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General stability for the Kirchhoff-type equation with memory boundary and acoustic boundary conditions

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Abstract

In this paper we consider the existence and general energy decay rate of global solution to the mixed problem for the Kirchhoff-type equation with memory boundary and acoustic boundary conditions. In order to prove the existence of solutions, we employ the Galerkin method and compactness arguments. Besides, we establish an explicit and general decay rate result using the perturbed modified energy method and some properties of the convex functions. Our result is obtained without imposing any restrictive assumptions on the behavior of the relaxation function at infinity. These general decay estimates extend and improve some earlier results, i.e., exponential or polynomial decay rates.

MSC: 35B40; 74D99

Keywords: Kirchhoff-type equation; general decay; acoustic boundary; relaxation function; memory boundary; convexity

1 Introduction

In this paper, we are concerned with the general decay of solutions to the Kirchhoff-type equation with memory boundary and acoustic boundary conditions:

$$u'' - M(\|\nabla u\|^2)\Delta u - \Delta u' + f(u) = 0 \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$u + \int_0^t g(t-s)\left(M(\|\nabla u(s)\|^2)\frac{\partial u}{\partial \nu}(s) + \frac{\partial u'}{\partial \nu}(s)\right) ds = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \tag{1.2}$$

$$M(\|\nabla u\|^2)\frac{\partial u}{\partial \nu} + \frac{\partial u'}{\partial \nu} = y' \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.3}$$

$$u' + p(x)y' + q(x)y = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.4}$$

$$u(0) = u_0, \qquad u'(0) = u_1 \quad \text{in } \Omega, \tag{1.5}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with sufficiently smooth boundary Γ, ν represents the outward unit normal vector to $\Gamma = \Gamma_0 \cup \Gamma_1$ for Γ_0 and Γ_1 be closed and disjoint. The relaxation function g is positive and nondecreasing, the function $f \in C^1(\mathbb{R})$ and $M \in C^1([0,\infty[)$, and p, q are functions satisfying some conditions to be specified later.

On the other hand, problem (1.1) with $u = 0$ on $\partial\Omega$ has its origin in the mathematical description of small amplitude vibrations of an elastic string. The existence of global solutions and exponential decay to this problem has been studied by many authors (see [1–4]). In fact, a mathematical model for the deflection of an elastic string of length $L > 0$ is given by the mixed problem for the nonlinear wave equation

$$\rho h u'' = \left(p_0 + \frac{Eh}{2L}\int_0^L u_x^2\,dx\right)u_{xx} \quad \text{for } 0 < x < L, t \geq 0, \tag{1.6}$$

where u is the lateral deflection, x is the space coordinate, t is the time, ρ denotes the mass density, h is the cross section area, p_0 is the initial axial tension and E is the Young modulus. Eq. (1.6) was introduced by Kirchhoff [5] as a nonlinear model of the free transversal vibrations of a clamped string.

The asymptotic behavior of solutions for nonlinear wave and plate equations with memory boundary condition has been proved by many authors [6–12]. In the aforementioned results, denoting by k the resolvent kernel of $-\frac{g'}{g(0)}$, they showed that the energy of the solution decays exponentially (polynomially) to zero provided k decays exponentially (polynomially) to zero. The decay result of Santos [9] was generalized by Messaoudi and Soufyane [13] without assuming the exponential (polynomial) decay of k. They obtained general stability for a wave equation under weaker condition on the resolvent kernel k such as

$$k'(t) \leq 0, \qquad k''(t) \geq \gamma(t)\bigl(-k'(t)\bigr), \tag{1.7}$$

where γ is a nonincreasing and positive function. Kang [14], Mustafa and Messaoudi [15] and Santos and Soufyane [16] investigated the general decay for the Kirchhoff plates, the Timoshenko system and the von Karman plate system with viscoelastic boundary conditions under condition (1.7), respectively. Recently, Kang [17] established a more general decay result of the differential inclusion of Kirchhoff type with strong damping term and boundary condition of memory type when a relaxation function satisfies the condition (1.7). This result improved the earlier decay results of Santos et al. [11]. More precisely, we studied that the energy decays at the rate similar to the relaxation functions, which are not necessarily decaying like polynomial or exponential functions.

Moreover, Beale and Rosencrans [18] introduced acoustic boundary conditions of the general form, and then Beale [19, 20] proved global existence and regularity of solutions for wave equations with acoustic boundary conditions. In these cases, the solution u of the wave equation is the velocity potential of a fluid undergoing acoustic wave motion and y is the normal displacement to the boundary at time t with the boundary point x. Recently, wave equations with acoustic boundary conditions have been treated by many authors [21–24]. They considered the existence of solutions, but gave no decay rate for solutions. As regards uniform decay rates for solutions to problems with acoustic boundary conditions, there is not much literature [25–29]. Most of these are concerned with exponential decay rates of solutions.

Motivated by these results, we study the stability for the Kirchhoff-type equation (1.1), which contains both memory boundary conditions and acoustic boundary conditions for resolvent kernel k satisfying

$$k''(t) \geq H\bigl(-k'(t)\bigr), \quad \forall t \geq 0, \tag{1.8}$$

where H is a positive function, with $H(0) = H'(0) = 0$, and H is linear or strictly increasing and strictly convex on $(0, r]$ for some $0 < r < 1$. Recently, Mustafa and Abusharkh [30] and Kang [31] showed the general decay result for plate equations and von Karman plate system with viscoelastic boundary damping when a relaxation function satisfies (1.8) and $u_0 \equiv 0$ on Γ_0, respectively. We obtain an explicit and general decay of the solution for the Kirchhoff-type equation without assuming that $u_0 \equiv 0$ on Γ_0 when relaxation function satisfies (1.8). Since problem (1.1) does not have a homogeneous Dirichlet condition on portion of the boundary, we introduce a close subspace $\tilde{V}$ of $H^1(\Omega)$, as in [24], where Poincaré's inequality is satisfied. Moreover, to prove the existence of a weak solution to the problem, we use the Galerkin method and compactness arguments. After this, we obtain the general decay rates by employing the multiplier method and some properties of convex functions including the use of general Young's inequality and Jensen's inequality.

The paper is organized as follows. In Section 2 we give some notations and material needed for our work and state the main results. In Section 3 we consider the existence of global weak solution for problem (1.1)-(1.5). In Section 4 we show the general decay of the solutions to the Kirchhoff-type equation with memory boundary and acoustic boundary conditions.

2 Statement of main results

In this section, we provide some material needed in the proof of our main result and state main results. Let us consider the Hilbert spaces $L^2(\Omega)$ and $L^2(\Gamma)$ endowed with the inner products

$$(u, v) = \int_\Omega u(x)v(x)\,dx, \qquad (u, v)_\Gamma = \int_\Gamma u(x)v(x)\,d\Gamma,$$

and the corresponding norms $\|u\|^2_{L^2(\Omega)} = (u, u)$ and $\|u\|^2_{L^2(\Gamma)} = (u, u)_\Gamma$, respectively. For simplicity, we denote $\|\cdot\|^2_{L^2(\Omega)}$ and $\|\cdot\|^2_{L^2(\Gamma)}$ by $\|\cdot\|$ and $\|\cdot\|_\Gamma$, respectively.

Following the idea in [24], we consider

$$V = \bigcup_{x\in\Gamma} V_x,$$

where, for each point x_0 fixed in Γ,

$$V_{x_0} = \{u \in C^1(\bar{\Omega}) : u(x_0) = 0\}.$$

Then Poincaré's inequality holds in V. From density, we see that Poincaré's inequality still holds in $H^1(\Omega)$ closure of V which we denote by $W = \bar{V}^{H^1(\Omega)}$. Let λ and λ_1 be the smallest positive constants such that

$$\lambda\|u\|^2 \le \|\nabla u\|^2, \qquad \lambda_1\|u\|^2_\Gamma \le \|\nabla u\|^2, \quad \forall u \in W. \tag{2.1}$$

Moreover, we let x^0 be a fixed point in $\mathbb{R}^n$, $m(x) = x - x^0$ and $R = \max\{|x - x^0| : x \in \bar{\Omega}\}$, and assume that

$$m(x) \cdot \nu(x) > 0, \quad x \in \Gamma. \tag{2.2}$$

We formulate the following hypotheses.

(H1) With respect to $M \in C^1([0,\infty[)$, we assume that

$$0 < m_0 \leq M(\zeta), \qquad \hat{M}(\zeta) \leq M(\zeta)\zeta, \quad \forall \zeta \geq 0, \tag{2.3}$$

where $\hat{M}(\zeta) = \int_0^\zeta M(s)\,ds$.

(H2) Let $f \in C^1(\mathbb{R})$ satisfy $f(s)s \geq 0$, $\forall s \in \mathbb{R}$. We suppose that f is superlinear, that is,

$$f(s)s \geq (2+\delta)F(s), \qquad F(z) = \int_0^z f(s)\,ds, \quad \forall s \in \mathbb{R} \tag{2.4}$$

for some $\delta > 0$ with the following growth condition:

$$\bigl|f(x) - f(y)\bigr| \leq c_0\bigl(1 + |x|^{\rho-1} + |y|^{\rho-1}\bigr)|x-y|, \quad \forall x,y \in \mathbb{R} \tag{2.5}$$

for some $c_0 > 0$ and $\rho \geq 1$ such that $(n-2)\rho \leq n$.

(H3) For the functions p and q, we assume that $p, q \in C(\Gamma_1)$ and $p(x) > 0$ and $q(x) > 0$ for all $x \in \Gamma_1$. It implies that there exist positive constants p_i, q_i $(i = 0,1)$ such that

$$p_0 \leq p(x) \leq p_1, \qquad q_0 \leq q(x) \leq q_1, \quad \forall x \in \Gamma_1. \tag{2.6}$$

(H4) In addition, we assume that $k : \mathbb{R}_+ \to \mathbb{R}_+$ is the resolvent kernel of $-\frac{g'}{g(0)}$, which is a twice differentiable function such that

$$k(0) > 0, \qquad \lim_{t\to\infty} k(t) = 0, \qquad k'(t) \leq 0, \tag{2.7}$$

and there exists a positive function $H \in C^1(\mathbb{R}_+)$ and H is a linear or strictly increasing and strictly convex C^2 function on $(0,r]$, $r < 1$, with $H(0) = H'(0) = 0$, such that

$$k''(t) \geq H\bigl(-k'(t)\bigr), \quad t > 0. \tag{2.8}$$

To simplify calculation in our analysis, we introduce the following notation:

$$(g * v)(t) = \int_0^t g(t-s)v(s)\,ds,$$
$$(g \Box v)(t) = \int_0^t g(t-s)\bigl|v(t) - v(s)\bigr|^2\,ds.$$

First, we shall use Eq. (1.2) to estimate the term $M(\|\nabla u\|^2)\frac{\partial u}{\partial \nu} + \frac{\partial u'}{\partial \nu}$. Differentiating Eq. (1.2), we get the following Volterra equation:

$$M\bigl(\|\nabla u\|^2\bigr)\frac{\partial u}{\partial \nu} + \frac{\partial u'}{\partial \nu} + \frac{1}{g(0)}g' * \left(M\bigl(\|\nabla u\|^2\bigr)\frac{\partial u}{\partial \nu} + \frac{\partial u'}{\partial \nu}\right) = -\frac{1}{g(0)}u'.$$

Using the Volterra inverse operator, we have

$$M\bigl(\|\nabla u\|^2\bigr)\frac{\partial u}{\partial \nu} + \frac{\partial u'}{\partial \nu} = -\frac{1}{g(0)}\bigl\{u' + k * u'\bigr\},$$

where the resolvent kernel k is given by the solution of

$$k + \frac{1}{g(0)} g' * k = -\frac{1}{g(0)} g'.$$

Denoting $\tau = \frac{1}{g(0)}$, we obtain

$$M\big(\|\nabla u\|^2\big)\frac{\partial u}{\partial \nu} + \frac{\partial u'}{\partial \nu} = -\tau\big\{u' + k(0)u - k(t)u_0 + k' * u\big\},$$

which is equivalent to condition (1.2). Then we get the following equivalent problem:

$$u'' - M\big(\|\nabla u\|^2\big)\Delta u - \Delta u' + f(u) = 0 \quad \text{in } \Omega \times (0,\infty), \tag{2.9}$$

$$M\big(\|\nabla u\|^2\big)\frac{\partial u}{\partial \nu} + \frac{\partial u'}{\partial \nu} = -\tau\big\{u' + k(0)u - k(t)u_0 + k' * u\big\} \quad \text{on } \Gamma_0 \times (0,\infty), \tag{2.10}$$

$$M\big(\|\nabla u\|^2\big)\frac{\partial u}{\partial \nu} + \frac{\partial u'}{\partial \nu} = y' \quad \text{on } \Gamma_1 \times (0,\infty), \tag{2.11}$$

$$u' + p(x)y' + q(x)y = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{2.12}$$

$$u(0) = u_0, \qquad u'(0) = u_1 \quad \text{in } \Omega. \tag{2.13}$$

By differentiating the term $g\Box v$, we have the following lemma.

Lemma 2.1 *If* $g, v \in C^1([0,\infty):\mathbb{R})$, *then*

$$(g * v)v' = -\frac{1}{2}g(t)\big|v(t)\big|^2 + \frac{1}{2}g'\Box v - \frac{1}{2}\frac{d}{dt}\left[g\Box v - \left(\int_0^t g(s)\,ds\right)|v|^2\right].$$

The energy of system (2.9)-(2.13) is given by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\big\|u'\big\|^2 + \frac{1}{2}\hat{M}\big(\|\nabla u\|^2\big) + \int_\Omega F(u)\,dx + \frac{\tau}{2}k(t)\|u\|_{\Gamma_0}^2 \\ & - \frac{\tau}{2}\int_{\Gamma_0} k'\Box u\,d\Gamma + \frac{1}{2}\int_{\Gamma_1} q(x)|y|^2\,d\Gamma. \end{aligned}$$

Now, we are ready to state our main results.

Theorem 2.1 *Suppose that* (H1)-(H4) *hold. If* $(u_0, u_1) \in (W \cap H^2(\Omega)) \times W$ *and satisfy the compatibility condition*

$$M\big(\|\nabla u_0\|^2\big)\frac{\partial u_0}{\partial \nu} + \frac{\partial u_1}{\partial \nu} + \tau u_1 = 0 \quad \text{on } \Gamma_0, \tag{2.14}$$

then, for all $T > 0$, *there exists a unique pair of functions* (u, y), *which is a solution of system* (2.9)-(2.13) *satisfying*

$$\begin{aligned} & u \in L^\infty\big(0,T; W \cap H^2(\Omega)\big), \qquad u' \in L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2(0,T;W), \\ & u'' \in L^\infty\big(0,T;L^2(\Omega)\big), \\ & y \in L^\infty\big(0,T;L^2(\Gamma_1)\big), \qquad y' \in L^2\big(0,T;L^2(\Gamma_1)\big). \end{aligned}$$

Theorem 2.2 *Suppose that* (H1)-(H4) *hold. Assume that D is a positive C^1 function, with* $D(0) = 0$, *for which H_0 is a strictly increasing and strictly convex C^2 function on* $(0, r]$ *and*

$$\int_0^{+\infty} \frac{-k'(s)}{H_0^{-1}(k''(s))}\, ds < +\infty. \tag{2.15}$$

Therefore, there exist positive constants c_1, c_2, c_3 and ϵ_0 such that the solution of (2.9)-(2.13) *satisfies*

$$E(t) \le c_3 H_1^{-1}(c_1 t + c_2), \quad \forall t \ge 0, \tag{2.16}$$

where

$$H_1(t) = \int_t^1 \frac{1}{sH_0'(\epsilon_0 s)}\, ds \quad \text{and} \quad H_0(t) = H\bigl(D(t)\bigr).$$

Furthermore, if $\int_0^1 H_1(t)\, dt < +\infty$, for some choice of D, then we obtain

$$E(t) \le c_3 G^{-1}(c_1 t + c_2), \tag{2.17}$$

where

$$G(t) = \int_t^1 \frac{1}{sH'(\epsilon_0 s)}\, ds. \tag{2.18}$$

In particular, (2.17) *is valid for the special case $H(t) = ct^p$ for $1 \le p < \frac{3}{2}$.*

Remark 2.1 For large $t_0 > 0$, there exists a constant $d_0 > 0$ such that

$$k''(t) \ge -d_0 k'(t), \quad \forall t \in [0, t_0]. \tag{2.19}$$

Indeed, from (H4), we find that $\lim_{t\to+\infty}(-k'(t)) = 0$. This implies that $\lim_{t\to+\infty} k''(t)$ cannot be equal to a positive number, and so it is natural to assume that $\lim_{t\to+\infty} k''(t) = 0$. Then there is $t_0 > 0$ large enough such that $k'(t_0) < 0$ and

$$\max\bigl\{k(t), -k'(t), k''(t)\bigr\} < \min\bigl\{r, H(r), H_0(r)\bigr\}, \quad \forall t \ge t_0. \tag{2.20}$$

Because k' is nondecreasing, $k'(0) < 0$ and $k'(t_0) < 0$, we get

$$0 < -k'(t_0) \le -k'(t) \le -k'(0), \quad \forall t \in [0, t_0]. \tag{2.21}$$

From H is a positive continuous function, we have for some positive constants d_1 and d_2,

$$d_1 \le H\bigl(-k'(t)\bigr) \le d_2, \quad \forall t \in [0, t_0]. \tag{2.22}$$

Therefore, by (2.8), (2.21) and (2.22), we see that (2.19) holds.

The well-known Jensen's inequality will be of essential use in establishing our main result.

Remark 2.2 If F_0 is a convex function on $[a,b]$, $f:\Omega\to[a,b]$ and h are integrable functions on Ω, $h(x)\geq 0$, and $\int_\Omega h(x)\,dx = h_0 > 0$, then Jensen's inequality states that

$$F_0\left(\frac{1}{h_0}\int_\Omega f(x)h(x)\,dx\right)\leq \frac{1}{h_0}\int_\Omega F_0\big(f(x)\big)h(x)\,dx. \tag{2.23}$$

3 Proof of Theorem 2.1

In this section, we study the existence of a global weak solution for problem (2.9)-(2.13) using Faedo-Galerkin's approximation. Since the problem is not normal, we cannot apply directly Caratheodory's theorem. So we use a degenerated second order equation on Γ_1. To this end, let $\{w_j\}_{j\in\mathbb{N}}$ and $\{z_j\}_{j\in\mathbb{N}}$ be orthonormal bases of W and $L^2(\Gamma)$, respectively. For each $m\in\mathbb{N}$, let $W_m = \operatorname{span}\{w_1,w_2,\ldots,w_m\}$ and $Z_m = \operatorname{span}\{z_1,z_2,\ldots,z_m\}$. For each $\epsilon\in(0,1)$ and any $T<0$, standard results on ordinary differential equations guarantee that there exists only one local solution for $0<T_m\leq T$

$$u_{m\epsilon}(x,t) = \sum_{j=1}^m a_{jm}(t)w_j(x), \quad x\in\Omega,$$

$$y_{m\epsilon}(x,t) = \sum_{j=1}^m b_{jm}(t)z_j(x), \quad x\in\Gamma_1,$$

satisfying the approximate perturbed problem

$$\begin{aligned}
&\big(u''_{m\epsilon},w_j\big) + \big(M\big(\|\nabla u_{m\epsilon}\|^2\big)\nabla u_{m\epsilon},\nabla w_j\big) + \big(\nabla u'_{m\epsilon},\nabla w_j\big) + \big(f(u_{m\epsilon}),w_j\big) - \big(y'_{m\epsilon},w_j\big)_{\Gamma_1}\\
&\quad + \tau\big(u'_{m\epsilon} + k(0)u_{m\epsilon} - k(t)u_{m\epsilon}(0) + k' * u_{m\epsilon},w_j\big)_{\Gamma_0} = 0,\\
&\epsilon\big(y''_{m\epsilon},z_j\big)_{\Gamma_1} + \big(u'_{m\epsilon} + py'_{m\epsilon} + qy_{m\epsilon},z_j\big)_{\Gamma_1} = 0,\\
&u_{m\epsilon}(0) = u_{0m} = \sum_{j=1}^m (u_0,w_j)w_j, \qquad u'_{m\epsilon}(0) = u_{1m} = \sum_{j=1}^m (u_1,w_j)w_j,\\
&y_{m\epsilon}(0) = y_{0m} = -\left(\frac{u_{1m} + py'_{m\epsilon}(0)}{q}\right), \qquad y'_{m\epsilon}(0) = y_{1m} = M\big(\|\nabla u_{0m}\|^2\big)\frac{\partial u_{0m}}{\partial\nu} + \frac{\partial u_{1m}}{\partial\nu},
\end{aligned} \tag{3.1}$$

for $j=1,2,\ldots,m$. Now we need estimates which allow us to extend the solutions to the whole interval $[0,T]$ and pass to limit as $m\to\infty$ and $\epsilon\to 0$. Hence, uniform estimates with respect to m and ϵ are needed. Indeed, from (3.1), we obtain the approximate equations

$$\begin{aligned}
&\big(u''_{m\epsilon},w\big) + \big(M\big(\|\nabla u_{m\epsilon}\|^2\big)\nabla u_{m\epsilon},\nabla w\big) + \big(\nabla u'_{m\epsilon},\nabla w\big) + \big(f(u_{m\epsilon}),w\big) - \big(y'_{m\epsilon},w\big)_{\Gamma_1}\\
&\quad + \tau\big(u'_{m\epsilon} + k(0)u_{m\epsilon} - k(t)u_{m\epsilon}(0) + k' * u_{m\epsilon},w\big)_{\Gamma_0} = 0, \quad \forall w\in W_m,\\
&\epsilon\big(y''_{m\epsilon},z\big)_{\Gamma_1} + \big(u'_{m\epsilon} + py'_{m\epsilon} + qy_{m\epsilon},z\big)_{\Gamma_1} = 0, \quad \forall z\in Z_m.
\end{aligned} \tag{3.2}$$

Estimate I Taking $w = u'_{m\epsilon}$ and $z = y'_{m\epsilon}$ in (3.2) and integrating over $(0,t)$, we get from Lemma 2.1

$$\begin{aligned}
&\frac{d}{dt}E(t,u_{m\epsilon}) + \frac{\epsilon}{2}\frac{d}{dt}\big\|y'_{m\epsilon}\big\|^2_{\Gamma_1} + \big\|\nabla u'_{m\epsilon}\big\|^2 + \int_{\Gamma_1} p(x)\big|y'_{m\epsilon}\big|^2\,d\Gamma + \tau\big\|u'_{m\epsilon}\big\|^2_{\Gamma_0}\\
&\quad = \frac{\tau}{2}k'(t)\|u_{m\epsilon}\|^2_{\Gamma_0} + \tau\int_{\Gamma_0} k(t)u_{0m}u'_{m\epsilon}\,d\Gamma - \frac{\tau}{2}\int_{\Gamma_0} k''\Box u_{m\epsilon}\,d\Gamma.
\end{aligned} \tag{3.3}$$

Using Young's inequality and (2.7), we have

$$\tau \int_{\Gamma_0} k(t) u_{0m} u'_{m\epsilon}\, d\Gamma \le \frac{\tau}{2} \|u'_{m\epsilon}\|_{\Gamma_0}^2 + \frac{\tau}{2} k^2(0) \|u_{0m}\|_{\Gamma_0}^2. \tag{3.4}$$

From (2.6)-(2.8), (3.3) and (3.4), we obtain

$$\begin{aligned} &\frac{d}{dt} E(t, u_{m\epsilon}) + \frac{d}{dt} \frac{\epsilon}{2} \|y'_{m\epsilon}\|_{\Gamma_1}^2 + \|\nabla u'_{m\epsilon}\|^2 + p_0 \|y'_{m\epsilon}\|_{\Gamma_1}^2 + \frac{\tau}{2} \|u'_{m\epsilon}\|_{\Gamma_0}^2 \\ &\quad \le \frac{\tau}{2} k^2(0) \|u_{0m}\|_{\Gamma_0}^2. \end{aligned}$$

Then, employing Gronwall's inequality, we conclude that there exists a constant $C = C(T)$, independent of m, ϵ and $t \in [0, T]$, such that

$$\begin{aligned} &\|u'_{m\epsilon}\|^2 + \hat{M}(\|\nabla u_{m\epsilon}\|^2) + \int_\Omega F(u_{m\epsilon})\, dx + \tau k(t) \|u_{m\epsilon}\|_{\Gamma_0}^2 \\ &\quad - \tau \int_{\Gamma_0} k' \Box u_{m\epsilon}\, d\Gamma + q_0 \|y_{m\epsilon}\|_{\Gamma_1}^2 \\ &\quad + \epsilon \|y'_{m\epsilon}\|_{\Gamma_1}^2 + \int_0^t \|\nabla u'_{m\epsilon}\|^2\, ds + p_0 \int_0^t \|y'_{m\epsilon}\|_{\Gamma_1}^2\, ds \le C. \end{aligned} \tag{3.5}$$

Estimate II First, we will estimate $\|u''_{m\epsilon}(0)\|^2$ and $\|y''_{m\epsilon}(0)\|_{\Gamma_1}^2$. Taking $t = 0$ in (3.2), replacing w and z by $u''_{m\epsilon}(0)$ and $y''_{m\epsilon}(0)$, respectively, and using (2.14), we get

$$\begin{aligned} \|u''_{m\epsilon}(0)\|^2 &= \int_\Omega M(\|\nabla u_{0m}\|^2) \Delta u_{0m} u''_{m\epsilon}(0)\, dx + \int_\Omega \Delta u_{1m} u''_{m\epsilon}(0)\, dx \\ &\quad - \int_\Omega f(u_{0m}) u''_{m\epsilon}(0)\, dx + \int_{\Gamma_1} y_{1m} u''_{m\epsilon}(0)\, d\Gamma \end{aligned} \tag{3.6}$$

and

$$\epsilon \|y''_{m\epsilon}(0)\|_{\Gamma_1}^2 + (u_{1m} + p y_{1m} + q y_{0m}, y''_{m\epsilon}(0))_{\Gamma_1} = 0. \tag{3.7}$$

From the assumptions on the initial data, f and M, we have that there exists a constant $C > 0$, independent of ϵ and m, such that

$$\|u''_{m\epsilon}(0)\|^2 \le C, \qquad \|y''_{m\epsilon}(0)\|_{\Gamma_1}^2 \le C. \tag{3.8}$$

Differentiating (3.2) with respect to t and substituting w and z by $u''_{m\epsilon}$ and $y''_{m\epsilon}$, respectively, we see that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left[\|u''_{m\epsilon}\|^2 + \epsilon \|y''_{m\epsilon}\|_{\Gamma_1}^2 + \int_{\Gamma_1} q(x) |y'_{m\epsilon}|^2 d\Gamma \right] + \|\nabla u''_{m\epsilon}\|^2 + \int_{\Gamma_1} p(x) |y''_{m\epsilon}|^2 d\Gamma \\ &\quad = -M(\|\nabla u_{m\epsilon}\|^2) \int_\Omega \nabla u'_{m\epsilon} \nabla u''_{m\epsilon}\, dx \\ &\qquad - 2M'(\|\nabla u_{m\epsilon}\|^2) \left(\int_\Omega \nabla u_{m\epsilon} \nabla u'_{m\epsilon}\, dx \right) \left(\int_\Omega \nabla u_{m\epsilon} \nabla u''_{m\epsilon}\, dx \right) \end{aligned}$$

$$- \int_\Omega f'(u_{m\epsilon})u'_{m\epsilon}u''_{m\epsilon}\,dx - \tau\|u''_{m\epsilon}\|^2_{\Gamma_0} - \tau\int_{\Gamma_0} k(0)u'_{m\epsilon}u''_{m\epsilon}\,d\Gamma$$
$$+ \tau\int_{\Gamma_0} k'(t)u_{m\epsilon}(0)u''_{m\epsilon}\,d\Gamma - \tau\int_{\Gamma_0}\big(k' * u_{m\epsilon}\big)'u''_{m\epsilon}\,d\Gamma. \tag{3.9}$$

From the first estimate, assumption on M and Young's inequality, we obtain

$$M\big(\|\nabla u_{m\epsilon}\|^2\big)\int_\Omega \nabla u'_{m\epsilon}\nabla u''_{m\epsilon}\,dx \le c\big\|\nabla u'_{m\epsilon}\big\|^2 + \frac{1}{4}\big\|\nabla u''_{m\epsilon}\big\|^2 \tag{3.10}$$

and

$$2M'\big(\|\nabla u_{m\epsilon}\|^2\big)\left(\int_\Omega \nabla u_{m\epsilon}\nabla u'_{m\epsilon}\,dx\right)\left(\int_\Omega \nabla u_{m\epsilon}\nabla u''_{m\epsilon}\,dx\right)$$
$$\le c\big\|\nabla u'_{m\epsilon}\big\|^2 + \frac{1}{4}\big\|\nabla u''_{m\epsilon}\big\|^2. \tag{3.11}$$

Using generalized Hölder's inequality, assumption (2.5), (3.5), the Sobolev imbedding and Young's inequality, we find that

$$\begin{aligned}
&\int_\Omega f'(u_{m\epsilon})u'_{m\epsilon}u''_{m\epsilon}\,dx \\
&\quad\le c\int_\Omega \big(1 + 2|u_{m\epsilon}|^{\rho-1}\big)\big|u'_{m\epsilon}\big|\big|u''_{m\epsilon}\big|\,dx \\
&\quad\le c\left(\int_\Omega \big(1 + 2|u_{m\epsilon}|^{\rho-1}\big)^n\,dx\right)^{\frac{1}{n}}\left(\int_\Omega \big|u'_{m\epsilon}\big|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{2n}}\left(\int_\Omega \big|u''_{m\epsilon}\big|^2\,dx\right)^{\frac{1}{2}} \\
&\quad\le c\left(\int_\Omega \big(1 + 2|\nabla u_{m\epsilon}|^2\big)\,dx\right)^{\frac{\rho-1}{2}}\left(\int_\Omega \big|\nabla u'_{m\epsilon}\big|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega \big|u''_{m\epsilon}\big|^2\,dx\right)^{\frac{1}{2}} \\
&\quad\le c\big\|\nabla u'_{m\epsilon}\big\|^2 + c\big\|u''_{m\epsilon}\big\|^2.
\end{aligned} \tag{3.12}$$

Noting that

$$\big(k' * u_{m\epsilon}\big)' = k'(t)u_{m\epsilon}(0) + k' * u'_{m\epsilon},$$

and using Lemma 2.1, we get

$$\begin{aligned}
&\tau\int_{\Gamma_0}\big(k' * u_{m\epsilon}\big)'u''_{m\epsilon}\,d\Gamma \\
&\quad= \tau\int_{\Gamma_0} k'(t)u_{m\epsilon}(0)u''_{m\epsilon}\,d\Gamma - \frac{\tau}{2}k'(t)\big\|u'_{m\epsilon}\big\|^2_{\Gamma_0} + \frac{\tau}{2}\int_{\Gamma_0} k''\Box u'_{m\epsilon}\,d\Gamma \\
&\qquad- \tau\int_{\Gamma_0} k(0)u'_{m\epsilon}u''_{m\epsilon}\,d\Gamma - \frac{\tau}{2}\frac{d}{dt}\left[\int_{\Gamma_0} k'\Box u'_{m\epsilon}\,d\Gamma - k(t)\big\|u'_{m\epsilon}\big\|^2_{\Gamma_0}\right].
\end{aligned} \tag{3.13}$$

Combining (3.10)-(3.13) with (3.9), we deduce that

$$\frac{1}{2}\frac{d}{dt}\left[\big\|u''_{m\epsilon}\big\|^2 + \epsilon\big\|y''_{m\epsilon}\big\|^2_{\Gamma_1} + \int_{\Gamma_1} q(x)\big|y'_{m\epsilon}\big|^2 d\Gamma + \tau k(t)\big\|u'_{m\epsilon}\big\|^2_{\Gamma_0} - \tau\int_{\Gamma_0} k'\Box u'_{m\epsilon}\,d\Gamma\right]$$
$$+ \frac{1}{2}\big\|\nabla u''_{m\epsilon}\big\|^2 + \int_{\Gamma_1} p(x)\big|y''_{m\epsilon}\big|^2 d\Gamma$$

$$\begin{aligned}&\le 3c\|\nabla u'_{m\epsilon}\|^2 + c\|u''_{m\epsilon}\|^2 - \tau\|u''_{m\epsilon}\|^2_{\Gamma_0} + \frac{\tau}{2}k'(t)\|u'_{m\epsilon}\|^2_{\Gamma_0}\\&\quad - \frac{\tau}{2}\int_{\Gamma_0} k''\Box u'_{m\epsilon}\,d\Gamma.\end{aligned}\tag{3.14}$$

Integrating (3.14) over $[0,t]$ and applying Gronwall's inequality and (H4), we conclude that there exists a constant C, independent of ϵ and m, such that

$$\|u''_{m\epsilon}\|^2 + \epsilon\|y''_{m\epsilon}\|^2_{\Gamma_1} + q_0\|y'_{m\epsilon}\|^2_{\Gamma_1} + \int_0^t \|\nabla u''_{m\epsilon}\|^2\,ds + p_0\int_0^t \|y''_{m\epsilon}\|^2_{\Gamma_1}\,ds \le C. \tag{3.15}$$

From (3.5) and (3.15) and Lions-Aubin's compactness theorem [32], we can pass to the limit in (3.1). This completes the proof of Theorem 2.1.

4 Proof of Theorem 2.2

In this section, we shall prove the general decay rates in Theorem 2.2. Let us consider the following binary operator:

$$(k\circ v)(t) := \int_0^t k(t-s)\big(v(t)-v(s)\big)\,ds.$$

Then, using Hölder's inequality for $0\le\alpha\le 1$, we have

$$\big|(k\circ v)(t)\big|^2 \le \left[\int_0^t \big|k(s)\big|^{2(1-\alpha)}\,ds\right]\big(|k|^{2\alpha}\Box v\big)(t). \tag{4.1}$$

Lemma 4.1 *The energy E satisfies, along the solution of* (2.9)-(2.13),

$$\begin{aligned}E'(t) &\le -\|\nabla u'\|^2 - \frac{\tau}{2}\|u'\|^2_{\Gamma_0} + \frac{\tau}{2}k^2(t)\|u_0\|^2_{\Gamma_0} + \frac{\tau}{2}k'(t)\|u\|^2_{\Gamma_0}\\&\quad - \frac{\tau}{2}\int_{\Gamma_0} k''\Box u\,d\Gamma - \int_{\Gamma_1} p(x)\big|y'\big|^2\,d\Gamma.\end{aligned}\tag{4.2}$$

Proof Multiplying Eq. (2.9) by u' and integrating by parts over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|u'\|^2 + \hat{M}\big(\|\nabla u\|^2\big) + 2\int_\Omega F(u)\,dx + \int_{\Gamma_1} q(x)|y|^2\,d\Gamma\right] + \|\nabla u'\|^2 + \int_{\Gamma_1} p(x)\big|y'\big|^2\,d\Gamma\\&\quad = -\tau\int_{\Gamma_0}\big(u' + k(0)u - k(t)u_0 + k' * u\big)u'\,d\Gamma.\end{aligned}$$

From Lemma 2.1 and Young's inequality, we get estimate (4.2). □

To this system, we introduce the functional

$$\Phi(t) := \int_\Omega \left\{m\cdot\nabla u + \left(\frac{n}{2}-\theta\right)u\right\}u'\,dx + \int_{\Gamma_1} uy\,d\Gamma + \frac{1}{2}\int_{\Gamma_1} p(x)|y|^2\,d\Gamma,$$

where θ is a small positive constant. The following lemma plays an important role in the construction of the Lyapunov functional.

Lemma 4.2 *There exists* $C > 0$ *such that*

$$\begin{aligned}\Phi'(t) &\le -\theta\|u'\|^2 - (1-\theta-\epsilon c-\epsilon_2)M(\|\nabla u\|^2)\|\nabla u\|^2 + \left(c_\epsilon + \frac{R}{2\lambda_1}\right)\|\nabla u'\|^2 \\ &\quad - \left(\frac{n}{2}\delta - \delta\theta - 2\theta\right)\int_\Omega F(u)\,dx + C\int_{\Gamma_0}\left(|u'|^2 + |k(t)u|^2 - k'\Box u + |k(t)u_0|^2\right)d\Gamma \\ &\quad + \left(c_{\epsilon_1} + \frac{c_{\epsilon_2}}{m_0\lambda_1}\right)\|y'\|^2_{\Gamma_1} - \int_{\Gamma_1} q(x)|y|^2\,d\Gamma. \end{aligned} \tag{4.3}$$

Proof Direct computations and (2.9) yield

$$\begin{aligned}\Phi'(t) &= \int_\Omega u'(m\cdot\nabla u')\,dx + \left(\frac{n}{2}-\theta\right)\int_\Omega |u'|^2\,dx + \int_{\Gamma_1} uy'\,d\Gamma - \int_{\Gamma_1} q(x)|y|^2\,d\Gamma \\ &\quad + \int_\Omega\left[(m\cdot\nabla u) + \left(\frac{n}{2}-\theta\right)u\right]\left\{M(\|\nabla u\|^2)\Delta u + \Delta u' - f(u)\right\}dx.\end{aligned}$$

Integrating by parts and using Young's inequality, we have

$$\begin{aligned}\Phi'(t) &\le \frac{1}{2}\int_\Gamma (m\cdot\nu)|u'|^2\,d\Gamma - \theta\|u'\|^2 - (1-\theta-\epsilon c)M(\|\nabla u\|^2)\|\nabla u\|^2 + c_\epsilon\|\nabla u'\|^2 \\ &\quad + \int_\Gamma\left(M(\|\nabla u\|^2)\frac{\partial u}{\partial\nu} + \frac{\partial u'}{\partial\nu}\right)\left(m\cdot\nabla u + \left(\frac{n}{2}-\theta\right)u\right)d\Gamma \\ &\quad - \frac{1}{2}\int_\Gamma (m\cdot\nu)M(\|\nabla u\|^2)|\nabla u|^2\,d\Gamma \\ &\quad + n\int_\Omega F(u)\,dx - \left(\frac{n}{2}-\theta\right)\int_\Omega f(u)u\,dx + \int_{\Gamma_1} uy'\,d\Gamma - \int_{\Gamma_1} q(x)|y|^2\,d\Gamma.\end{aligned} \tag{4.4}$$

We know that

$$\left(k' * u\right)(t) = -\left(k'\circ u\right)(t) + u(t)\left[k(t) - k(0)\right]. \tag{4.5}$$

From (4.5), the boundary condition (2.10) can be written as

$$M(\|\nabla u\|^2)\frac{\partial u}{\partial\nu} + \frac{\partial u'}{\partial\nu} = -\tau\left\{u' + k(t)u - k'\circ u - k(t)u_0\right\}. \tag{4.6}$$

Applying Young's and Poincaré's inequalities, (2.11), (4.6) and (4.1) with $\alpha = \frac{1}{2}$, we obtain, for $\epsilon_1 > 0$,

$$\begin{aligned}&\int_\Gamma\left(M(\|\nabla u\|^2)\frac{\partial u}{\partial\nu} + \frac{\partial u'}{\partial\nu}\right)\left(m\cdot\nabla u + \left(\frac{n}{2}-\theta\right)u\right)d\Gamma \\ &\quad\le \epsilon_1\int_\Gamma\left(|m\cdot\nabla u|^2 + \left(\frac{n}{2}-\theta\right)^2|u|^2\right)d\Gamma + c_{\epsilon_1}\int_\Gamma\left|\left(M(\|\nabla u\|^2)\frac{\partial u}{\partial\nu} + \frac{\partial u'}{\partial\nu}\right)\right|^2 d\Gamma \\ &\quad\le \epsilon_1 c\int_\Gamma (m\cdot\nu)M(\|\nabla u\|^2)|\nabla u|^2\,d\Gamma + c_{\epsilon_1}\|y'\|^2_{\Gamma_1} \\ &\qquad + C\int_{\Gamma_0}\left(|u'|^2 + |k(t)u|^2 - k'\Box u + |k(t)u_0|^2\right)d\Gamma.\end{aligned} \tag{4.7}$$

By (2.1), (2.3) and Young's inequality, we get, for $\epsilon_2 > 0$,

$$\int_{\Gamma_1} uy'\,d\Gamma \le \epsilon_2 m_0 \|\nabla u\|^2 + \frac{c_{\epsilon_2}}{m_0\lambda_1}\|y'\|_{\Gamma_1}^2 \le \epsilon_2 M(\|\nabla u\|^2)\|\nabla u\|^2 + \frac{c_{\epsilon_2}}{m_0\lambda_1}\|y'\|_{\Gamma_1}^2 \tag{4.8}$$

and

$$\frac{1}{2}\int_{\Gamma}(m\cdot\nu)|u'|^2\,d\Gamma \le \frac{R}{2\lambda_1}\|\nabla u'\|^2. \tag{4.9}$$

Substituting (4.7)-(4.9) into (4.4) and using (2.4), we deduce that

$$\begin{aligned}\Phi'(t) \le{}& -\theta\|u'\|^2 - (1-\theta-\epsilon c-\epsilon_2)M(\|\nabla u\|^2)\|\nabla u\|^2 + \left(c_\epsilon + \frac{R}{2\lambda_1}\right)\|\nabla u'\|^2 \\ &- \left(\frac{n}{2}\delta - \delta\theta - 2\theta\right)\int_\Omega F(u)\,dx + C\int_{\Gamma_0}\left(|u'|^2 + |k(t)u|^2 - k'\Box u + |k(t)u_0|^2\right)d\Gamma \\ &+ \left(c_{\epsilon_1} + \frac{c_{\epsilon_2}}{m_0\lambda_1}\right)\|y'\|_{\Gamma_1}^2 - \int_{\Gamma_1} q(x)|y|^2\,d\Gamma \\ &- \left(\frac{1}{2} - \epsilon_1 c\right)\int_\Gamma (m\cdot\nu)M(\|\nabla u\|^2)|\nabla u|^2\,d\Gamma.\end{aligned}$$

Using (2.2) and choosing ϵ_1 small enough, we have estimate (4.3). □

Proof of Theorem 2.2 Let us introduce the Lyapunov functional

$$L(t) := NE(t) + \Phi(t)$$

with $N > 0$. From (4.2) and (4.3), we obtain, for all $t \ge t_0$,

$$\begin{aligned}L'(t) \le{}& -\theta\|u'\|^2 - (1-\theta-\epsilon c-\epsilon_2)M(\|\nabla u\|^2)\|\nabla u\|^2 - \left(\frac{n}{2}\delta - \delta\theta - 2\theta\right)\int_\Omega F(u)\,dx \\ &+ \left(\frac{\tau N}{2}k'(t) + Ck^2(t)\right)\|u\|_{\Gamma_0}^2 - \left(N - c_\epsilon - \frac{R}{2\lambda_1}\right)\|\nabla u'\|^2 - \left(\frac{\tau N}{2} - C\right)\|u'\|_{\Gamma_0}^2 \\ &+ \left(\frac{\tau N}{2} + C\right)k^2(t)\|u_0\|_{\Gamma_0}^2 - \frac{\tau N}{2}\int_{\Gamma_0} k''\Box u\,d\Gamma - C\int_{\Gamma_0} k'\Box u\,d\Gamma \\ &- \int_{\Gamma_1} q(x)|y|^2\,d\Gamma - \left(p_0 N - c_{\epsilon_1} - \frac{c_{\epsilon_2}}{m_0\lambda_1}\right)\|y'\|_{\Gamma_1}^2.\end{aligned}$$

We take θ, ϵ and $\epsilon_2 > 0$ so small that

$$\left(\frac{n}{2} - \theta\right)\delta - 2\theta > 0, \qquad 1 - \theta - \epsilon c - \epsilon_2 > 0.$$

And then, choosing N large, for some positive constant θ_0, we have

$$L'(t) \le -\theta_0 E(t) + \left(\frac{\tau N}{2} + C\right)k^2(t)\|u_0\|_{\Gamma_0}^2 - C\int_{\Gamma_0} k'\Box u\,d\Gamma,$$

which, using the fact that $\lim_{t\to\infty} k(t) = 0$, yields, for large t_0,

$$L'(t) \le -\theta_0 E(t) - C\int_{\Gamma_0} k'\Box u\, d\Gamma, \quad \forall t \ge t_0. \tag{4.10}$$

Meanwhile, we can choose N even larger so that

$$L(t) \sim E(t). \tag{4.11}$$

Therefore, from (2.19), (4.2) and (4.10), we get

$$\begin{aligned} L'(t) &\le -\theta_0 E(t) + \frac{C}{d_0}\int_0^{t_0} k''(s)\int_{\Gamma_0}\bigl|u(t)-u(t-s)\bigr|^2\, d\Gamma\, ds \\ &\quad - C\int_{t_0}^{t} k'(s)\int_{\Gamma_0}\bigl|u(t)-u(t-s)\bigr|^2\, d\Gamma\, ds \\ &\le -\theta_0 E(t) - \frac{2C}{d_0\tau}E'(t) - C\int_{t_0}^{t} k'(s)\int_{\Gamma_0}\bigl|u(t)-u(t-s)\bigr|^2\, d\Gamma\, ds, \quad \forall t \ge t_0. \end{aligned} \tag{4.12}$$

We take $\mathcal{L}(t) = L(t) + \frac{2C}{d_0\tau}E(t)$, which is clearly equivalent to $E(t)$. Then by (4.12) we arrive at

$$\mathcal{L}'(t) \le -\theta_0 E(t) - C\int_{t_0}^{t} k'(s)\int_{\Gamma_0}\bigl|u(t)-u(t-s)\bigr|^2\, d\Gamma\, ds. \tag{4.13}$$

(A) The special case $H(t) = ct^p$ and $1 \le p < \frac{3}{2}$:
Case 1. $p = 1$: From (2.8), (4.2) and (4.13), we have

$$\mathcal{L}'(t) \le -\theta_0 E(t) - \frac{2C}{c\tau}E'(t), \quad \forall t \ge t_0,$$

which yields

$$\left(\mathcal{L} + \frac{2C}{c\tau}E\right)'(t) \le -\theta_0 E(t), \quad \forall t \ge t_0.$$

By (4.11), we find that $\mathcal{L} + \frac{2C}{c\tau}E \sim E$. Then we easily obtain

$$E(t) \le c'e^{-ct} = c'G^{-1}(t),$$

where

$$G(t) = \int_t^1 \frac{1}{sH'(\epsilon_0 s)}\, ds = \int_t^1 \frac{1}{sc}\, ds = -\frac{\ln t}{c}.$$

Case 2. $1 < p < \frac{3}{2}$: We see that

$$\int_0^\infty \bigl(-k'(s)\bigr)^{1-\delta_0}\, ds < \infty \tag{4.14}$$

for any $\delta_0 < 2 - p$. Using (4.14) and taking t_0 even larger if needed, we get, for all $t \geq t_0$,

$$I(t) := \int_{t_0}^{t} \big(-k'(s)\big)^{1-\delta_0} \int_{\Gamma_0} \big|u(t) - u(t-s)\big|^2 \, d\Gamma \, ds \leq cE(0) \int_{t_0}^{t} \big(-k'(s)\big)^{1-\delta_0} \, ds < 1. \tag{4.15}$$

From Hölder's inequality, (2.8), (2.23), (4.2) and (4.15), we deduce that

$$\begin{aligned}
&\int_{t_0}^{t} \big(-k'(s)\big) \int_{\Gamma_0} \big|u(t) - u(t-s)\big|^2 \, d\Gamma \, ds \\
&\quad = \int_{t_0}^{t} \big(-k'(s)\big)^{(p-1+\delta_0)(\frac{\delta_0}{p-1+\delta_0})} \big(-k'(s)\big)^{1-\delta_0} \int_{\Gamma_0} \big|u(t) - u(t-s)\big|^2 \, d\Gamma \, ds \\
&\quad \leq \left(\int_{t_0}^{t} \big(-k'(s)\big)^{p-1+\delta_0} \big(-k'(s)\big)^{1-\delta_0} \int_{\Gamma_0} \big|u(t) - u(t-s)\big|^2 \, d\Gamma \, ds \right)^{\frac{\delta_0}{p-1+\delta_0}} \\
&\qquad \times \left(\int_{t_0}^{t} \big(-k'(s)\big)^{1-\delta_0} \int_{\Gamma_0} \big|u(t) - u(t-s)\big|^2 \, d\Gamma \, ds \right)^{\frac{p-1}{p-1+\delta_0}} \\
&\quad = I(t) \left(\frac{1}{I(t)} \int_{t_0}^{t} \big(-k'(s)\big)^{p-1+\delta_0} \big(-k'(s)\big)^{1-\delta_0} \int_{\Gamma_0} \big|u(t) - u(t-s)\big|^2 \, d\Gamma \, ds \right)^{\frac{\delta_0}{p-1+\delta_0}} \\
&\quad \leq \left(\int_{t_0}^{t} \big(-k'(s)\big)^{p} \int_{\Gamma_0} \big|u(t) - u(t-s)\big|^2 \, d\Gamma \, ds \right)^{\frac{\delta_0}{p-1+\delta_0}} \\
&\quad \leq \left(\frac{1}{c} \right)^{\frac{\delta_0}{p-1+\delta_0}} \big(-E'(t)\big)^{\frac{\delta_0}{p-1+\delta_0}}.
\end{aligned} \tag{4.16}$$

Hence, by (4.16), estimate (4.13) yields, for $\delta_0 = \frac{1}{2}$,

$$\mathcal{L}'(t) \leq -\theta_0 E(t) + \frac{C}{c^{\frac{1}{2p-1}}} \big(-E'(t)\big)^{\frac{1}{2p-1}}. \tag{4.17}$$

Multiplying (4.17) by $E^{\gamma}(t)$, with $\gamma = 2p - 2$, and using Young's inequality, we have

$$\big(\mathcal{L}E^{\gamma}\big)'(t) \leq -\theta_0 E^{\gamma+1}(t) + \frac{C}{c^{\frac{1}{\gamma+1}}} E^{\gamma}(t) \big(-E'(t)\big)^{\frac{1}{\gamma+1}} \leq -\theta_0 E^{\gamma+1}(t) + \varepsilon E^{\gamma+1}(t) + C_{\varepsilon}\big(-E'(t)\big),$$

where we have used the fact $E'(t) \leq 0$, $\forall t \geq t_0$. Then, taking $\varepsilon < \theta_0$, we obtain, for some $C_1 > 0$,

$$L_0'(t) \leq -C_1 L_0^{\gamma+1}(t),$$

where $L_0 = \mathcal{L}E^{\gamma} + C_{\varepsilon}E \sim E$. Hence we get

$$E(t) \leq \frac{c}{(c' + c''t)^{\frac{1}{\gamma}}}. \tag{4.18}$$

On the other hand, using (4.18), we have, for $p < \frac{3}{2}$,

$$\int_{0}^{t} \int_{\Gamma_0} \big|u(t) - u(t-s)\big|^2 \, d\Gamma \, ds \leq c \int_{0}^{t} E(s) \, ds \leq \int_{0}^{\infty} \frac{c}{(c' + c''t)^{\frac{1}{2p-2}}} \, dt < +\infty. \tag{4.19}$$

Therefore, by Hölder's inequality, (2.8), (4.2) and (4.19), estimate (4.13) yields

$$\begin{aligned}
\mathcal{L}'(t) &\le -\theta_0 E(t) + C\left(\int_0^t \int_{\Gamma_0} \left|u(t)-u(t-s)\right|^2 \,d\Gamma\,ds\right)^{\frac{p-1}{p}} \left(\left(-k'(s)\right)^p \Box u\right)^{\frac{1}{p}} \\
&\le -\theta_0 E(t) + c\left(k''\Box u\right)^{\frac{1}{p}} \le -\theta_0 E(t) + c\left(-E'(t)\right)^{\frac{1}{p}}.
\end{aligned} \tag{4.20}$$

Then, multiplying (4.20) by $E^\gamma(t)$ with $\gamma = p-1$ and repeating the above steps, we see that

$$E(t) \le \frac{c}{(c' + c''t)^{\frac{1}{\gamma}}} = cG^{-1}\left(a' + a''t\right),$$

where

$$G(t) = \frac{1}{cp\epsilon_0^{p-1}} \int_t^1 \frac{1}{s^p}\,ds = \frac{1}{cp(p-1)\epsilon_0^{p-1}}\left(\frac{1}{t^{p-1}} - 1\right).$$

(B) The general case: Because of the ideas presented in [30, 31, 33], this case is obtained as follows. We define $\eta(t)$ by

$$\eta(t) := \int_{t_0}^t \frac{-k'(s)}{H_0^{-1}(k''(s))} \int_{\Gamma_0} \left|u(t)-u(t-s)\right|^2 \,d\Gamma\,ds,$$

where H_0 satisfies (2.15). Like in (4.15), we see that $\eta(t)$ satisfies

$$\eta(t) < 1, \quad \forall t \ge t_0. \tag{4.21}$$

Moreover, we define $\kappa(t)$ by

$$\kappa(t) := \int_{t_0}^t k''(s)\frac{-k'(s)}{H_0^{-1}(k''(s))} \int_{\Gamma_0} \left|u(t)-u(t-s)\right|^2 \,d\Gamma\,ds.$$

Because $H_0(0) = 0$ and H_0 is strictly convex on $(0, r]$, then

$$H_0(\lambda x) \le \lambda H_0(x) \tag{4.22}$$

provided $0 \le \lambda \le 1$ and $x \in (0, r]$. From (2.23), (4.21) and (4.22), we get

$$\int_{t_0}^t -k'(s) \int_{\Gamma_0} \left|u(t)-u(t-s)\right|^2 \,d\Gamma\,ds \le H_0^{-1}\left(\kappa(t)\right). \tag{4.23}$$

Indeed,

$$\begin{aligned}
\kappa(t) &= \frac{1}{\eta(t)} \int_{t_0}^t \eta(t) H_0\left[H_0^{-1}\left(k''(s)\right)\right] \frac{-k'(s)}{H_0^{-1}(k''(s))} \int_{\Gamma_0} \left|u(t)-u(t-s)\right|^2 \,d\Gamma\,ds \\
&\ge \frac{1}{\eta(t)} \int_{t_0}^t H_0\left[\eta(t) H_0^{-1}\left(k''(s)\right)\right] \frac{-k'(s)}{H_0^{-1}(k''(s))} \int_{\Gamma_0} \left|u(t)-u(t-s)\right|^2 \,d\Gamma\,ds \\
&\ge H_0\left(\int_{t_0}^t -k'(s) \int_{\Gamma_0} \left|u(t)-u(t-s)\right|^2 \,d\Gamma\,ds\right).
\end{aligned}$$

Then by (4.23) estimate (4.13) yields

$$\mathcal{L}'(t) \le -\theta_0 E(t) + CH_0^{-1}\big(\kappa(t)\big), \quad \forall t \ge t_0. \tag{4.24}$$

Now, for $\epsilon_0 < r$ and $\alpha_0 > 0$, we define the functional

$$F_1(t) := H_0'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{L}(t) + \alpha_0 E(t),$$

which satisfies, for some $\alpha_1, \alpha_2 > 0$,

$$\alpha_1 F_1(t) \le E(t) \le \alpha_2 F_1(t). \tag{4.25}$$

By using a similar analysis as in [30, 31], we can compute to find

$$F_1'(t) \le -\big(\theta_0 E(0) - \epsilon_0 C\big)\frac{E(t)}{E(0)} H_0'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) - CE'(t) + \alpha_0 E'(t).$$

Therefore, with a suitable choice of ϵ_0 and α_0, we have, for all $t \ge t_0$,

$$F_1'(t) \le -\alpha_3\left(\frac{E(t)}{E(0)}\right) H_0'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) = -\alpha_3 H_2\left(\frac{E(t)}{E(0)}\right), \tag{4.26}$$

where $\alpha_3 > 0$ and $H_2(t) = tH_0'(\epsilon_0 t)$. From

$$H_2'(t) = H_0'(\epsilon_0 t) + \epsilon_0 t H_0''(\epsilon_0 t)$$

and the strict convexity of H_0 on $(0, r]$, we find that $H_2'(t), H_2(t) > 0$ on $(0, 1]$. We denote

$$J(t) = \frac{\alpha_1 F_1(t)}{E(0)},$$

which is clearly equivalent to $E(t)$. From (4.25) and (4.26), we obtain

$$J'(t) \le -\frac{\alpha_1 \alpha_3}{E(0)} H_2\left(\frac{E(t)}{E(0)}\right) \le -k_0 H_2\big(J(t)\big), \quad \forall t \ge t_0,$$

where $k_0 = \frac{\alpha_1 \alpha_3}{E(0)} > 0$. Consequently, a simple integration gives, for some $k_1, k_2 > 0$,

$$J(t) \le H_1^{-1}(k_1 t + k_2), \quad \forall t \ge t_0, \tag{4.27}$$

where $H_1(t) = \int_t^1 \frac{1}{H_2(s)}\, ds$. Here, we have used the properties of H_2 and the fact that $\lim_{t\to 0} H_1(t) = +\infty$ and H_1 is a strictly decreasing function on $(0, 1]$. Using (4.27), we see that (2.16) holds.

Furthermore, if $\int_0^t H_1(t)\, dt < +\infty$, then $\int_0^{+\infty} H_1^{-1}(t)\, dt < +\infty$. From (2.16), we get $\int_0^{+\infty} E(t)\, dt < \infty$ and

$$\int_0^t \int_{\Gamma_0} \big|u(t) - u(t-s)\big|^2 \, d\Gamma\, ds \le c \int_0^t E(s)\, ds < +\infty.$$

Analogously, we define, for large t_0,

$$\xi(t) := \int_{t_0}^{t} \int_{\Gamma_0} \left| u(t) - u(t-s) \right|^2 d\Gamma\, ds < 1$$

and

$$\chi(t) := \int_{t_0}^{t} k''(s) \int_{\Gamma_0} \left| u(t) - u(t-s) \right|^2 d\Gamma\, ds.$$

Using (2.8), (2.23) and the strict convexity of H, we find that

$$\int_{t_0}^{t} \bigl(-k'(s)\bigr) \int_{\Gamma_0} \left| u(t) - u(t-s) \right|^2 d\Gamma\, ds \le H^{-1}\bigl(\chi(t)\bigr). \tag{4.28}$$

Indeed,

$$\begin{aligned}
\chi(t) &\ge \frac{1}{\xi(t)} \int_{t_0}^{t} \xi(t) H\bigl(-k'(s)\bigr) \int_{\Gamma_0} \left| u(t) - u(t-s) \right|^2 d\Gamma\, ds \\
&\ge \frac{1}{\xi(t)} \int_{t_0}^{t} H\bigl(-\xi(t) k'(s)\bigr) \int_{\Gamma_0} \left| u(t) - u(t-s) \right|^2 d\Gamma\, ds \\
&\ge H\biggl(\int_{t_0}^{t} -k'(s) \int_{\Gamma_0} \left| u(t) - u(t-s) \right|^2 d\Gamma\, ds \biggr).
\end{aligned}$$

Hence, by (4.28), estimate (4.13) becomes

$$\mathcal{L}'(t) \le -\theta_0 E(t) + C H^{-1}\bigl(\chi(t)\bigr), \quad \forall t \ge t_0.$$

Consequently, repeating the same procedures, we deduce that for some c_1, c_2 and $c_3 > 0$,

$$E(t) \le c_3 G^{-1}(c_1 t + c_2),$$

where $G(t) = \int_t^1 \frac{1}{sH'(\epsilon_0 s)}\, ds$. □

Examples We give some examples to explain the energy decay rates given by Theorem 2.2.

(1) As in [30], let $0 < q < 1$

$$k'(t) = -\exp\bigl(-t^q\bigr),$$

then $k''(t) = H(-k'(t))$, where $H(t) = \frac{qt}{[\ln(1/t)]^{\frac{1}{q}-1}}$ for $t \in (0, r]$, $r < 1$. Therefore,

$$E(t) \le c \exp\bigl(-\omega t^q\bigr).$$

(2) As in [31], if

$$k'(t) = -\frac{1}{a + t^q}$$

for $q > 3$ and $a > 1$, then $k''(t) = H(-k'(t))$, where

$$H(t) = qt^2\left(\frac{1}{t} - a\right)^{1-\frac{1}{q}}.$$

Since

$$H'(t) = \frac{q(1 + \frac{1}{q} - 2at)}{(\frac{1}{t} - a)^{\frac{1}{q}}}, \qquad H''(t) = \frac{\frac{2a^2q}{t^2}(t - \frac{1+q-\sqrt{q^2-1}}{2aq})(t - \frac{1+q+\sqrt{q^2-1}}{2aq})}{(\frac{1}{t} - a)^{1+\frac{1}{q}}},$$

then the function H satisfies hypothesis (H4) on the interval $(0, r]$ for any $0 < r < \frac{1+q-\sqrt{q^2-1}}{2aq}$. By taking $D(t) = t^\alpha$, (2.15) is satisfied for any $\alpha > \frac{q}{q-1}$. Then an explicit rate of decay can be obtained by Theorem 2.2. The function $H_0(t) = H(t^\alpha)$ has derivative

$$H_0'(t) = \frac{q\alpha t^{\alpha-1}[1 + \frac{1}{q} - 2at^\alpha]}{(\frac{1}{t^\alpha} - a)^{\frac{1}{q}}}.$$

Therefore,

$$H_1(t) = \int_t^1 \frac{1}{sH_0'(\epsilon_0 s)}\,ds = \int_t^1 \frac{[\frac{1}{(\epsilon_0 s)^\alpha} - a]^{\frac{1}{q}}}{q\alpha s(\epsilon_0 s)^{\alpha-1}[1 + \frac{1}{q} - 2a(\epsilon_0 s)^\alpha]}\,ds.$$

Now, we see that if $\alpha < \frac{2q}{1+q}$,

$$\begin{aligned}\int_0^1 H_1(t)\,dt &\leq \frac{\epsilon_0^{\frac{q-\alpha-\alpha q}{q}}}{\alpha(\alpha - q + \alpha q)[1 + \frac{1}{q} - 2a\epsilon_0^\alpha]}\int_0^1 \left[t^{\frac{q-\alpha-\alpha q}{q}} - 1\right]dt \\ &= \frac{\epsilon_0^{\frac{q-\alpha-\alpha q}{q}}}{\alpha(2q - \alpha - \alpha q)[1 + \frac{1}{q} - 2a\epsilon_0^\alpha]} < +\infty.\end{aligned}$$

Choosing $\frac{1}{\epsilon_0 s} = \nu$ and $\epsilon_0 < a^{-1}$, we have

$$\begin{aligned}G(t) &= \int_t^1 \frac{1}{sH'(\epsilon_0 s)}\,ds = \int_t^1 \frac{(\frac{1}{\epsilon_0 s} - a)^{\frac{1}{q}}}{sq(1 + \frac{1}{q} - 2a\epsilon_0 s)}\,ds = \int_{\frac{1}{\epsilon_0}}^{\frac{1}{\epsilon_0 t}} \frac{(\nu - a)^{\frac{1}{q}}\nu^{-1}}{q(1 + \frac{1}{q} - \frac{2a}{\nu})}\,d\nu \\ &\leq \frac{1}{q(1 + \frac{1}{q} - 2a\epsilon_0)}\int_{\frac{1}{\epsilon_0}}^{\frac{1}{\epsilon_0 t}} \nu^{\frac{1}{q}-1}\,d\nu = \frac{1}{1 + \frac{1}{q} - 2a\epsilon_0}\left[\left(\frac{1}{\epsilon_0 t}\right)^{\frac{1}{q}} - \left(\frac{1}{\epsilon_0}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Hence,

$$G^{-1}(t) \leq \frac{1}{\epsilon_0[(\frac{1}{\epsilon_0})^{\frac{1}{q}} + (1 + \frac{1}{q} - 2a\epsilon_0)t]^q}.$$

Consequently, we can use (2.17) to conclude that the energy decays

$$E(t) \leq \frac{\tilde{c}_1}{\tilde{c}_2 + \tilde{c}_3 t^q},$$

where $\tilde{c}_i$ $(i = 1, 2, 3)$ are constants.

Competing interests
The author declares that she has no competing interests.

Author's contributions
The work was realized by the author.

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