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Existence,multiplicity, and nonexistence of solutions for a p-Kirchhoff elliptic equation on $\mathbb{R}^N$

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Abstract

In this paper, we study the multiplicity of solutions for the following nonhomogeneous p-Kirchhoff elliptic equation:

$$\left(a+\lambda\left(\int_{\mathbb{R}^N}(|\nabla u|^p+|u|^p)\,dx\right)^m\right)(-\Delta_p u+|u|^{p-2}u)=f(u)+h(x),\quad x\in\mathbb{R}^N,\tag{0.1}$$

with $a,\lambda,m>0$ and $1<p<N$. By variational methods we prove that problem (0.1) admits at least two solutions under appropriate assumptions on $f(u)$ and $h(x)$. The main difficulty to overcome is the lack of an a priori bound for Palais-Smale sequence. Motivated by Jeanjean (Proc. R. Soc. Edinb., Sect. A 129:787-809, 1999), we use a cut-off functional to obtain a bounded (PS) sequence. Also, if $f(u)=|u|^{q-2}u$, $p<q<\min\{p(m+1),p^*=\frac{pN}{N-p}\}$, and $h(x)=0$, then we prove that problem (0.1) has at least one nontrivial solution for any $\lambda\in(0,\lambda^*]$ and has no nontrivial weak solutions for any $\lambda\in(\lambda^*,+\infty)$.

Keywords: p-Kirchhoff elliptic equation; bounded potential; variational methods; mountain pass lemma

1 Introduction

In this paper, we are interested in the multiplicity of solutions to the following nonhomogeneous p-Kirchhoff elliptic problem:

$$\left(a+\lambda\left(\int_{\mathbb{R}^N}(|\nabla u|^p+|u|^p)\,dx\right)^m\right)(-\Delta_p u+|u|^{p-2}u)=f(u)+h(x),\quad x\in\mathbb{R}^N,\tag{1.1}$$

where $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, and the nontrivial function $h(x)$ can be seen as a perturbation term. Problem (1.1) is a generalization of the model introduced by Kirchhoff [2]. More precisely, Kirchhoff proposed the model given by the equation

$$\rho_{tt}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L u_x^2\,dx\right)u_{xx}=0,\quad 0<x<L,t>0,\tag{1.2}$$

which takes into account the changes in length of string produced by transverse vibration. The parameters in (1.2) have the following meaning: L is the length of the string, h is the area of cross-section, E is the Young modulus of material, ρ is the mass density, and P_0 is the initial tension.

The equation

$$\rho_{tt} - M\big(\|\nabla u\|_2^2\big)\Delta u = f(x,u), \quad x \in \Omega, t > 0, \tag{1.3}$$

generalizes equation (1.2), where $M : \mathbb{R}^+ \to \mathbb{R}$ is a given function, Ω is a domain of $\mathbb{R}^N$. The stationary counterpart of (1.3) is the Kirchhoff-type elliptic equation

$$-M\big(\|\nabla u\|_2^2\big)\Delta u = f(x,u), \quad x \in \Omega, t > 0. \tag{1.4}$$

Some classical and interesting results on Kirchhoff-type elliptic equations can be found, for example, in [3–9].

Particularly, Li et al. [10] considered the Kirchhoff-type problem

$$\left(a + \lambda\left(\int_{\mathbb{R}^N} \big(|\nabla u|^2 + b|u|^2\big)\,dx\right)\right)(-\Delta u + bu) = f(u), \quad x \in \mathbb{R}^N, \tag{1.5}$$

where $N \geq 3$, with constants $a, b > 0$ and $\lambda \geq 0$ under the following assumptions:

(H_1) $f \in C(\mathbb{R}^+, \mathbb{R}^+)$, $|f(t)| \leq C(1 + t^{q-1})$ for all $t \in \mathbb{R}^+ = [0, +\infty)$ and some $q \in (2, 2^*)$, where $2^* = \frac{2N}{N-2}$ for $N \geq 3$;

(H_2) $\lim_{t\to 0} \frac{f(t)}{t} = 0$; $\lim_{t\to +\infty} \frac{f(t)}{t} = +\infty$.

It is easy to see that $f(u) = |u|^{q-2}u$, $2 < q < 4$, and $N = 3$ satisfy these conditions. They obtained that there exists $\lambda_0 > 0$ such that, for any $\lambda \in [0, \lambda_0)$, problem (1.5) has at least one positive solution in $W^{1,2}(\mathbb{R}^N)$. The λ_0 depends on f, a, b, the Sobolev constant, and several test functions in [10]; it is not very clear whether the the existence of solutions for (1.5) still holds for large $\lambda > 0$. Recently, Chen et al. [11] studied the existence of positive solutions to the p-Kirchhoff problem

$$\begin{cases} (a + \lambda(\int_{\mathbb{R}^N} (|\nabla u|^p + b|u|^p)\,dx)^\tau)(-\Delta_p u + b|u|^{p-2}u) \\ \quad = |u|^{m-2}u + \mu|u|^{q-2}u, \quad x \in \mathbb{R}^N, \\ u(x) > 0, \quad x \in \mathbb{R}^N, \qquad u(x) \in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{1.6}$$

where $a, b > 0$, $\tau, \lambda \geq 0$, $\mu \in \mathbb{R}$, and $1 < p < N$. By the Nehari manifold method, they proved that problem (1.6) admits at least a positive ground state solution for any $\lambda > 0$ when $p(\tau + 1) < q < m < p^* = \frac{pN}{N-p}$. However, does the existence of solutions for (1.5) still hold for any $\lambda > 0$ when $p < q < p(\tau + 1)$ and $\mu = 0$? This is a interesting problem. In this paper, we answer positively this question. More interesting results for Kirchhoff-type problems can be found in [1, 2, 5–7, 10–14].

In the present paper, we are ready to extend the analysis to the nonhomogeneous p-Kirchhoff-type equation of (1.1) in $\mathbb{R}^N$ with the nonlinearity $f(u)$ satisfying the following conditions:

(F_1) $f \in C(\mathbb{R}^+, \mathbb{R}^+)$, $|f(t)| \leq C(t^{p-1} + t^{q-1})$ for all $t \in \mathbb{R}^+$ and some $q \in (p, p^*)$, where $p^* = pN/(N-p)$, $1 < p < N$;
(F_2) $\lim_{t\to 0^+} \frac{f(t)}{t^{p-1}} = 0$;
(F_3) $\lim_{t\to +\infty} \frac{f(t)}{t^{p-1}} = +\infty$.

In addition, we suppose that the nontrivial and nonnegative function $h(x) \equiv h(|x|) \in C^1(\mathbb{R}^N) \cap L^{p'}(\mathbb{R}^N)$ satisfies

(H) there exists $\xi(x) \in L^{p'}(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ such that

$$\left|\nabla h(x) \cdot x\right| \leq \xi^{p'}(x), \quad \forall x \in \mathbb{R}^N, \tag{1.7}$$

with $p' = \frac{p}{p-1}$.

We will use the Ekeland variational principle [15] and a version of the mountain pass theorem in [1] to study the existence of multiple solutions of problem (1.1) in $\mathbb{R}^N$. It is well known that an important technical condition to get a bounded (PS) sequence is the following Ambrosetti-Rabinowitz-type condition (AR): there exists $\theta > p$ such that $0 < \theta F(s) \leq sf(s)$ for $s > 0$. The loss of (AR) condition renders variational techniques more delicate. Inspired by [1, 10], we use a cut-off functional and obtain a bounded (PS) sequence.

In order to state our main result, we introduce some Sobolev spaces and norms. Let $W^{1,p}(\mathbb{R}^N)$ be the usual Sobolev space with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^p + |u|^p \, dx\right)^{\frac{1}{p}}, \quad 1 < p < \infty. \tag{1.8}$$

We denote by $\|\cdot\|_q$ the usual $L^q(\mathbb{R}^N)$ norm. Then it well known that the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for $q \in (p, p^*]$ and there exists a constant S_q such that

$$\|u\|_q \leq S_q \|u\|, \quad \forall u \in W^{1,p}\big(\mathbb{R}^N\big). \tag{1.9}$$

Let $X = W_r^{1,p}(\mathbb{R}^N)$ be the subspace of $W^{1,p}(\mathbb{R}^N)$ containing only the radial functional. Then by the Lemma 2.2 in [11] we have that the embedding $X \hookrightarrow L^q(\mathbb{R}^N)$ is compact for $q \in (p, p^*)$.

A function $u \in X$ is said to be a weak solution of (1.1) if for all $v \in X$,

$$\big(a + \lambda \|u\|^{pm}\big) \int_{\mathbb{R}^N} \big(|\nabla u|^{p-2} \nabla u \nabla v + |u|^{p-2} uv\big) \, dx = \int_{\mathbb{R}^N} \big(f(u) + h\big) v \, dx. \tag{1.10}$$

Let $I(u) : X \to \mathbb{R}$ be the energy functional associated with problem (1.1) defined by

$$I(u) = \frac{a}{p} \|u\|^p + \frac{\lambda}{p(m+1)} \|u\|^{p(m+1)} - \int_{\mathbb{R}^N} \big(F(u) + hu\big) \, dx, \tag{1.11}$$

where $F(u) = \int_0^u f(s) \, ds$. It is easy to see that the functional $I \in C^1(X, \mathbb{R})$ and its Gateaux derivative is given by

$$\begin{aligned} I'(u)v = {} & \big(a + \lambda \|u\|^{pm}\big) \int_{\mathbb{R}^N} \big(|\nabla u|^{p-2} \nabla u \nabla v + |u|^{p-2} uv\big) \, dx \\ & - \int_{\mathbb{R}^N} \big(f(u) + h\big) v \, dx, \quad \forall v \in X. \end{aligned} \tag{1.12}$$

Clearly, we see that a weak solution of (1.1) corresponds to a critical point of the functional.

The main result in this paper is as follows.

Theorem 1.1 *Let* (F_1)-(F_3) *and* (H) *hold. Then, there exist* $\lambda_0, \widetilde{m}_0 > 0$ *such that, for any* $\lambda \in [0, \lambda_0)$, (1.1) *has at least two nontrivial solutions in* X *when* $\|h\|_{p'} < \widetilde{m}_0$.

Furthermore, consider $h(x) = 0$ and $f(x,u) = |u|^{q-2}u$, $p < q < \min\{p(m+1), p^*\}$, that is,

$$\left(a + \lambda\left(\int_{\mathbb{R}^N} \left(|\nabla u|^p + |u|^p\right) dx\right)^m\right)\left(-\Delta_p u + |u|^{p-2}u\right) = |u|^{q-2}u, \quad x \in \mathbb{R}^N. \tag{1.13}$$

We can now state the second main result.

Theorem 1.2 *Let* $a > 0$ *and* $p < q < \min\{p(m+1), p^*\}$. *Then there exists* $\lambda^* > 0$ *such that problem* (1.13) *has at least one nontrivial solution for any* $\lambda \in (0, \lambda^*]$ *and has no nontrivial weak solutions for any* $\lambda \in (\lambda^*, +\infty)$.

Remark 1.3 In [11], Chen and Zhu considered the case $p < p(m+1) < q < p^*$. They proved that problem (1.1) admits at least one positive solution for any $\lambda > 0$.

2 Proof of Theorem 1.1

In this section, we first establish some properties of the functional I and then prove Theorem 1.1. Throughout the paper, we denote by C or C_i s positive constants that may vary from line to line and are not essential to the problem.

Lemma 2.1 *If assumptions* (F_1)-(F_3) *hold and* $h(x) \in L^{p'}(\mathbb{R}^N)$, *then there exist* $\rho, \alpha, m_0 > 0$ *such that* $I(u) \geq \alpha > 0$ *with* $\|u\| = \rho$ *and* $\|h\|_{p'} < m_0$.

Proof It follows from (F_1)-(F_2) that

$$F(s) \leq \varepsilon |s|^p + C_\varepsilon |s|^q, \quad \forall s \in \mathbb{R}, \tag{2.1}$$

with $\varepsilon > 0$. By the Hölder inequality we have

$$\left|\int_{\mathbb{R}^N} hu\, dx\right| \leq S_q^{-1} \|h\|_{p'} \|u\| \leq \epsilon \|u\|^p + C_\epsilon \|h\|_{p'}^{p'}. \tag{2.2}$$

Thus,

$$\begin{aligned} I(u) &\geq \frac{a}{p}\|u\|^p - \varepsilon\|u\|^p - C_\varepsilon\|u\|^q - \epsilon\|u\|^p - C_\epsilon\|h\|_{p'}^{p'} \\ &\geq \frac{a}{2p}\|u\|^p - C_1\|u\|^q - C_2\|h\|_{p'}^{p'}, \end{aligned} \tag{2.3}$$

where $\varepsilon = \epsilon = \frac{a}{4p}$, C_1, C_2 are some positive constants. Let

$$z(t) = \frac{a}{2p}t^p - C_1 t^q, \quad t \geq 0. \tag{2.4}$$

We see that there exists $\rho > 0$ such that $\max_{t\geq 0} z(t) = z(\rho) \equiv m_0 > 0$. Then it follows from (2.3) that there exists $\alpha > 0$ such that $I(u) \geq \alpha$ with $\|u\| = \rho$ and $\|h\|_{p'} < m_0$. This ends the proof of Lemma 2.1. □

We denote by B_r the open ball in X centered at the origin with radius r. By Ekland's variational principle [15] we get the following lemma, which implies that there exists a function u_0 such that $I'(u_0) = 0$ and $I(u_0) < 0$ if $\|h\|_{p'}$ is small.

Lemma 2.2 *Let assumptions* (F_1)-(F_3) *hold, and* $h(x) \in L^{p'}(\mathbb{R}^N)$, $h(x) \not\equiv 0$, *with* $\|h\|_{p'} < m_0$. *Then there exists a function* $u_0 \in X$ *such that*

$$I(u_0) = \inf\{I(u) : u \in \overline{B}_\rho\} < 0, \tag{2.5}$$

and u_0 *is a nontrivial weak solution of problem* (1.1).

Proof Choose a function $\phi \in C_0^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} h(x)\phi(x)\,dx > 0$. Then

$$I(t\phi) \le \frac{a}{p} t^p \|\phi\|^p + \frac{\lambda}{p(m+1)} t^{p(m+1)} \|\phi\|^{p(m+1)} - t \int_{\mathbb{R}^N} h(x)\phi\,dx < 0 \tag{2.6}$$

for small $t > 0$ and thus for any open ball $B_\kappa \subset X$ such that $-\infty < c_\kappa = \inf_{\overline{B}_\kappa} I(u) < 0$. Thus,

$$c_\rho = \inf_{u \in \overline{B}_\rho} I(u) < 0 \quad \text{and} \quad \inf_{u \in \partial B_\rho} I(u) > 0, \tag{2.7}$$

where ρ is given in Lemma 2.1. Let $\varepsilon_n \downarrow 0$ be such that

$$0 < \varepsilon_n < \inf_{u \in \partial B_\rho} I(u) - \inf_{u \in B_\rho} I(u). \tag{2.8}$$

Then, by Ekland's variational principle [15] there exists $\{u_n\} \subset \overline{B}_\rho$ such that

$$c_\rho \le I(u_n) < c_\rho + \varepsilon_n \tag{2.9}$$

and

$$I(u_n) < I(u) + \varepsilon_n \|u_n - u\| \qquad \text{for all } u \in \overline{B_\rho}, u_n \neq u. \tag{2.10}$$

Then, it follows from (2.8)-(2.10) that

$$I(u_n) < c_\rho + \varepsilon_n \le \inf_{u \in B_\rho} I(u) + \varepsilon_n < \inf_{u \in \partial B_\rho} I(u). \tag{2.11}$$

So $u_n \in B_\rho$, and we now consider the function $F : \overline{B}_\rho \to \mathbb{R}$ given by

$$F(u) = I(u) + \varepsilon_n \|u_n - u\|, \quad u \in \overline{B}_\rho. \tag{2.12}$$

Then (2.10) shows that $F(u_n) < F(u)$, $u \in \overline{B}_\rho$, $u_n \neq u$, and thus u_n is a strict local minimum of F. Moreover,

$$t^{-1}\big(F(u_n + tv) - F(u_n)\big) \ge 0 \quad \text{for small } t > 0, \forall v \in B_1. \tag{2.13}$$

Hence,

$$t^{-1}\big(I(u_n+tv)-I(u_n)\big)+\varepsilon_n\|v\|\geq 0. \tag{2.14}$$

Passing to the limit as $t\to 0^+$, it follows that

$$I'(u_n)v+\varepsilon_n\|v\|\geq 0,\quad \forall v\in B_1. \tag{2.15}$$

Replacing v in (2.15) by $-v$, we get

$$-I'(u_n)v+\varepsilon_n\|v\|\geq 0,\quad \forall v\in B_1, \tag{2.16}$$

so that $\|I'(u_n)\|\leq\varepsilon_n$. Therefore, there is a sequence $\{u_n\}\in B\rho$ such that $I(u_n)\to c_\rho<0$ and $I'(u_n)\to 0$ in X^* as $n\to\infty$. In the following, we will prove that $\{u_n\}$ has a convergent subsequence in X. Indeed, since $\|u_n\|<\rho$, by the reflexivity of X and compact embedding $X\hookrightarrow L^q$ for all $q\in(p,p^*)$, passing to a subsequence, we can assume that

$$u_n\rightharpoonup u_0,\quad \text{in } X;\qquad u_n\to u_0,\quad L^q(\mathbb{R}^N);\qquad u_n\to u_0,\quad \text{a.e. in } \mathbb{R}^N. \tag{2.17}$$

By (1.12) we can get

$$\big(I(u_n)-I(u_0)\big)'(u_n-u_0)=P_n+Q_n+K_n, \tag{2.18}$$

where

$$\begin{aligned}
P_n&=\big(a+\lambda\|u_n\|^{pm}\big)\int_{\mathbb{R}^N}\big(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\big)\nabla(u_n-u_0)\\
&\quad+\big(|u_n|^{p-2}u_n-u_0^{p-2}u_0\big)(u_n-u_0)\,dx,\\
Q_n&=\lambda\big(\big(\|u_n\|^{pm}-\|u_0\|^{pm}\big)\big)\int_{\mathbb{R}^N}|\nabla u_0|^{p-2}\nabla u_0\nabla(u_n-u_0)\\
&\quad+|u_0|^{p-2}u_0(u_n-u_0)\,dx,\\
K_n&=\int_{\mathbb{R}^N}\big(f(u_n)-f(u_0)\big)(u_n-u_0)\,dx.
\end{aligned} \tag{2.19}$$

It is clear that

$$\big(I(u_n)-I(u_0)\big)'(u_n-u_0)\to 0\quad \text{as } n\to\infty. \tag{2.20}$$

By (F_1) and (F_2), for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$\big|f(t)\big|\leq\varepsilon|t|^{p-1}+C_\varepsilon|t|^{q-1},\quad t\in\mathbb{R}. \tag{2.21}$$

Hence,

$$\begin{aligned}
|K_n|&=\left|\int_{\mathbb{R}^N}\big(f(u_n)-f(u_0)\big)(u_n-u_0)\,dx\right|\\
&\leq\varepsilon\big(\|u_n\|^{p-1}+\|u_0\|^{p-1}\big)\|u_n-u_0\|+C_\varepsilon\big(\|u_n\|_q^{q-1}+\|u_0\|_q^{q-1}\big)\|u_n-u_0\|_q\\
&\to 0\quad \text{as } n\to\infty.
\end{aligned} \tag{2.22}$$

Define the linear function $g : X \to \mathbb{R}$ by

$$g(\omega) = \int_{\mathbb{R}^N} |\nabla u_0|^{p-2}\nabla u_0 \nabla \omega + |u_0|^{p-2}u_0\omega \, dx. \tag{2.23}$$

Noticing that $|g(\omega)| \leq 2\|u_0\|^{p-1}\|\omega\|$, we can deduce that g is continuous on X. Using $u_n \rightharpoonup u_0$ in X, we have

$$\begin{aligned} g(u_n - u_0) &= \int_{\mathbb{R}^N} |\nabla u_0|^{p-2}\nabla u_0 \nabla(u_n - u_0) + |u_0|^{p-2}u_0(u_n - u_0)\, dx \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.24}$$

Since $\|u_n\| < \rho$, we deduce that $|Q_n| \to 0$ as $n \to \infty$.

Combining the above results, we have $|P_n| \to 0$ as $n \to \infty$, Then, using the standard inequalities in $\mathbb{R}^N$

$$\begin{aligned} &\big\langle |x|^{p-2}x - |y|^{p-2}y, x - y\big\rangle \geq C_p|x - y|^p, \quad p \geq 2, \\ &\big\langle |x|^{p-2}x - |y|^{p-2}y, x - y\big\rangle \geq \frac{C_p|x - y|^p}{|x| + |y|^{2-p}}, \quad 2 > p > 1, \end{aligned} \tag{2.25}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^N$, we can show that $u_n \to u_0$ in X. Thus, u_0 is a nontrivial weak solution of problem (1.1). The proof is completed. □

Next, we prove that problem (1.1) has a mountain-pass-type solution. To overcome the difficulty of finding a bounded (PS) sequence for the associated functional I, motivated by [1, 10], we use a cut-off function $\psi \in C_0^1(\mathbb{R}^+)$ that satisfies

$$\begin{aligned} &\psi(t) = 1, \quad \forall t \in [0, 1]; \qquad 0 \leq \psi \leq 1, \quad \forall t \in (1, 2); \\ &\psi(t) \equiv 0, \quad \forall t \in [2, +\infty); \qquad \big\|\psi'\big\|_\infty \leq 2, \end{aligned} \tag{2.26}$$

and study the following modified functional I^T defined by

$$I^T(u) = \frac{a}{p}\|u\|^p + \frac{\lambda}{p(m+1)}\eta_T(u)\|u\|^{p(m+1)} - \int_{\mathbb{R}^N} \big(F(u) + hu\big)\, dx, \quad u \in X, \tag{2.27}$$

where $T > 0$ and $\eta_T(u) = \psi(\frac{\|u\|^p}{T^p})$. For $T > 0$ sufficiently large and λ sufficiently small, we will prove that there exists a critical point $\tilde{u}_0$ of I_T such that $\|\tilde{u}_0\| \leq T$, and so $\tilde{u}_0$ is also a critical point of I. For this purpose, we use the following theorem given in [1].

Lemma 2.3 (see[1]) *Let X be a Banach space with norm $\|\cdot\|_X$, and $K \subset \mathbb{R}^+$ be an interval. Consider the family of C^1 functionals on X*

$$I_\mu(u) = A(u) - \mu B(u), \quad \mu \in K, \tag{2.28}$$

with B nonnegative and either $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\|_X \to \infty$ and $I_\mu(0) = 0$. For any $\mu \in K$, we set

$$\Gamma_\mu = \big\{\gamma \in \big(C[0, 1], X\big) : \gamma(0) = 0, I_\mu\big(\gamma(1)\big) < 0\big\}. \tag{2.29}$$

If for any $\mu \in K$, the set Γ_μ is nonempty, and

$$c_\mu = \inf_{\gamma\in\Gamma_\mu} \max_{t\in[0,1]} I_\mu\big(\gamma(t)\big) > 0, \tag{2.30}$$

then, for almost every $\mu \in K$, there is a sequence $\{u_n\} \subset X$ such that (i) *$\{u_n\}$ is bounded*; (ii) $I_\mu(u_n) \to c_\mu$; (iii) $I'_\mu(u_n) \to 0$ *in* X^{-1}.

In our case,

$$A(u) = \frac{a}{p}\|u\|^p + \frac{\lambda}{p(m+1)}\eta_T(u)\|u\|^{p(m+1)}, \qquad B(u) = \int_{\mathbb{R}^N}\big(F(u) + hu\big)\,dx. \tag{2.31}$$

So the perturbed functional we study is

$$I_\mu^T(u) = \frac{a}{p}\|u\|^p + \frac{\lambda}{p(m+1)}\eta_T(u)\|u\|^{p(m+1)} - \mu\int_{\mathbb{R}^N}\big(F(u) + hu\big)\,dx, \tag{2.32}$$

and

$$\big(I_\mu^T(u)\big)'v = \widehat{M}\big(\|u\|\big)\int_{\mathbb{R}^N}\big(|\nabla u|^{p-2}\nabla u\nabla v + |u|^{p-2}uv\big)\,dx - \mu\int_{\mathbb{R}^N}\big(f(u) + h\big)v\,dx, \tag{2.33}$$

where $\widehat{M}(\|u\|) = (a + \lambda\eta_T(u)\|u\|^{pm} + \frac{\lambda}{(m+1)T^p}\eta'_T(u)\|u\|^{p(m+1)})$. The following lemmas, Lemma 2.4 and Lemma 2.5, imply that I_μ^T satisfies the conditions of Lemma 2.3.

Lemma 2.4 *Let* (F_1)-(F_3) *hold, Then $\Gamma_\mu \neq \emptyset$ for all $\mu \in [\frac{1}{2}, 1]$.*

Proof Choose $\beta(x) \in C_0^1(\mathbb{R}^N)$ with $\beta(x) \geq 0$ in $\mathbb{R}^N$, $\|\beta\| = 1$, and $\operatorname{supp}(\beta) \subset B_R$ for some $R > 0$. By (F_3) we have that, for any $C_3 > 0$ with $C_3/2\int_{B_R}\beta^p\,dx > a/p$, there exists $C_4 > 0$ such that

$$F(t) \geq C_3|t|^p - C_4, \quad t \in \mathbb{R}^+. \tag{2.34}$$

Then, for $t^p > 2T^p$,

$$\begin{aligned} I_\mu^T(t\beta) &= \frac{a}{p}\|t\beta\|^p + \frac{\lambda}{p(m+1)}\psi\left(\frac{\|t\beta\|^p}{T^p}\right)\|t\beta\|^{p(m+1)} - \mu\int_{\mathbb{R}^N}\big(F(t\beta) + ht\beta\big)\,dx \\ &= \frac{a}{p}\|t\beta\|^p - \mu\int_{\mathbb{R}^N}\big(F(t\beta) + ht\beta\big)\,dx \leq \left(\frac{a}{p} - \frac{C_3}{2}\int_{B_R}\beta^p\,dx\right)t^p + C_5. \end{aligned} \tag{2.35}$$

It follows that we can choose $t > 0$ large enough such that $I_\mu^T(t\beta) < 0$. The proof is completed. □

Lemma 2.5 *Let* (F_1)-(F_3) *hold. Then there exists a constant $c > 0$ such that $c_\mu \geq c > 0$ for all $\mu \in [\frac{1}{2}, 1]$ if $\|h\|_{p'} < m_1$.*

Proof Similarly as in the proof of Lemma 2.1, we can show that, for every $\mu \in [\frac{1}{2}, 1]$, there exists $c > 0$ such that $I_\mu^T(u) \geq c$ with $\|u\| = \tilde{\rho}$ and $\|h\|_{p'} < m_1$. Fix $\mu \in [\frac{1}{2}, 1]$ and $\gamma \in \Gamma_\mu$. By the definition of Γ_μ, $\|\gamma(1)\| > \tilde{\rho}$. By the continuity we deduce that there exists $t_\gamma \in (0,1)$ such that $\|\gamma(t_\gamma)\|_E = \tilde{\rho}$. Therefore, for any $\mu \in [\frac{1}{2}, 1]$,

$$c_\mu = \inf_{\gamma\in\Gamma_\mu} \max_{t\in[0,1]} I_\mu^T\big(\gamma(t)\big) \geq \inf_{\gamma\in\Gamma_\mu} I_\mu^T\big(\gamma(t_\gamma)\big) \geq c > 0, \tag{2.36}$$

which completes the proof. □

Lemma 2.6 *For any $\mu \in [\frac{1}{2}, 1]$ and $a > 2^{m+1}(\frac{m+3}{m+1})\lambda T^{pm}$, each bounded (PS) sequence of the functional I_μ^T admits a convergent subsequence.*

Proof By Lemmas 2.3-2.5, we obtain that, for a.e. $\mu \in [1/2, 1]$, there is a bounded sequence $\{u_n\}$ in X that satisfies

$$I_\mu^T(u_n) \to c_\mu, \qquad \big(I_\mu^T(u_n)\big)' \to 0 \quad \text{in } X^*, \quad \text{and} \quad \sup_n \|u_n\| < T. \tag{2.37}$$

Since the embedding $X \hookrightarrow L^q(\mathbb{R}^N)$ is compact for $q \in (p, p^*)$, passing to a subsequence, we can assume that

$$u_n \rightharpoonup u, \quad \text{in } X; \qquad u_n \to u, \quad L^q\big(\mathbb{R}^N\big); \qquad u_n \to u, \quad \text{a.e. in } \mathbb{R}^N. \tag{2.38}$$

By (2.16) we can get

$$\big(I_\mu^T(u_n) - I_\mu^T(u)\big)'(u_n - u) = A_n + B_n + \mu C_n, \tag{2.39}$$

where

$$\begin{aligned}
A_n &= \widehat{M}(u_n) \int_{\mathbb{R}^N} \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big)\nabla(u_n - u) \\
&\quad + \big(|u_n|^{p-2}u_n - u^{p-2}u\big)(u_n - u)\,dx, \\
B_n &= \big(\widehat{M}(u_n) - \widehat{M}(u)\big) \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla(u_n - u) + |u|^{p-2}u(u_n - u)\,dx, \\
C_n &= \int_{\mathbb{R}^N} \big(f(u_n) - f(u)\big)(u_n - u)\,dx.
\end{aligned} \tag{2.40}$$

It is clear that

$$\big(I_\mu^T(u_n) - I_\mu^T(u)\big)'(u_n - u) \to 0 \quad \text{as } n \to \infty. \tag{2.41}$$

An analogous argument as in (2.22) and (2.25) gives us that

$$B_n \to 0 \quad \text{and} \quad C_n \to 0 \quad \text{as } n \to \infty. \tag{2.42}$$

Combining the above results and $a > 2^{m+1}(\frac{m+3}{m+1})\lambda T^{pm}$, we have that $|A_n| \to 0$ as $n \to \infty$. Then, using a standard equality ([3], Lemma 2.1), we can show that $u_n \to u$ in X. The proof is completed. □

Lemma 2.7 *Assume* (F_1)-(F_3) *and* $a > 2^{m+1}(\frac{m+3}{m+1})\lambda T^{pm}$. *Then, for almost every* $\mu \in [\frac{1}{2},1]$, *there exist* $u^\mu \in X \setminus \{0\}$ *such that* $(I_\mu^T)'(u^\mu) = 0$ *and* $I_\mu^T(u^\mu) = c_\mu$ *with* $\|h\|_{p'} < m_1$.

Proof It follows from Lemmas 2.3-2.5 that, for every $\mu \in [\frac{1}{2},1]$, there exists a bounded sequence $\{u_n^\mu\} \subset X$ such that

$$I_\mu^T(u_n^\mu) \to c_\mu \quad \text{and} \quad (I_\mu^T)'(u_n^\mu) \to 0 \quad \text{as } n \to \infty.$$

By Lemma 2.6 we can suppose that $u^\mu \in X$ and $u_n^\mu \to u^\mu$ in X. The proof is completed. □

According to Lemma 2.6, there exists a sequence $\{\mu_n\} \subset [\frac{1}{2},1]$ with $\mu_n \to 1$ and $\{u_n\} \subset X$ as $n \to \infty$ such that $I_{\mu_n}^T(u_n) = c_{\mu_n}$, $(I_{\mu_n}^T)'(u_n) = 0$, and u_n is a positive solution of

$$\widehat{M}(\|u\|)(-\Delta_p u + |u|^{p-2}u) = \mu_n(f(u) + h(x)). \tag{2.43}$$

In the following, to obtain $\|u_n\| < T$, we establish an identity that extends the Kazin-Pohozav identity in ([13], Thm. 29.4) with $p = 2$.

Lemma 2.8 *Assume that* $f(x,u) : \mathbb{R}^N \times \mathbb{R}^1 \to \mathbb{R}^1$ *is a Caretheódary function,* $u \in C^2_{\mathrm{loc}}(\mathbb{R}^N)$ *is a solution of*

$$\begin{cases} -\Delta_p u + f(x,u) = 0 & in\ \mathbb{R}^N, \\ u(x) \to 0 & as\ \to 0, \end{cases} \tag{2.44}$$

$\frac{\partial u}{\partial x_i} \in L^p(\mathbb{R}^N)$, $i = 1,2,\ldots$, *and* $F(x,u), F_1(x,u) \in L^1(\mathbb{R}^N)$. *Then*

$$\frac{N-p}{p}\int_{\mathbb{R}^N} |\nabla u|^p\,dx + \int_{\mathbb{R}^N} (NF(x,u) + F_1(x,u))\,dx = 0, \tag{2.45}$$

where $F(x,u) = \int_0^u f(x,s)\,ds$ *and* $F_1(x,u) = \sum_{i=1}^N x_i \frac{\partial F(x,u)}{\partial x_i}$.

Proof Multiplying equation (2.44) by $x \cdot \nabla u$ and integrating over the ball B_R, we obtain

$$\int_{B_R} f(x,u)x \cdot \nabla u\,dx = \int_{B_R} \operatorname{div}(|\nabla u|^{p-2}\nabla u)x \cdot \nabla u\,dx. \tag{2.46}$$

Then

$$\begin{aligned} \int_{B_R} f(x,u)x \cdot \nabla u\,dx &= \sum_{i=1}^N \int_{B_R} x_i f(x,u)\frac{\partial u}{\partial x_i}\,dx \\ &= \sum_{i=1}^N \int_{B_R} \left(\frac{\partial}{\partial x_i}(x_i F(x,u)) - \left(F(x,u) + x_i\frac{\partial F(x,u)}{\partial x_i}\right)\right)dx \\ &= \sum_{i=1}^N \int_{\partial B_R} F(x,u)x_i n_i\,ds - \int_{B_R} (NF(x,u) + F_1(x,u))\,dx \\ &= R\int_{\partial B_R} F(x,u)\,ds - \int_{B_R} (NF(x,u) + F_1(x,u))\,dx, \end{aligned} \tag{2.47}$$

where n_i are the components of the unit outward normal to ∂B_R, and ds is an area element. On the other hand, integrating by parts, we obtain

$$\begin{aligned}
&\int_{B_R} \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)x\cdot\nabla u\,dx \\
&= \sum_{j=1}^{N}\int_{B_R}\frac{\partial}{\partial x_j}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\right)\sum_{i=1}^{N}x_i\frac{\partial u}{\partial x_i}\,dx \\
&= \sum_{j=1}^{N}\int_{B_R}\left(\frac{\partial}{\partial x_j}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\sum_{i=1}^{N}x_i\frac{\partial u}{\partial x_i}\right)-|\nabla u|^{p-2}\frac{\partial u}{\partial x_j}\left(\frac{\partial}{\partial x_j}\sum_{i=1}^{N}x_i\frac{\partial u}{\partial x_i}\right)\right)dx \\
&= \int_{\partial B_R}|\nabla u|^{p-2}\frac{\partial u}{\partial n}x\cdot\nabla u\,ds-\int_{B_R}|\nabla u|^{p}\,dx \\
&\quad-\int_{B_R}\sum_{j=1}^{N}|\nabla u|^{p-2}\left(\sum_{i=1}^{N}x_i\frac{\partial^2 u}{\partial x_i\,\partial x_j}\frac{\partial u}{\partial x_j}\right)dx.
\end{aligned}\tag{2.48}$$

On B_R, we have $\nabla u=\frac{\partial u}{n}\cdot\vec{n}=\frac{\partial u}{\partial n}\frac{x}{R}$ and

$$\int_{\partial B_R}|\nabla u|^{p-2}\frac{\partial u}{\partial n}x\cdot\nabla u\,dx=R\int_{\partial B_R}|\nabla u|^{p}\,ds.\tag{2.49}$$

Further, we have

$$\begin{aligned}
&\int_{B_R}\sum_{j=1}^{N}|\nabla u|^{p-2}\left(\sum_{i=1}^{N}x_i\frac{\partial^2 u}{\partial x_i\,\partial x_j}\frac{\partial u}{\partial x_j}\right)dx \\
&= \frac{1}{p}\sum_{i=1}^{N}\int_{B_R}\left(\frac{\partial}{\partial x_i}\big(x_i|\nabla u|^{p}\big)-|\nabla u|^{p}\right)dx \\
&= \frac{R}{p}\int_{\partial B_R}|\nabla u|^{p}\,ds-\frac{N}{p}\int_{B_R}|\nabla u|^{p}\,dx.
\end{aligned}\tag{2.50}$$

Therefore, we obtain

$$R\int_{\partial B_R}\left(F-\left(1-\frac{1}{p}\right)|\nabla u|^{p}\right)ds+\left(1-\frac{N}{p}\right)\int_{B_R}|\nabla u|^{p}\,dx-\int_{B_R}(NF+F_1)\,dx=0.\tag{2.51}$$

Since $F(x,u)\in L^1(\mathbb{R}^N)$ and $u\in X$, we claim that

$$\liminf_{n\to\infty}R\int_{\partial B_R}\big(\big|F(x,u)\big|+|\nabla u|^{p}\big)\,dS=0.\tag{2.52}$$

Indeed, otherwise,

$$\liminf_{n\to\infty}R\int_{\partial B_R}\big(\big|F(x,u)\big|+|\nabla u|^{p}\big)\,dS=a_0>0.\tag{2.53}$$

Then, there exists $R_0>0$ such that, for $R\geq R_0$,

$$R\int_{\partial B_R}\big(\big|F(x,u)\big|+|\nabla u|^{p}\big)\,dS\geq\frac{a_0}{2}.\tag{2.54}$$

Let $R_n = R_0 + n$, $n = 1, 2, \ldots$. Then $R_n \to \infty$ as $n \to \infty$. It follows from the integral mean theorem that there is $\xi_n \in (R_{n-1}, R_n)$ and $\xi_n \geq R_0$ such that, for $R \geq R_0$,

$$\int_{R_{n-1}}^{R_n} \int_{\partial B_R} \left(|F| + |\nabla u|^p\right) ds\, dR = \xi_n \int_{\partial B_{\xi_n}} \left(|F| + |\nabla u|^p\right) ds \geq \frac{a_0}{2}, \tag{2.55}$$

and thus

$$\int_{R_0}^{\infty} \int_{\partial B_R} \left(|F| + |\nabla u|^p\right) ds\, dR \geq \sum_{n=2}^{\infty} \int_{R_{n-1}}^{R_n} \int_{\partial B_R} \left(|F| + |\nabla u|^p\right) ds\, dR = \infty. \tag{2.56}$$

This contradicts the fact

$$\int_{\mathbb{R}^N} \left(|F| + |\nabla u|^p\right) dx = \int_0^{\infty} \int_{\partial B_R} \left(|F| + |\nabla u|^p\right) ds\, dR < \infty. \tag{2.57}$$

Therefore, (2.52) is true. Thus, letting $R \to \infty$ in (2.51), we have

$$\frac{N-p}{p} \int_{\mathbb{R}^N} |\nabla u|^p\, dx + \int_{\mathbb{R}^N} \left(NF(x,u) + F_1(x,u)\right) dx = 0. \tag{2.58}$$

Then, we finish the proof of Lemma 2.8. □

Lemma 2.9 *Let $a > 2^{m+1}(\frac{m+3}{m+1})\lambda T^{pm}$, and let $u \in X$ be a weak solution of*

$$\widehat{M}\left(\|u\|\right)\left(-\Delta_p u + |u|^{p-2}u\right) = \mu\left(f(u) + h(x)\right), \tag{2.59}$$

where $\widehat{M}(\|u\|) = (a + \lambda\eta_T(u)\|u\|^{pm} + \frac{\lambda}{(m+1)T^p}\eta'_T(u)\|u\|^{p(m+1)})$. Then the following identity holds:

$$\widehat{M}\left(\|u\|\right)\left(\frac{N-p}{p} \int_{\mathbb{R}^N} |\nabla u|^p\, dx + \frac{N}{p} \int_{\mathbb{R}^N} |u|^p\, dx\right)$$
$$= N\mu \int_{\mathbb{R}^N} \left(F(u) + hu\right) dx + \mu \int_{\mathbb{R}^N} \nabla h \cdot xu\, dx. \tag{2.60}$$

Proof Since $u \in X$ is a weak solution of (2.59), by standard regularity results, $u \in C^2_{\mathrm{loc}}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$. Let

$$g(x,u) = \frac{\mu(f(u) + h(x))}{\widehat{M}(\|u\|)} - |u|^{p-2}u. \tag{2.61}$$

Then $u \in X$ is also a solution of

$$-\Delta_p u = g(x,u). \tag{2.62}$$

By Lemma 2.8,

$$\frac{N-p}{p} \int_{\mathbb{R}^N} |\nabla u|^p\, dx = \int_{\mathbb{R}^N} \left(NG(u) + G_1(x,u)\right) dx, \tag{2.63}$$

where $G(x,u) = \int_0^u g(x,s)\, ds$ and $G_1(x,u) = \sum_{i=1}^N x_i \frac{\partial G(x,u)}{\partial x_i}$. Then the conclusion holds. □

Lemma 2.10 *Assume that (F_1)-(F_3) and (H) hold and that $\|h\|_{p'} < m_1$ for m_1 given in Lemma 2.6. Let u_n be a critical point of $I^T_{\mu_n}$ at level c_{μ_n}. Then for T sufficiently large, there exists $\lambda_0 = \lambda_0(T)$ with $\lambda_0 < a(\frac{m+1}{m+3})T^{-pm}$ such that, for any $\lambda \in [0, \lambda_0)$, subject to a subsequence, $\|u_n\| < T$ for all $n \in \mathbb{N}$.*

Proof Since $(I^T_{\mu_n})'(u_n) = 0$, by Lemma 2.9 u_n satisfies

$$\begin{aligned} &\widehat{M}\big(\|u\|\big)\left(\frac{N}{p}\|u\|^p + \int_{\mathbb{R}^N} |\nabla u|^p\,dx\right) \\ &= N\mu_n \int_{\mathbb{R}^N} \big(F(u_n) + hu_n\big)\,dx + \mu_n \int_{\mathbb{R}^N} \nabla h \cdot xu_n\,dx. \end{aligned} \tag{2.64}$$

Using $I^T_{\mu_n}(u_n) = c_{\mu_n}$, we have

$$\frac{aN}{p}\|u_n\|^p + \frac{\lambda N}{p(m+1)}\eta_T(u_n)\|u_n\|^{p(m+1)} = N\mu_n \int_{\mathbb{R}^N} \big(F(u_n) + hu_n\big)\,dx + Nc_{\mu_n}. \tag{2.65}$$

Therefore, by (2.64), (2.65) and $a > 2^{m+1}(\frac{m+3}{m+1})\lambda T^{pm}$ we deduce that

$$\begin{aligned} &\frac{a}{2}\int_{\mathbb{R}^N} |\nabla u_n|^p\,dx \\ &\le \widehat{M}\big(\|u_n\|\big)\int_{\mathbb{R}^N} |\nabla u_n|^p\,dx \\ &= Nc_{\mu_n} + N\left(\widehat{M}\big(\|u_n\|\big) - \frac{a}{p}\right)\|u\|^p - \frac{\lambda N}{p(m+1)}\eta_T(u_n)\|u_n\|^{p(m+1)} - \mu_n \int_{\mathbb{R}^N} \nabla hx \cdot u_n\,dx \\ &\le Nc_{\mu_n} + \frac{\lambda Nm}{p(m+1)}\eta_T(u_n)\|u_n\|^{p(m+1)} + \frac{\lambda N}{p(m+1)T^p}\eta'_T(u_n)\|u_n\|^{p(m+2)} \\ &- \mu_n \int_{\mathbb{R}^N} \nabla hx \cdot u_n\,dx. \end{aligned} \tag{2.66}$$

By the min-max definition of the mountain pass level, Lemma 2.5, and (2.35) we have

$$\begin{aligned} c_{\mu_n} &\le \max_t I^T_{\mu_n}(t\beta) \\ &\le \max_t \left\{\left(\frac{a}{p} - \frac{C_3}{2}\int_{B_R} |\beta|^p\,dx\right)t^p + C_5\right\} + \max_t \frac{\lambda}{p(m+1)}\psi\left(\frac{t^p}{T^p}\right)t^{p(m+1)} \\ &\le \frac{\lambda 2^{m+1}}{p(m+1)}T^{p(m+1)} + C_5. \end{aligned} \tag{2.67}$$

Using (H) and the Young equality, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla h \cdot xu_n\,dx &\le \frac{1}{p'}\int_{\mathbb{R}^N} |\xi|^{p'}\,dx + \frac{1}{p}\int_{\mathbb{R}^N} |\xi|^{p'}|u_n|^p\,dx \\ &\le \frac{1}{p}\int_{\mathbb{R}^N} |\xi|^{p'}|u_n|^p\,dx + C_6. \end{aligned} \tag{2.68}$$

We can easily calculate that

$$\eta_T(u_n)\|u_n\|^{p(m+1)} \le 2^{m+1}T^{p(m+1)}, \qquad \eta'(u_n)\|u_n\|^{p(m+2)} \le 2^{m+2}T^{p(m+2)}. \tag{2.69}$$

Combining the above estimates, we see that

$$\frac{a}{2}\int_{\mathbb{R}^N} |\nabla u_n|^p\,dx \le \frac{\lambda N(m+5)}{p(m+1)}2^{m+1}T^{p(m+1)} + \frac{1}{p}\int_{\mathbb{R}^N} |\xi|^{p'}|u_n|^p\,dx + C_7. \tag{2.70}$$

Since $\xi(x) \in L^{p'}(\mathbb{R}^N) \cap W^{1,\infty}$, we see that $\xi^{p'}u_n \in X$. It follows from $(I^T_{\mu_n}(u_n))'(\xi^{p'}u_n) = 0$ that

$$\begin{aligned}
&\widehat{M}\bigl(\|\xi^{p'}u_n\|\bigr)\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n\nabla\bigl(\xi^{p'}u_n\bigr) + |u_n|^{p-2}u\bigl(\xi^{p'}u_n\bigr)\,dx \\
&\quad = \mu_n\int_{\mathbb{R}^N}\bigl(f(u_n)+h\bigr)\xi^{p'}u_n\,dx.
\end{aligned} \tag{2.71}$$

Since $a > 2^{m+1}(\frac{m+3}{m+1})\lambda T^{pm}$, we have $(3a/2) \ge \widehat{M}(\|\xi^{p'}u_n\|)$, and it follows from (2.69) and (2.71) that

$$(3a/2)\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n\nabla\bigl(\xi^{p'}u_n\bigr) + |u_n|^p\xi^{p'}\,dx \ge (1/2)\int_{\mathbb{R}^N} f(u_n)u_n\xi^{p'}\,dx. \tag{2.72}$$

From (2.70) by the Hölder inequality we deduce that

$$\begin{aligned}
&3a\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n\nabla\bigl(\xi^{p'}u_n\bigr)\,dx \\
&\quad \le 3a\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n\bigl(p'\xi^{p'-1}u_n\nabla\xi + \xi^{p'}\nabla u_n\bigr)\,dx \\
&\quad \le 3\bigl(\|\xi\|^\infty_{p'} + \|\nabla\xi\|^\infty_{p'}\bigr)\left(a\int_{\mathbb{R}^N} |\nabla u_n|^p\,dx\right) + 3a(p-1)^{-1}\int_{\mathbb{R}^N} \xi^{p'}|u_n|^p\,dx \\
&\quad \le C\lambda T^{p(m+1)} + C\int_{\mathbb{R}^N} \xi^{p'}|u_n|^p\,dx + C,
\end{aligned} \tag{2.73}$$

where C is a constant independent of λ and T.

By (F_3), for any $L > 0$, there exists $C(L) > 0$ such that

$$f(s)s \ge Ls^p - C(L) \quad \text{for all } s > 0. \tag{2.74}$$

Combining (2.72)-(2.74), we get

$$\left(\frac{1}{2}L - C\right)\int_{\mathbb{R}^N} \xi^{p'}|u_n|^p\,dx \le C\lambda T^{p(m+1)} + C. \tag{2.75}$$

For $L > 0$ large enough, we obtain

$$\int_{\mathbb{R}^N} \xi^{p'}|u_n|^p\,dx \le C\lambda T^{p(m+1)} + C. \tag{2.76}$$

It follows from (2.70) and (2.76) that

$$\int_{\mathbb{R}^N} |\nabla u_n|^p\,dx \le C\lambda T^{p(m+1)} + C. \tag{2.77}$$

On the other hand,

$$\begin{aligned}
&a\|u_n\|^p + \eta_T(u_n)\|u_n\|^{p(m+1)} + \frac{\lambda}{m+1}\eta_T'(u_n)\|u_n\|^{p(m+2)} \\
&= \mu_n \int_{\mathbb{R}^N} \big(f(u_n)u_n + hu_n\big)\,dx \\
&\leq \varepsilon\|u_n\|^p + C_\varepsilon\|u_n\|_{p^*}^{p^*} + \frac{1}{p'}\|h\|_{p'}^{p'} + \frac{1}{p}\|u\|^p.
\end{aligned} \tag{2.78}$$

By (2.77) and (2.78) we have

$$\begin{aligned}
(a-\varepsilon-1/p)\|u_n\|^p &\leq C_\varepsilon\|u_n\|_{p*}^{p*} - \lambda/\big((m+1)T^p\big)\eta_T'(u_n)\|u_n\|^{p(m+2)} + C \\
&\leq C\|\nabla u_n\|_p^{p*} + \lambda 2^{m+2}(m+1)^{-1}T^{p(m+1)} + C \\
&\leq C\lambda T^{p^*(m+1)} + C\lambda T^{p(m+1)} + C.
\end{aligned} \tag{2.79}$$

Suppose that $\|u_n\| > T$ for $n \in \mathbb{N}$ and T large enough. Then

$$T^p < \|u_n\|^p \leq C\lambda T^{p^*(m+1)} + C\lambda T^{p(m+1)} + C, \tag{2.80}$$

which is not true if we choose T large and λ small enough. So by setting $\lambda(T)$ small we obtain that the sequence $\{u_n\}$ is bounded for any $\lambda \in [0,\lambda_0)$, and the conclusion holds. □

Lemma 2.11 *Let T, λ_0 be defined by Lemma* 2.10, *and u_n be the critical point of $I_{\mu_n}^T$ at level c_{μ_n}. Then the sequence $\{u_n\}$ is also a* (*PS*) *sequence for I.*

Proof From the proof of Lemma 2.10 we may assume that $\|u_n\| \leq T$. So

$$I(u_n) = I_{\mu_n}^T(u_n) + (\mu_n - 1)\int_{\mathbb{R}^N} \big(F(u_n) + hu_n\big)\,dx. \tag{2.81}$$

Since $\mu_n \to 1$, we can show that $\{u_n\}$ is a (PS) sequence of I. Indeed, the boundedness of $\{u_n\}$ implies that $\{I_{\mu_n}^T\}$ is bounded. Also,

$$I'(u_n)v = \big(I_{\mu_n}^T\big)'(u_n, v) + (\mu_n - 1)\int_{\mathbb{R}^N} \big(f(u_n) + h(u_n)\big)v\,dx, \quad v \in X. \tag{2.82}$$

Thus, $I'(u_n) \to 0$, and $\{u_n\}$ is a bounded (PS) sequence of I. By Lemma 2.5, $\{u_n\}$ has a convergent subsequence. We may assume that $u_n \to \tilde{u}_0$. Consequently, $I'(\tilde{u}_0) = 0$. According to Lemma 2.4, we have that $I(\tilde{u}_0) = \lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} I_{\mu_n}^T(u_n) \geq c > 0$ and $\tilde{u}_0$ is a solution of problem (1.1). Thus, we completed the proof. □

Proof of Theorem 1.1 By Lemma 2.2 the problem has a solution $u_0 \in X$ with $I(u_0) < 0$. From Lemma 2.9 we know that problem (1.1) possesses a second solution $\tilde{u}_0 \in X$ with $I(\tilde{u}_0) \geq c > 0$. Hence, $u_0 \neq \tilde{u}_0$, and we complete the proof of Theorem 1.1. □

3 Proof of Theorem 1.2

Let $I_\lambda(u) : X \to \mathbb{R}$ be the energy functional associated with problem (1.13) defined by

$$I_\lambda(u) = \frac{a}{p}\|u\|^p + \frac{\lambda}{p(m+1)}\|u\|^{p(m+1)} - \frac{1}{q}\|u\|_q^q, \tag{3.1}$$

where $F(u) = \int_0^u f(s)\,ds$. It is easy to see that the functional $I \in C^1(E,\mathbb{R})$ and its Gateaux derivative is given by

$$\begin{aligned} I'_\lambda(u)v &= \left(a + \lambda\|u\|^{pm}\right)\int_{\mathbb{R}^N}\left(|\nabla u|^{p-2}\nabla u\nabla v + |u|^{p-2}uv\right)dx \\ &\quad - \int_{\mathbb{R}^N}|u|^{q-2}uv\,dx, \quad \forall v \in E. \end{aligned} \tag{3.2}$$

Clearly, we see that a weak solution of (1.13) corresponds to a critical point of the functional.

In this part, we first proof the nonexistence for problem (1.13) for large $\lambda > \lambda^*$. which means that if a solution exists, then λ must sufficiently small. Secondly, we obtain that there exists λ^{**} such that problem (1.1) has at least one solution for any $0 < \lambda < \lambda^{**}$. Finally, by the properties of λ^* and λ^{**} we deduce that $\lambda^* = \lambda^{**}$. We will break the proof into six steps.

Proof of Theorem 1.2 *Step 1. Nonexistence for large* $\lambda > 0$. It is sufficient to show that if u is a nontrivial solution of problem (1.13), then $\lambda > 0$ must be small. Assume that u is a nontrivial solution of problem (1.1). Then we get $I'_\lambda(u)u = 0$, that is,

$$a\|u\|^p + \lambda\|u\|^{p(m+1)} = \|u\|_q^q. \tag{3.3}$$

Since $p < q < \min\{p(m+1), p^*\}$, applying the Young inequality and (1.9), we deduce that

$$a\|u\|^p + \lambda\|u\|^{p(m+1)} = \|u\|_q^q \le S_q^q\|u\|_E^q \le a\|u\|_E^p + \lambda_1\|u\|_E^{p(m+1)}, \tag{3.4}$$

which implies that $\lambda \le \lambda_1 = (S_q^q)^{\frac{pm}{q-p}} a^{-\frac{p(m+1)-q}{q-p}}$. On the other hand, if $\lambda^* \ge \lambda_1$, then we conclude that problem (1.1) has no solution for any $\lambda \in (\lambda^*, +\infty)$.

Step 2. Coercivity of $I_\lambda(u)$. Indeed, for any $u \in E$ and all $\lambda > 0$,

$$\begin{aligned} I_\lambda(u) &= \frac{a}{p}\|u\|^p + \frac{\lambda}{p(m+1)}\|u\|^{p(m+1)} - \frac{1}{q}\|u\|_q^q \\ &\ge \frac{a}{p}\|u\|^p + \frac{\lambda}{2p(m+1)}\|u\|^{p(m+1)} + \frac{\lambda}{2p(m+1)}\|u\|^{p(m+1)} - \frac{S_q^q}{q}\|u\|^q. \end{aligned} \tag{3.5}$$

Since $q < p(m+1)$, there exists $C_1 = C_1(\lambda, q, m, S_q)$ such that

$$\frac{S_q^q}{q}\|u\|^q \le \frac{\lambda}{2p(m+1)}\|u\|^{p(m+1)} + C_1. \tag{3.6}$$

It follows that

$$I_\lambda(u) \ge \frac{a}{p}\|u\|^p + \frac{\lambda}{2p(m+1)}\|u\|^{p(m+1)} - C_1. \tag{3.7}$$

This implies that $I_\lambda(u)$ is coercive.

Step 3. The infimum of I_λ *is attained.* Let $\{u_n\}$ be a minimizing sequence of I_λ. Then from Step 2 we immediately see that $\{u_n\}$ is bounded in X . Therefore, without loss of generality, we may assume that $\{u_n\}$ is nonnegative and converges weakly and pointwise to some u in X.

Using the compact embedding $X \hookrightarrow L^q(\mathbb{R}^N)$, we have

$$\|u\|_q = \lim_{n\to\infty} \|u_n\|_q \quad \text{and} \quad \|u\| \le \liminf_{n\to\infty} \|u_n\| \tag{3.8}$$

by the weak lower semicontinuity of the norm $\|\cdot\|$. Thus,

$$\begin{aligned} I_\lambda(u) &= \frac{a}{p}\|u\|^p + \frac{\lambda}{p(m+1)}\|u\|^{p(m+1)} - \frac{1}{q}\|u\|_q^q \\ &\le \liminf_{n\to\infty}\left(\frac{a}{p}\|u_n\|^p + \frac{\lambda}{p(m+1)}\|u_n\|^{p(m+1)}\right) - \frac{1}{q}\lim_{n\to\infty}\|u_n\|_q^q \\ &\le \liminf_{n\to\infty}\left(\frac{a}{p}\|u_n\|^p + \frac{\lambda}{p(m+1)}\|u_n\|^{p(m+1)} - \frac{1}{q}\|u_n\|_q^q\right) = \liminf_{n\to\infty} I_\lambda(u_n). \end{aligned} \tag{3.9}$$

Therefore, u is a global minimum for I_λ, and hence it is a critical point, namely a weak solution to problem (1.1).

Step 4. The weak solution u is nontrivial if $\lambda > 0$ *is sufficiently small.* Clearly, $I_\lambda(0) = 0$. Therefore, it is sufficient to show that there exists $\lambda_0 > 0$ such that

$$\inf_{u\in E} I_\lambda(u) < 0, \quad \text{for any } \lambda \in (0, \lambda_0). \tag{3.10}$$

Choose $u_0 \in C_0^\infty(\mathbb{R}^N)$, $u_0 \not\equiv 0$, such that $\|u_0\|_E = 1$. Denote

$$I_\lambda(tu_0) = t^p s(t), \qquad s(t) = B_1 + \lambda B_2 t^{pm} - B_3 t^{q-p}, \quad t \ge 0, \tag{3.11}$$

where

$$B_1 = \frac{a}{p}, \qquad B_2 = \frac{1}{p(m+1)} > 0, \qquad B_3 = \frac{1}{q}\int_{\mathbb{R}^N} |u_0|^q\,dx > 0.$$

Then there exist $\lambda_0 > 0$ and large $t_\lambda > 0$ such that $I_\lambda(t_\lambda u_0) < 0$ for $\lambda \in (0, \lambda_0]$. Let $e = t_\lambda u_0$. Then $\|e\| = t_\lambda$ and $I_\lambda(e) < 0$. This implies that (3.10) is true. So the weak solution u is nontrivial if $\lambda > 0$ is sufficiently small.

Now, we define

$$\lambda^{**} = \sup\{\lambda > 0, \text{problem (1.13) admits a nontrival weak solution}\},$$

$$\lambda^{*} = \inf\{\lambda > 0, \text{problem (1.13) does not admit any nontrival weak solution}\}.$$

Clearly, $\lambda^{**} \ge \lambda^*$. To complete the proof of Theorem 1.2, it suffices to prove the following facts: (a) problem (1.13) has a weak solution for any $\lambda < \lambda^{**}$; (b) $\lambda^{**} = \lambda^*$, and problem (1.13) admits a weak solution when $\lambda = \lambda^*$.

*Step 5. Problem (*1.13*) has a solution for any* $\lambda < \lambda^{**}$ *and* $\lambda^* = \lambda^{**}$. Fix $\lambda < \lambda^{**}$. By the definition of λ^{**}, there exists $\mu \in (\lambda, \lambda^{**})$ such that I_λ has a nontrivial critical point $u_\mu \in E$. Clearly, we have

$$\left(a + \lambda\left(\int_{\mathbb{R}^N} \left(|\nabla u_m u|^p + |u_\mu|^p\right)dx\right)^m\right)\left(-\Delta_p u_\mu + |u_\mu|^{p-2}u_\mu\right) \le |u_\mu|^{q-2}u_\mu. \tag{3.12}$$

This implies that u_μ is a subsolution of problem (1.13). In order to find a supsolution of (1.13) that dominates u_μ, we consider the constrained minimization problem

$$\inf\left\{\frac{a}{p}\|\omega\|^p + \frac{\lambda}{p(m+1)}\|\omega\|^{p(m+1)} - \frac{1}{q}\|\omega\|_q^q : \omega \in E, \|\omega\|_q^q = q \text{ and } \omega \geq u_\mu\right\}. \tag{3.13}$$

Arguments similar to those used in Step 3 and Step 4 show that the above minimization has a solution $u_\lambda \geq u_\mu$, which is also a weak solution of problem (1.13). Hence, problem (1.13) admits a weak solution for any $\lambda \in [0, \lambda^{**})$, This means that $\lambda^* \geq \lambda^{**}$ by the definition of λ^*. But we already know that $\lambda^{**} \geq \lambda^*$, and therefore $\lambda^{**} = \lambda^*$.

*Step 6. Problem (*1.13*) admits a nontrivial solution when $\lambda = \lambda^*$.* Let $\{\lambda_n\}$ be a increasing sequence converging to λ^*, and $\{u_n\}$ be a sequence of solutions of (1.1) corresponding to λ_n. By Step 2, $\{u_n\}$ is bounded in X, and without loss of generality we may assume that $u_n \rightharpoonup u$ in X, $u_n \to u$ in $L^q(\mathbb{R}^N)$, and $u_n \to u^*$ a.e. in X. It follows from $I_\lambda(u_n)v = 0$ that, for any $v \in X$,

$$\left(a + \lambda_n\|u_n\|^{pm}\right)\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p-2}\nabla u_n \nabla v + |u_n|^{p-2}u_n v\right)dx = \int_{\mathbb{R}^N}|u_n|^{q-2}u_n v\,dx. \tag{3.14}$$

Then, passing to the limit as $n \to \infty$, we deduce that u^* satisfies $I_\lambda(u^*)v = 0$ when $\lambda = \lambda^*$. Now, it remains to prove that u^* is a nontrivial critical point for I_{λ^*}. From $I'_\lambda(u_n)u_n = 0$ it is easy to deduce that $\|u_n\| \geq (\lambda_n S_q^{-q})^{1/(q-p(m+1))}$, which implies that u_n has a lower bound. Next, since $\lambda_n \nearrow \lambda^*$ as $n \to \infty$, it suffices to show that $\|u_n - u^*\| \to 0$ as $n \to \infty$.

Since u_n and u^* are the solutions of (1.1) corresponding to λ_n and λ^*, we see that

$$0 = \left(I'_{\lambda_n}(u_n) - I'_{\lambda^*}\left(u^*\right)\right)(u_n - u) = X_n + Y_n - Z_n, \tag{3.15}$$

where

$$\begin{aligned}
X_n &= \left(a + \lambda_n\|u_n\|^{pm}\right)\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p-2}\nabla u_n - \left|\nabla u^*\right|^{p-2}\nabla u^*\right)\nabla\left(u_n - u^*\right)dx \\
&\quad + \left(|u_n|^{p-2}u_n - \left|u^*\right|^{p-2}u^*\right)\left(u_n - u^*\right)dx; \\
Y_n &= \left(\lambda_n\|u_n\|^{pm} - \lambda^*\left\|u^*\right\|^{pm}\right)\int_{\mathbb{R}^N}\left|\nabla u^*\right|^{p-2}\nabla u^*\nabla\left(u_n - u^*\right) \\
&\quad + \left|u^*\right|^{p-2}u^*\left(u_n - u^*\right)dx; \\
Z_n &= \int_{\mathbb{R}^N}\left(|u_n|^{q-2}u_n - \left|u^*\right|^{q-2}u^*\right)\left(u_n - u^*\right)dx.
\end{aligned}$$

By the Hölder inequality and compact embedding $u_n \to u$ in $L^q(\mathbb{R}^N, H)$ we have

$$\begin{aligned}
|X_n| &= \left|\int_{\mathbb{R}^N}\left(|u_n|^{q-2}u_n - \left|u^*\right|^{q-2}u^*\right)\left(u_n - u^*\right)dx\right| \\
&\leq \int_{\mathbb{R}^N}\left(|u_n|^{q-1} + \left|u^*\right|^{q-1}\right)\left|u_n - u^*\right|dx \\
&\leq C\left(\|u_n\|^{q-1} + \left\|u^*\right\|^{q-1}\right)\left\|u_n - u^*\right\|_q \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.16}$$

Next, consider the functional $j : X \to \mathbb{R}$ defined by

$$j(\omega) = \int_{\mathbb{R}^N} \left|\nabla u^*\right|^{p-2} \nabla u^* \nabla \omega + \left|u^*\right|^{p-2} u^* \omega \, dx. \tag{3.17}$$

Since $|j(\omega)| \leq 2\|u^*\|^{p-1}\|\omega\|$, j is continuous on X. Using $u_n \rightharpoonup u^*$ and the boundedness of u_n and u^* in X, we have that

$$|Y_n| \leq \left(\|u_n\|^{pm} + \left\|u^*\right\|^{pm}\right)\left|g\left(u_n - u^*\right)\right| \to 0 \quad \text{as } n \to \infty. \tag{3.18}$$

Combining (3.15), (3.16), and (3.18), this forces $X_n \to 0$ as $n \to \infty$. Then, using the standard inequality (2.25) in $\mathbb{R}^N$, we have that $\|u_n - u^*\| \to 0$ as $n \to \infty$, and thus u^* is a nontrivial weak solution of problem (1.13) corresponding to $\lambda = \lambda^*$. This completes the proof of Theorem 1.2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors contributed to each part of this study equally. Both authors read and proved the final vision of the manuscript.

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