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Dynamic behaviors of a local modified stochastic Swift-Hohenberg equation with multiplicative noise

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Abstract
In this paper, we investigate a global random attractor for a stochastic local modified Swift-Hohenberg equation with multiplicative noise in Stratonovich sense. Through the Ornstein-Uhlenbeck (O-U) transformation, we obtain the random dynamical system associated with the stochastic local modified Swift-Hohenberg equation. Using the properties of the O-U process, we derive the specific uniform *a priori* estimates, using which we prove the existence of global random attractor for the corresponding random dynamical system.

Keywords: local modified stochastic Swift-Hohenberg equation; uniform *a priori* estimates; random attractor

1 Introduction

Swift and Hohenberg [1] proposed a model for the convective instability in the Rayleigh-Bénard convection, also known as the Swift-Hohenberg (S-H) equation, which is included as an important equation in different branches of physics, such as Taylor-Couette flow [2, 3], the study of lasers [4], and so on. After that, Doelman and Standstede [5] proposed the following modified Swift-Hohenberg equation for a pattern formation system near the onset to instability:

$$u_t + \Delta^2 u + 2\Delta u + \alpha u + b|\nabla u|^2 + u^3 = 0, \tag{1.1}$$

where α and b are arbitrary constants. In the case of $b = 0$, it is the usual Swift-Hohenberg equation. The additional term $b|\nabla u|^2$ arises in the study of various pattern formation phenomena involving some kind of phase turbulence or phase transition that breaks the symmetry $u \to -u$. For more references, one can see [6–8] and the references therein. The dynamical properties of the S-H equation are important for the studies of pattern formation and global attractors, and the stability of stationary solution and pattern selections of the S-H equation have been extensively investigated; see [9–11].

It turns out that a stochastic equation can conform to physical phenomena better in some cases. These random perturbations are intrinsic effects in a variety of settings and spatial scales. In fact, when the Rayleigh number is near thermal equilibrium, the influence of small noise or molecular noise is detected in various convection experiments [12, 13].

As the effect of thermal fluctuations on the onset of convective motion into the Bénard system is considered, the stochastic local S-H equation with additive noise [1] is proposed:

$$u_t = \mu u - (1 + \partial_{xx})^2 u - u^3 + \sigma\xi. \tag{1.2}$$

Furthermore, a local stochastic S-H equation driven by multiplicative noise arises when the effects of small possible noise from μ is considered [14]:

$$u_t = \mu u - (1 + \partial_{xx})^2 u - u^3 + \sigma u \circ \xi, \tag{1.3}$$

where $\sigma > 0$, and $\xi = \frac{dW}{dt}$ is the generalized derivative of a real-valued Brownian motion.

There have been a lot of outstanding work and important results related to the existence and uniqueness of solution and attractors for stochastic partial differential equations. For research progress on these aspects, we refer to [15–19]. Until now, there are few results on the dynamics behaviors of the stochastic local modified Swift-Hohenberg equation with multiplicative noise in Stratonovich sense. This is our main purpose. After making use of the O-U transform and changing the stochastic equation into the corresponding deterministic equation with random parameter, we obtain uniform *a priori* estimates under some additional assumptions and prove the existence of global random attractor for the random dynamical system associated with the stochastic local modified Swift-Hohenberg equation. It allows us to overcome the computational difficulties according to the properties of the local modified Swift-Hohenberg equation. In particular, showing the existence of a random attractor needs a lot of technical skills to obtain the desired results.

In this paper, we consider the following one-dimensional stochastic local modified Swift-Hohenberg equation with multiplicative noise:

$$du + \left(\Delta^2 u + 2\Delta u + \alpha u + b|u_x|^2 - u^3\right) dt = \sigma u \circ dW(t) \tag{1.4}$$

with initial condition

$$u(x,0) = u_0(x), \quad x \in D, \tag{1.5}$$

and the boundary conditions

$$u|_{\partial D} = \Delta u|_{\partial D} = 0, \quad x \in \partial D, \tag{1.6}$$

where D is a bounded open interval, $|b| \ll 1$ is a constant, Δu means u_{xx}, and $\Delta^2 u$ means u_{xxxx}.

An outline of this paper is as follows. We devote Section 2 to recall some definitions and results referred to global random attractors and to present some notation. In Section 3, we not only introduce the O-U transformer, but also obtain uniform *a priori* estimates of the solution for the stochastic local modified Swift-Hohenberg equation. Finally, the proof of the main theorem on the existence of global random attractor is presented in Section 4.

2 Preliminaries

There are many research results on random attractors and related issues. For simplicity of the structure of the article, we only list the definitions; for the relevant theorems, we refer to [20, 21], and so on. Let $(X, \|\cdot\|_X)$ be a completely separable Hilbert space with Borel σ-algebra $\mathscr{B}(X)$, and let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be an ergodic metric dynamical system.

Definition 2.1 (See [15, 16]) A measurable mapping

$$\varphi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x),$$

has the cocycle property:

(1) $\varphi(0, \omega, x)$ is the identity mapping on X;

(2) $\varphi(t + s, \omega, x) = \varphi(t, \theta_s\omega, \varphi(s, \omega, x))$ for all $s, t \geq 0$, $x \in X$, and $\omega \in \Omega$.

We call φ a random dynamical system (RDS) on X over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$. Furthermore, the RDS φ is continuous if $\varphi(t, \omega, \cdot) = \varphi(t, \omega) : X \to X$ is continuous for all $t \geq 0$.

Definition 2.2 (See [15, 16]) A random compact set $\mathcal{A}(\omega)$ is said to be a random attractor for RDS φ if the following conditions hold:

(1) pullback attracting property:

$$\lim_{t\to+\infty} d\big(\varphi\big(t, \theta_{-t}\omega, \mathscr{D}(\theta_{-t}\omega)\big); \mathcal{A}(\omega)\big) = 0,$$

(2) the invariance property: $\varphi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all $t \geq 0$.

Remark Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with Wiener measure $\mathbb{P}$. The Wiener shift $(\theta_t)_{t\in\mathbb{R}}$ is defined by

$$\theta_s\omega(t) = \omega(t + s) - \omega(s), \quad t, s \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is an ergodic metric dynamical system; see [20].

For the convenience of the following contents, we introduce some functional spaces and some notations. $L^q(D)$ is the Lebesgue space with norm $\|\cdot\|_{L^q}$, and $\|\cdot\|_{L^2} = \|\cdot\|$. Particularly, $\|u\|_{L^\infty} = \operatorname{ess\,sup}_{x\in D} |u(x)|$ for $q = \infty$.

$H^\sigma(D)$ is the Sobolev space $\{u \in L^2(D), D^k u \in L^2(D), k \leq \sigma\}$ with norm $\|\cdot\|_{H^\sigma} = \|\cdot\|_\sigma$. Especially, $H_0^2(D)$ is the Sobolev space $\{u \in L^2(D), D^k u \in L^2(D), k \leq 2, \frac{\partial u}{\partial n}|_{x\in\partial D} = 0\}$.

For notational simplicity, C is a generic constant that may take various values from line to line; Δv means v_{xx}, and $\Delta^2 v$ means v_{xxxx}.

3 Uniform *a priori* estimates of solution

In this section, we mainly show uniform *a priori* estimates of a solution for the stochastic local modified Swift-Hohenberg equation.

The original equation (1.4) can be rewritten as follows:

$$u(t) = u_0 - \int_0^t \big(\alpha u + 2\Delta u + \Delta^2 u + b|u_x|^2 - u^3\big)\, ds + \sigma \int_0^t u(s) \circ dW(s).$$

We now introduce an O-U process $z(\theta_t\omega)$ that solves the following Itô equation:

$$dz + z\, dt = \sigma\, dW(t).$$

By [20, 22], the random variable $z(\theta_t\omega)$ is tempered, and for every $\omega \in \tilde{\Omega}$, $t \mapsto z(\theta_t\omega)$ is continuous with respect to t. Especially, the properties $\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0$ and $\lim_{t\to\pm\infty} \frac{\int_0^t z(\theta_s\omega)\,ds}{t} = 0$ hold.

Let $v = e^{-z(\theta_t\omega)}u(t)$. Using the Itô equation, combined with the original equation (1.4), we get

$$\begin{aligned} dv(t) &= e^{-z(\theta_t\omega)}\,du(t) - u(t)e^{-z(\theta_t\omega)} \circ d\big(z(\theta_t\omega)\big) \\ &= -\big(\Delta^2 v + 2\Delta v + \alpha v + e^{2z(\theta_t\omega)}v^3 + be^{z(\theta_t\omega)}v_x^2\big)\,dt + z(\theta_t\omega)v(t)\,dt. \end{aligned}$$

Thus, we consider the following system:

$$\frac{dv}{dt} + \big(\alpha - z(\theta_t\omega)\big)v + 2\Delta v + \Delta^2 v + e^{2z(\theta_t\omega)}v^3 + be^{z(\theta_t\omega)}v_x^2 = 0, \tag{3.1}$$

$$v(x,0) = e^{-z(\omega)}u_0, \tag{3.2}$$

$$v|_{\partial D} = \Delta v|_{\partial D} = 0. \tag{3.3}$$

Similarly to [8, 23], by the Galerkin method and some *a priori* estimates we can prove that $v(t,\omega,v_0)$ is unique and continuous with respect to initial value v_0 in $H_0^2(D)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, where $v(t,\omega,v_0)$ is the solution of system (3.1)-(3.3). Define the continuous random dynamical system $\{\psi(t)\}_{t\geq 0}$ by

$$\psi(t,\omega,v_0) = v(t,\omega,v_0)$$

for all $v_0 \in H_0^2(D)$, $t \geq 0$, and $\omega \in \Omega$. Furthermore, setting $u(t,\omega,u_0) = e^{z(\theta_t\omega)}v(t,\omega,v_0)$, we have

$$\phi(t,\omega,u_0) = u(t,\omega,u_0) = e^{z(\theta_t\omega)}v(t,\omega,v_0) = e^{z(\theta_t\omega)}\psi(t,\omega,v_0).$$

Then ϕ is a continuous random dynamical system on $H_0^2(D)$. It is straightforward to show that the existence of a random attractor for ϕ is equivalent to the existence of a random attractor for ψ.

In the following, provided that $\mathscr{D}$ is a collection of tempered random subsets of $H_0^2(D)$, we will prove the existence of an absorbing set in $H_0^2(D)$.

Lemma 3.1 *Provided that $v_0 \in B = \{B(\omega)\}_\omega \subset \mathscr{D}$, there exist a random radius $\rho_1(\omega) > 0$ and $T_{1B}(\omega) > 0$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ such that*

$$\big\|v\big(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\big\|^2 \leq \rho_1(\omega), \quad t > T_{1B}(\omega). \tag{3.4}$$

Proof Taking the inner product of equation (3.1) with v, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v\|^2 + \big(\alpha - z(\theta_t\omega)\big)\big\|v(t)\big\|^2 + (2\Delta v, v) + \|\Delta v\|^2 + \big(e^{2z(\theta_t\omega)}v^3, v\big) \\ &\quad + \big(be^{z(\theta_t\omega)}v_x^2, v\big) = 0. \end{aligned} \tag{3.5}$$

Noticing that $e^{2z(\theta_t\omega)}(v^3, v) = e^{2z(\theta_t\omega)}\|v\|_{L^4}^4$ and applying the Hölder inequality and ϵ-Young inequality, we get

$$\left|(2\Delta v, v)\right| \le \frac{1}{4}\|\Delta v\|^2 + 4\|v\|^2.$$

Now, we deal with the last term on the left side of equation (3.5). By integration by parts we have

$$\int_D b e^{z(\theta_t\omega)} v_x^2 v\,dx = -b e^{z(\theta_t\omega)}\int_D v(v_x v)_x\,dx = -b e^{z(\theta_t\omega)}\int_D \left(v^2 v_{xx} + v v_x^2\right)dx,$$

and thus

$$b e^{z(\theta_t\omega)}\int_D v_x^2 v\,dx = -\frac{b}{2}e^{z(\theta_t\omega)}\int_D v^2 v_{xx}\,dx.$$

Applying the Hölder inequality and ϵ-Young inequality again, we get

$$\left|b e^{z(\theta_t\omega)}\int_D v_x^2 v\,dx\right| \le bCe^{z(\theta_t\omega)}\|v\|_{L^4}^2\|v_{xx}\| \le \frac{1}{4}\|v_{xx}\|^2 + b^2Ce^{2z(\theta_t\omega)}\|v\|_{L^4}^4.$$

Taking b small enough such that $b^2C \le \frac{1}{2}$, we obtain

$$\left|b e^{z(\theta_t\omega)}\int_D v_x^2 v\,dx\right| \le \frac{1}{4}\|v_{xx}\|^2 + \frac{1}{2}e^{2z(\theta_t\omega)}\|v\|_{L^4}^4.$$

Putting all these inequalities together, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v\|^2 + \frac{1}{2}\|\Delta v\|^2 + \left(\beta - z(\theta_t\omega)\right)\|v\|^2 + (\alpha - \beta - 4)\|v\|^2\\&\quad + \frac{1}{2}e^{2z(\theta_t\omega)}\|v\|_{L^4}^4 \le 0,\end{aligned} \tag{3.6}$$

where $\beta > 0$ is a constant such that $\alpha - \beta - 4 < 0$.

By the Sobolev imbedding $L^4(D) \subset L^2(D)$ we get

$$\begin{aligned}&\frac{d}{dt}\|v\|^2 + 2\left(\beta - z(\theta_t\omega)\right)\|v\|^2 + \|\Delta v\|^2\\&\quad \le -e^{2z(\theta_t\omega)}\|v\|_{L^4}^4 - 2C(\alpha - \beta - 4)\|v\|_{L^4}^2.\end{aligned}$$

We can change the right-hand side of this inequality as follows:

$$\begin{aligned}&-e^{2z(\theta_t\omega)}\|v\|_{L^4}^4 - 2C(\alpha - \beta - 4)\|v\|_{L^4}^2\\&\quad = -e^{2z(\theta_t\omega)}\left(\|v\|_{L^4}^2 + Ce^{-2z(\theta_t\omega)}(\alpha - \beta - 4)\right)^2 + C(\alpha - \beta - 4)^2e^{-2z(\theta_t\omega)}.\end{aligned}$$

Then we have

$$\frac{d}{dt}\|v\|^2 + 2\left(\beta - z(\theta_t\omega)\right)\|v\|^2 + \|\Delta v\|^2 \le C(\alpha - \beta - 4)^2e^{-2z(\theta_t\omega)}. \tag{3.7}$$

By the Gronwall inequality we have

$$\begin{aligned}\big\|v\big(t,\omega,v_0(\omega)\big)\big\|^2 &\le e^{-2\beta t+\int_0^t 2z(\theta_\tau\omega)\,d\tau}\big\|v_0(\omega)\big\|^2\\ &\quad + C(\alpha-\beta-4)^2\int_0^t e^{-2z(\theta_s\omega)-2\beta(t-s)+\int_s^t 2z(\theta_\tau\omega)\,d\tau}\,ds. \end{aligned}\tag{3.8}$$

Furthermore, replacing ω with $\theta_{-t}\omega$ in (3.8), then we have

$$\begin{aligned}&\big\|v\big(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\big\|^2\\ &\quad\le e^{-2t(\beta-\frac{\int_{-t}^0 2z(\theta_\tau\omega)\,d\tau}{t})}\big\|v_0(\theta_{-t}\omega)\big\|^2\\ &\qquad + C(\alpha-\beta-4)^2\int_0^t e^{-2z(\theta_{s-t}\omega)-2\beta(t-s)+\int_{s-t}^0 2z(\theta_\tau\omega)\,d\tau}\,ds\\ &\quad= e^{-2t(\beta-\frac{\int_{-t}^0 2z(\theta_\tau\omega)\,d\tau}{t})}\big\|v_0(\theta_{-t}\omega)\big\|^2 + C(\alpha-\beta-4)^2\int_{-t}^0 e^{-2z(\theta_s\omega)+2\beta s+\int_s^0 2z(\theta_\tau\omega)\,d\tau}\,ds\\ &\quad\le e^{-2t(\beta-\frac{\int_{-t}^0 2z(\theta_\tau\omega)\,d\tau}{t})}\big\|v_0(\theta_{-t}\omega)\big\|^2\\ &\qquad + C(\alpha-\beta-4)^2\int_{-\infty}^0 e^{2s(\beta-\frac{z(\theta_s\omega)}{s}+\frac{\int_s^0 z(\theta_\tau\omega)\,d\tau}{s})}\,ds. \end{aligned}\tag{3.9}$$

Because of the properties of $z(\theta_t\omega)$, there exists $T_{1B}(\omega)>0$ such that, for all $t\ge T_{1B}(\omega)$,

$$\frac{\int_{-t}^0 2z(\theta_\tau\omega)\,d\tau}{t}\le\frac{\beta}{2}.$$

It follows that

$$e^{-2t(\beta-\frac{\int_{-t}^0 2z(\theta_\tau\omega)\,d\tau}{t})}\big\|v_0(\theta_{-t}\omega)\big\|^2\le e^{-\beta t}\big\|v_0(\theta_{-t}\omega)\big\|^2.$$

The random set $\mathscr{D}$ is tempered, which implies the boundedness of the first term on the right-hand side of (3.9). The second term on the right-hand side of (3.9) is convergent.

Thus, there exist $T_{1B}(\omega)>0$ and a random variable $\rho_1(\omega)$ such that, for $\mathbb{P}$-a.e. $\omega\in\Omega$ and all $t>T_{1B}(\omega)$,

$$\big\|v\big(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\big\|^2\le\rho_1(\omega).$$

□

Lemma 3.2 *Provided that $v_0\in B=\{B(\omega)\}_\omega\subset\mathscr{D}$, there exist a random radius $\rho_2(\omega)>0$ and $T_{2B}(\omega)>0$ for $\mathbb{P}$-a.e. $\omega\in\Omega$ such that*

$$\int_t^{t+1}\big\|\Delta v\big(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big)\big\|^2\,ds\le\rho_2(\omega),\quad t>T_{2B}(\omega).\tag{3.10}$$

Proof Multiplying equation (3.7) by $e^{2\beta t-\int_0^t 2z(\theta_\tau\omega)\,d\tau}$, we get

$$\begin{aligned}&\frac{d}{dt}\big[e^{2\beta t-\int_0^t 2z(\theta_\tau\omega)\,d\tau}\|v\|^2\big]+e^{2\beta t-\int_0^t 2z(\theta_\tau\omega)\,d\tau}\|\Delta v\|^2\\ &\quad\le C(\alpha-\beta-4)^2e^{-2z(\theta_t\omega)+2\beta t-\int_0^t 2z(\theta_\tau\omega)\,d\tau}. \end{aligned}\tag{3.11}$$

Set $\widehat{T} \le t \le t+1$. Integrating from $\widehat{T}$ to t, we have

$$\begin{aligned}
&e^{2\beta t-\int_0^t 2z(\theta_\tau\omega)\,d\tau}\left\|v(t)\right\|^2+\int_{\widehat{T}}^t e^{2\beta s-\int_0^s 2z(\theta_\tau\omega)\,d\tau}\left\|\Delta v(s)\right\|^2 ds\\
&\quad\le C(\alpha-\beta-4)^2\int_{\widehat{T}}^t e^{-2z(\theta_s\omega)+2\beta s-\int_0^s 2z(\theta_\tau\omega)\,d\tau}\,ds+e^{2\beta\widehat{T}-\int_0^{\widehat{T}} 2z(\theta_\tau\omega)\,d\tau}\left\|v\left(\widehat{T},\omega,v_0(\omega)\right)\right\|^2.
\end{aligned}$$

Multiplying this inequality by $e^{-2\beta t+\int_0^t 2z(\theta_\tau\omega)\,d\tau}$ and getting rid of the first term, we obtain

$$\begin{aligned}
&\int_{\widehat{T}}^t e^{2\beta(s-t)+\int_s^t 2z(\theta_\tau\omega)\,d\tau}\left\|\Delta v(s)\right\|^2 ds\\
&\quad\le e^{2\beta(\widehat{T}-t)+\int_{\widehat{T}}^t 2z(\theta_\tau\omega)\,d\tau}\left\|v\left(\widehat{T},\omega,v_0(\omega)\right)\right\|^2\\
&\qquad+C(\alpha-\beta-4)^2\int_{\widehat{T}}^t e^{-2z(\theta_s\omega)+2\beta(s-t)+\int_s^t 2z(\theta_\tau\omega)\,d\tau}\,ds.
\end{aligned}\tag{3.12}$$

Now, substituting t for $\widehat{T}$ in (3.8), we obtain

$$\begin{aligned}
\left\|v\left(\widehat{T},\omega,v_0(\omega)\right)\right\|^2&\le e^{-2\beta\widehat{T}+\int_0^{\widehat{T}} 2z(\theta_\tau\omega)\,d\tau}\left\|v_0(\omega)\right\|^2\\
&\quad+C(\alpha-\beta-4)^2\int_0^{\widehat{T}} e^{-2z(\theta_s\omega)-2\beta(\widehat{T}-s)+\int_s^{\widehat{T}} 2z(\theta_\tau\omega)\,d\tau}\,ds.
\end{aligned}\tag{3.13}$$

If we plug (3.13) back into (3.12), we have

$$\begin{aligned}
&\int_{\widehat{T}}^t e^{2\beta(s-t)+\int_s^t 2z(\theta_\tau\omega)\,d\tau}\left\|\Delta v\left(s,\omega,v_0(\omega)\right)\right\|^2 ds\\
&\quad\le e^{-2\beta t+\int_0^t 2z(\theta_\tau\omega)\,d\tau}\left\|v_0(\omega)\right\|^2\\
&\qquad+C(\alpha-\beta-4)^2\int_0^t e^{-2z(\theta_s\omega)-2\beta(t-s)+\int_s^t 2z(\theta_\tau\omega)\,d\tau}\,ds.
\end{aligned}\tag{3.14}$$

Replacing ω with $\theta_{-t}\omega$ in (3.14), we get

$$\begin{aligned}
&\int_{\widehat{T}}^t e^{2\beta(s-t)+\int_s^t 2z(\theta_{\tau-t}\omega)\,d\tau}\left\|\Delta v\left(s,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\right)\right\|^2 ds\\
&\quad\le C(\alpha-\beta-4)^2\int_0^t e^{-2z(\theta_{s-t}\omega)-2\beta(t-s)+\int_s^t 2z(\theta_{\tau-t}\omega)\,d\tau}\,ds\\
&\qquad+e^{-2\beta t+\int_0^t 2z(\theta_{\tau-t}\omega)\,d\tau}\left\|v_0(\theta_{-t}\omega)\right\|^2.
\end{aligned}\tag{3.15}$$

In order to obtain the result, we need to substitute $\widehat{T}$ for t and t for $t+1$ in (3.15) as follows:

$$\begin{aligned}
&\int_t^{t+1} e^{2\beta(s-t-1)+\int_s^{t+1} 2z(\theta_{\tau-t-1}\omega)\,d\tau}\left\|\Delta v\left(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\right)\right\|^2 ds\\
&\quad\le e^{-2\beta(t+1)+\int_0^{t+1} 2z(\theta_{\tau-t-1}\omega)\,d\tau}\left\|v_0(\theta_{-t-1}\omega)\right\|^2\\
&\qquad+C(\alpha-\beta-4)^2\int_0^{t+1} e^{-2z(\theta_{s-t-1}\omega)-2\beta(t+1-s)+\int_s^{t+1} 2z(\theta_{\tau-t-1}\omega)\,d\tau}\,ds,
\end{aligned}$$

that is,

$$\begin{aligned}
&\int_t^{t+1} e^{2\beta(s-t-1)+\int_{s-t-1}^0 2z(\theta_\tau\omega)\,d\tau} \left\| \Delta v\big(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\right\|^2 ds \\
&\quad\le e^{-2(t+1)(\beta-\frac{\int_{-t-1}^0 2z(\theta_\tau\omega)\,d\tau}{2(t+1)})} \left\| v_0(\theta_{-t-1}\omega)\right\|^2 \\
&\qquad + C(\alpha-\beta-4)^2 \int_{-t-1}^0 e^{-2z(\theta_s\omega)+2\beta s+\int_s^0 2z(\theta_\tau\omega)\,d\tau}\, ds \\
&\quad\le e^{-2(t+1)(\beta-\frac{\int_{-t-1}^0 2z(\theta_\tau\omega)\,d\tau}{2(t+1)})} \left\| v_0(\theta_{-t-1}\omega)\right\|^2 \\
&\qquad + C(\alpha-\beta-4)^2 \int_{-\infty}^0 e^{-2z(\theta_s\omega)+2\beta s+\int_s^0 2z(\theta_\tau\omega)\,d\tau}\, ds.
\end{aligned} \tag{3.16}$$

According to the properties of $z(\theta_t\omega)$, when $-1 \le s-t-1 \le 0$, we can deduce

$$\begin{aligned}
&\int_t^{t+1} e^{2\beta(s-t-1)+\int_{s-t-1}^0 2z(\theta_\tau\omega)\,d\tau} \left\| \Delta v\big(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\right\|^2 ds \\
&\quad\ge e^{-2\beta-2\max_{-1\le\tau\le 0}|z(\theta_\tau\omega)|} \int_t^{t+1} \left\| \Delta v\big(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\right\|^2 ds.
\end{aligned} \tag{3.17}$$

Combining estimates (3.16) and (3.17), we prove that there exist a random variable $\rho_2(\omega)$ and $T_{2B}(\omega) > 0$ such that, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all $t > T_{2B}(\omega)$,

$$\int_t^{t+1} \left\| \Delta v\big(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\right\|^2 ds \le \rho_2(\omega).$$

The proof of the lemma is now complete. □

4 The existence of random attractor

Lemma 4.1 *Provided that $v_0 \in B = \{B(\omega)\}_\omega \subset \mathscr{D}$, there exist a random radius $\rho_3(\omega) > 0$ and $T_{3B}(\omega) > 0$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ such that*

$$\left\| \Delta v\big(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\right\|^2 \le \rho_3(\omega), \quad t > T_{3B}(\omega). \tag{4.1}$$

Proof Taking the inner product of equation (3.1) with $\Delta^2 v$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta v\|^2 + \left\|\Delta^2 v\right\|^2 + \left(2\Delta v, \Delta^2 v\right) + \left(\alpha - z(\theta_t\omega)\right)\|\Delta v\|^2 \\
&\quad + \left(e^{2z(\theta_t\omega)} v^3, \Delta^2 v\right) + \left(b e^{z(\theta_t\omega)}(v_x)^2, \Delta^2 v\right) = 0.
\end{aligned} \tag{4.2}$$

Applying the Hölder inequality and ϵ-Young inequality, we get

$$\left|\left(2\Delta v, \Delta^2 v\right)\right| \le \frac{1}{4}\left\|\Delta^2 v\right\|^2 + 4\|\Delta v\|^2.$$

By the Gagliardo-Nirenberg inequality $\|v\|_{L^6} \le C\|v\|^{\frac{11}{12}}\|\Delta^2 v\|^{\frac{1}{12}}$ (see [8]) and the ϵ-Young inequality we have

$$\begin{aligned}\left|\left(e^{2z(\theta_t\omega)}v^3, \Delta^2 v\right)\right| &\le Ce^{2z(\theta_t\omega)}\|v\|_{L^6}^3\left\|\Delta^2 v\right\| \le Ce^{2z(\theta_t\omega)}\left\|\Delta^2 v\right\|\cdot\|v\|^{\frac{11}{4}}\left\|\Delta^2 v\right\|^{\frac{1}{4}}\\ &\le \eta_1\left\|\Delta^2 v\right\|^2 + C(\eta_1)e^{\frac{16z(\theta_t\omega)}{3}}\|v\|^{\frac{22}{3}}.\end{aligned}$$

Similarly, by the Gagliardo-Nirenberg inequality $\|v_x\|_{L^4} \le C\|v\|^{\frac{11}{16}}\|\Delta^2 v\|^{\frac{5}{16}}$ and the ϵ-Young inequality again, we obtain

$$\begin{aligned}\left|\left(be^{z(\theta_t\omega)}v_x^2, \Delta^2 v\right)\right| &\le |b|Ce^{z(\theta_t\omega)}\left\|\Delta^2 v\right\|\|v_x\|_{L^4}^2 \le |b|Ce^{z(\theta_t\omega)}\left\|\Delta^2 v\right\|^{\frac{13}{8}}\|v\|^{\frac{11}{8}}\\ &\le \eta_2\left\|\Delta^2 v\right\|^2 + C(\eta_2)e^{\frac{16}{3}z(\theta_t\omega)}\|v\|^{\frac{22}{3}}.\end{aligned}$$

Taking $\eta_1 = \eta_2 = \frac{1}{4}$, we deduce

$$\begin{aligned}&\frac{d}{dt}\|\Delta v\|^2 + 2\big(\beta - z(\theta_t\omega)\big)\|\Delta v\|^2 + \frac{1}{2}\left\|\Delta^2 v\right\|^2\\ &\quad\le Ce^{\frac{16}{3}z(\theta_t\omega)}\|v\|^{\frac{22}{3}} + 2(4-\alpha+\beta)\|\Delta v\|^2.\end{aligned}\tag{4.3}$$

Integrating (4.3) from s to $t+1$ with respect to t, we have

$$\begin{aligned}&\left\|\Delta v\big(t+1,\omega,v_0(\omega)\big)\right\|^2\\ &\quad\le \left\|\Delta v\big(s,\omega,v_0(\omega)\big)\right\|^2 + 2\int_s^{t+1}\big(z(\theta_\tau\omega)-\beta\big)\left\|\Delta v\big(\tau,\omega,v_0(\omega)\big)\right\|^2 d\tau\\ &\qquad+\int_s^{t+1}\Big[Ce^{\frac{16}{3}z(\theta_\tau\omega)}\left\|v\big(\tau,\omega,v_0(\omega)\big)\right\|^{\frac{22}{3}} + 2(4-\alpha+\beta)\left\|\Delta v\big(\tau,\omega,v_0(\omega)\big)\right\|^2\Big]d\tau.\end{aligned}$$

Integrating from t to $t+1$ with respect to s again, we get

$$\begin{aligned}&\left\|\Delta v\big(t+1,\omega,v_0(\omega)\big)\right\|^2\\ &\quad\le \int_t^{t+1}\left\|\Delta v\big(s,\omega,v_0(\omega)\big)\right\|^2 ds + 2\int_t^{t+1}\left|\big(z(\theta_\tau\omega)-\beta\big)\right|\left\|\Delta v\big(\tau,\omega,v_0(\omega)\big)\right\|^2 d\tau\\ &\qquad+\int_t^{t+1}Ce^{\frac{16}{3}z(\theta_\tau\omega)}\left\|v\big(\tau,\omega,v_0(\omega)\big)\right\|^{\frac{22}{3}}d\tau + 2(4-\alpha+\beta)\int_t^{t+1}\left\|\Delta v\big(\tau,\omega,v_0(\omega)\big)\right\|^2 d\tau\\ &\quad\le (9-2\alpha+4\beta)\int_t^{t+1}\left\|\Delta v\big(s,\omega,v_0(\omega)\big)\right\|^2 ds + 2\int_t^{t+1}\left|z(\theta_\tau\omega)\right|\left\|\Delta v\big(\tau,\omega,v_0(\omega)\big)\right\|^2 d\tau\\ &\qquad+\int_t^{t+1}Ce^{\frac{16}{3}z(\theta_\tau\omega)}\left\|v\big(\tau,\omega,v_0(\omega)\big)\right\|^{\frac{22}{3}}d\tau.\end{aligned}\tag{4.4}$$

Replacing ω with $\theta_{-t-1}\omega$ in (4.4), we have

$$\begin{aligned}&\left\|\Delta v\big(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big)\right\|^2\\ &\quad\le (9-2\alpha+4\beta)\int_t^{t+1}\left\|\Delta v\big(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big)\right\|^2 ds\end{aligned}$$

$$+ 2\int_t^{t+1} \left|z(\theta_{\tau-t-1}\omega)\right| \left\| \Delta v\big(\tau,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\right\|^2 d\tau$$
$$+ \int_t^{t+1} Ce^{\frac{16}{3}z(\theta_{\tau-t-1}\omega)} \left\| v\big(\tau,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\right\|^{\frac{22}{3}} d\tau. \tag{4.5}$$

According to Lemma 3.2, the first term on the right-hand side of the above inequality is bounded:

$$(9-2\alpha+4\beta)\int_t^{t+1} \left\| \Delta v\big(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\right\|^2 ds \le |9-2\alpha+4\beta|\rho_2(\omega).$$

For the second term, taking into account the properties of $z(\theta_t\omega)$, we have $|z(\theta_t\omega)| \le e^{|t|}r(\omega)$. Then

$$2\int_t^{t+1} \left|z(\theta_{\tau-t-1}\omega)\right| \left\| \Delta v\big(\tau,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\right\|^2 d\tau$$
$$\le 2\int_t^{t+1} e^{|\tau-t-1|}r(\omega) \left\| \Delta v\big(\tau,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\right\|^2 d\tau.$$

Noticing that $|\tau - t - 1| < 1$ and applying Lemma 3.2, we obtain

$$2\int_t^{t+1} \left|z(\theta_{\tau-t-1}\omega)\right| \left\| \Delta v\big(\tau,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\right\|^2 d\tau \le 2er(\omega)\rho_2(\omega).$$

Now, we estimate the last term. Replacing t with τ in (3.8), we get

$$\left\| v\big(\tau,\omega, v_0(\omega)\big)\right\|^2 \le e^{-2\beta\tau+\int_0^\tau 2z(\theta_r\omega)\,dr}\left\| v_0(\omega)\right\|^2$$
$$+ C(\alpha-\beta-4)^2\int_0^\tau e^{-2z(\theta_s\omega)-2\beta(\tau-s)+\int_s^\tau 2z(\theta_r\omega)\,dr}\,ds. \tag{4.6}$$

Substituting ω for $\theta_{-t-1}\omega$ in (4.6), we obtain

$$\left\| v\big(\tau,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\right\|^2$$
$$\le e^{-2\beta\tau+\int_0^\tau 2z(\theta_{r-t-1}\omega)\,dr}\left\| v_0(\theta_{-t-1}\omega)\right\|^2$$
$$+ C(\alpha-\beta-4)^2\int_0^\tau e^{-2z(\theta_{s-t-1}\omega)-2\beta(\tau-s)+\int_s^\tau 2z(\theta_{r-t-1}\omega)\,dr}\,ds. \tag{4.7}$$

If we plug (4.7) back into (4.5), we obtain

$$\int_t^{t+1} Ce^{\frac{16}{3}z(\theta_{\tau-t-1}\omega)}\bigg[e^{-2\beta\tau+\int_0^\tau 2z(\theta_{r-t-1}\omega)\,dr}\left\| v_0(\theta_{-t-1}\omega)\right\|^2$$
$$+ C(\alpha-\beta-4)^2\int_0^\tau e^{-2z(\theta_{s-t-1}\omega)-2\beta(\tau-s)+\int_s^\tau 2z(\theta_{r-t-1}\omega)\,dr}\,ds\bigg]^{\frac{11}{3}} d\tau$$
$$\le \int_t^{t+1} Ce^{\frac{16}{3}z(\theta_{\tau-t-1}\omega)}\bigg[\Big(e^{-2\beta\tau+\int_0^\tau 2z(\theta_{r-t-1}\omega)\,dr}\left\| v_0(\theta_{-t-1}\omega)\right\|^2\Big)^{\frac{11}{3}}$$
$$+ C(\alpha-\beta-4)^{\frac{22}{3}}\bigg(\int_0^\tau e^{-2z(\theta_{s-t-1}\omega)-2\beta(\tau-s)+\int_s^\tau 2z(\theta_{r-t-1}\omega)\,dr}\,ds\bigg)^{\frac{11}{3}}\bigg] d\tau$$

$$
\begin{aligned}
&= \int_t^{t+1} Ce^{\frac{16}{3} z(\theta_{\tau-t-1}\omega)} \bigl(e^{-2\beta\tau + \int_0^\tau 2z(\theta_{r-t-1}\omega)\,dr} \bigl\| v_0(\theta_{-t-1}\omega)\bigr\|^2\bigr)^{\frac{11}{3}}\, d\tau \\
&\quad + C(\alpha-\beta-4)^{\frac{22}{3}} \\
&\quad \times \int_t^{t+1} e^{\frac{16}{3} z(\theta_{\tau-t-1}\omega)} \left(\int_0^\tau e^{-2z(\theta_{s-t-1}\omega)-2\beta(\tau-s)+\int_s^\tau 2z(\theta_{r-t-1}\omega)\,dr}\, ds\right)^{\frac{11}{3}} d\tau.
\end{aligned}
\tag{4.8}
$$

Next, we estimate each term on the right-hand side of the last inequality. For the first term, we have

$$
\begin{aligned}
&C\int_t^{t+1} \bigl(e^{-2\beta\tau + \int_0^\tau 2z(\theta_{r-t-1}\omega)\,dr} \bigl\| v_0(\theta_{-t-1}\omega)\bigr\|^2\bigr)^{\frac{11}{3}} e^{\frac{16}{3} z(\theta_{\tau-t-1}\omega)}\, d\tau \\
&= C\int_t^{t+1} \bigl(e^{-2\beta\tau + \int_{-t-1}^{\tau-t-1} 2z(\theta_r\omega)\,dr} \bigl\| v_0(\theta_{-t-1}\omega)\bigr\|^2\bigr)^{\frac{11}{3}} e^{\frac{16}{3} z(\theta_{\tau-t-1}\omega)}\, d\tau \\
&= C\int_{-1}^{0} \bigl(e^{-2\beta(\tau+1+t) + \int_{-t-1}^{\tau} 2z(\theta_r\omega)\,dr} \bigl\| v_0(\theta_{-t-1}\omega)\bigr\|^2\bigr)^{\frac{11}{3}} e^{\frac{16}{3} z(\theta_{\tau}\omega)}\, d\tau \\
&\le Ce^{\frac{16}{3}\max_{-1\le\tau\le 0}|z(\theta_\tau\omega)|} \int_{-1}^{0} \bigl(e^{-2\beta(\tau+t+1) + \int_{-t-1}^{0} 2|z(\theta_r\omega)|\,dr}\bigr)^{\frac{11}{3}}\, d\tau \\
&= Ce^{\frac{16}{3}\max_{-1\le\tau\le 0}|z(\theta_\tau\omega)|} \int_{-1}^{0} \bigl(e^{-2\beta(t+1) + \int_{-t-1}^{0} 2|z(\theta_r\omega)|\,dr}\bigr)^{\frac{11}{3}} \cdot e^{-\frac{22}{3}\beta\tau}\, d\tau \\
&= Ce^{\frac{16}{3}\max_{-1\le\tau\le 0}|z(\theta_\tau\omega)|} \bigl(e^{-2\beta(t+1) + \int_{-t-1}^{0} 2|z(\theta_r\omega)|\,dr}\bigr)^{\frac{11}{3}} \cdot \int_{-1}^{0} e^{-\frac{22}{3}\beta\tau}\, d\tau \\
&\le Ce^{\frac{16}{3}\max_{-1\le\tau\le 0}|z(\theta_\tau\omega)|} \bigl(e^{-2\beta(t+1) + \int_{-t-1}^{0} 2|z(\theta_r\omega)|\,dr}\bigr)^{\frac{11}{3}},
\end{aligned}
\tag{4.9}
$$

where the first inequality is due to the properties of $z(\theta_t\omega)$.

For the second term, similarly to the above method, as $t < \tau < t+1$, we have

$$
\begin{aligned}
&C(\alpha-\beta-4)^{\frac{22}{3}} \int_t^{t+1} e^{\frac{16}{3} z(\theta_{\tau-t-1}\omega)} \left(\int_0^\tau e^{-2z(\theta_{s-t-1}\omega)-2\beta(\tau-s)+\int_s^\tau 2z(\theta_{r-t-1}\omega)\,dr}\, ds\right)^{\frac{11}{3}} d\tau \\
&\le C(\alpha-\beta-4)^{\frac{22}{3}} \int_t^{t+1} e^{\frac{16}{3} z(\theta_{\tau-t-1}\omega)} \left(\int_0^\tau e^{-2z(\theta_{s-t-1}\omega)+\int_{s-t-1}^{0} 2|z(\theta_r\omega)|\,dr-2\beta(\tau-s)}\, ds\right)^{\frac{11}{3}} d\tau \\
&\le C(\alpha-\beta-4)^{\frac{22}{3}} \int_t^{t+1} e^{\frac{16}{3} z(\theta_{\tau-t-1}\omega)} \left(\int_{-t-1}^{\tau-t-1} e^{-2z(\theta_s\omega)+\int_s^0 2|z(\theta_r\omega)|\,dr-2\beta(\tau-t-1-s)}\, ds\right)^{\frac{11}{3}} d\tau \\
&\le C(\alpha-\beta-4)^{\frac{22}{3}} \\
&\quad \times \int_t^{t+1} e^{\frac{16}{3} z(\theta_{\tau-t-1}\omega)} \left(\int_{-t-1}^{0} e^{-2z(\theta_s\omega)+2\beta s+\int_s^0 2|z(\theta_r\omega)|\,dr+2\beta}\, ds\right)^{\frac{11}{3}} d\tau,
\end{aligned}
\tag{4.10}
$$

where by the condition $t < \tau < t+1$, $-2\beta(\tau-t) < 0$, we get rid of the term $e^{-2\beta(\tau-t)}$. Then the last inequality can be estimated as follows:

$$
\begin{aligned}
&C(\alpha-\beta-4)^{\frac{22}{3}} \int_t^{t+1} e^{\frac{16}{3} z(\theta_{\tau-t-1}\omega)} \left(\int_0^\tau e^{-2z(\theta_{s-t-1}\omega)-2\beta(\tau-s)+\int_s^\tau 2z(\theta_{r-t-1}\omega)\,dr}\, ds\right)^{\frac{11}{3}} d\tau \\
&\le C(\alpha-\beta-4)^{\frac{22}{3}} e^{\frac{22}{3}\beta} \int_t^{t+1} e^{\frac{16}{3} z(\theta_{\tau-t-1}\omega)} \left(\int_{-\infty}^{0} e^{-2z(\theta_s\omega)+2\beta s+\int_s^0 2|z(\theta_r\omega)|\,dr}\, ds\right)^{\frac{11}{3}} d\tau
\end{aligned}
$$

$$\le C(\alpha-\beta-4)^{\frac{22}{3}} e^{\frac{22}{3}\beta} \int_{-1}^{0} e^{\frac{16}{3}z(\theta_\tau\omega)} \left(\int_{-\infty}^{0} e^{-2z(\theta_s\omega)+2\beta s+\int_s^0 2|z(\theta_r\omega)|\,dr}\,ds \right)^{\frac{11}{3}} d\tau$$

$$\le C(\alpha-\beta-4)^{\frac{22}{3}} e^{\frac{22}{3}\beta} e^{\frac{16}{3}\max_{-1\le\tau\le0}|z(\theta_\tau\omega)|} \left(\int_{-\infty}^{0} e^{-2z(\theta_s\omega)+2\beta s+\int_s^0 2|z(\theta_r\omega)|\,dr}\,ds \right)^{\frac{11}{3}}. \quad (4.11)$$

In summary, from estimates of (4.9) and (4.10) we obtain that the terms on the right-hand of inequality (4.8) are bounded. Therefore, we prove that there exist a random variable $\rho_3(\omega)$ and $T_{3B}(\omega) > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all $t > T_{3B}(\omega)$,

$$\left\| \Delta v\big(t+1, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big) \right\|^2 \le \rho_3(\omega).$$

This completes the proof. □

Based on the above arguments, it is easy to deduce that there exists a random absorbing set for the random dynamical system generated by system (3.1)-(3.3) in $H_0^2(D)$.

Theorem 4.1 *Assume that* $v_0 \in B = \{B(\omega)\}_\omega \subset \mathscr{D}$. *There exists a random absorbing set* $B^*(\omega)$ *for the random dynamical system associated with system* (3.1)-(3.3) *in* $H_0^2(D)$.

Proof We can take $T = \max\{T_{1B}, T_{2B}, T_{3B}\}$ and $\rho(\omega) = \max\{\rho_1(\omega), \rho_2(\omega), \rho_3(\omega)\}$. Then, for all $t \ge T$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists a random absorbing set $B^*(\omega)$ for the random dynamical system associated with system (3.1)-(3.3) in $H_0^2(D)$. □

Based on the above results, by Lemma 2.8 in [10] we claim that ψ is asymptotically compact. Therefore, the existence of a random attractor for ψ follows immediately from Theorem 2.2 in [15].

Theorem 4.2 *Provided that* $v_0 \in B = \{B(\omega)\}_\omega \subset \mathscr{D}$, *there exists a global random attractor in* $H_0^2(D)$ *for the random dynamical system associated with system* (3.1)-(3.3).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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