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Boundary value problems on part of a level-n Sierpinski gasket

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Abstract

We study the boundary value problems for the Laplacian on a sequence of domains constructed by cutting level-n Sierpinski gaskets properly. Under proper assumptions on these domains, we manage to give an explicit Poisson integral formula to obtain a series of solutions subject to the boundary data. In particular, it is proved that there exists a unique solution continuous on the closure of the domain for a given sequence of convergent boundary values.

MSC: 28A80; 35J25

Keywords: boundary value problems; level-n Sierpinski gasket; harmonic functions; postcritically finite; fractal Laplacian

1 Introduction

The study of boundary value problems on the domains of Sierpinski gasket (SG) was initiated by [1]. Since then, two natural choices have been considered, namely the upper part of SG cut by a horizontal line (cf. [1, 2]) and half Sierpinski gasket constructed by cutting SG with a vertical line in the middle (cf. [3]). For more related works see, for example, [4–8]. This work is strongly motivated by [3].

In this work, we will introduce a new class of domains on level-n Sierpinski gasket and prove the exact form of the solution to the boundary value problems on these domains. Note that these domains are new examples of non-p.c.f. (postcritically finite) type fractals (can also be viewed as fractafold in [9, 10]) where harmonic functions can be well defined.

We follow [11, 12] by recalling that the fractal K is the invariant set for a finite iterated function systems (IFS) of contractive similarities in the Euclidean space $\mathbb{R}^2$. We denote the mappings $\{F_i\}_{i=0,\dots,N-1}$ for some positive integer N. Then K is the unique nonempty compact set satisfying

$$K = \bigcup_{i=0}^{N-1} F_i(K). \tag{1.1}$$

For $m \geq 1$, we define the space of words of length m by

$$W_m^N = \{0,1,2,\dots,N-1\}^m = \{w_1 w_2 \dots w_m : w_i \in \{0,1,2,\dots,N-1\}\}.$$

$w \in W_m^N$ is called a word of length m with symbols $\{0,1,2,\ldots,N-1\}$. We also set $W_*^N = \bigcup_{m\geq 0} W_m^N$ and denote the length of $w \in W_*^N$ by $|w|$.

Recall that K is called *postcritically finite* (p.c.f.) if K is connected and there exists a finite set $V_0 \subseteq K$ called the *boundary* such that

$$F_w K \cap F_{w'} K \subseteq F_w V_0 \cap F_{w'} V_0 \quad \text{for } w \neq w' \text{ with } |w| = |w'|, \tag{1.2}$$

with the intersection disjoint from V_0. Set $V_0 = \{q_0, q_1, \ldots, q_{N_0}\}$ for $N_0 < N$. We require that each boundary point is the fixed point of one of the mappings $\{F_i\}$ and that

$$F_i(q_i) = q_i \quad \text{for } 0 \leq i \leq N_0. \tag{1.3}$$

The standard SG is the unique nonempty compact set K satisfying (1.1) with the boundary set $V_0 = \{q_0, q_1, q_2\}$, where the contractive mappings $\{F_i\}_{i=0,1,2}$ are given by

$$F_i(x) = \frac{1}{2}(x - q_{i-1}) + q_{i-1}.$$

Similarly, the *level-3 Sierpinski gasket* SG_3 is the unique nonempty compact set K satisfying (1.1) with the boundary set $V_0 = \{q_0, q_1, q_2\}$, where $\{F_i\}_{i=0\ldots,5}$ are given by

$$F_i(x) = \frac{1}{3}(x - q_i) + q_i. \tag{1.4}$$

Here $q_3 = \frac{q_1+q_2}{2}$, $q_4 = \frac{q_0+q_2}{2}$, $q_5 = \frac{q_0+q_1}{2}$. See Figure 2 for an illustration.

As above, we can define *level-n Sierpinski gasket* in a similar way.

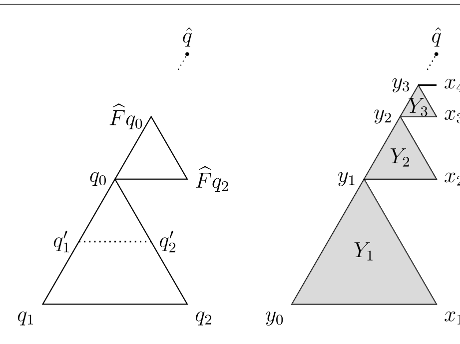
Inspired by [3] we will construct a new class of domains in the following statement.

1.1 Description of the general domains

Let $K = SG_n$ and $\tilde{K} = \frac{1}{n}K$, that is, shrinking K n times. Denote by $\widetilde{E}$ the compact triangular domain with boundary set $\{q_0, q_1, q_2\}$ which is constructed by gluing finite copies of $\tilde{K}$ at boundary (see Example 1.3 below). Assume that the compact triangular domain $\widetilde{E}_1$ with boundary $\{q_0, q_1', q_2'\}$ (as a part of $\widetilde{E}$) satisfies $\widetilde{E}_1 = F_0(\widetilde{E})$, where $F_0(x) := q_0 + \frac{x-q_0}{L}$ for some constant L (see Figure 1(a)). Pick some point $\hat{q}$ such that the contractive map

$$\widehat{F}(x) := \widehat{q} + \frac{x - \widehat{q}}{L}$$

Figure 1 General domain.



satisfies $\widehat{F}(q_1) = q_0$. Set $E = \widetilde{E}\backslash\{q_1, q_2\}$, $y_0 = q_1$, $y_m = \widehat{F}^m(q_1)$, $x_m = \widehat{F}^{m-1}(q_2)$, $Y_m := \widehat{F}^{m-1}(E)$ for integer $m > 0$ (see Figure 1(b)). Let $X = \bigcup_{m=1}^{\infty}\{x_m\}$. Set

$$\partial Y_m = \{y_{m-1}, y_m, x_m\}, \qquad \overline{Y}_m = Y_m \cup \partial Y_m.$$

For each $\overline{Y}_m$ define mapping F_m as

$$F_m(z) = y_m + \frac{z - y_m}{L}. \tag{1.5}$$

Then by assumption $\overline{Y}_m$ is self-similar with respect to F_m. Define

$$\Omega := \bigcup_{m=0}^{\infty} Y_m, \qquad \overline{\Omega} := \bigcup_{m=0}^{\infty} \overline{Y}_m. \tag{1.6}$$

We say that the set Ω is a DCPB *(domain of countable-point boundary)* with boundary $\partial\Omega := X \cup y_0 \cup \hat{q}$ for $K = \mathrm{SG}_n$.

Remark 1.1 In application, we need the constant $L = n^k$ with $k > 0$ to ensure that $\widetilde{E}$ is self-similar with respect to the map F_0, and thus $F_m(\overline{Y}_m)$ is exactly a copy of $\overline{Y}_{m+1}$. This property is useful in constructing harmonic functions on these domains in a sequel. The description of those domains will be justified by the examples below.

1.2 Examples

Example 1.2 For $K = \mathrm{SG}_3$, let $\widetilde{E} = F_1(K)$ be the compact triangular domain with boundary set $\{q_1, p_1, p_6\}$ (see Figure 2). Let $E = \widetilde{E}/\{q_1, p_1\}$, set $y_0 = q_1$, $y_m = F_5(q_1)$, $x_m = F_5^{m-1}(p_1)$ and $Y_m = F_5^{m-1}(E)$ for all positive integers m, where F_5 is as defined in (1.4). Set $\widehat{F} = F_5$, $\hat{q} = q_5$. Then Ω (green part) can be well established as in (1.6) .

Example 1.3 For $K = \mathrm{SG}_3$, $\widetilde{E} = F_1(K) \cup F_3(K) \cup F_5(K)$ is the compact triangular domain with boundary set $\{q_1, p_2, p_5\}$ (see Figure 2). Let $E = \widetilde{E}/\{q_1, p_2\}$, set $y_0 = q_1$, $y_m = F_0^m(q_1)$, $x_m = F_0^{m-1}(p_2)$ and $Y_m = F_0^{m-1}(E)$ for all positive integers m. Let $X = \bigcup_{m=1}^{\infty}\{x_m\}$. Setting $\widehat{F} = F_0$, $\hat{q} = q_0$, we obtain Ω (gray part) as a DCPB by (1.6).

Example 1.4 For $K = \mathrm{SG}_2$, $\widetilde{E} = F_1(K)$ is the compact triangular domain with boundary points $\{q_1, p_2, p_0\}$ (see Figure 3). Let $q_4 = \frac{2p_2+q_0}{3}$ and define the contractive mapping

$$F_4(x) = \frac{x - q_4}{4} + q_4. \tag{1.7}$$

Figure 2 SG$_3$.

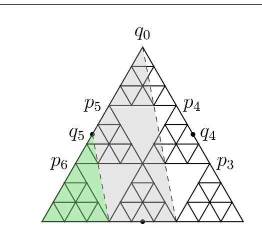
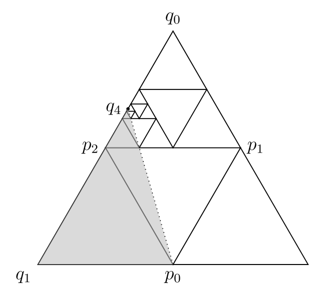


Figure 3 Domain in SG_2.



Let $E = \widetilde{E}/\{q_1, p_0\}$, set $y_0 = q_1$, $y_m = F_4^m(q_1)$, $x_m = F_4^{m-1}(p_0)$ and $Y_m = F_4^{m-1}(E)$ for all $m > 0$. Let $X = \bigcup_{m=1}^{\infty}\{x_m\}$. Set $\widehat{F} = F_4$, $\hat{q} = q_4$. Now we can define the desired domain Ω by (1.6). Note that F_4 is not one of the contractive mappings for standard SG.

In the following section, we construct a solution to the boundary value problem using harmonic extension algorithm. Denote by $C(U)$ the space of all continuous functions on some set U. We will see that the space of $C(\Omega)$-solutions to the boundary value problem is one-dimensional, but in general, the solution blows up at $\hat{q}$. We show that if the boundary data on X converges, there exists a unique $C(\overline{\Omega})$-solution.

2 Main results

The Laplacian on the standard SG was first constructed as a generator of a stochastic process by Goldstein [13] and Kusuoka [14]. Kigami [15, 16] developed an analytical version of the Laplacian for SG, and then generalized it to any p.c.f. self-similar set (see [11], Definition 3.7.1, p.108).

We now study the boundary value problems on Ω as a DCPB defined in Section 1.1:

$$\begin{cases} \triangle u = 0 & \text{on } \Omega, \\ u(y_0) = a_0, \quad u(x_m) = a_m & \text{on } \partial\Omega, \end{cases} \tag{BVP}$$

where $\triangle$ denotes the Kigami's Laplacian for $K = SG_n$ with respect to the standard self-similar measure, $u : \overline{\Omega} \to \mathbb{R}$ is the unknown, and $\{a_m\}_{m=0}^{\infty}$ is the boundary data. Note that the Laplacian $\triangle$ here is well defined for all cells Y_m, hence the whole Ω by recalling that every cell $\overline{Y}_m$ of Ω can be viewed as a part of $K = SG_n$ or gluing several copies of it.

Harmonic extension algorithm is the simplest tool for constructing harmonic functions subject to boundary value problems on SG_n. In fact, we can apply this algorithm infinitely many times and obtain a function harmonic on SG_n.

Using this, we will give an explicit solution to (BVP) based on the following assumption.

Assumption 0 Let Ω be a DCPB for $K = SG_n$. For each cell Y_m with boundary set $\partial\overline{Y}_m = \{y_{m-1}, y_m, x_m\}$, if some function u is harmonic on Y_m and satisfies that

$$u(y_m) = c_1, \qquad u(y_{m-1}) = c_2, \qquad u(x_m) = c_3 \tag{2.1}$$

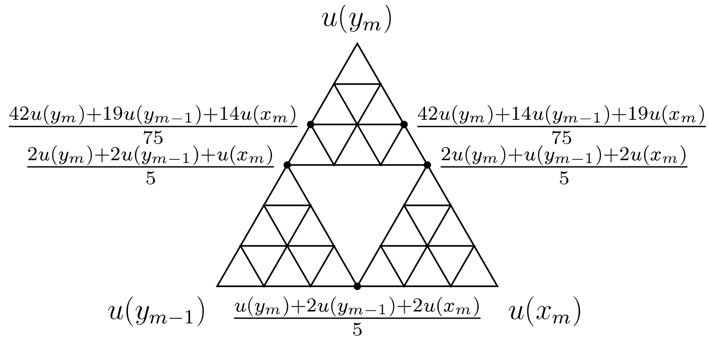


Figure 4 Values for Assumption 0 in Example 1.3.

for some real constants c_1, c_2, c_3, then

$$\begin{bmatrix} u(y_m) \\ u(y'_{m-1}) \\ u(x'_m) \end{bmatrix} = M_0 \begin{bmatrix} u(y_m) \\ u(y_{m-1}) \\ u(x_m) \end{bmatrix} = M_0 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \quad M_0 = \begin{bmatrix} 1 & 0 & 0 \\ \theta_1 & \theta_2 & \theta_3 \\ \theta_1 & \theta_3 & \theta_2 \end{bmatrix} \tag{2.2}$$

for some positive constants $\theta_1, \theta_2, \theta_3$ satisfying that

$$\theta_1 + \theta_2 + \theta_3 = 1, \tag{2.3}$$

where $y'_{m-1} = F_m(y_{m-1})$, $x'_m = F_m(x_m)$ with F_m given by (1.5).

Note that this assumption can be easily verified by harmonic extension algorithm. In Example 1.3, we have (see Figure 4)

$$M_0 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{42}{75} & \frac{19}{75} & \frac{14}{75} \\ \frac{42}{75} & \frac{14}{75} & \frac{19}{75} \end{bmatrix}.$$

We set, for Assumption 0,

$$\Theta_0 = \theta_2 + \theta_3, \qquad \Theta_1 = 2(2 - \theta_1),$$
$$T_+ = \frac{\Theta_1 + K}{2}, \qquad T_- = \frac{\Theta_1 - K}{2}, \qquad K = \sqrt{\Theta_1^2 - 4\Theta_0}.$$

Theorem 2.1 *For every choice of the convergent boundary data* $\{a_m\}$ *for some* Ω *as a DCPB* (*defined in Section* 1.1) *satisfying Assumption* 0, *there exists a one-dimensional space of* $C(\Omega)$ *solutions to the* (BVP). *For each real constant* λ, *there exists a unique solution to the* (BVP) u_λ *such that* $u_\lambda(y_1) = \lambda$ *and that* $u_\lambda(x_m) = a_m$ *for* $m \geq 1$. *Furthermore, for* $m \geq 2$

$$u_\lambda(y_m) = K^{-1}\left\{T_+^m \phi_m^+(\lambda) - T_-^m \phi_m^-(\lambda)\right\}, \tag{2.4}$$

where

$$\phi_m^+(\lambda) = \lambda - T_+^{-1}\Theta_0(a_0 + a_1) - (\Theta_0 + T_+)\sum_{k=2}^{m} T_+^{-k} a_k,$$

$$\phi_m^-(\lambda) = \lambda - T_-^{-1}\Theta_0(a_0 + a_1) - (\Theta_0 + T_-)\sum_{k=2}^{m} T_-^{-k} a_k.$$

Proof For fixed $m \geq 2$, let u be a continuous piecewise harmonic function for (BVP). In view of Assumption 0, it is easy to adapt the argument for [3], proof of Lemma 2.1, to obtain that $\triangle u(y_m) = 0$ holds if and only if

$$u(y_m) = \Theta_1 u(y_{m-1}) - \Theta_0 u(y_{m-2}) - a_m - \Theta_0 a_{m-1}. \tag{2.5}$$

The rest is trivial algebra as in [3], proof of Theorem 2.2. □

The theorem below can be obtained by following the argument in [3], proof of Theorem 2.4, Corollary 2.5. We include a brief proof for the readers' convenience.

Theorem 2.2 *If $a_m \to A$ as $m \to \infty$ for some constant A, there exists a unique solution to* (BVP) $u \in C(\overline{\Omega})$ *which satisfies that*

$$u(y_1) = T_+^{-1}\Theta_0(a_0 + a_1) + (\Theta_0 + T_+)\sum_{k=2}^{\infty} T_+^{-k} a_k, \tag{2.6}$$

and for $m \geq 2$

$$\begin{aligned} u(y_m) = {} & K^{-1}(\Theta_0 + T_+)\left\{\sum_{k=1}^{\infty} T_+^{-k} a_{m+k} - T_-^{m}\sum_{k=2}^{\infty} T_+^{-k} a_k\right\} \\ & + T_-^{m}\left\{a_0 + a_1 + K^{-1}(\Theta_0 + T_-)\sum_{k=2}^{m} T_-^{-k} a_k\right\}. \end{aligned} \tag{2.7}$$

Proof We first prove the theorem for the case $A = 0$.

Substituting (2.6) into (2.4) yields (2.7).

By using the triangle inequality, we have

$$\begin{aligned} \left|u(y_m)\right| \leq {} & K^{-1}(\Theta_0 + T_+)\left\{\sum_{k=1}^{\infty} T_+^{-k} |a_{m+k}| - T_-^{m}\sum_{k=2}^{\infty} T_+^{-k} |a_k|\right\} \\ & + T_-^{m}\left\{|a_0| + |a_1| + K^{-1}(\Theta_0 + T_-)\sum_{k=2}^{m} T_-^{-k} |a_k|\right\}. \end{aligned} \tag{2.8}$$

From this and $a_m \to 0$ we can easily see that $u(y_m) \to 0$. Thus, by the definition of BVP,

$$\lim_{m\to\infty} u(x_m) = \lim_{m\to\infty} u(y_m) = 0.$$

It follows by [3], Lemma 2.3, that $u \in C(\overline{\Omega})$. Since harmonic functions that are continuous to the boundary satisfy the maximum principle [17], we obtain the uniqueness by the standard argument for linear differential equations satisfying the maximum principle.

For the case $A \neq 0$, we consider the modified BVP:

$$\begin{cases} \triangle u = 0 & \text{on } \Omega, \\ u(y_0) = a_0 - A, \quad u(x_m) = a_m - A & \text{on } \partial\Omega. \end{cases}$$

Noting that $a_m - A \to 0$, the rest of the proof can be done by using the result from the last part of the proof and the maximum principle. □

Remark 2.3 The results in [3], Section 2, reduce to a special case of our theorems above with parameters $\theta_0 = 2/5$, $\theta_1 = 2/5$, $\theta_2 = 1/5$. Indeed, [3] proved many more results on half SG. Many of them are highly dependent on the fact that we can obtain SG from half SG by reflection, and thus the top point enjoys many more nice properties than our domains. We will touch on that elsewhere.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author read and approved the final manuscript.

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