Jiang et al. *Boundary Value Problems* (2016) 2016:175
DOI 10.1186/s13661-016-0680-x

Boundary Value Problems
a SpringerOpen Journal

RESEARCH | Open Access

The existence of positive solutions for p-Laplacian boundary value problems at resonance

Weihua Jiang*, Jiqing Qiu and Caixia Yang

*Correspondence:
weihuajiang@hebust.edu.cn
College of Sciences, Hebei University of Science and Technology, Shijiazhuang, Hebei 050018, P.R. China

Abstract

By using the Leggett-Williams norm-type theorem due to O'Regan and Zima and constructing suitable Banach spaces and operators, we investigate the existence of positive solutions for fractional p-Laplacian boundary value problems at resonance. An example is given to illustrate the main results.

MSC: 34B15

Keywords: positive solutions; p-Laplacian operator; boundary value problem; resonance; Fredholm operator

1 Introduction

Boundary value problems at resonance have attracted more and more attention. Many authors studied the existence of *solutions* for these problems by using Mawhin's continuous theorem [1] and its extension obtained by Ge and Ren [2]; see [3–23] and the references cited therein. By using Leggett-Williams norm-type theorems due to O'Regan and Zima [24], the existence of *positive solutions* for the boundary value problems at resonance with a *linear derivative operator* has been investigated (see [25–28]). To the best of our knowledge, there is no paper to show the existence of a *positive solution* for boundary value problems with a *nonlinear derivative operator* (for instance, p-Laplacian operator) at resonance by using Leggett-Williams norm-type theorems. Motivated by the excellent results mentioned above, we will discuss the existence of *positive solutions* for the p-Laplacian boundary value problem

$$\begin{cases} {}^{C}D_{0+}^{\beta}[\varphi_p({}^{C}D_{0+}^{\alpha}x)](t) = f(t,({}^{C}D_{0+}^{\alpha}x)(t)), \quad t\in(0,1), \\ ({}^{C}D_{0+}^{\alpha}x)(0) = ({}^{C}D_{0+}^{\alpha}x)(1), \qquad x^{(i)}(0)=0, \quad i=0,1,2,\ldots,n-1, \end{cases} \tag{1.1}$$

where $n-1<\alpha\le n$, $0<\beta<1$, $\varphi_p(s)=|s|^{p-2}s$, $p>1$, ${}^{C}D_{0+}^{\beta}$ is the Caputo fractional derivative (see [29, 30]).

2 Preliminaries

For convenience, we introduce some notations and a theorem. For more details see [24].

Assume that X, Y are real Banach spaces. A linear mapping $L:\operatorname{dom}L\subset X\to Y$ is a Fredholm operator of index zero (*i.e.* $\dim\operatorname{Ker}L=\operatorname{codim}\operatorname{Im}L<+\infty$ and $\operatorname{Im}L$ is closed in

Y) and an operator $N : X \to Y$ is nonlinear. $P : X \to X$ and $Q : Y \to Y$ are projectors with $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is a isomorphism since $\dim \operatorname{Im} Q = \dim \operatorname{Ker} L$. Denote by L_P the restriction of L to $\operatorname{Ker} P \cap \operatorname{dom} L \to \operatorname{Im} L$ and its inverse by K_P. So, x is a solution of $Lx = Nx$ if and only if it satisfies $x = (P + JQN)x + K_P(I - Q)Nx$.

Let $C \subset X$ be a cone, $\gamma : X \to C$ be a retraction, $\Psi := P + JQN + K_P(I - Q)N$ and $\Psi_\gamma := \Psi \circ \gamma$.

Theorem 2.1 [24] *Let Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that $L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator of index zero and the following conditions are satisfied.*

- (C1) *$QN : X \to Y$ is continuous and bounded and $K_P(I - Q)N : X \to X$ is compact on every bounded subset of X;*
- (C2) *$Lx \neq \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap \operatorname{dom} L$ and $\lambda \in (0, 1)$;*
- (C3) *γ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C;*
- (C4) *$d_B([I - (P + JQN)\gamma]|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_2, 0) \neq 0$, where d_B stands for the Brouwer degree;*
- (C5) *there exists $u_0 \in C \setminus \{0\}$ such that $\|x\| \le \sigma(u_0)\|\Psi x\|$ for $x \in C(u_0) \cap \partial\Omega_1$, where $C(u_0) = \{x \in C : \mu u_0 \preceq x \text{ for some } \mu > 0\}$ and $\sigma(u_0)$ is such that $\|x + u_0\| \ge \sigma(u_0)\|x\|$ for every $x \in C$;*
- (C6) *$(P + JQN)\gamma(\partial\Omega_2) \subset C$;*
- (C7) *$\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$.*

Then the equation $Lx = Nx$ has at least one solution in the set $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Now, we present some fundamental facts on the fractional calculus theory which can be found in [29, 30].

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to R$ is given by

$$I_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y(s)\, ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 The Caputo fractional derivative of order $\delta > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$${}^{\mathrm{C}}D_{0^+}^{\delta} y(t) = \frac{1}{\Gamma(n - \delta)} \int_0^t (t - s)^{n - \delta - 1} y^{(n)}(s)\, ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$, where $n = [\delta] + 1$.

Lemma 2.1 [29, 30] *Assume $f \in L[0, 1]$, $q > p \ge 0$, $q > 1$, then*

$${}^{\mathrm{C}}D_{0^+}^{p} I_{0^+}^{q} f(t) = I_{0^+}^{q - p} f(t), \qquad {}^{\mathrm{C}}D_{0^+}^{p} I_{0^+}^{p} f(t) = f(t).$$

Lemma 2.2 [29, 30] *Assume $p > 0$, then*

$$I_{0^+}^{p}\, {}^{\mathrm{C}}D_{0^+}^{p} f(t) = f(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},$$

where n is an integer and $n - 1 < p \le n$.

Since ${}^{C}D^{\beta}_{0^+}[\varphi_p({}^{C}D^{\alpha}_{0^+}\cdot)]$ is a nonlinear operator, we cannot solve the problem (1.1) by Theorem 2.1. Based on this, we prove the following lemma.

Lemma 2.3 *$u(t)$ is a solution of the following problem*:

$$\begin{cases} ({}^{C}D^{\beta}_{0^+}u)(t) = f(t, \varphi_q(u(t))), \quad t \in [0,1], \\ u(0) = u(1), \end{cases} \tag{2.1}$$

if and only if $x(t)$ is a solution of (1.1), *where* $x(t) = I^{\alpha}_{0^+}\varphi_q(u(t))$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Assume that $u(t)$ is a solution of the problem (2.1) and $x(t) = I^{\alpha}_{0^+}\varphi_q(u(t))$. Then $u(t) = [\varphi_p({}^{C}D^{\alpha}_{0^+}x)](t)$ and $x^{(i)}(0) = 0$, $i = 0, 1, 2, \ldots, n-1$. Replaces $u(t)$ with $[\varphi_p({}^{C}D^{\alpha}_{0^+}x)](t)$ in (2.1), we can see that $x(t)$ is a solution of (1.1).

On the other hand, if $x(t)$ is a solution of (1.1) and $u(t) = [\varphi_p({}^{C}D^{\alpha}_{0^+}x)](t)$, substituting $u(t)$ for $[\varphi_p({}^{C}D^{\alpha}_{0^+}x)](t)$ in (1.1), we can see that $u(t)$ satisfies (2.1). □

In this paper, we will always suppose that $f \in [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $p > 1$, $\varphi_p(s) = s \cdot |s|^{p-2}$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$, $0 < \beta < 1$.

3 Main result

Let $X = Y = C[0,1]$ with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Take a cone

$$C = \{u(t) \in X \mid u(t) \geq 0, t \in [0,1]\}.$$

Define operator $L : \operatorname{dom} L \subset X \to Y$ and $N : X \to Y$ as follows:

$$(Lu)(t) = ({}^{C}D^{\beta}_{0^+}u)(t), \qquad (Nu)(t) = f(t, \varphi_q(u(t))),$$

where

$$\operatorname{dom} L = \{u(t) \mid u(t), {}^{C}D^{\beta}_{0^+}u(t) \in X, u(0) = u(1)\}.$$

Then the problem (2.1) can be written by

$$Lu = Nu, \quad u \in \operatorname{dom} L.$$

Lemma 3.1 *L is a Fredholm operator of index zero. K_P is the inverse of $L|_{\operatorname{dom} L\cap \operatorname{Ker} P}$, where $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ is given by*

$$K_P y(t) = \frac{1}{\Gamma(\beta)}\left[\int_0^t (t-s)^{\beta-1}y(s)\,ds - \frac{1}{\beta}\int_0^1 (1-s)^{\beta}y(s)\,ds\right].$$

Proof It is easy to see that

$$\operatorname{Ker} L = \{c \mid c \in \mathbb{R}\}, \qquad \operatorname{Im} L = \left\{y \in Y \,\middle|\, \int_0^1 (1-s)^{\beta-1}y(s)\,ds = 0\right\},$$

and $\operatorname{Im} L \subset Y$ is closed.

Define $P: X \to X$, $Q: Y \to Y$ as

$$Pu = \int_0^1 u(t)\,dt, \qquad Qy = \beta \int_0^1 (1-s)^{\beta-1} y(s)\,ds.$$

Obviously, $P: X \to X$, $Q: Y \to Y$ are projectors and $\operatorname{Im} P = \operatorname{Ker} L$, $X = \operatorname{Ker} P \oplus \operatorname{Ker} L$.

It is easy to see that $\operatorname{Im} L \subset \operatorname{Ker} Q$. Conversely, if $y(t) \in \operatorname{Ker} Q$, take $u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\,ds$. Then $u(t) \in \operatorname{dom} L$ and $Lu = {}^{\mathrm{C}}D_{0^+}^{\beta} u(t) = y(t)$. These imply $\operatorname{Ker} Q \subset \operatorname{Im} L$. Therefore $\operatorname{Im} L = \operatorname{Ker} Q$. For $y \in Y$, $y = (y - Qy) + Qy \in \operatorname{Im} L + \operatorname{Im} Q$. If $y \in \operatorname{Im} L \cap \operatorname{Im} Q$, then $y = Qy$ and $y \in \operatorname{Im} L = \operatorname{Ker} Q$. This means that $y = 0$, *i.e.* $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. So, $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 1 < +\infty$. L is a Fredholm operator of index zero.

For $y \in \operatorname{Im} L$, it is clear that $K_P y \in \operatorname{dom} L \cap \operatorname{Ker} P$ and $LK_P y = y$. On the other hand, if $u \in \operatorname{dom} L \cap \operatorname{Ker} P$, by Lemma 2.2, we get

$$\begin{aligned} K_P Lu(t) &= \frac{1}{\Gamma(\beta)} \left[\int_0^t (t-s)^{\beta-1} Lu(s)\,ds - \frac{1}{\beta} \int_0^1 (1-s)^{\beta} Lu(s)\,ds \right] \\ &= I_{0^+}^{\beta}\, {}^{\mathrm{C}}D_{0^+}^{\beta} u(t) - I_{0^+}^{\beta+1}\, {}^{\mathrm{C}}D_{0^+}^{\beta} u(1) \\ &= u(t) + c - I_{0^+}^{\beta+1}\, {}^{\mathrm{C}}D_{0^+}^{\beta} u(1). \end{aligned}$$

Thus, $\int_0^1 K_P Lu(t)\,dt = \int_0^1 u(t)\,dt + c - I_{0^+}^{\beta+1}\, {}^{\mathrm{C}}D_{0^+}^{\beta} u(1)$. It follows from $u \in \operatorname{Ker} P$ and $K_P Lu \in \operatorname{Ker} P$ that $c - I_{0^+}^{\beta+1}\, {}^{\mathrm{C}}D_{0^+}^{\beta} u(1) = 0$. So, we have $K_P Lu = u$, $u \in \operatorname{dom} L \cap \operatorname{Ker} P$. □

Define $J: \operatorname{Im} Q \to \operatorname{Ker} L$ as $J(c) = c$, $c \in \mathbb{R}$.

Thus, $JQN + K_P(I-Q)N: X \to X$ is given by

$$\bigl[JQN + K_P(I-Q)N\bigr] u(t) = \int_0^1 G(t,s) f\bigl(s, \varphi_q\bigl(u(s)\bigr)\bigr)\,ds, \tag{3.1}$$

where

$$G(t,s) = \begin{cases} \beta(1-s)^{\beta-1}\bigl(1 - \frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta+2)}\bigr) - \frac{(1-s)^{\beta}}{\Gamma(\beta+1)} + \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \le s < t \le 1, \\ \beta(1-s)^{\beta-1}\bigl(1 - \frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta+2)}\bigr) - \frac{(1-s)^{\beta}}{\Gamma(\beta+1)}, & 0 \le t \le s < 1. \end{cases}$$

Lemma 3.2 *$QN: X \to Y$ is continuous and bounded and $K_P(I-Q)N: \overline{\Omega} \to X$ is compact, where $\Omega \subset X$ is bounded.*

Proof Assume that $\Omega \subset X$ is bounded. There exists a constant $M > 0$, such that $|Nu| = |f(t, \varphi_q(u(t)))| \le M$, $t \in [0,1]$, $u \in \overline{\Omega}$. So, $|QNu| \le M$, $u \in \overline{\Omega}$, *i.e.* $QN(\overline{\Omega})$ is bounded. Based on the definition of Q and the continuity of f we know that $QN: X \to Y$ is continuous.

For $u \in \overline{\Omega}$, we have

$$\begin{aligned} &\bigl|K_P(I-Q)Nu(t)\bigr| \\ &\quad = \left| \frac{1}{\Gamma(\beta)} \left[\int_0^t (t-s)^{\beta-1} (I-Q)Nu(s)\,ds - \frac{1}{\beta} \int_0^1 (1-s)^{\beta} (I-Q)Nu(s)\,ds \right] \right| \\ &\quad \le \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \bigl|Nu(s)\bigr|\,ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \bigl|QNu(s)\bigr|\,ds \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\beta\Gamma(\beta)} \int_0^1 (1-s)^\beta \left|Nu(s)\right| ds + \frac{1}{\beta\Gamma(\beta)} \int_0^1 (1-s)^\beta \left|QNu(s)\right| ds \\
&\le \frac{4M}{\Gamma(\beta+1)} < +\infty.
\end{aligned}
$$

Thus, $|K_P(I-Q)N(\overline{\Omega})$ is bounded.

For $u \in \overline{\Omega}$, $0 \le t_1 < t_2 \le 1$, we get

$$
\begin{aligned}
&\left|K_P(I-Q)Nu(t_2) - K_P(I-Q)Nu(t_1)\right| \\
&= \frac{1}{\Gamma(\beta)} \left| \int_0^{t_2} (t_2-s)^{\beta-1}(I-Q)Nu(s)\,ds - \int_0^{t_1} (t_1-s)^{\beta-1}(I-Q)Nu(s)\,ds \right| \\
&= \frac{1}{\Gamma(\beta)} \left| \int_0^{t_1} \left[(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}\right](I-Q)Nu(s)\,ds + \int_{t_1}^{t_2} (t_2-s)^{\beta-1}(I-Q)Nu(s)\,ds \right| \\
&\le \frac{2M}{\Gamma(\beta)} \left[\int_0^{t_1} \left[(t_1-s)^{\beta-1} - (t_2-s)^{\beta-1}\right] ds + \int_{t_1}^{t_2} (t_2-s)^{\beta-1}\,ds \right] \\
&= \frac{2M}{\Gamma(\beta+1)} \left[t_1^\beta - t_2^\beta + 2(t_2-t_1)^\beta\right].
\end{aligned}
$$

It follows from the uniform continuity of t^β and t on $[0,1]$ that $K_P(I-Q)N(\overline{\Omega})$ are equicontinuous on $[0,1]$. By the Arzela-Ascoli theorem, we see that $K_P(I-Q)N(\overline{\Omega})$ is compact. □

In order to prove our main result, we need the following conditions.

(H_1) There exists a constant $R_0 > 0$, such that $f(t,u) < 0$, $t \in [0,1]$, $u > R_0$.

(H_2) There exist nonnegative functions $a(t)$, $b(t)$ with $\max_{t\in[0,1]} \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{(\beta-1)} a(s)\,ds := A < +\infty$, $\max_{t\in[0,1]} \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{(\beta-1)} b(s)\,ds := B < 1/2$, such that

$$
\left|f(t,u)\right| \le a(t) + b(t)\varphi_p\left(|u|\right).
$$

(H_3) $f(t,u) \ge -(1-t)^{1-\beta}\varphi_p(u)/\beta$, $t \in [0,1]$, $u > 0$.

(H_4) There exist $r > 0$, $t_0 \in [0,1]$, and $M_0 \in (0,1)$ such that

$$
G(t_0,s)f(s,u) \ge \frac{1-M_0}{M_0}\varphi_p(u), \quad s \in [0,1), M_0 r \le u \le r.
$$

(H_5) $G(t,s)f(s,u) \ge -\varphi_p(u)$, $s \in [0,1)$, $t \in [0,1]$, $u \ge 0$.

Lemma 3.3 *If the conditions* (H_1) *and* (H_2) *hold, the set*

$$
\Omega_0 = \left\{u(t) \mid (Lu)(t) = \lambda Nu(t), u(t) \in C \cap \operatorname{dom} L, \lambda \in (0,1)\right\}
$$

is bounded.

Proof For $u(t) \in \Omega_0$, we get $QNu(t) = 0$ and $u(t) = \lambda I_{0^+}^\beta Nu(t) + u(0)$. By ($\mathrm{H}_1$) and $QNu(t) = 0$, there exists $t_0 \in [0,1]$ such that $\varphi_q(u(t_0)) \le R_0$. This, together with $u(t) = \lambda I_{0^+}^\beta Nu(t) + u(0)$, means

$$
u(0) \le u(t_0) + \left|\lambda I_{0^+}^\beta Nu(t_0)\right| \le \varphi_p(R_0) + \left|I_{0^+}^\beta Nu(t_0)\right|.
$$

Thus, we have

$$u(t) \leq u(0) + \left|\lambda I_{0^+}^{\beta} Nu(t)\right| \leq \varphi_p(R_0) + \left|I_{0^+}^{\beta} Nu(t_0)\right| + \left|I_{0^+}^{\beta} Nu(t)\right|. \tag{3.2}$$

It follows from (H_2) that

$$\begin{aligned} u(t) &\leq \varphi_p(R_0) + \frac{1}{\Gamma(\beta)} \int_0^{t_0} (t_0 - s)^{\beta-1} \left|f\left(s, \varphi_q\left(u(s)\right)\right)\right| ds \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^{t} (t - s)^{\beta-1} \left|f\left(s, \varphi_q\left(u(s)\right)\right)\right| ds \\ &\leq \varphi_p(R_0) + \frac{1}{\Gamma(\beta)} \int_0^{t_0} (t_0 - s)^{\beta-1} \left[a(s) + b(s)u(s)\right] ds \\ &\quad + \frac{1}{\Gamma(\beta)} \int_0^{t} (t - s)^{\beta-1} \left[a(s) + b(s)u(s)\right] ds \\ &\leq \varphi_p(R_0) + 2\left(A + B\|u\|\right). \end{aligned}$$

Therefore,

$$\|u\| \leq \frac{\varphi_p(R_0) + 2A}{1 - 2B} < +\infty.$$

This means that Ω_0 is bounded. □

Theorem 3.1 *Assume that the conditions* (H_1)-(H_5) *hold. Then the boundary value problem* (1.1) *has at least one positive solution.*

Proof Set

$$\Omega_1 = \left\{u \in X \mid M_0\|u\| < \left|u(t)\right| < r < R, t \in [0,1]\right\}, \qquad \Omega_2 = \left\{u \in X \mid \|u\| < R\right\},$$

where $R > \max\{\varphi_p(R_0), \Gamma(\beta+1)A\}$ is large enough such that $\Omega_2 \supset \Omega_0$. Clearly, Ω_1 and Ω_2 are open bounded sets of X, $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$.

In view of Lemmas 3.1, 3.2, and 3.3, L is a Fredholm operator of index zero and the conditions (C1), (C2) of Theorem 2.1 are fulfilled.

Define $\gamma : X \to C$ as $(\gamma u)(t) = |u(t)|$, $u(t) \in X$. Then $\gamma : X \to C$ is a retraction and (C3) holds.

Let $u(t) \in \operatorname{Ker} L \cap \partial\Omega_2$, then $u(t) \equiv c = \pm R$, $t \in [0,1]$. Define

$$H(c,\lambda) = c - \lambda|c| - \lambda\beta \int_0^1 (1-s)^{\beta-1} f\left(s, \varphi_q\left(|c|\right)\right) ds.$$

If $c = R$, $\lambda \in [0,1]$, by (H_1), we get

$$H(R,\lambda) = R - \lambda R - \lambda\beta \int_0^1 (1-s)^{\beta-1} f\left(s, \varphi_q(R)\right) ds > 0.$$

If $c = -R$, $\lambda \in [0,1]$, by (H_3), we obtain

$$H(-R,\lambda) = -R - \lambda R - \lambda\beta \int_0^1 (1-s)^{\beta-1} f\left(s, \varphi_q(R)\right) ds < -(1+\lambda)R + \lambda R = -R.$$

So, we have $H(u,\lambda) \neq 0$, $u \in \operatorname{Ker} L \cap \partial\Omega_2$, $\lambda \in [0,1]$.

Therefore,

$$\begin{aligned}
&d_B\big(\big[I-(P+JQN)\gamma\big]|_{\operatorname{Ker}L}, \operatorname{Ker}L\cap\Omega_2, 0\big)\\
&\quad= d_B\big(H(\cdot,1)|_{\operatorname{Ker}L}, \operatorname{Ker}L\cap\Omega_2, 0\big) = d_B\big(H(\cdot,0)|_{\operatorname{Ker}L}, \operatorname{Ker}L\cap\Omega_2, 0\big)\\
&\quad= d_B(I|_{\operatorname{Ker}L}, \operatorname{Ker}L\cap\Omega_2, 0) = 1 \neq 0.
\end{aligned}$$

Thus, (C4) holds.

Set $u_0(t) = 1, t \in [0,1]$, then $u_0 \in C\setminus\{0\}$, $C(u_0) = \{u \in C \mid u(t) > 0, t \in [0,1]\}$. Take $\sigma(u_0) = 1$ and $u \in C(u_0)\cap\partial\Omega_1$. Then $M_0 r \leq u(t) \leq r$, $t \in [0,1]$. By (H_4), we get

$$\begin{aligned}
\Psi u(t_0) &= \int_0^1 u(s)\,ds + \int_0^1 G(t_0,s)f\big(s,\varphi_q\big(u(s)\big)\big)\,ds\\
&\geq \int_0^1 u(s)\,ds + \int_0^1 \frac{1-M_0}{M_0}u(s)\,ds\\
&\geq M_0 r + (1-M_0)r = r = \|u\|.
\end{aligned}$$

Thus, $\|u\| \leq \sigma(u_0)\|\Psi u\|$, for $u \in C(u_0)\cap\partial\Omega_1$. So, (C5) holds.

For $u(t) \in \partial\Omega_2$, $t \in [0,1]$, by the condition (H_3), we have

$$\begin{aligned}
(P+JQN)\gamma(u) &= \int_0^1 \big|u(s)\big|\,ds + \beta\int_0^1 (1-s)^{\beta-1}f\big(s,\varphi_q\big(\big|u(s)\big|\big)\big)\,ds\\
&\geq \int_0^1 \big|u(s)\big|\,ds - \int_0^1 \big|u(s)\big|\,ds = 0.
\end{aligned}$$

So, $(P+JQN)\gamma(\partial\Omega_2) \subset C$. Hence, (C6) holds.

For $u(t) \in \overline{\Omega}_2\setminus\Omega_1$, $t \in [0,1]$, it follows from (H_5) that

$$(\Psi_\gamma u)(t) = \int_0^1 \big|u(s)\big|\,ds + \int_0^1 G(t,s)f\big(s,\varphi_q\big(\big|u(s)\big|\big)\big)\,ds \geq \int_0^1 \big|u(s)\big|\,ds - \int_0^1 \big|u(s)\big|\,ds = 0.$$

So, (C7) is satisfied.

By Theorem 2.1, we confirm that the equation $Lu = Nu$ has a positive solution u. Based on Lemma 2.3, the problem (1.1) has at least one positive solution. □

4 Examples

To illustrate our main result, we present an example.

Let us consider the following boundary value problem:

$$\begin{cases}
{}^{C}D_{0^+}^{\frac{3}{4}}[\varphi_{\frac{5}{4}}({}^{C}D_{0^+}^{\frac{1}{2}}x)](t) = \frac{1}{4}(1-t)^{\frac{1}{4}} - \frac{1}{20}(1-t)^{\frac{1}{4}}|{}^{C}D_{0^+}^{\frac{1}{2}}x(t)|^{\frac{1}{4}}, \quad t \in (0,1),\\
x(0) = 0, \qquad ({}^{C}D_{0^+}^{\frac{1}{2}}x)(0) = ({}^{C}D_{0^+}^{\frac{1}{2}}x)(1).
\end{cases} \tag{4.1}$$

On the basis of Lemma 2.3, it is sufficient to examine the issue

$$\begin{cases}
{}^{C}D_{0^+}^{\frac{3}{4}}u(t) = \frac{1}{4}(1-t)^{\frac{1}{4}} - \frac{1}{20}(1-t)^{\frac{1}{4}}|u(t)|, \quad t \in [0,1],\\
u(0) = u(1).
\end{cases} \tag{4.2}$$

Corresponding to the problem (2.1), we have $f(t,u) = \frac{1}{4}(1-t)^{\frac{1}{4}} - \frac{1}{20}(1-t)^{\frac{1}{4}}|u|^{\frac{1}{4}}$, $p = \frac{5}{4}$, $q = 5$, $\alpha = \frac{1}{2}$, $\beta = \frac{3}{4}$. So,

$$G(t,s) = \begin{cases} \frac{3}{4}(1-s)^{-\frac{1}{4}}(1 - \frac{t^{\frac{3}{4}}}{\Gamma(\frac{7}{4})} + \frac{1}{\Gamma(\frac{11}{4})}) - \frac{(1-s)^{\frac{3}{4}}}{\Gamma(\frac{7}{4})} + \frac{(t-s)^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})}, & 0 \le s < t \le 1, \\ \frac{3}{4}(1-s)^{-\frac{1}{4}}(1 - \frac{t^{\frac{3}{4}}}{\Gamma(\frac{7}{4})} + \frac{1}{\Gamma(\frac{11}{4})}) - \frac{(1-s)^{\frac{3}{4}}}{\Gamma(\frac{7}{4})}, & 0 \le t \le s < 1. \end{cases}$$

Take $R_0 = 625$, $a(t) = 1$, $b(t) = \frac{1}{4}$, $r = 0.006$, $t_0 = 0$, and $M_0 = 0.95$.

Clearly, (H_1) holds. By simple calculations, we can see that

$$\begin{aligned} &\bigl|f(t,u)\bigr| \le a(t) + b(t)\varphi_p\bigl(|u|\bigr), \\ &A = \max_{t\in[0,1]} \frac{1}{\Gamma(\frac{3}{4})} \int_0^t (t-s)^{-\frac{1}{4}}\,ds = \frac{4}{3.6762} < +\infty, \\ &B = \max_{t\in[0,1]} \frac{1}{\Gamma(\frac{3}{4})} \int_0^t (t-s)^{-\frac{1}{4}} \cdot \frac{1}{4}\,ds = \frac{1}{3.6762} < \frac{1}{2}, \\ &f(t,u) \ge -\frac{4}{3}(1-t)^{\frac{1}{4}}u^{\frac{1}{4}}, \quad u > 0, \\ &G(t_0,s)f(s,u) \ge \frac{0.12828103}{4} - \frac{1.21630192}{20}u^{\frac{1}{4}} \\ &\qquad\qquad\qquad \ge \frac{0.05}{0.95}u^{\frac{1}{4}}, \quad 0.0057 \le u \le 0.006, s \in [0,1), \\ &G(t,s)f(s,u) \ge -u^{\frac{1}{4}}, \quad u \ge 0, s \in [0,1), t \in [0,1]. \end{aligned}$$

So, the conditions (H_1)- (H_5) hold. By Theorem 3.1, we can conclude that the problem (4.1) has at least one positive solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All results belong to WJ, JQ, and CY. All authors read and approved the final manuscript.

Acknowledgements

This work is supported by the Natural Science Foundation of Hebei Province (A2017208101).

Received: 11 March 2016 Accepted: 13 September 2016 Published online: 27 September 2016

References

1. Mawhin, J: Topological Degree Methods in Nonlinear Boundary Value Problems. NSFCBMS Regional Conference Series in Mathematics. Am. Math. Soc., Providence (1979)
2. Ge, W, Ren, J: An extension of Mawhin's continuation theorem and its application to boundary value problems with a *p*-Laplacian. Nonlinear Anal. TMA **58**, 477-488 (2004)
3. Ma, R: Existence results of a *m*-point boundary value problem at resonance. J. Math. Anal. Appl. **294**, 147-157 (2004)
4. Xue, C, Ge, W: The existence of solutions for multi-point boundary value problem at resonance. Acta Math. Sin. **48**, 281-290 (2005)
5. Du, Z, Lin, X, Ge, W: Some higher-order multi-point boundary value problems at resonance. J. Comput. Appl. Math. **177**, 55-65 (2005)
6. Feng, W, Webb, JRL: Solvability of *m*-point boundary value problems with nonlinear growth. J. Math. Anal. Appl. **212**, 467-480 (1997)
7. Lian, H, Pang, H, Ge, W: Solvability for second-order three-point boundary value problem at resonance on a half-line. J. Math. Anal. Appl. **337**, 1171-1181 (2008)
8. Zhang, X, Feng, M, Ge, W: Existence result of second-order differential equations with integral boundary conditions at resonance. J. Math. Anal. Appl. **353**, 311-319 (2009)
9. Liu, B, Li, J, Liu, L: Existence and uniqueness for an *m*-point boundary value problem at resonance on infinite intervals. Comput. Math. Appl. **64**, 1677-1690 (2012)

10. Bai, C, Fang, J: Existence of positive solutions for three-point boundary value problems at resonance. J. Math. Anal. Appl. **291**, 538-549 (2004)
11. Kosmatov, N: Multi-point boundary value problems on an unbounded domain at resonance. Nonlinear Anal. **68**, 2158-2171 (2008)
12. Kosmatov, N: A boundary value problem of fractional order at resonance. Electron. J. Differ. Equ. **2010**, 135 (2010)
13. Liu, B: Solvability of multi-point boundary value problem at resonance (II). Appl. Math. Comput. **136**, 353-377 (2003)
14. Liu, B, Yu, J: Solvability of multi-point boundary value problem at resonance (III). Appl. Math. Comput. **129**, 119-143 (2002)
15. Liu, Y, Ge, W: Solvability of nonlocal boundary value problems for ordinary differential equations of higher order. Nonlinear Anal. **57**, 435-458 (2004)
16. Jiang, W: Solvability for a coupled system of fractional differential equations at resonance. Nonlinear Anal., Real World Appl. **13**, 2285-2292 (2012)
17. Meng, F, Du, Z: Solvability of a second-order multi-point boundary value problem at resonance. Appl. Math. Comput. **208**, 23-30 (2009)
18. Hu, L, Zhang, S, Shi, A: Existence result for nonlinear fractional differential equation with a *p*-Laplacian operator at resonance. J. Appl. Math. Comput. **48**, 519-532 (2015)
19. Jiang, W: Solvability of fractional differential equations with a *p*-Laplacian at resonance. Appl. Math. Comput. **260**, 48-56 (2015)
20. Lu, S: Homoclinic solutions for a class of second-order *p*-Laplacian differential systems with delay. Nonlinear Anal., Real World Appl. **12**, 780-788 (2011)
21. Du, B: Homoclinic solutions for a kind of neutral differential systems. Nonlinear Anal., Real World Appl. **13**, 168-175 (2012)
22. Feng, H, Lian, H, Gao, W: A symmetric solution of a multipoint boundary value problem with one-dimensional *p*-Laplacian at resonance. Nonlinear Anal. TMA **69**, 3964-3972 (2008)
23. Lu, S: Periodic solutions to a second order *p*-Laplacian neutral functional differential system. Nonlinear Anal. TMA **69**, 4215-4229 (2008)
24. O'Regan, D, Zima, M: Leggett-Williams norm-type theorems for coincidences. Arch. Math. **87**, 233-244 (2006)
25. Infante, G, Zima, M: Positive solutions of multi-point boundary value problems at resonance. Nonlinear Anal. **69**, 2458-2465 (2008)
26. Chen, Y, Tang, X: Positive solutions of fractional differential equations at resonance on the half-line. Bound. Value Probl. **2012**, 64 (2012)
27. Zhang, H, Sun, J: Positive solutions of third-order nonlocal boundary value problems at resonance. Bound. Value Probl. **2012**, 102 (2012)
28. Zima, M, Drygas, P: Existence of positive solutions for a kind of periodic boundary value problem at resonance. Bound. Value Probl. **2013**, 19 (2013)
29. Podlubny, I: Fractional Differential Equations. Academic Press, New York (1999)
30. Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)