Dou et al. *Boundary Value Problems* (2016) 2016:166
DOI 10.1186/s13661-016-0674-8

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence and multiplicity of positive solutions to a fourth-order impulsive integral boundary value problem with deviating argument

Jian Dou, Dongyuan Zhou and Huihui Pang*

*Correspondence: phh2000@163.com
College of Science, China Agricultural University, Beijing, 100083, China

Abstract
In this paper, we study the existence of multiple positive solutions for fourth-order impulsive differential equation with integral boundary conditions and deviating argument. The main tool is based on the Avery and Peterson fixed point theorem. Meanwhile, an example to demonstrate the main results is given.

Keywords: fourth-order; impulsive differential equation; integral boundary condition; fixed point theorem

1 Introduction

As an important area of investigation, the theory and applications of the fourth-order ordinary differential equations are emerging. The study concerns mainly the description of the deformations of an elastic beam by means of a fourth-order differential equation boundary value problem (BVP for short). Owing to its various applications in physics, engineering, and material mechanics, a lot of attention has been received by fourth-order differential equation BVPs. For more information, see [1–8].

In the meantime, many very interesting and significant cases of BVPs are constituted by integral boundary conditions. They include two-, three-, multi-point, and nonlocal BVPs as special cases. Therefore, in recent years, increasing attention has been given to integral boundary conditions [9–15] of fourth-order BVPs. Especially, we intend to mention some recent results.

In [12], Ma studied the following fourth-order BVP:

$$\begin{cases} u^{(4)}(t) = h(t)f(t,u), \quad 0 < t < 1, \\ u(0) = u(1) = \int_0^1 p(s)u(s)\,ds, \\ u''(0) = u''(1) = \int_0^1 q(s)u(s)\,ds, \end{cases}$$

where $p, q \in L^1[0,1]$, h and f are continuous. From an application of the fixed point index in cones, the existence of at least one symmetric positive solution could be obtained.

In 2015, the authors [13] investigated the fourth-order differential equation with integral boundary conditions,

$$\begin{cases} y^{(4)}(t) = \omega(t)F(t,y(t),y''(t)), \quad 0 < t < 1, \\ y(0) = y(1) = \int_0^1 h(s)y(s)\,ds, \\ ay''(0) - by'''x(0) = \int_0^1 g(s)y''(s)\,ds, \\ ay''(1) + by'''x(1) = \int_0^1 g(s)y''(s)\,ds. \end{cases}$$

By using a novel technique and fixed point theories, they showed the existence and multiplicity of positive solutions.

Unlike [12] and [13], a class of fourth-order differential equations with advanced or delayed argument were considered by the author of [14],

$$x^4(t) = h(t)f\bigl(t,x(t),x\bigl(\alpha(t)\bigr)\bigr), \quad t \in (0,1),$$

subject to the boundary conditions

$$\begin{cases} x(0) = \gamma x'(0) - \int_0^1 g(s)x(s)\,ds, \\ x(1) = \beta x(\eta), \qquad x''(0) = x''(1) = 0, \end{cases}$$

or

$$\begin{cases} x(0) = \beta x(\eta), \qquad x''(0) = x''(1) = 0, \\ x(1) = \gamma x'(0) - \int_0^1 g(s)x(s)\,ds. \end{cases}$$

The existence of multiple positive solutions is obtained by using a fixed point theorem due to Avery and Peterson.

In addition, to deviating arguments, the authors of [15] studied a fourth-order impulsive BVP as follows:

$$\begin{cases} (\phi_p(y''(t)))'' = \lambda\omega(t)f(t,y(\alpha(t))), \quad t \in (0,1)/(t_1,t_2,\ldots,t_n), \\ \triangle y'_{t_k} = -\mu I_k(t_k,y(t_k)), \quad k = 1,2,\ldots,n, \\ ay(0) - by'(0) = \int_0^1 g(s)y(s)\,ds, \\ ay(1) + by'(1) = \int_0^1 g(s)y(s)\,ds, \\ \phi_p(y''(0)) = \phi_p(y''(1)) = \int_0^1 h(t)\phi_p(y''(t))\,dt. \end{cases}$$

The boundary conditions above are special Sturm-Liouville integral boundary conditions, since $ay(0) - by'(0) = ay(1) + by'(1) = \int_0^1 g(s)y(s)\,ds$. Several existence and multiplicity results were derived by using inequality techniques and fixed point theories. For most research papers on impulsive differential equation BVPs, see [16–19] and the references therein.

Motivated by the mentioned results, we investigate a fourth-order impulsive differential equation Sturm-Liouville integral BVP with deviating argument,

$$x^{(4)}(t) = h(t)f\bigl(t,x(t),x\bigl(\alpha(t)\bigr)\bigr), \quad t \in J_0, \tag{1.1}$$

subject to

$$\begin{cases} x(0) = x(1) = \int_0^1 g_0(s)x(s)\,ds, \\ \triangle x'_{t_k} = -I_k(t_k, x(t_k)), \quad k = 1, 2, \dots, m, \\ x''(0) - \xi x'''(0) = \int_0^1 g_1(s)x''(s)\,ds, \\ x''(1) + \eta x'''(1) = \int_0^1 g_2(s)x''(s)\,ds, \end{cases} \tag{1.2}$$

where $\xi, \eta > 0$. Compared with [13–15], in this paper, (1.2) contains the general Sturm-Liouville integral boundary conditions, where $g_1(s)$, $g_2(s)$ could be two different functions in $L^1[0,1]$. In this case, we have to establish a more complicated expression of operator T and to find the proper lower and upper bounds of Green's functions (Lemma 2.2 and Lemma 2.4). Further, by using the fixed point theorem due to Avery and Peterson, the existence and multiplicity of positive solutions are obtained.

In (1.2), t_k ($k = 1, 2, \dots, m$) are fixed points with $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$, $\triangle x'_{t_k} = x'(t_k^+) - x'(t_k^-)$, $x'(t_k^+)$ and $x'(t_k^-)$ represent the right-hand limit and the left-hand limit of $x'(t_k)$ at $t = t_k$, respectively.

Throughout the paper, we always assume that $J = [0,1]$, $J_0 = (0,1)\backslash\{t_1, t_2, \dots, t_m\}$, $\mathbb{R}^+ = [0, +\infty)$, $J_k = (t_k, t_{k+1}]$, $k = 0, 1, \dots, m-1$. $\alpha : J \to J$ is continuous and:

- (H_1) $f \in C(J \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, with $f(t, u, v) > 0$ for $t \in J$, $u > 0$, and $v > 0$;
- (H_2) h is a nonnegative continuous function defined on $(0,1)$; h is not identically zero on any subinterval on J_0;
- (H_3) $I_k : J \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous with $I_k(t, u) > 0$ ($k = 1, 2, \dots, m$) for all $t \in J$ and $u > 0$;
- (H_4) $g_0, g_1, g_2 \in L^1[0,1]$ are nonnegative and $\gamma = \int_0^1 g_0(s)\,ds \in (0,1)$.

2 Expression and properties of Green's function

For $v(t) \in C(J)$, we consider the equation

$$x^4(t) = v(t), \quad 0 < t < 1, \tag{2.1}$$

with boundary conditions (1.2).

We shall reduce BVP (2.1) and (1.2) to two second-order problems. To this goal, first, by means of the transformation

$$x''(t) = -y(t), \tag{2.2}$$

we convert problem (2.1) and (1.2) into

$$\begin{cases} y''(t) = -v(t), \\ y(0) - \xi y'(0) = \int_0^1 g_1(s)y(s)\,ds, \\ y(1) + \eta y'(1) = \int_0^1 g_2(s)y(s)\,ds, \end{cases} \tag{2.3}$$

and

$$\begin{cases} x''(t) = -y(t), \\ x(0) = x(1) = \int_0^1 g_0(s)x(s)\,ds, \\ \triangle x'_{t_k} = -I_k, \quad k = 1, 2, \dots, m. \end{cases} \tag{2.4}$$

Lemma 2.1 *$\lambda(t)$ is the solution of $x''(t) = 0$, $x(0) = \xi$, $x'(0) = 1$. $\mu(t)$ is the solution of $y''(t) = 0$, $y(1) = \eta$, $y'(1) = -1$. Then $\lambda(t)$ is strictly increasing on J, $\lambda(t) > 0$ on $(0,1]$; $\mu(t)$ is strictly decreasing on J, and $\mu(t) > 0$ on $[0,1)$. For any $v(t) \in C(J)$, then the BVP (2.3) has a unique solution as follows*:

$$y(t) = (Fv)(t) + A(v)\lambda(t) + B(v)\mu(t), \tag{2.5}$$

where

$$(Fv)(t) = \int_0^1 G_1(t,s)v(s)\,ds, \tag{2.6}$$

$$G_1(t,s) = \begin{cases} \frac{1}{\rho}(\eta + 1 - t)(s + \xi), & 0 \le s \le t \le 1, \\ \frac{1}{\rho}(\eta + 1 - s)(t + \xi), & 0 \le t \le s \le 1, \end{cases} \tag{2.7}$$

$$\rho = 1 + \xi + \eta, \qquad \lambda(t) = t + \xi, \qquad \mu(t) = \eta + 1 - t, \tag{2.8}$$

and

$$A(v) = \frac{1}{\Delta}\begin{vmatrix} \alpha[Fv] & \rho - \alpha[\mu] \\ \beta[Fv] & -\beta[\mu] \end{vmatrix}, \qquad B(v) = \frac{1}{\Delta}\begin{vmatrix} -\alpha[\lambda] & \alpha[Fv] \\ \rho - \beta[\lambda] & \beta[Fv] \end{vmatrix}, \tag{2.9}$$

where

$$\begin{gathered} \Delta = \begin{vmatrix} -\alpha[\lambda] & \rho - \alpha[\mu] \\ \rho - \beta[\lambda] & -\beta[\mu] \end{vmatrix}, \\ \alpha[v] = \int_0^1 g_1(s)v(s)\,ds, \qquad \beta[v] = \int_0^1 g_2(s)v(s)\,ds. \end{gathered} \tag{2.10}$$

Proof Since λ and μ are two linearly independent solutions of the equation $y''(t) = 0$, we know that any solution of $y'' = v(t)$ can be represented by (2.5).

It is easy to check that the function defined by (2.5) is a solution of (2.3) if A and B are as in (2.9), respectively.

Now we show that the function defined by (2.5) is a solution of (2.3) only if A and B are as in (2.9), respectively.

Let $y(t) = (Fv)(t) + A\lambda(t) + B\mu(t)$ be a solution of (2.3), then we have

$$\begin{aligned} y(t) &= \int_0^t \frac{1}{\rho}(\eta + 1 - t)(s + \xi)v(s)\,ds + \int_t^1 \frac{1}{\rho}(\eta + 1 - s)(t + \xi)v(s)\,ds + A\lambda(t) + B\mu(t), \\ y'(t) &= -\int_0^t \frac{1}{\rho}(s + \xi)v(s)\,ds + \int_t^1 \frac{1}{\rho}(\eta + 1 - s)v(s)\,ds + A\lambda'(t) + B\mu'(t), \end{aligned}$$

and

$$y''(t) = -\frac{1}{\rho}(t + \xi)v(t) - \frac{1}{\rho}(\eta + 1 - t)v(t) + A\lambda''(t) + B\mu''(t).$$

Thus, by (2.8), we can obtain

$$y'' = -v(t).$$

Since

$$y(0) = \int_0^1 \frac{\xi}{\rho}(\eta + 1 - s)v(s)\,ds + A\lambda(0) + B\mu(0),$$
$$y'(0) = \int_0^1 \frac{1}{\rho}(\eta + 1 - s)v(s)\,ds + A\lambda'(0) + B\mu'(0),$$

we have

$$A\left(-\int_0^1 g_1(s)\lambda(s)\,ds\right) + B\left(\rho - \int_0^1 g_1(s)\mu(s)\,ds\right) = \int_0^1 g_1(s)(Fv)(s)\,ds. \tag{2.11}$$

Since

$$y(1) = \int_0^1 \frac{\eta}{\rho}(s + \xi)v(s)\,ds + A\lambda(1) + B\mu(1),$$
$$y'(1) = -\int_0^1 \frac{1}{\rho}(s + \xi)v(s)\,ds + A\lambda'(1) + B\mu'(1),$$

we have

$$A\left(\rho - \int_0^1 g_2(s)\lambda(s)\,ds\right) + B\left(-\int_0^1 g_2(s)\mu(s)\,ds\right) = \int_0^1 g_2(s)(Fv)(s)\,ds. \tag{2.12}$$

From (2.11) and (2.12), we get

$$\begin{pmatrix} -\alpha[\lambda] & \rho - \alpha[\mu] \\ \rho - \beta[\lambda] & -\beta[\mu] \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \alpha[Fv] \\ \beta[Fv] \end{pmatrix},$$

which implies that A and B satisfy (2.9), respectively. □

Assume that

(H_5) $\Delta < 0, \alpha[\mu] < \rho, \beta[\lambda] < \rho$.

Lemma 2.2 *Denote $e_1(t) = G_1(t,t)$, $\hat{e}_1(t) = \frac{1}{\rho}(1-t)(t+\xi)$, for $t \in J$. Let $\kappa[v] = \int_0^1 e_1(s)v(s)\,ds$ and $\hat{\kappa}[v] = \int_0^1 \hat{e}_1(s)v(s)\,ds$, for $v \in C(J, \mathbb{R}^+)$. If* (H_5) *is satisfied, then the following results are true*:

(1) $\hat{e}_1(s)\hat{e}_1(t) \le G_1(t,s) \le e_1(s)$, *for* $t, s \in J$;

(2) $0 \le \underline{A}\hat{\kappa}[v] \le A(v) \le \overline{A}\kappa[v]$, *for* $v \in C(J, \mathbb{R}^+)$;

(3) $0 \le \underline{B}\hat{\kappa}[v] \le B(v) \le \overline{B}\kappa[v]$, *for* $v \in C(J, \mathbb{R}^+)$,

where

$$\overline{A} = \frac{1}{\Delta}\begin{vmatrix} \alpha[1] & \rho - \alpha[\mu] \\ \beta[1] & -\beta[\mu] \end{vmatrix}, \qquad \overline{B} = \frac{1}{\Delta}\begin{vmatrix} -\alpha[\lambda] & \alpha[1] \\ \rho - \beta[\lambda] & \beta[1] \end{vmatrix},$$
$$\underline{A} = \frac{1}{\Delta}\begin{vmatrix} \alpha[\hat{e}_1] & \rho - \alpha[\mu] \\ \beta[\hat{e}_1] & -\beta[\mu] \end{vmatrix}, \qquad \underline{B} = \frac{1}{\Delta}\begin{vmatrix} -\alpha[\lambda] & \alpha[\hat{e}_1] \\ \rho - \beta[\lambda] & \beta[\hat{e}_1] \end{vmatrix}.$$

Proof Now we show that (1) is true. Obviously, $G_1(t,s) \le e_1(s)$ for $t,s \in J$.

In fact, $\hat{e}_1(s)\hat{e}_1(t) = \frac{1}{\rho^2}(1-s)(s+\xi)(1-t)(t+\xi)$, for $s,t \in J$.

For $0 \le s \le t \le 1$, we notice that

$$\frac{\hat{e}_1(s)\hat{e}_1(t)}{G_1(t,s)} = \frac{(1-s)(1-t)(t+\xi)}{\rho(\eta+1-t)} = \frac{(1-s)(1-t)(t+\xi)}{(1+\xi+\eta)(\eta+1-t)}.$$

It is easy to see that $1-s \le 1$, $1-t \le \eta+1-t$, $t+\xi \le 1+\xi+\eta$, for $s,t \in J$, $\xi,\eta > 0$, which implies

$$\frac{(1-s)(1-t)(t+\xi)}{(1+\xi+\eta)(\eta+1-t)} \le 1.$$

Hence, we have

$$\hat{e}_1(s)\hat{e}_1(t) \le G_1(t,s), \quad \text{for } 0 \le s \le t \le 1.$$

Similarly, we can obtain

$$\hat{e}_1(s)\hat{e}_1(t) \le G_1(t,s), \quad \text{for } 0 \le t \le s \le 1.$$

In the following we show (2) and (3) hold. In view of (H_5), for $v \in C(J,\mathbb{R}^+)$, we have

$$\begin{aligned} A(v) &= -\frac{1}{\Delta}\bigl(\alpha[Fv]\beta[\mu] + \beta[Fv]\bigl(\rho-\alpha[\mu]\bigr)\bigr) \\ &\le -\frac{1}{\Delta}\bigl(\alpha[1]\beta[\mu] + \beta[1]\bigl(\rho-\alpha[\mu]\bigr)\bigr)\kappa[v] = \overline{A}\kappa[v], \\ A(v) &= -\frac{1}{\Delta}\bigl(\alpha[Fv]\beta[\mu] + \beta[Fv]\bigl(\rho-\alpha[\mu]\bigr)\bigr) \\ &\ge -\frac{1}{\Delta}\bigl(\alpha[\hat{e}_1]\beta[\mu] + \beta[\hat{e}_1]\bigl(\rho-\alpha[\mu]\bigr)\bigr)\hat{\kappa}[v] = \underline{A}\hat{\kappa}[v]. \end{aligned}$$

In the same way, we have $B(v) \le \overline{B}\kappa[v]$, $B(v) \ge \underline{B}\hat{\kappa}[v]$, for $v \in C(J,\mathbb{R}^+)$. □

Analogously to Lemma 2.1 in [12], we obtain the following result; we omit the proof.

Lemma 2.3 *If* (H_4) *holds, for any* $y \in C(J)$, *the problem* (2.4) *has a unique solution* x *expressed in the form*

$$x(t) = \int_0^1 H(t,s)y(s)\,ds + \sum_{k=1}^{m} H(t,t_k)I_k, \tag{2.13}$$

where

$$H(t,s) = G_2(t,s) + \frac{1}{1-\gamma}\int_0^1 G_2(s,\tau)g_0(\tau)\,d\tau, \tag{2.14}$$

$$G_2(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{2.15}$$

From (2.14) and (2.15), we can prove that $H(t,s)$ and $G_2(t,s)$ have the following properties.

Lemma 2.4 *Let* $H(t,s)$, $G_2(t,s)$ *be given as in Lemma* 2.3. *Assume that* (H_4) *holds, then the following results are true*:

(1) $e_2(s)e_2(t) \le G_2(t,s) \le e_2(s)$, *for* $t,s \in J$;

(2) $\frac{\Gamma}{1-\gamma}e_2(s) \le H(t,s) \le \frac{1}{1-\gamma}e_2(s)$, *for* $t,s \in J$,

where

$$\Gamma = \int_0^1 e_2(s)g_0(s)\,ds, \qquad e_2(t) = G_2(t,t), \quad \text{for } t \in J.$$

Proof It is easy to see that (1) holds. In the following, we prove that (2) is satisfied:

$$\begin{aligned} H(t,s) &= G(t,s) + \frac{1}{1-\gamma}\int_0^1 G_2(s,\tau)g_0(\tau)\,d\tau \\ &\le e_2(s) + \frac{1}{1-\gamma}\int_0^1 e_2(s)g_0(\tau)\,d\tau \\ &= \frac{1}{1-\gamma}e_2(s), \quad \text{for } s \in J, \end{aligned}$$

and

$$\begin{aligned} H(t,s) &\ge e_2(s)e_2(t) + \frac{1}{1-\gamma}\int_0^1 e_2(s)e_2(\tau)g_0(\tau)\,d\tau \\ &= e_2(s)\left[e_2(t) + \frac{1}{1-\gamma}\int_0^1 e_2(\tau)g_0(\tau)\,d\tau\right] \\ &= \frac{\Gamma}{1-\gamma}e_2(s), \quad \text{for } t,s \in J. \end{aligned}$$

□

Lemma 2.5 *Assume that* (H_1)-(H_4) *hold. Then problem* (2.1) *and* (1.2) *has a unique solution* x *given by*

$$x(t) = \int_0^1 H(t,s)\big[(Fv)(s) + A(v)\lambda(s) + B(v)\mu(s)\big]\,ds + \sum_{k=1}^{m} H(t,t_k)I_k. \tag{2.16}$$

Lemma 2.6 *Assume that* (H_3)-(H_5) *hold, for* $v \in C(J,\mathbb{R}^+)$, *the unique solution* x *of problem* (2.1) *and* (1.2) *satisfies* $x(t) \ge 0$ *on* J.

Proof By Lemma 2.4, we can obtain $H(t,s) \ge 0$ for $t,s \in J$. Hence, from Lemma 2.5, combining with Lemma 2.2 and (H_3), we can obtain

$$x(t) \ge 0, \quad \text{for } t \in J.$$

This completes the proof. □

3 Background materials and definitions

Now, we present the prerequisite definitions in Banach spaces from the theory of cones.

Definition 3.1 Let E be a real Banach space. A nonempty convex closed set $P \subset E$ is said to be a cone if

(i) $ku \in P$ for all $u \in P$ and all $k \geq 0$, and

(ii) $u, -u \in P$ implies $u = 0$.

Definition 3.2 On a cone P of a real Banach space E, a map Λ is said to be a nonnegative continuous concave functional if $\Lambda : P \to \mathbb{R}^+$ is continuous and

$$\Lambda\bigl(tx + (1-t)y\bigr) \geq t\Lambda(x) + (1-t)\Lambda(y),$$

for all $x, y \in P$ and $t \in J$.

At the same time, on a cone P of a real Banach space E, a map φ is said to be a nonnegative continuous convex functional if φ: $P \to \mathbb{R}^+$ is continuous and

$$\varphi\bigl(tx + (1-t)y\bigr) \leq t\varphi(x) + (1-t)\varphi(y),$$

for all $x, y \in P$ and $t \in J$.

Definition 3.3 If it is continuous and maps bounded sets into pre-compact sets, an operator is called completely continuous.

Let φ and Θ be nonnegative continuous convex functionals on P, Λ be a nonnegative continuous concave functional on P, and Ψ be a nonnegative continuous functional on P. Then for positive numbers a, b, c, and d, we define the following sets:

$$P(\varphi, d) = \bigl\{x \in P : \varphi(x) < d\bigr\},$$
$$P(\varphi, \Lambda, b, d) = \bigl\{x \in P : b \leq \Lambda(x), \varphi(x) \leq d\bigr\},$$
$$P(\varphi, \Theta, \Lambda, b, c, d) = \bigl\{x \in P : b \leq \Lambda(x), \Theta(x) \leq c, \varphi(x) \leq d\bigr\},$$

and

$$R(\varphi, \Psi, a, d) = \bigl\{x \in P : a \leq \Psi(x), \varphi(x) \leq d\bigr\}.$$

We will make use of the following fixed point theorem of Avery and Peterson to establish multiple positive solutions to problem (1.1) and (1.2).

Theorem 3.1 (See [20]) *Let P be a cone in a real Banach space E. Let φ and Θ be nonnegative continuous convex functionals on P, Λ be nonnegative continuous concave functional on P, and Ψ be a nonnegative continuous functional on P satisfying $\Psi(kx) \leq k\Psi(x)$ for $0 \leq k \leq 1$, such that for some positive numbers M and d,*

$$\Lambda(x) \leq \Psi(x) \quad \textit{and} \quad \|x\| \leq M\varphi(x),$$

for all $x \in \overline{P(\varphi, d)}$. Suppose

$$T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)},$$

is completely continuous and there exist positive numbers a, b, and c with $a < b$ such that

(S_1) $\{x \in P(\varphi, \Theta, \Lambda, b, c, d) : \Lambda(x) > b\} \neq \emptyset$ *and* $\Lambda(Tx) > b$ *for* $x \in P(\varphi, \Theta, \Lambda, b, c, d)$;
(S_2) $\Lambda(Tx) > b$ *for* $x \in P(\varphi, \Lambda, b, d)$ *with* $\Theta(Tx) > c$;
(S_3) $0 \notin R(\varphi, \Psi, a, d)$ *and* $\Psi(Tx) < a$ *for* $x \in R(\varphi, \Psi, a, d)$ *with* $\Psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\varphi, d)}$, *such that*

$$\varphi(x_i) \le d, \quad \text{for } i = 1, 2, 3,$$
$$b < \Lambda(x_1), \qquad a < \Psi(x_2) \quad \text{with } \Lambda(x_2) < b,$$

and

$$\Psi(x_3) < a.$$

4 Existence result for the case of $\alpha(t) \ge t$ on J

Function $h(t)$ in (1.1) satisfies (H_2). We introduce the notations

$$l_1 = \kappa[h] = \int_0^1 e_1(s)h(s)\,ds, \qquad \hat{l}_1 = \int_0^1 \hat{e}_1(s)h(s)\,ds.$$

Let $X = C(J, \mathbb{R})$ be our Banach space with the maximum norm $\|x\| = \max_{t\in J}|x|$.

Set

$$P = \{x \in X : x \text{ is nonnegative, concave and } x(t) \ge \Gamma\|x\|, t \in J\},$$
$$\overline{P_r} = \{x \in P : \|x\| \le r\}, \tag{4.1}$$

where Γ is defined as in Lemma 2.4. We define the nonnegative continuous concave functional $\Lambda = \Lambda_1$ on P by

$$\Lambda_1(x) = \min_{t\in[\delta,1]} |x(t)|,$$

where $\delta \in (0,1)$ is such that $0 < \delta < 1 - \delta < 1$. Set $J_{\delta_1} = [\delta, 1]$.

Note that $\Lambda_1(x) \le \|x\|$. Put $\Psi(x) = \Theta(x) = \varphi(x) = \|x\|$.

Theorem 4.1 *Let assumptions* (H_1)-(H_5) *hold and* $\alpha(t) \ge t$ *on J. In addition, we assume that there exist positive constants a, b, c, d, ω, L with a < b such that*

$$\omega > \frac{1}{1-\gamma}\left[\frac{l_1}{6} + \left(\frac{1}{12} + \frac{\xi}{6}\right)l_1\overline{A} + \left(\frac{1}{12} + \frac{\eta}{6}\right)l_1\overline{B} + \sum_{k=1}^m t_k(1-t_k)\right],$$
$$0 < L < \frac{\Gamma}{(1-\gamma)}\left[\left(\frac{1}{30} + \frac{\xi}{12}\right)\hat{l}_1 + \left(\frac{1}{12} + \frac{\xi}{6}\right)\hat{l}_1\underline{A} + \left(\frac{1}{12} + \frac{\eta}{6}\right)\hat{l}_1\underline{B} + \sum_{k=1}^m t_k(1-t_k)\right], \tag{4.2}$$

and

(A_1) $f(t,u,v) \le \frac{d}{\omega}$, *for* $(t,u,v) \in J \times [0,d] \times [0,d]$, $I_k(t,u) \le \frac{d}{\omega}$, *for* $(t,u) \in J_k \times [0,d]$;
(A_2) $f(t,u,v) \ge \frac{b}{L}$, *for* $(t,u,v) \in J_{\delta_1} \times [b, \frac{b}{\Gamma}] \times [b, \frac{b}{\Gamma}]$, $I_k(t,u) \ge \frac{b}{L}$, *for* $(t,u) \in J_{\delta_1} \cap J_k \times [b, \frac{b}{\Gamma}]$;
(A_3) $f(t,u,v) \le \frac{a}{\omega}$, *for* $(t,u,v) \in J \times [0,a] \times [0,a]$, $I_k(t,u) \le \frac{a}{\omega}$, *for* $(t,u) \in J_k \times [0,a]$.

Then problem (1.1) *and* (1.2) *has at least three positive solutions* x_1, x_2, x_3 *satisfying* $\|x_i\| \leq d$, $i = 1, 2, 3$, *and*

$$b \leq \Lambda_1(x_1), \qquad a < \|x_2\| \quad \text{with } \Lambda_1(x_2) < b$$

and

$$\|x_3\| < a.$$

Proof For any $x \in C(J, \mathbb{R}^+)$, we define operator T by

$$\begin{aligned}(Tx)(t) = {} & \int_0^1 H(t,s)\bigl[F\bigl(hf_x(s)\bigr) + A\bigl(hf_x(s)\bigr)\lambda(s) + B\bigl(hf_x(s)\bigr)\mu(s)\bigr]\,ds \\ & + \sum_{k=1}^{m} H(t,t_k)I_k\bigl(t_k, x(t_k)\bigr),\end{aligned} \tag{4.3}$$

where $hf_x(s) = h(s)f(s, x(s), x(\alpha(s)))$. Indeed, $T : X \to X$. Problem (1.1) and (1.2) has a solution x if and only if x solves the operator equation $x = Tx$.

We need to prove the existence of at least three fixed points of T by verifying that operator T satisfies the Avery-Peterson fixed point theorem.

From the definition of T, we can obtain

$$(Tx)''(t) = -F\bigl(hf_x(s)\bigr) - A\bigl(hf_x(s)\bigr)\lambda(t) - B\bigl(hf_x(s)\bigr)\mu(t). \tag{4.4}$$

In view of (H_1), (H_2), Lemma 2.1, and Lemma 2.2, we have

$$(Tx)''(t) \leq 0, \quad \text{for } t \in J.$$

So Tx is concave on J. From (4.3) and (4.4), combining with Lemma 2.4 and (H_3), we can obtain

$$(Tx)(t) \geq 0, \quad \text{for } t \in J.$$

Noting (2) in Lemma 2.4, it follows that

$$\begin{aligned}\|Tx\| &= \max_{t\in J}\left\{\int_0^1 H(t,s)\bigl[F\bigl(hf_x(s)\bigr) + A\bigl(hf_x(s)\bigr)\lambda(s) + B\bigl(hf_x(s)\bigr)\mu(s)\bigr]\,ds \right. \\ &\qquad \left. + \sum_{k=1}^{m} H(t,t_k)I_k\bigl(t_k, x(t_k)\bigr)\right\} \\ &\leq \frac{1}{1-\gamma}\left\{\int_0^1 e_2(s)\bigl[F\bigl(hf_x(s)\bigr) + A\bigl(hf_x(s)\bigr)\lambda(s) + B\bigl(hf_x(s)\bigr)\mu(s)\bigr]\,ds \right. \\ &\qquad \left. + \sum_{k=1}^{m} e_2(t_k)I_k\bigl(t_k, x(t_k)\bigr)\right\}.\end{aligned} \tag{4.5}$$

On the other hand, from the properties of $H(t,s)$, we have

$$\begin{aligned}(Tx)(t) &= \int_0^1 H(t,s)\big[F\big(hf_x(s)\big) + A\big(hf_x(s)\big)\lambda(s) + B\big(hf_x(s)\big)\mu(s)\big]\,ds \\ &\quad + \sum_{k=1}^{m} H(t,t_k)I_k\big(t_k,x(t_k)\big) \\ &\geq \frac{\Gamma}{1-\gamma}\left\{\int_0^1 e_2(s)\big[F\big(hf_x(s)\big) + A\big(hf_x(s)\big)\lambda(s) + B\big(hf_x(s)\big)\mu(s)\big]\,ds \right.\\ &\quad \left. + \sum_{k=1}^{m} e_2(t_k)I_k\big(t_k,x(t_k)\big)\right\} \\ &\geq \Gamma\|Tx\|. \end{aligned} \tag{4.6}$$

This proves that $TP \subset P$.

Now we prove that the operator $T: P \to P$ is completely continuous. Let $x \in \overline{P_r}$, then $\|x\| \leq r$. Note that h and f are continuous, so h is bounded on J and f is bounded on $J \times [-r,r]$. It means that there exists a constant $K > 0$ such that $\|Tx\| \leq K$. This proves that $T\overline{P}$ is uniformly bounded. On the other hand, for $t_1, t_2 \in J$ there exists a constant $L_1 > 0$ such that

$$\big|(Tx)(t_1) - (Tx)(t_2)\big| \leq L_1|t_1 - t_2|.$$

This shows that $T\overline{P}$ is equicontinuous on J, so T is completely continuous.

Let $x \in \overline{P(\varphi,d)}$, so $0 \leq x(t) \leq d$, $t \in J$, and $\|x\| \leq d$. Note that also $0 \leq x(\alpha(t)) \leq d$, $t \in J$ because $0 \leq t \leq \alpha(t) \leq 1$ on J. Hence

$$\varphi(Tx) = \|Tx\| = \max_{t\in J}\big|(Tx)(t)\big| = \max_{t\in J}(Tx)(t).$$

By (4.2), Lemma 2.2, Lemma 2.4, and (A_1), we have

$$\begin{aligned}\varphi(Tx) &= \max_{t\in J}\left\{\int_0^1 H(t,s)\big[F\big(hf_x(s)\big) + A\big(hf_x(s)\big)\lambda(s) + B\big(hf_x(s)\big)\mu(s)\big]\,ds \right.\\ &\quad \left. + \sum_{k=1}^{m} H(t,t_k)I_k\big(t_k,x(t_k)\big)\right\} \\ &\leq \frac{1}{1-\gamma}\left[\int_0^1\int_0^1 e_2(s)e_1(\tau)h(\tau)f_x(\tau)\,d\tau\,ds + \int_0^1 e_2(s)A\big(hf_x(s)\big)(s+\xi)\,ds \right.\\ &\quad \left. + \int_0^1 e_2(s)B\big(hf_x(s)\big)(\eta+1-s)\,ds + \sum_{k=1}^{m} e_2(t_k)I_k\big(t_k,x(t_k)\big)\right] \\ &\leq \frac{d}{\omega(1-\gamma)}\left[\int_0^1\int_0^1 e_2(s)e_1(\tau)h(\tau)\,d\tau\,ds + \overline{A}\int_0^1 \kappa[h]e_2(s)(s+\xi)\,ds \right.\\ &\quad \left. + \overline{B}\int_0^1 \kappa[h]e_2(s)(\eta+1-s)\,ds + \sum_{k=1}^{m} t_k(1-t_k)\right] \end{aligned}$$

$$= \frac{d}{\omega(1-\gamma)}\left[\frac{l_1}{6} + \left(\frac{1}{12} + \frac{\xi}{6}\right)l_1\overline{A} + \left(\frac{1}{12} + \frac{\eta}{6}\right)l_1\overline{B} + \sum_{k=1}^{m} t_k(1-t_k)\right]$$
$$< d.$$

This shows that $T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$.

To check condition (S_1) we choose

$$x(t) = \frac{1}{2}\left(b + \frac{b}{\Gamma}\right), \quad t \in J.$$

Then

$$\|x\| = \frac{b(\Gamma+1)}{2\Gamma} < \frac{b}{\Gamma},$$

so

$$\Lambda_1(x) = \min_{t\in[\delta,1]} x(t) = \frac{b(\Gamma+1)}{2\Gamma} > b = \frac{b}{\Gamma}\Gamma \geq \Gamma\|x\|.$$

It proves that

$$\left\{x \in P\left(\varphi, \Theta, \Lambda_1, b, \frac{b}{\Gamma}, d\right) : b < \Lambda_1(x)\right\} \neq \emptyset.$$

Let $b \leq x(t) \leq \frac{b}{\Gamma}$ for $t \in [\delta, 1]$, then $\delta \leq t \leq \alpha(t) \leq 1$ on $[\delta, 1]$. It yields $b \leq x(\alpha(t)) \leq \frac{b}{\Gamma}$ on $[\delta, 1]$. It gives

$$\Lambda_1(Tx) = \min_{t\in[\delta,1]} (Tx)(t).$$

By (4.2), Lemma 2.2, Lemma 2.4, and (A_2), we have

$$\begin{aligned}
\Lambda_1(Tx) &= \min_{t\in[\delta,1]}\left\{\int_0^1 H(t,s)\left[F\left(hf_x(s)\right) + A\left(hf_x(s)\right)\lambda(s) + B\left(hf_x(s)\right)\mu(s)\right] ds \right.\\
&\quad \left. + \sum_{k=1}^{m} H(t,t_k)I_k\left(t_k, x(t_k)\right)\right\}\\
&\geq \frac{\Gamma}{1-\gamma}\left[\int_0^1\int_0^1 e_2(s)\hat{e}_1(s)\hat{e}_1(\tau)h(\tau)f_x(\tau)\,d\tau\,ds\right.\\
&\quad + \int_0^1 e_2(s)A\left(hf_x(s)\right)(s+\xi)\,ds\\
&\quad \left. + \int_0^1 e_2(s)B\left(hf_x(s)\right)(\eta+1-s)\,ds + \sum_{k=1}^{m} e_2(t_k)I_k\left(t_k, x(t_k)\right)\right]\\
&\geq \frac{b\Gamma}{L(1-\gamma)}\left[\int_0^1\int_0^1 e_2(s)\hat{e}_1(s)\hat{e}_1(\tau)h(\tau)\,d\tau\,ds + \underline{A}\int_0^1 \hat{\kappa}[h]e_2(s)(s+\xi)\,ds\right.\\
&\quad \left. + \underline{B}\int_0^1 \hat{\kappa}[h]e_2(s)(\eta+1-s)\,ds + \sum_{k=1}^{m} t_k(1-t_k)\right]
\end{aligned}$$

$$= \frac{b\Gamma}{L(1-\gamma)}\left[\left(\frac{1}{30}+\frac{\xi}{12}\right)\hat{l}_1+\left(\frac{1}{12}+\frac{\xi}{6}\right)\hat{l}_1\underline{A}\right.$$
$$\left.+\left(\frac{1}{12}+\frac{\eta}{6}\right)\hat{l}_1\underline{B}+\sum_{k=1}^{m}t_k(1-t_k)\right]$$
$$> b.$$

It proves condition (S_1) holds.

Now we need to prove that condition (S_2) is satisfied. Take

$$x \in P(\varphi, \Lambda_1, b, d) \quad \text{with } \|Tx\| > \frac{b}{\Gamma} = c.$$

Then

$$\Lambda_1(Tx) = \min_{t\in[\delta,1]}(Tx)(t) \geq \Gamma\|Tx\| > \Gamma\frac{b}{\Gamma} = b.$$

So condition (S_2) holds.

We finally show that condition (S_3) also holds. Clearly, as $\Psi(0) = 0 < a$, so $0 \notin R(\varphi, \Psi, a, d)$. Suppose that $x \in R(\varphi, \Psi, a, d)$ with $\Psi(x) = \|x\| = a$.

Similarly, by (4.2), Lemma 2.2, Lemma 2.4, and (A_3), we have

$$\Psi(Tx) = \|Tx\| = \max_{t\in J}(Tx)(t)$$
$$= \max_{t\in J}\left\{\int_0^1 H(t,s)\big[F\big(hf_x(s)\big)+A\big(hf_x(s)\big)\lambda(s)+B\big(hf_x(s)\big)\mu(s)\big]\,ds\right.$$
$$\left.+\sum_{k=1}^{m}H(t,t_k)I_k\big(t_k,x(t_k)\big)\right\}$$
$$\leq \frac{a}{\omega}\left\{\frac{1}{1-\gamma}\left[\frac{l_1}{6}+\left(\frac{1}{12}+\frac{\xi}{6}\right)l_1\overline{A}+\left(\frac{1}{12}+\frac{\eta}{6}\right)l_1\overline{B}+\sum_{k=1}^{m}t_k(1-t_k)\right]\right\}$$
$$< a.$$

It proves that condition (S_3) is satisfied.

By Theorem 4.1, there exist at least three positive solutions x_1, x_2, x_3 of problem (1.1) and (1.2) such that $\|x_i\| \leq d$ for $i = 1, 2, 3$,

$$b \leq \min_{t\in[\delta,1]} x_1(t), \qquad a < \|x_2\| \quad \text{with } \min_{t\in[\delta,1]} x_2(t) < b,$$

and $\|x_3\| < a$. This ends the proof. □

Example We consider the following BVP:

$$\begin{cases} x^{(4)}(t) = h(t)f(x(\alpha(t))), & t \in J_0, \\ x(0) = x(1) = \int_0^1 \frac{s}{2}x(s)\,ds, & \triangle x'_{t_1} = -I_1(t_1, x(t_1)), \\ x''(0) - \frac{1}{2}x'''(0) = \int_0^1 sx''(s)\,ds, & x''(1) + \frac{1}{2}x'''(1) = \int_0^1 s^2x''(s)\,ds, \end{cases} \tag{4.7}$$

with $\alpha \in C(J,J)$, $h(t) = Dt$, $\alpha(t) \geq t$, $\xi = \eta = \frac{1}{2}$, $\rho = 2$, and $t_1 = \frac{1}{12}$. It follows that $\mu(t) = \frac{3}{2} - t$, $\lambda(t) = t + \frac{1}{2}$, for $t \in J$, and

$$f(v) = I_1(t,v) = \begin{cases} \frac{v^2}{250}, & 0 \leq v \leq 1, \\ \frac{1}{250}(748v - 747), & 1 \leq v \leq \frac{3}{2}, \\ \frac{13}{69}v + \frac{28}{23}, & \frac{3}{2} \leq v \leq 36, \\ 8, & v \geq 36. \end{cases}$$

Note that $f \in C(\mathbb{R}^+, \mathbb{R}^+)$. As a function α we can take, for example, $\alpha(t) = \sqrt{t}$.

In this case we have $\gamma = \frac{1}{4}$, $\Delta = -\frac{85}{36}$, $\overline{A} = \frac{47}{170}$, $\overline{B} = \frac{71}{170}$, $\underline{A} = \frac{269}{6{,}800}$, $\underline{B} = \frac{457}{6{,}800}$, $l_1 = \frac{11}{48}D$, $\hat{l}_1 = \frac{1}{12}D$, $\Gamma = \frac{1}{24}$. Let $a = 1$, $b = \frac{3}{2}$, $c = 36$, $d \geq 2{,}000$, $D = 2{,}784$. In this case we can take $\omega = 250$, $L = 1$. We see that all assumptions of Theorem 4.1 hold, so BVP (4.7) has at least three positive solutions.

5 Existence result for the case of $\alpha(t) \leq t$ on J

The cone P is defined as in (4.1). We define the nonnegative continuous concave functional $\Lambda = \Lambda_2$ on P by

$$\Lambda_2(x) = \min_{t\in[0,1-\delta]} |x(t)|,$$

where $\delta \in (0,1)$ satisfying $0 < \delta < 1 - \delta < 1$. Set $J_{\delta_2} = [0, 1-\delta]$. Similar to the proof of Theorem 4.1, we have the following result.

Theorem 5.1 *Let assumptions* (H_1)-(H_5) *hold and* $\alpha(t) \leq t$ *on* J. *In addition, we assume that there exist positive constants* a, b, c, d, ω, L *with* $a < b$ *such that* (4.2) *holds and*

(A_1) $f(t,u,v) \leq \frac{d}{\omega}$, *for* $(t,u,v) \in J \times [0,d] \times [0,d]$, $I_k(t,u) \leq \frac{d}{\omega}$, *for* $(t,u) \in J_k \times [0,d]$;
(A_2) $f(t,u,v) \geq \frac{b}{L}$, *for* $(t,u,v) \in J_{\delta_2} \times [b, \frac{b}{\Gamma}] \times [b, \frac{b}{\Gamma}]$, $I_k(t,u) \geq \frac{b}{L}$, *for* $(t,u) \in J_{\delta_2} \cap J_k \times [b, \frac{b}{\Gamma}]$;
(A_3) $f(t,u,v) \leq \frac{a}{\omega}$, *for* $(t,u,v) \in J \times [0,a] \times [0,a]$, $I_k(t,u) \leq \frac{a}{\omega}$, *for* $(t,u) \in J_k \times [0,a]$.

Then problem (1.1) *and* (1.2) *has at least three positive solutions* x_1, x_2, x_3 *satisfying* $\|x_i\| \leq d$, $i = 1,2,3$, *and*

$$b \leq \Lambda_2(x_1), \qquad a < \|x_2\| \quad \textit{with } \Lambda_2(x_2) < b,$$

and

$$\|x_3\| < a.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
The work is supported by Chinese Universities Scientific Fund (Project No. 2016 LX002). The authors would like to thank the anonymous referees very much for helpful comments and suggestions, which lead to the improvement of presentation and quality of the work.

Received: 25 April 2016 Accepted: 2 September 2016 Published online: 13 September 2016

References

1. Sun, J, Wang, X: Monotone positive solutions for an elastic beam equation with nonlinear boundary conditions. Math. Probl. Eng. **2011**, Article ID 609189 (2011)
2. Yao, Q: Positive solutions of nonlinear beam equations with time and space singularities. J. Math. Anal. Appl. **374**, 681-692 (2011)
3. O'Regan, D: Solvability of some fourth (and higher) order singular boundary value problems. J. Math. Anal. Appl. **161**, 78-116 (1991)
4. Zhang, X: Existence and iteration of monotone positive solutions for an elastic beam equation with a corner. Nonlinear Anal. **10**, 2097-2103 (2009)
5. Bonanno, G, Bella, BD: A boundary value problem for fourth-order elastic beam equations. J. Math. Anal. Appl. **343**, 1166-1176 (2008)
6. Cabada, A, Tersian, S: Existence and multiplicity of solutions to boundary value problems for fourth-order impulsive differential equations. Bound. Value Probl. **2014**, 105 (2014)
7. Amster, P, Alzate, PPC: A shooting method for a nonlinear beam equation. Nonlinear Anal. **68**, 2072-2078 (2008)
8. Zhao, Y, Huang, L, Zhang, Q: Existence results for an impulsive Sturm-Liouville boundary value problems with mixed double parameters. Bound. Value Probl. **2015**, 150 (2015)
9. Kang, P, Wei, Z, Xu, J: Positive solutions to fourth-order singular boundary value problems with integral boundary conditions in abstract spaces. Appl. Math. Comput. **206**, 245-256 (2008)
10. Zhang, X, Feng, M: Positive solutions for classes of multi-parameter fourth-order impulsive differential equations with one-dimensional singular *p*-Laplacian. Bound. Value Probl. **2014**, 112 (2014)
11. Lv, X, Wang, L, Pei, M: Monotone positive solution of a fourth-order BVP with integral boundary conditions. Bound. Value Probl. **2015**, 172 (2015)
12. Ma, H: Symmetric positive solutions for nonlocal boundary value problems of fourth order. Nonlinear Anal. **68**, 645-651 (2008)
13. Zhang, X, Feng, M: Positive solutions of singular beam equations with the bending term. Bound. Value Probl. **2015**, 84 (2015)
14. Jankowski, T: Positive solutions for fourth-order differential equations with deviating arguments and integral boundary conditions. Nonlinear Anal. **73**, 1289-1299 (2010)
15. Feng, M, Qiu, J: Multi-parameter fourth order impulsive integral boundary value problems with one-dimensional *m*-Laplacian and deviating arguments. J. Inequal. Appl. **2015**, 64 (2015)
16. Afrouzi, GA, Hadjian, A, Radulescu, VD: Variational approach to fourth-order impulsive differential equations with two control parameters. Results Math. **65**, 371-384 (2014)
17. Sun, JT, Chen, HB, Yang, L: Variational methods to fourth-order impulsive differential equations. J. Appl. Math. Comput. **35**, 232-340 (2011)
18. Jankowski, T: Positive solutions for second order impulsive differential equations involving Stieltjes integral conditions. Nonlinear Anal. **74**, 3775-3785 (2011)
19. Ding, W, Wang, Y: New result for a class of impulsive differential equation with integral boundary conditions. Commun. Nonlinear Sci. Numer. Simul. **18**, 1095-1105 (2013)
20. Avery, RI, Peterson, AC: Three positive fixed points of nonlinear operators on ordered Banach spaces. Comput. Math. Appl. **42**, 313-322 (2001)