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Existence of solutions for a class of fractional differential equations with integral and anti-periodic boundary conditions

Yan Qiao and Zongfu Zhou*

*Correspondence:
zhouzf12@126.com
School of Mathematical Sciences, Anhui University, Hefei, Anhui 230601, China

Abstract
In this paper, we use the Banach contraction mapping principle and Leray-Schauder degree theory to obtain some results of the existence and uniqueness of solution for a class of fractional boundary value problem with integral and anti-periodic boundary conditions.

Keywords: existence and uniqueness; fixed point theorem; Leray-Schauder degree theory; fractional boundary value problem

1 Introduction

Recently, fractional differential equations have been proved to be significance tools in the fields of economics, science and engineering such as materials and mechanical systems, control and robotics, *etc.* (see [1–6] and the references therein). It is found that fractional differential equations are applied in modeling for physical phenomena such as fluid flow, and signal and image processing. Boundary value problems of fractional equations have been considered in many papers (see [7–10] and the references therein).

In [11], Cababa and Wang considered a fractional boundary value problem with one integral and two zero initial conditions, the existence of positive solution is obtained by constructing a proper cone.

In [12], Agrawal and Ahmad discussed the following problem:

$$\begin{cases} {}^cD^q y(t) = f(t, y(t)), \quad t \in [0, T], T > 0, 3 < q \le 4, \\ y(0) = -y(T), \qquad y'(0) = -y'(T), \qquad y''(0) = -y''(T), \qquad y'''(0) = -y'''(T). \end{cases} \tag{1.1}$$

They got some existence results via topological degree theory.

In [13], Xu researched the following problem:

$$\begin{cases} {}^cD^q u(t) = f(t, u(t)), \quad t \in [0, 1], 1 < q < 2, \\ u(1) = \mu \int_0^1 u(s)\,ds, \qquad u'(0) + u'(1) = 0. \end{cases} \tag{1.2}$$

Based on the above work, we are interested in the following fractional differential problem:

$$\begin{cases} {}^cD^\alpha_{0+} u(t) = f(t, u(t)), \quad t \in [0, 1], \\ D^\beta_{0+} u(1) = \int_0^1 l(s)u(s)\,ds, \qquad u'(0) + u'(1) = 0, \end{cases} \tag{1.3}$$

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where $1 < \alpha < 2$, $0 \leq \beta < 1$, ${}^cD_{0^+}^{\alpha}$ denotes the Caputo fractional derivative, $D_{0^+}^{\beta}$ is the Riemann-Liouville fractional derivative, $f \in C([0,1] \times R, R)$, $l \in L[0,1]$ with $\Gamma(1-\beta)\int_0^1 l(s)\,ds < 1$. The existence and uniqueness of solutions for (1.3) will be derived by the Banach contraction mapping principle and Leray-Schauder degree theory.

Compared with the previous research problem, question (1.3) has wider and more general boundary conditions. It contains the situations which are in the above papers. It is necessary to study problem (1.3).

In the present paper, we present some important lemmas and theorems (in Section 2). Furthermore, we utilize the fixed point theorem and Leray-Schauder degree theory to study the existence of solutions for boundary value problem (1.3) (in Section 3). At last, we will give an example to illustrate our main results (in Section 4).

2 Preliminaries and relevant lemmas

In this section, we will recall some classic results on fractional calculus. In order to avoid redundance, as regards the definitions of the Riemann-Liouville fractional integral, the Riemann-Liouville fractional derivative, and the Caputo fractional derivative, we recommend the reader to refer to [14].

Lemma 2.1 ([14]) *Let $p > 0$, $x(t) \in AC^n[0,1]$, then*

$$I_{0^+}^{p}\left({}^cD_{0^+}^{p}\right)x(t) = x(t) - x(0) - x'(0)t - \frac{x''(0)}{2!}t^2 - \cdots - \frac{x^{(n-1)}(0)}{(n-1)!}t^{n-1},$$

where n is the smallest integer greater than or equal to p.

Lemma 2.2 *Given $h \in C(0,1)$, $1 < \alpha < 2$, $0 \leq \beta < 1$, $l \in L[0,1]$ with $\Gamma(1-\beta)\int_0^1 l(s)\,ds < 1$, then the unique solution of the following problem*:

$$\begin{cases} {}^cD_{0^+}^{\alpha}u(t) = h(t), \quad t \in (0,1), \\ D_{0^+}^{\beta}u(1) = \int_0^1 l(s)u(s)\,ds, \qquad u'(0) + u'(1) = 0, \end{cases} \tag{2.1}$$

is given by

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds - \frac{t}{2\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}h(s)\,ds \\ & - \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}h(s)\,ds \\ & + \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha-1)\Gamma(2-\beta)}\int_0^1 (1-s)^{\alpha-2}h(s)\,ds \\ & + \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha)}\int_0^1 \left(\int_s^1 l(\tau)(\tau-s)^{\alpha-1}\,d\tau\right)h(s)\,ds \\ & - \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha-1)}\int_0^1 \left(\int_0^1 \tau l(\tau)(1-s)^{\alpha-2}\,d\tau\right)h(s)\,ds, \end{aligned}$$

where $\Delta = 1 - \Gamma(1-\beta)\int_0^1 l(s)\,ds$.

Proof From Lemma 2.1, we can get

$$\begin{aligned} u(t) &= I_{0^+}^{\alpha} h(t) + u(0) + u'(0)t \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds + u(0) + u'(0)t. \end{aligned}$$

Then

$$u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} h(s)\, ds + u'(0),$$

by $u'(0) + u'(1) = 0$, we have

$$u'(0) + u'(1) = \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} h(s)\, ds + 2u'(0) = 0$$

and this yields

$$u'(0) = -\frac{1}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} h(s)\, ds.$$

Moreover,

$$\begin{aligned} D_{0^+}^{\beta} u(t) &= I_{0^+}^{\alpha-\beta} h(t) + D_{0^+}^{\beta} u(0) + D_{0^+}^{\beta} u'(0)t \\ &= \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} h(s)\, ds + \frac{u(0)t^{-\beta}}{\Gamma(1-\beta)} + \frac{u'(0)t^{1-\beta}}{\Gamma(2-\beta)}. \end{aligned}$$

From $D_{0^+}^{\beta} u(1) = \int_0^1 l(s)u(s)\, ds$, we have

$$\begin{aligned} D_{0^+}^{\beta} u(1) &= \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} h(s)\, ds + \frac{u(0)}{\Gamma(1-\beta)} \\ &\quad - \frac{1}{2\Gamma(\alpha-1)\Gamma(2-\beta)} \int_0^1 (1-s)^{\alpha-2} h(s)\, ds \\ &= \int_0^1 l(s) \biggl[\frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} h(\tau)\, d\tau + u(0) \\ &\quad - \frac{s}{2\Gamma(\alpha-1)} \int_0^1 (1-\tau)^{\alpha-2} h(\tau)\, d\tau \biggr] ds, \end{aligned}$$

then

$$\begin{aligned} u(0) &= \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha)} \int_0^1 \biggl(\int_0^s (s-\tau)^{\alpha-1} l(s) h(\tau)\, d\tau \biggr) ds \\ &\quad - \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha-1)} \int_0^1 \biggl(\int_0^1 s(1-\tau)^{\alpha-2} l(s) h(\tau)\, d\tau \biggr) ds \\ &\quad - \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} h(s)\, ds \\ &\quad + \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha-1)\Gamma(2-\beta)} \int_0^1 (1-s)^{\alpha-2} h(s)\, ds, \end{aligned}$$

which implies that the solution of (2.1) is

$$\begin{aligned}u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds - \frac{t}{2\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}h(s)\,ds \\ &\quad - \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}h(s)\,ds \\ &\quad + \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha-1)\Gamma(2-\beta)}\int_0^1 (1-s)^{\alpha-2}h(s)\,ds \\ &\quad + \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha)}\int_0^1 \left(\int_s^1 l(\tau)(\tau-s)^{\alpha-1}\,d\tau\right)h(s)\,ds \\ &\quad - \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha-1)}\int_0^1 \left(\int_0^1 \tau l(\tau)(1-s)^{\alpha-2}\,d\tau\right)h(s)\,ds.\end{aligned}$$

This completes the proof. □

Theorem 2.1 ([15]) *Let X be a Banach space, assume that Ω is an open bounded subset of X with $\theta \in \Omega$ and let $A : \overline{\Omega} \to X$ be a completely continuous operator such that*

$$\|Ax\| \le \|x\|, \quad \forall x \in \partial\Omega.$$

Then A has a fixed point in $\overline{\Omega}$.

3 Main results

In this section we will show the existence and uniqueness of solutions for the problem (1.3).

Let $\Lambda = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha)} + \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha-\beta+1)} + \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha)\Gamma(2-\beta)} + \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha+1)}\int_0^1 l(s)\,ds + \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha)}\int_0^1 l(s)\,ds$.

Now, we introduce the following hypotheses:

(H_1) f satisfies the Lipschitz condition

$$\bigl|f(t,x) - f(t,y)\bigr| \le L|x-y|, \quad \forall t \in [0,1], x, y \in R.$$

(H_2) $\lim_{u\to 0}\frac{f(t,u)}{u} = 0$ is uniformly respect to $t \in [0,1]$.

(H_3) There exist $0 \le c < \frac{1}{\Lambda}$, $K > 0$, such that $|f(t,u)| \le c|u| + K$, for $t \in [0,1]$, $u \in R$.

Let $E = C[0,1]$ denote the Banach space endowed with the norm given by $\|u\| = \max_{0\le t\le 1}|u(t)|$.

Define an operator $A : E \to E$ by

$$\begin{aligned}(Au)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\bigl(s,u(s)\bigr)\,ds \\ &\quad - \frac{t}{2\Gamma(\alpha-1)}\int_o^1 (1-s)^{\alpha-2}f\bigl(s,u(s)\bigr)\,ds \\ &\quad - \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}f\bigl(s,u(s)\bigr)\,ds \\ &\quad + \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha-1)\Gamma(2-\beta)}\int_0^1 (1-s)^{\alpha-2}f\bigl(s,u(s)\bigr)\,ds\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha)} \int_0^1 \left(\int_s^1 l(\tau)(\tau-s)^{\alpha-1}\,d\tau \right) f\big(s,u(s)\big)\,ds \\
&- \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha-1)} \int_0^1 \left(\int_0^1 \tau l(\tau)(1-s)^{\alpha-2}\,d\tau \right) f\big(s,u(s)\big)\,ds, \quad t\in[0,1]. \qquad (3.1)
\end{aligned}
$$

It is easy to prove that the solution for (1.3) is equivalent to the fixed point of A.

Lemma 3.1 *If A is defined in* (3.1), *then $A: E \to E$ is completely continuous.*

Proof Obviously, $A: E \to E$ is continuous. For any bounded set $\Omega \subset E$, since $f(t,u)$ is continuous on $[0,1]\times R$, there exists a positive constant Q such that $|f(t,u(t))| \le Q$, for all $t\in[0,1]$ and $u\in\Omega$. Thus, we can obtain

$$
\begin{aligned}
\big|(Au)(t)\big| \le{}& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big|f\big(s,u(s)\big)\big|\,ds \\
&+ \frac{t}{2\Gamma(\alpha-1)} \int_o^1 (1-s)^{\alpha-2} \big|f\big(s,u(s)\big)\big|\,ds \\
&+ \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} \big|f\big(s,u(s)\big)\big|\,ds \\
&+ \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha-1)\Gamma(2-\beta)} \int_0^1 (1-s)^{\alpha-2} \big|f\big(s,u(s)\big)\big|\,ds \\
&+ \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha)} \int_0^1 \left(\int_s^1 l(\tau)(\tau-s)^{\alpha-1}\,d\tau \right) \big|f\big(s,u(s)\big)\big|\,ds \\
&+ \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha-1)} \int_0^1 \left(\int_0^1 \tau l(\tau)(1-s)^{\alpha-2}\,d\tau \right) \big|f\big(s,u(s)\big)\big|\,ds \\
\le{}& Q\left[\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t}{2\Gamma(\alpha)} + \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha-\beta+1)} + \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha)\Gamma(2-\beta)} \right. \\
&\left. + \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha+1)} \int_0^1 t^\alpha l(t)\,dt + \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha)} \int_0^1 tl(t)\,dt \right] \\
\le{}& Q\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha)} + \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha-\beta+1)} + \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha)\Gamma(2-\beta)} \right. \\
&\left. + \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha+1)} \int_0^1 l(s)\,ds + \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha)} \int_0^1 l(s)\,ds \right] \\
={}& Q\Lambda,
\end{aligned}
$$

which means $A\Omega$ is uniformly bounded.

Furthermore, for $0 \le t_1 < t_2 \le 1$, by a simple computation

$$
\begin{aligned}
&\big|(Au)(t_2)-(Au)(t_1)\big| \\
&\quad= \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} f\big(s,u(s)\big)\,ds - \frac{t_2}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f\big(s,u(s)\big)\,ds \right. \\
&\qquad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f\big(s,u(s)\big)\,ds + \frac{t_1}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f\big(s,u(s)\big)\,ds \right| \\
&\quad= \left| \frac{1}{\Gamma(\alpha)} \left\{ \int_0^{t_1} \big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big] f\big(s,u(s)\big)\,ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f\big(s,u(s)\big)\,ds \right\} \right.
\end{aligned}
$$

$$\left. + \frac{\int_0^1 (1-s)^{\alpha-2} f(s,u(s))\,ds}{2\Gamma(\alpha-1)}(t_1-t_2) \right|$$
$$\le \frac{Q}{\Gamma(\alpha+1)}\left|t_2^\alpha - t_1^\alpha\right| + \frac{Q}{2\Gamma(\alpha)}|t_1-t_2|,$$

which implies that $A\Omega$ is equicontinuous. Thus, by the Arzelà-Ascoli theorem, $A: E \to E$ is completely continuous.

The proof is completed. □

Theorem 3.1 *Suppose that* (H_1) *is satisfied, and* $L\Lambda < 1$. *Then* (1.3) *has a unique solution.*

Proof Define $N = \max_{t\in[0,1]} |f(t,0)|$, and select $\sigma \ge \frac{N\Lambda}{1-L\Lambda}$, define a closed ball as $B_\sigma = \{u \in E : \|u\| \le \sigma\}$, from the proof of Lemma 3.1, we derive

$$\begin{aligned}
\left|(Au)(t)\right| \le{}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\left|f\left(s,u(s)\right)-f(s,0)\right| + \left|f(s,0)\right|\right) ds \\
&+ \frac{t}{2\Gamma(\alpha-1)}\int_o^1 (1-s)^{\alpha-2}\left(\left|f\left(s,u(s)\right)-f(s,0)\right| + \left|f(s,0)\right|\right) ds \\
&+ \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}\left(\left|f\left(s,u(s)\right)-f(s,0)\right| + \left|f(s,0)\right|\right) ds \\
&+ \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha-1)\Gamma(2-\beta)}\int_0^1 (1-s)^{\alpha-2}\left(\left|f\left(s,u(s)\right)-f(s,0)\right| + \left|f(s,0)\right|\right) ds \\
&+ \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha)}\int_0^1 \left(\int_s^1 l(\tau)(\tau-s)^{\alpha-1}\,d\tau\right) \\
&\times \left(\left|f\left(s,u(s)\right)-f(s,0)\right| + \left|f(s,0)\right|\right) ds \\
&+ \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha-1)}\int_0^1 \left(\int_0^1 \tau l(\tau)(1-s)^{\alpha-2}\,d\tau\right) \\
&\times \left(\left|f\left(s,u(s)\right)-f(s,0)\right| + \left|f(s,0)\right|\right) ds \\
\le{}& (L\sigma+N)\left[\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t}{2\Gamma(\alpha)} + \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha-\beta+1)} + \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha)\Gamma(2-\beta)}\right. \\
&\left. + \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha+1)}\int_0^1 t^\alpha l(t)\,dt + \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha)}\int_0^1 tl(t)\,dt\right] \\
\le{}& (L\sigma+N)\Lambda \le \sigma,
\end{aligned}$$

which means that $\|Au\| \le \sigma$, that is, $A(B_\sigma) \subset B_\sigma$.

In the following, for $x, y \in E$, for each $t \in [0,1]$, we compute

$$\begin{aligned}
&\left|(Ax)(t)-(Ay)(t)\right| \\
&\quad\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\left|f\left(s,x(s)\right)-f\left(s,y(s)\right)\right|\right) ds \\
&\qquad+ \frac{t}{2\Gamma(\alpha-1)}\int_o^1 (1-s)^{\alpha-2}\left(\left|f\left(s,x(s)\right)-f\left(s,y(s)\right)\right|\right) ds \\
&\qquad+ \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}\left(\left|f\left(s,x(s)\right)-f\left(s,y(s)\right)\right|\right) ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha-1)\Gamma(2-\beta)}\int_0^1 (1-s)^{\alpha-2}\bigl(\bigl|f\bigl(s,x(s)\bigr)-f\bigl(s,y(s)\bigr)\bigr|\bigr)\,ds \\
&+\frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha)}\int_0^1\left(\int_s^1 l(\tau)(\tau-s)^{\alpha-1}\,d\tau\right)\bigl(\bigl|f\bigl(s,x(s)\bigr)-f\bigl(s,y(s)\bigr)\bigr|\bigr)\,ds \\
&+\frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha-1)}\int_0^1\left(\int_0^1 \tau l(\tau)(1-s)^{\alpha-2}\,d\tau\right)\bigl(\bigl|f\bigl(s,x(s)\bigr)-f\bigl(s,y(s)\bigr)\bigr|\bigr)\,ds \\
\leq{}& L\Lambda\|x-y\|,
\end{aligned}
$$

which implies A is a contraction. Thus, from the Banach contraction mapping principle, A has a unique fixed point, that is, (1.3) has a unique solution.

The proof is completed. □

Theorem 3.2 *Suppose that* (H_2) *is satisfied, then* (1.3) *has at least one solution.*

Proof Since $\lim_{u\to 0}\frac{f(t,u)}{u}=0$ is uniformly respect to $t\in[0,1]$, there exist constants $\epsilon>0$ ($\epsilon\leq\frac{1}{\Lambda}$) and $\delta>0$ such that $|f(t,u)|\leq\epsilon|u|$, for all $0<|u|<\delta$ and $0\leq t\leq 1$.

Define $D=\{u\in E:\|u\|<\delta\}$. Taking any $y\in\partial D$, then $\|y\|=\delta$. From the proof of Lemma 3.1, we know that $|(Ay)(t)|\leq\Lambda\epsilon\|y\|\leq\|y\|$, which implies that $\|Ay\|\leq\|y\|$. Moreover, from Lemma 3.1, A is completely continuous. Thus, by Theorem 2.1, A has at least one fixed point, that is, (1.3) has at least one solution.

The proof is completed. □

Theorem 3.3 *Suppose that* (H_3) *is satisfied, then* (1.3) *has at least one solution.*

Proof We consider the operator equation

$$u=Au. \tag{3.2}$$

We shall prove that there exists at least one point $u\in E$ satisfying (3.2).

Suppose a ball $B_{\sigma_0}\subset E$ and $B_{\sigma_0}=\{u\in E:\|u\|<\sigma_0\}$, with radius $\sigma_0>0$ calculated later. We will demonstrate that $A:\overline{B_{\sigma_0}}\to E$ satisfies the condition $u\neq\lambda Au$, $\forall u\in\partial B_{\sigma_0}$, $\forall\lambda\in[0,1]$.

Due to Lemma 3.1, we know that A is completely continuous, then it is not difficult to know that $g_\lambda(u)$ is also completely continuous, where $g_\lambda(u)$ is defined by

$$g_\lambda(u)=u-\lambda Au,\quad \forall u\in E,\lambda\in[0,1].$$

From the homotopy invariance of the topological degree in Leray-Schauder degree theory, we can see

$$\deg(g_\lambda,B_{\sigma_0},0)=\deg(g_1,B_{\sigma_0},0)=\deg(g_0,B_{\sigma_0},0)=\deg(I,B_{\sigma_0},0)=1\neq 0,$$

where I denotes the unit operator.

According to the nonzero property of Leray-Schauder degree, $g_1(u)=u-Au=0$ for at least one $u\in B_{\sigma_0}$.

Assume that $u = \lambda Au$ for some $\lambda \in [0,1]$ and $u \in E$, then for $t \in [0,1]$, similar to proof of the Lemma 3.1, we have

$$\begin{aligned}
|u(t)| &= |\lambda Au(t)| \leq |Au(t)| \\
&\leq (c\|u\| + K)\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds + \frac{t}{2\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\,ds\right. \\
&\quad + \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}\,ds + \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha-1)\Gamma(2-\beta)}\int_0^1 (1-s)^{\alpha-2}\,ds \\
&\quad + \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha)}\int_0^1 \left(\int_S^1 l(\tau)(\tau-s)^{\alpha-1}\,d\tau\right) ds \\
&\quad \left. + \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha-1)}\int_0^1 \left(\int_0^1 \tau l(\tau)(1-s)^{\alpha-2}\,d\tau\right) ds\right] \leq (c\|u\| + K)\Lambda.
\end{aligned}$$

So $\|u\| \leq (c\|u\| + K)\Lambda$. This yields

$$\|u\| \leq \frac{K\Lambda}{1-c\Lambda}.$$

If $\sigma_0 = \frac{K\Lambda}{1-c\Lambda} + 1$, then $u \neq \lambda Au$, for any $u \in \partial B_{\sigma_0}$, for all $\lambda \in [0,1]$. Thus, equation (3.2) has at least one solution in B_{σ_0}, that is, problem (1.1) has at least one solution.

The proof is completed. □

4 Examples

Example 4.1 Consider the following boundary value problem:

$$\begin{cases} {}^cD_{0^+}^{\frac{3}{2}} u(t) = \frac{2+u(t)\cos u(t)}{(4+t)^3}, & t \in (0,1), \\ D_{0^+}^{\frac{1}{2}} u(1) = \int_0^1 su(s)\,ds, & u'(0) + u'(1) = 0. \end{cases} \tag{4.1}$$

Let $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$, $l(t) = t$, $f(t,u) = \frac{2+u\cos u}{(4+t)^3}$. Obviously, $f \in C([0,1] \times R, R)$, $l \in L[0,1]$, $\Gamma(1-\beta)\int_0^1 l(s)\,ds = \Gamma(\frac{1}{2})\int_0^1 s\,ds \approx 0.8862 < 1$. It is not difficult to calculate that $\Delta = 1 - \Gamma(1-\beta)\int_0^1 l(s)\,ds \approx 0.1138$, $\Lambda = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{2\Gamma(\alpha)} + \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha-\beta+1)} + \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha)\Gamma(2-\beta)} + \frac{\Gamma(1-\beta)}{\Delta\Gamma(\alpha+1)}\int_0^1 l(s)\,ds + \frac{\Gamma(1-\beta)}{2\Delta\Gamma(\alpha)}\int_0^1 l(s)\,ds \approx 37.0518$, $\frac{1}{\Lambda} \approx 0.0270$. Then there exist $c = \frac{1}{64}$ $(0 \leq c < \frac{1}{\Lambda})$ and $K = \frac{1}{32}$, such that $|f(t,u)| = \frac{|2+u\cos u|}{(4+t)^3} \leq c|u| + K$. According to Theorem 3.3, we can see that (4.1) has at least one solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have equally made contributions to each part of this paper. All authors read and approved the final manuscript.

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