|eh\|_1 \le \|e\|_1 \|h\|_\infty.$$

Lemma 2.5 (See [28]; fixed point theorem of cone expansion and compression of norm type) *Let E be a Banach space, P be a cone in E. Assume that Ω_1, Ω_2 are bounded open subsets in E with $\theta \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$, where θ denotes zero operator. Suppose $A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$ is completely continuous such that either*

(i) $\|Ax\| \le \|x\|$, $\forall x \in P \cap \partial\Omega_1$; $\|Ax\| \ge \|x\|$, $\forall x \in P \cap \partial\Omega_2$;

(ii) $\|Ax\| \le \|x\|$, $\forall x \in P \cap \partial\Omega_2$; $\|Ax\| \ge \|x\|$, $\forall x \in P \cap \partial\Omega_1$.

Then A has a fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3 Main results

In this section, using Lemmas 2.1-2.5, we give our main results in the case $\omega \in L^P[0,1]$; $p > 1$, $p = 1$ and $p = \infty$.

For convenience, we write

$$D = \max\bigl\{\|G\|_q \|\omega\|_p, \|G\|_1 \|\omega\|_\infty, B\|\omega\|_1\bigr\}, \qquad \rho_0 = \min\left\{1, \frac{A}{\sinh\gamma}\right\}.$$

Firstly, we consider the case $p > 1$.

Theorem 3.1 *Assume that* (H_1)-(H_3) *hold. Let* $\{r_i\}_{i=1}^{\infty}$ *and* $\{R_i\}_{i=1}^{\infty}$ *be such that*

$$R_{i+1} < \sigma_{ik} r_i < r_i < L_0 r_i < R_i, \quad i = 1, 2, \dots, k = 1, 2, \dots, m,$$

where

$$L_0 = \max\left\{\frac{\gamma \sinh \gamma}{A(N+m)D_k'}, \frac{2\rho_0}{D+mB}, 2\right\}.$$

For each natural number i, we assume that f and I_k *satisfy*:

(H_4) *For any* $t \in J$, $x \in [0, R_i]$, $f(t,x) \leq M_0 R_i$, *and for any* $x \in [0, R_i]$, $k \in \{1, 2, \dots, m\}$, $I_k(x(t_k)) \leq M_0 R_i$, *where*

$$0 < M_0 \leq \frac{\rho_0}{D+mB}.$$

(H_5) *For any* $t \in J$, $x \in [\sigma_{ik} r_i, r_i]$, $f(t,x) \geq L_0 r_i$, *and for any* $x \in [\sigma_{ik} r_i, r_i]$, $k \in \{1, 2, \dots, m\}$, $I_k(x) \geq L_0 r_i$.

Then problem (1.1) *has denumerably many positive solutions* $\{x_i(t)\}_{i=1}^{\infty}$ *such that*

$$r_i \leq \|x_i\|_{PC^1} \leq R_i, \quad i = 1, 2, \dots.$$

Proof We consider the following open subset sequences $\{\Omega_{1,i}\}_{i=1}^{\infty}$ and $\{\Omega_{2,i}\}_{i=1}^{\infty}$ of $PC^1[0,1]$:

$$\{\Omega_{1,i}\}_{i=1}^{\infty} = \{x \in PC^1[0,1] : \|x\|_{PC^1} < R_i\};$$
$$\{\Omega_{2,i}\}_{i=1}^{\infty} = \{x \in PC^1[0,1] : \|x\|_{PC^1} < r_i\}.$$

Let $\{\tau_i\}_{i=1}^{\infty}$ be as in the hypothesis and note that $0 < t_{i+1}' < \tau_i < t_i' < \delta$, $i = 1, 2, \dots$. For fixed i, we assume that $x \in K_{\tau_i} \cap \partial\Omega_{2,i}$, then for any $t \in J$

$$r_i = \|x\|_{PC^1} \geq x(t) \geq \min_{t \in [\tau_i, \tau_i' k]} x(t) \geq \sigma_{ik} \|x\|_{PC^1} = \sigma_{ik} r_i.$$

Noticing (2.5) and (2.14), for all $x \in K_{\tau_i} \cap \partial\Omega_{2,i}$, by (H_1) and (H_5), we have

$$\begin{aligned}
(Tx)(t) &= \int_0^1 G(t,s)\omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^m G(t,t_k) I_k\bigl(x(t_k)\bigr) \\
&\geq \min_{t \in [\tau, \tau_k']} \left[\int_0^1 G(t,s)\omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^m G(t,t_k) I_k\bigl(x(t_k)\bigr)\right] \\
&\geq \frac{D_k'}{\gamma \sinh \gamma}\left[\int_0^1 \omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^m I_k\bigl(x(t_k)\bigr)\right] \\
&\geq \frac{D_k'}{\gamma \sinh \gamma}\left[N \int_{\tau_i}^{\tau_{ik}'} f\bigl(s, x(s)\bigr)\, ds + \min_{t_k \in [\tau_i, \tau_{ik}']} \sum_{k=1}^m I_k\bigl(x(t_k)\bigr)\right]
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{D'_k}{\gamma \sinh \gamma} L_0(N+m) r_i \\
&\geq r_i = \|x\|_{PC^1},
\end{aligned}
$$

which shows that

$$
\|Tx\|_{PC^1} \geq \|x\|_{PC^1}, \quad \forall x \in K_{\tau_i} \cap \partial \Omega_{2,i}. \tag{3.1}
$$

On the other hand, for all $t \in J$, $x \in P_i \cap \partial \Omega_{1,i}$, we have $x(t) \leq \|x\|_{PC^1} = R_i$. Noticing (2.4) and (2.14), for all $t \in J$, $x \in K_{\tau_i} \cap \partial \Omega_{1,i}$, by (H_4), we have

$$
\begin{aligned}
(Tx)(t) &= \int_0^1 G(t,s)\omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^m G(t,t_k) I_k\bigl(x(t_k)\bigr) \\
&\leq M_0 R_i \int_0^1 G(s,s)\omega(s)\, ds + M_0 R_i \sum_{k=1}^m G(t,t_k) \\
&\leq M_0 R_i \|G\|_q \|\omega\|_p + M_0 R_i m B \\
&\leq M_0 (D + mB) R_i \\
&\leq R_i = \|x\|_{PC^1}.
\end{aligned} \tag{3.2}
$$

Moreover, by (2.6), (2.16) and (H_4), we have

$$
\begin{aligned}
\bigl|(Tx)'(t)\bigr| &\leq \int_0^1 \bigl|G'_t(t,s)\bigr| \omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^m \bigl|G'_t(t,t_k)\bigr| I_k\bigl(x(t_k)\bigr) \\
&\leq \sinh \gamma \left[\int_0^1 \omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^m I_k\bigl(x(t_k)\bigr) \right] \\
&\leq \frac{\sinh \gamma}{A} \left[\int_0^1 G(s,s)\omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^m G(s,s) I_k\bigl(x(t_k)\bigr) \right] \\
&\leq \frac{\sinh \gamma}{A} \left[\int_0^1 \|G\|_q \|\omega\|_p f\bigl(s, x(s)\bigr)\, ds + B \sum_{k=1}^m I_k\bigl(x(t_k)\bigr) \right] \\
&\leq \frac{\sinh \gamma}{A} \bigl(M_0 R_i \|G\|_q \|\omega\|_p + B m M_0 R_i \bigr) \\
&\leq M_0 \frac{\sinh \gamma}{A} (D + mB) R_i \\
&\leq R_i = \|x\|_{PC^1}.
\end{aligned} \tag{3.3}
$$

From (3.2) and (3.3), we have

$$
\|Tx\|_{PC^1} \leq \|x\|_{PC^1}, \quad \forall x \in K_{\tau_i} \cap \partial \Omega_{1,i}. \tag{3.4}
$$

Applying Lemma 2.5 to (3.1) and (3.4) shows that the operator T has a fixed point $x_i \in K_{\tau_i} \cap (\bar{\Omega}_{2,i}/\Omega_{1,i})$ such that $r_i \leq \|x_i\| \leq R_i$. Since $i \in \mathrm{N}$ was arbitrary, the proof is complete. □

The following results deal with the case $p = \infty$.

Theorem 3.2 *Assume that* (H_1)-(H_3) *hold. Let* $\{a_i\}_{i=1}^{\infty}$ *and* $\{b_i\}_{i=1}^{\infty}$ *be such that*

$$a_{i+1} < \sigma_{ik} b_i < b_i < L_0 b_i < a_i, \quad i = 1, 2, \dots, k = 1, 2, \dots, m.$$

For each natural number i, we assume that f and I_k *satisfy* (H_4) *and* (H_5)*, then problem* (1.1) *has denumerably many positive solutions* $\{x_i(t)\}_{i=1}^{\infty}$ *such that*

$$r_i \leq \|x_i\|_{PC^1} \leq R_i, \quad i = 1, 2, \dots.$$

Proof Let $\|G\|_1 \|\omega\|_\infty$ replace $\|G\|_q \|\omega\|_p$ and repeat the previous argument. □

Finally, we consider the case of $p = 1$.

Theorem 3.3 *Assume that* (H_1)-(H_3) *hold. Let* $\{a_i\}_{i=1}^{\infty}$ *and* $\{b_i\}_{i=1}^{\infty}$ *be such that*

$$a_{i+1} < \sigma_{ik} b_i < b_i < L_0 b_i < a_i, \quad i = 1, 2, \dots, k = 1, 2, \dots, m.$$

For each natural number i, we assume that f and I_k *satisfy* (H_4) *and* (H_5)*, then the problem* (1.1) *has denumerably many positive solutions* $\{x_i(t)\}_{i=1}^{\infty}$ *such that*

$$r_i \leq \|x_i\|_{PC^1} \leq R_i, \quad i = 1, 2, \dots.$$

Proof Similar to the proof of (3.2) and (3.3), for all $t \in [\tau_i, \delta - \tau_i]$, $x \in K_{\tau_i} \cap \partial\Omega_{1,i}$, then $x(t) \leq \|x\|_{PC^1} = R_i$.

Since (2.4) and (2.14), for all $x \in K_{\tau_i} \cap \partial\Omega_{1,i}$, by (H_4), we have

$$\begin{aligned}
(Tx)(t) &= \int_0^1 G(t,s)\omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^{m} G(t, t_k) I_k\bigl(t_k, x(t_k)\bigr) \\
&\leq B\|\omega\|_1 \int_0^1 f\bigl(s, x(s)\bigr)\, ds + B \sum_{k=1}^{m} I_k\bigl(t_k, x(t_k)\bigr) \\
&\leq M_0 R_i B \|\omega\|_1 + m M_0 R_i B \\
&\leq M_0 (D + mB) R_i \\
&\leq R_i = \|x\|_{PC^1},
\end{aligned} \tag{3.5}$$

and by (2.4), (2.7), (2.16) and (H_4),

$$\begin{aligned}
\bigl|(Tx)'(t)\bigr| &\leq \int_0^1 \bigl|G_t'(t,s)\bigr| \omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^{m} \bigl|G_t'(t, t_k)\bigr| I_k\bigl(t_k, x(t_k)\bigr) \\
&\leq \sinh\gamma \left[\int_0^1 \omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^{m} I_k\bigl(t_k, x(t_k)\bigr) \right] \\
&\leq \frac{\sinh\gamma}{A} \left[\int_0^1 G(s,s)\omega(s) f\bigl(s, x(s)\bigr)\, ds + \sum_{k=1}^{m} G(s,s) I_k\bigl(t_k, x(t_k)\bigr) \right]
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\sinh\gamma}{A}\left[\int_0^1 B\|\omega\|_1 f\bigl(s,x(s)\bigr)\,ds + B\sum_{k=1}^{m} I_k\bigl(t_k,x(t_k)\bigr)\right]\\
&\le \frac{\sinh\gamma}{A}\bigl(M_0R_iB\|\omega\|_1 + BmM_0R_i\bigr)\\
&\le M_0\frac{\sinh\gamma}{A}B(D+m)R_i\\
&\le R_i = \|x\|_{PC^1}.
\end{aligned}
\tag{3.6}
$$

From (3.5) and (3.6), we have

$$\|Tx\|_{PC^1} \le \|x\|_{PC^1}, \quad \forall x \in K_{\tau_i} \cap \partial\Omega_{1,i}.$$

Similarly to the proof of Theorem 3.1, we can finish the proof of Theorem 3.3. □

4 An example

From Section 3, it is not difficult to see that (H_3) plays an important role in the proof that problem (1.1) has denumerably many positive solutions. As an example, we consider a family of functions $\omega(t)$ as follows.

Example 4.1 Let $k = m = 1$, $t_1 = \frac{1}{3}$, and

$$t'_n = t_1 - \sum_{i=1}^{n} \frac{1}{(i+1)(i+2)(i+3)(i+4)}, \quad n = 1,2,\ldots.$$

It is easy to see that

$$
\begin{aligned}
&t'_1 = \frac{39}{120} < \frac{1}{3},\\
&t'_n - t'_{n+1} = \frac{1}{(n+2)(n+3)(n+4)(n+5)}, \quad n = 1,2,\ldots,
\end{aligned}
$$

and

$$t^* = \lim_{n\to\infty} t_n = t_1 - \sum_{i=1}^{\infty} \frac{1}{(i+1)(i+2)(i+3)(i+4)} = \frac{1}{3} - \frac{1}{72} = \frac{23}{72} > \frac{1}{4},$$

where $\sum_{i=1}^{\infty} \frac{1}{(i+1)(i+2)(i+3)(i+4)} = \frac{1}{72}$.

Let

$$\omega(t) = \sum_{i=1}^{\infty} \omega_n(t), \quad t \in J,$$

where

$$
\omega_n(t) = \begin{cases}
\frac{1}{2n^4(t'_n+t'_{n+1})}, & t \in [0, \frac{t'_n+t'_{n+1}}{2}),\\
\frac{1}{\sqrt{t'_n - t}}, & t \in [\frac{t'_n+t'_{n+1}}{2}, t'_n),\\
\frac{1}{\sqrt{t - t'_n}}, & t \in [t'_n, \frac{t'_n+t'_{n-1}}{2}],\\
\frac{2}{2n^4(2-t'_n-t'_{n-1})}, & t \in (\frac{t'_n+t'_{n-1}}{2}, 1].
\end{cases}
$$

From $\sum_{i=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$ and $\sum_{i=1}^{\infty}\frac{1}{n^2}=\frac{\pi^4}{6}$, we have

$$\begin{aligned}
\sum_{i=1}^{\infty}\int_0^1\omega_n(t)\,dt &= \sum_{i=1}^{\infty}\biggl\{\int_0^{(t_n'+t_{n+1}')/2}\frac{1}{2n^4(t_n'+t_{n+1}')}\,dt+\int_{(t_{n-1}'+t_n')/2}^{1}\frac{2}{2n^4(2-t_n'-t_{n-1}')}\,dt \\
&\quad+\int_{(t_n'+t_{n+1}')/2}^{t_n}\frac{1}{\sqrt{t_n'-t}}\,dt+\int_{t_n}^{(t_{n-1}'+t_n')/2}\frac{1}{\sqrt{t-t_n'}}\,dt\biggr\} \\
&= \sum_{i=1}^{\infty}\frac{1}{n^4}+\sqrt{2}\sum_{i=1}^{\infty}\Bigl(\sqrt{(t_n'-t_{n+1}')}+\sqrt{(t_{n-1}'-t_n')}\Bigr) \\
&= \frac{\pi^4}{90}+\sqrt{2}\sum_{i=1}^{\infty}\biggl[\frac{1}{(n+2)(n+3)(n+4)(n+5)}\biggr]^{\frac{1}{2}} \\
&\quad+\sqrt{2}\sum_{i=1}^{\infty}\biggl[\frac{1}{(n+1)(n+2)(n+3)(n+4)}\biggr]^{\frac{1}{2}} \\
&\leq \frac{\pi^4}{90}+\sqrt{2}\sum_{i=1}^{\infty}\frac{1}{n^2}+\sqrt{2}\sum_{i=1}^{\infty}\frac{1}{n^2} \\
&= \frac{\pi^4}{90}+\sqrt{2}\frac{\pi^2}{3}.
\end{aligned}$$

Thus, it is easy to see

$$\int_0^1\omega(t)\,dt=\int_0^1\sum_{i=1}^{\infty}\omega_n(t)\,dt=\sum_{i=1}^{\infty}\int_0^1\omega_n(t)\,dt<\infty.$$

Therefore, $\omega(t)\in L^1[0,1]$, which satisfies condition (H_3).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All results belong to MW and MF. All authors read and approved the final manuscript.

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