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Singularly perturbed semilinear elliptic boundary value problems with discontinuous source term

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Abstract

A class of singularly perturbed semilinear elliptic boundary value problems with discontinuous source term on two different domains is considered in this article. The formal asymptotic solution is constructed by using the method of boundary layer functions. Furthermore, the uniform validity of the solutions is proved by using the maximum principle. Finally, as an illustration, an example is presented.

MSC: 35B25; 35C20; 35J15

Keywords: singular perturbation; asymptotic expansion; boundary layer function; maximum principle

1 Introduction

We consider the following singularly perturbed elliptic boundary value problem:

$$\begin{cases} Lu(x,y) \equiv \varepsilon^2 \Delta u(x,y) = f(u,x,y), & (x,y) \in \Omega, \\ u(x,y)|_{\partial\Omega} = g(x,y), \end{cases} \tag{1.1}$$

where the bounded domain Ω is partitioned into two subdomains Ω^- and Ω^+ by a smooth curve Γ, that is, $\Omega = \Omega^- \cup \Gamma \cup \Omega^+$, and

$$f(u,x,y) = \begin{cases} f_1(u,x,y), & (x,y) \in \Omega^-, \\ f_2(u,x,y), & (x,y) \in \Omega^+, \end{cases}$$

with $f_1(u,x,y) \in C^2(\mathbb{R} \times \Omega^-)$, $f_2(u,x,y) \in C^2(\mathbb{R} \times \Omega^+)$ and $f_1(u,x,y) \neq f_2(u,x,y)$ for $(x,y) \in \Gamma$.

Partial differential equations are often used to describe plenty of phenomena in physics and engineering, thereby attracting much attention (see [1–3], for instance). As a branch of partial differential equations, singular perturbation problems for differential equations of elliptic type with smooth coefficients and smooth data, arising in many areas, such as fluid mechanics, heat and mass transfer in chemical engineering, theory of plates and shells, have been extensively studied since the 1960s; see [4] for instance and the references therein. Recently, due to the significance of interface problems appearing in many physical contexts with heterogeneous media, boundary value problems for elliptic and parabolic equations with discontinuous coefficients have attracted much attention (see

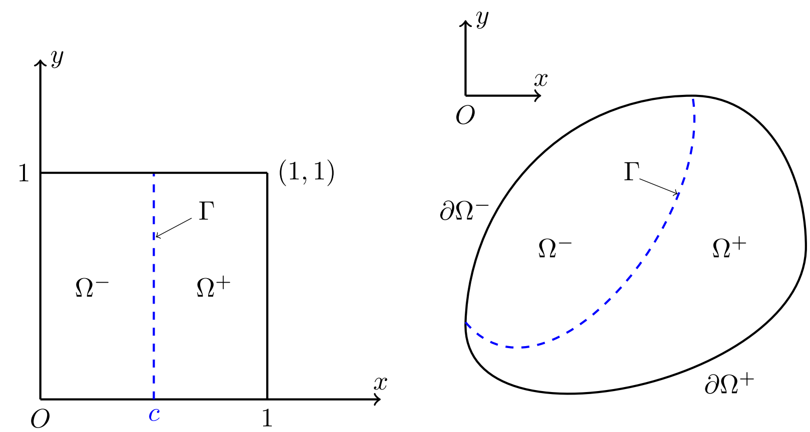


Figure 1 Two kinds of different regions. Left panel: a rectangular region. Right panel: a bounded region with smooth boundary.

[5–11], for example). In [5, 6], Babuška studied a kind of elliptic interface problems defined on a smooth domain with a smooth interface using the finite element method. In [7], Brayanov considered a mixed singularly perturbed parabolic-elliptic problem with discontinuous coefficients, which describes an electromagnetic field arising in the motion of a train on an air-pillow. O'Riordan has examined lately a particular class of singularly perturbed convection-diffusion problems with a discontinuous coefficient of the convective term [9].

In the present paper, we investigate the problem (1.1) with discontinuous source term $f(u,x,y)$ using the method of boundary layer functions, on two different regions: a rectangular region and a bounded region with smooth boundary. Usually, owing to the discontinuity of the coefficients at a curve, an interior layer around the interface may occur, besides a possible boundary layer at $\partial\Omega$. When the problem (1.1) has the property of axial symmetry, it can be reduced to an ordinary differential equation of second order with discontinuous source term and singular perturbation, which recently has been studied in [12] by using the method of lower and upper solutions. Therefore, the present work can be viewed as an extension of the corresponding smooth version or one dimensional case.

This paper is organized as follows. In Section 2, the formal asymptotic solutions of the problem (1.1) are constructed for two different bounded domains (see Figure 1). The uniform validity of the solutions is proved by using the maximum principle in Section 3. An example is presented to illustrate the main results in Section 4.

2 Formal asymptotic solutions

In this section, we construct the formal asymptotic solutions to the problem (1.1) on two kinds of regions: a rectangular region and a bounded region with smooth boundary (see Figure 1). The region Ω is divided by a smooth curve Γ into two subregions Ω^- and Ω^+, and $\partial\Omega = \partial\Omega^- \cup \partial\Omega^+$. In the first case, for the sake of clarity of presentation, the rectangular region Ω is given in the (x,y)-plane as $\Omega = (0,1) \times (0,1)$, the curve $\Gamma = \{(x,y) \mid x = c, 0 < y < 1, 0 < c < 1\}$, $\Omega^- = (0,c) \times (0,1)$, and $\Omega^+ = (c,1) \times (0,1)$.

We first make a basic assumption on the reduced problem.

(H_1) The reduced equation $f(u,x,y) = 0$ has a solution

$$u(x,y) = \begin{cases} \varphi(x,y), & (x,y) \in \Omega^-, \\ \psi(x,y), & (x,y) \in \Omega^+, \end{cases}$$

with $u(x,y) \in C^2(\Omega^- \cup \Omega^+)$, and there exist constants σ_1 and σ_2 such that

$$\frac{\partial f_1}{\partial u}(u,x,y) \geq \sigma_1 > 0, \quad (u,x,y) \in \mathbb{R} \times \Omega^-,$$
$$\frac{\partial f_2}{\partial u}(u,x,y) \geq \sigma_2 > 0, \quad (u,x,y) \in \mathbb{R} \times \Omega^+.$$

Considering that the function $f(u,x,y)$ is discontinuous at the curve Γ, the original problem (1.1) can be regarded as the coupling of the left problem

$$\begin{cases} \varepsilon^2 \Delta U^- = f_1(U^-,x,y), \\ U^-|\partial\Omega^- = g(x,y), \\ U^-|\Gamma = \gamma(\varepsilon,x,y), \end{cases} \tag{2.1}$$

and the right problem

$$\begin{cases} \varepsilon^2 \Delta U^+ = f_2(U^+,x,y), \\ U^+|\partial\Omega^+ = g(x,y), \\ U^+|\Gamma = \gamma(\varepsilon,x,y), \end{cases} \tag{2.2}$$

where $\gamma(\varepsilon,x,y) = \gamma_0(x,y) + \varepsilon\gamma_1(x,y) + \varepsilon^2\gamma_2(x,y) + \cdots$ is a smooth function which will be determined later on.

In the following, we will distinguish two cases in order to construct the formal asymptotic solutions.

Case 1 Ω is a rectangular region.

Let us look for the formal asymptotic solutions $U^{\mp}(x,y,\varepsilon)$ of the problems (2.1) and (2.2) in the form

$$\begin{aligned} U^-(x,y,\varepsilon) &= \bar{U}^-(x,y,\varepsilon) + L^-\left(\xi_1^-,y,\varepsilon\right) + Q^-\left(\xi_2^-,y,\varepsilon\right) \\ &\quad + W^-\left(x,\xi_3^-,\varepsilon\right) + \Pi^-\left(x,\xi_4^-,\varepsilon\right), \quad (x,y) \in \Omega^-, \\ U^+(x,y,\varepsilon) &= \bar{U}^+(x,y,\varepsilon) + L^+\left(\xi_1^+,y,\varepsilon\right) + Q^+\left(\xi_2^+,y,\varepsilon\right) \\ &\quad + W^+\left(x,\xi_3^+,\varepsilon\right) + \Pi^+\left(x,\xi_4^+,\varepsilon\right), \quad (x,y) \in \Omega^+, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \bar{U}^{\pm}(x,y,\varepsilon) &= \bar{U}_0^{\pm}(x,y) + \varepsilon\bar{U}_1^{\pm}(x,y) + \varepsilon^2\bar{U}_2^{\pm}(x,y) + \cdots, \\ L^{\pm}\left(\xi_1^{\pm},y,\varepsilon\right) &= L_0^{\pm}\left(\xi_1^{\pm},y\right) + \varepsilon L_1^{\pm}\left(\xi_1^{\pm},y\right) + \varepsilon^2 L_2^{\pm}\left(\xi_1^{\pm},y\right) + \cdots, \\ Q^{\pm}\left(\xi_2^{\pm},y,\varepsilon\right) &= Q_0^{\pm}\left(\xi_2^{\pm},y\right) + \varepsilon Q_1^{\pm}\left(\xi_2^{\pm},y\right) + \varepsilon^2 Q_2^{\pm}\left(\xi_2^{\pm},y\right) + \cdots, \\ W^{\pm}\left(x,\xi_3^{\pm},\varepsilon\right) &= W_0^{\pm}\left(x,\xi_3^{\pm}\right) + \varepsilon W_1^{\pm}\left(x,\xi_3^{\pm}\right) + \varepsilon^2 W_2^{\pm}\left(x,\xi_3^{\pm}\right) + \cdots, \\ \Pi^{\pm}\left(x,\xi_4^{\pm},\varepsilon\right) &= \Pi_0^{\pm}\left(x,\xi_4^{\pm}\right) + \varepsilon\Pi_1^{\pm}\left(x,\xi_4^{\pm}\right) + \varepsilon^2\Pi_2^{\pm}\left(x,\xi_4^{\pm}\right) + \cdots, \end{aligned}$$

and

$$\xi_1^- = \frac{x}{\varepsilon}, \qquad \xi_2^- = \frac{c-x}{\varepsilon}, \qquad \xi_1^+ = \frac{1-x}{\varepsilon},$$
$$\xi_2^+ = \frac{x-c}{\varepsilon}, \qquad \xi_3^\pm = \frac{y}{\varepsilon}, \qquad \xi_4^\pm = \frac{1-y}{\varepsilon}.$$

Here $\bar{U}^\pm$ are regular terms, $L^\pm$, $W^\pm$, $\Pi^\pm$ being boundary layer terms, and $Q^\pm$ are interior layer terms.

To determine the terms in the expansions (2.3) we represent the function $f_1(U^-, x, y)$ in a form which is similar to (2.3). According to the boundary layer function method [4, 13], $f_1(U^-(x,y,\varepsilon),x,y)$ will be represented in Ω^- in the form

$$\begin{aligned}
&f_1\bigl(U^-(x,y,\varepsilon),x,y\bigr)\\
&\quad= f_1\bigl(\bar{U}^- + L^- + Q^- + W^- + \Pi^-, x, y\bigr)\\
&\quad= f_1\bigl(\bar{U}^-, x, y\bigr) + \bigl(f_1\bigl(\bar{U}^- + L^-, x, y\bigr) - f_1\bigl(\bar{U}^-, x, y\bigr)\bigr)|_{x=\varepsilon\xi_1^-}\\
&\qquad+ \bigl(f_1\bigl(\bar{U}^- + L^- + Q^-, x, y\bigr) - f_1\bigl(\bar{U}^- + L^-, x, y\bigr)\bigr)|_{x=c-\varepsilon\xi_2^-}\\
&\qquad+ \bigl(f_1\bigl(\bar{U}^- + L^- + Q^- + W^-, x, y\bigr) - f_1\bigl(\bar{U}^- + L^- + Q^-, x, y\bigr)\bigr)|_{y=\varepsilon\xi_3^-}\\
&\qquad+ \bigl(f_1\bigl(\bar{U}^- + L^- + Q^- + W^- + \Pi^-, x, y\bigr) - f_1\bigl(\bar{U}^- + L^- + Q^- + W^-, x, y\bigr)\bigr)|_{y=1-\varepsilon\xi_4^-}.
\end{aligned}$$

Put the formal asymptotic solution (2.3) into the first equation of (2.1). For the sake of simplicity, we only consider the approximation of zeroth order. For $(x,y) \in \Omega^-$, concerning the regular part $\bar{U}^-$, we get

$$\varepsilon^2\left(\frac{\partial^2 \bar{U}^-}{\partial x^2} + \frac{\partial^2 \bar{U}^-}{\partial y^2}\right) = f_1\bigl(\bar{U}^-(x,y,\varepsilon),x,y\bigr). \tag{2.4}$$

For $\varepsilon = 0$, we get from (2.4) the degenerate equation

$$f_1\bigl(\bar{U}_0^-(x,y),x,y\bigr) = 0.$$

According to (H_1), we know

$$\bar{U}_0^-(x,y) = \varphi(x,y), \quad (x,y) \in \Omega^-.$$

Considering the boundary layer function L^-, we have

$$\frac{\partial^2 L_0^-}{\partial \xi_1^{-,2}} + \varepsilon \frac{\partial^2 L_1^-}{\partial \xi_1^{-,2}} + \cdots + \varepsilon^2 \frac{\partial^2 L_0^-}{\partial y^2} + \cdots = f_1\bigl(L^- + \bar{U}^-, \varepsilon\xi_1^-, y\bigr) - f_1\bigl(\bar{U}^-, \varepsilon\xi_1^-, y\bigr). \tag{2.5}$$

For the zeroth-order boundary layer function L_0^- we obtain from (2.5) the boundary value problem

$$\begin{cases} \frac{\partial^2 L_0^-}{\partial \xi_1^{-,2}} = f_1(\bar{U}_0^-(0,y) + L_0^-(\xi_1^-, y), 0, y), \\ L_0^-(0,y) = g(0,y) - \varphi(0,y), \qquad L_0^-(+\infty, y) = 0. \end{cases} \tag{2.6}$$

Concerning the continuity of $f_1(u,x,y)$, we denote

$$\sigma_3 = \max\left\{ \frac{\partial f_1}{\partial u}\bigl(\varphi(x,y) + \zeta_1, x, y\bigr) \Big| |\zeta_1| \le N_1^+ \right\}, \tag{2.7}$$

where $\zeta_1 = k_1 L^- + k_2 Q^- + k_3 W^- + k_4 \Pi^-$ for $0 \le k_i \le 1$, $i = 1,2$, and N_1^+ is large enough.

In order to study the asymptotic behavior of L_0^- with respect to the small parameter ε, we will show the following lemma.

Lemma 2.1 *Under the assumption* (H$_1$), *for sufficiently small* $\varepsilon > 0$, *problem* (2.6) *has a solution* $L_0^-(\xi_1^-, y)$ *and satisfies the estimate*

$$\bigl(g(0,y) - \varphi(0,y)\bigr)\exp\bigl(-\sqrt{\tilde{\sigma}_{l_1}}\xi_1^-\bigr) \le L_0^-\bigl(\xi_1^-, y\bigr) \le \bigl(g(0,y) - \varphi(0,y)\bigr)\exp\bigl(-\sqrt{\bar{\sigma}_{l_1}}\xi_1^-\bigr),$$

where

$$\tilde{\sigma}_{l_1} = \begin{cases} \sigma_3, & g(0,y) - \varphi(0,y) > 0, \\ \sigma_1, & g(0,y) - \varphi(0,y) < 0, \end{cases} \qquad \bar{\sigma}_{l_1} = \begin{cases} \sigma_1, & g(0,y) - \varphi(0,y) > 0, \\ \sigma_3, & g(0,y) - \varphi(0,y) < 0. \end{cases}$$

Proof We have

$$\begin{aligned} &f_1\bigl(\bar{U}_0^-(0,y) + L_0^-\bigl(\xi_1^-, y\bigr), 0, y\bigr) \\ &\quad = f_1\bigl(\varphi(0,y) + L_0^-\bigl(\xi_1^-, y\bigr), 0, y\bigr) - f_1\bigl(\varphi(0,y), 0, y\bigr) \\ &\quad = \frac{\partial f_1}{\partial u}\bigl(\varphi(0,y) + \theta L_0^-\bigl(\xi_1^-, y\bigr), 0, y\bigr) L_0^-\bigl(\xi_1^-, y\bigr), \quad \theta \in [0,1]. \end{aligned}$$

Rewriting the first equation of (2.6) into the equivalent equation, we have

$$\frac{\partial^2 L_0^-}{\partial \xi_1^{-,2}} = \frac{\partial f_1}{\partial u}\bigl(\varphi(0,y) + \theta L_0^-\bigl(\xi_1^-, y\bigr), 0, y\bigr) L_0^-\bigl(\xi_1^-, y\bigr).$$

Choose the barrier functions

$$\begin{aligned} \alpha\bigl(\xi_1^-, y\bigr) &= \bigl(g(0,y) - \varphi(0,y)\bigr)\exp\bigl(-\sqrt{\tilde{\sigma}_{l_1}}\xi_1^-\bigr), \\ \beta\bigl(\xi_1^-, y\bigr) &= \bigl(g(0,y) - \varphi(0,y)\bigr)\exp\bigl(-\sqrt{\bar{\sigma}_{l_1}}\xi_1^-\bigr), \end{aligned}$$

where

$$\tilde{\sigma}_{l_1} = \begin{cases} \sigma_3, & g(0,y) - \varphi(0,y) > 0, \\ \sigma_1, & g(0,y) - \varphi(0,y) < 0, \end{cases} \qquad \bar{\sigma}_{l_1} = \begin{cases} \sigma_1, & g(0,y) - \varphi(0,y) > 0, \\ \sigma_3, & g(0,y) - \varphi(0,y) < 0. \end{cases}$$

Treating y as a parameter, we have

$$\alpha\bigl(\xi_1^-, y\bigr) \le \beta\bigl(\xi_1^-, y\bigr), \qquad \xi_1^- \in \left[0, \frac{c}{\varepsilon}\right), \qquad \alpha(0,y) = g(0,y) = \beta(0,y),$$

and $\alpha(\frac{c}{\varepsilon}, y) \le \beta(\frac{c}{\varepsilon}, y)$.

By (2.7) and the assumption (H_1), we get

$$\begin{aligned}
&\frac{\partial^2\alpha}{\partial\xi_1^{-,2}} - \frac{\partial f_1}{\partial u}\big(\varphi(0,y) + \theta\alpha\big(\xi_1^-,y\big),0,y\big)\alpha\big(\xi_1^-,y\big) \\
&\quad = \tilde{\sigma}_{l_1}\big(g(0,y) - \varphi(0,y)\big)\exp\big(-\sqrt{\tilde{\sigma}_{l_1}}\xi_1^-\big) \\
&\qquad - \frac{\partial f_1}{\partial u}\big(\varphi(0,y) + \theta\alpha\big(\xi_1^-,y\big),0,y\big)\big(g(0,y) - \varphi(0,y)\big)\exp\big(-\sqrt{\tilde{\sigma}_{l_1}}\xi_1^-\big) \\
&\quad \geq 0
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial^2\beta}{\partial\xi_1^{-,2}} - \frac{\partial f_1}{\partial u}\big(\varphi(0,y) + \theta\beta\big(\xi_1^-,y\big),0,y\big)\beta\big(\xi_1^-,y\big) \\
&\quad = \bar{\sigma}_{l_1}\big(g(0,y) - \varphi(0,y)\big)\exp\big(-\sqrt{\bar{\sigma}_{l_1}}\xi_1^-\big) \\
&\qquad - \frac{\partial f_1}{\partial u}\big(\varphi(0,y) + \theta\beta\big(\xi_1^-,y\big),0,y\big)\big(g(0,y) - \varphi(0,y)\big)\exp\big(-\sqrt{\bar{\sigma}_{l_1}}\xi_1^-\big) \\
&\quad \leq 0.
\end{aligned}$$

It follows that the function $\alpha(\xi_1^-,y)$ and $\beta(\xi_1^-,y)$ are lower and upper solutions of the problem (2.6), respectively. By the theory of differential inequality, we see that problem (2.6) has a solution $L_0^-(\xi_1^-,y)$ which satisfies

$$\alpha\big(\xi_1^-,y\big) \leq L_0^-\big(\xi_1^-,y\big) \leq \beta\big(\xi_1^-,y\big). \qquad \square$$

Next, we consider the inner layer term of Q^-. We have in a similar manner

$$\frac{\partial^2 Q^-}{\partial\xi_2^{-,2}} + \varepsilon^2\frac{\partial^2 Q^-}{\partial y^2} = f_1\big(\bar{U}^- + L^- + Q^-, c - \varepsilon\xi_2^-, y\big) - f_1\big(\bar{U}^- + L^-, c - \varepsilon\xi_2^-, y\big). \tag{2.8}$$

Note that

$$\begin{aligned}
&f_1\big(\bar{U}^- + L^- + Q^-, x, y\big) - f_1\big(\bar{U}^- + L^-, x, y\big) \\
&\quad = f_1\big(\bar{U}^- + Q^-, x, y\big) - f_1\big(\bar{U}^-, x, y\big) + \Delta F_1 + \Delta F_2,
\end{aligned}$$

where

$$\begin{aligned}
&\Delta F_1 = f_1\big(\bar{U}^- + L^- + Q^-, x, y\big) - f_1\big(\bar{U}^- + Q^-, x, y\big) = \frac{\partial f_1}{\partial u}\big(\bar{U}^- + \theta_1 L^- + Q^-, x, y\big)L^-, \\
&\Delta F_2 = f_1\big(\bar{U}^- + L^-, x, y\big) - f_1\big(\bar{U}^-, x, y\big) = \frac{\partial f_1}{\partial u}\big(\bar{U}^- + \theta_2 L^-, x, y\big)L^-,
\end{aligned}$$

with $\theta_i \in [0,1], i = 1,2$. Here $\Delta F_1(\xi_1^-,y,\varepsilon)$ and $\Delta F_2(\xi_1^-,y,\varepsilon)$ decay exponentially by Lemma 2.1. The term of zeroth order Q_0^- satisfies

$$\begin{cases} \frac{\partial^2 Q_0^-}{\partial\xi_2^{-,2}} = f_1(\bar{U}_0^-(c,y) + Q_0^-(\xi_2^-,y), c, y), \\ Q_0^-(0,y) = \gamma_0(c,y) - \varphi(c,y), \qquad Q_0^-(+\infty,y) = 0. \end{cases} \tag{2.9}$$

In a similar way, we see that the boundary layer term W_0^- of the left problem solves

$$\begin{cases} \frac{\partial^2 W_0^-}{\partial \xi_3^{-,2}} = f_1(\bar{U}_0^-(x,0) + W_0^-(x,\xi_3^-), x, 0), \\ W_0^-(x,0) = g(x,0) - \varphi(x,0), \qquad W_0^-(x,+\infty) = 0, \end{cases} \tag{2.10}$$

and the boundary layer term Π_0^- of the left problem solves

$$\begin{cases} \frac{\partial^2 \Pi_0^-}{\partial \xi_4^{-,2}} = f_1(\bar{U}_0^-(x,1) + \Pi_0^-(x,\xi_4^-), x, 1), \\ \Pi_0^-(x,1) = g(x,1) - \varphi(x,1), \qquad \Pi_0^-(x,+\infty) = 0. \end{cases} \tag{2.11}$$

The following lemmas are related to the asymptotic behavior of the boundary layer terms for the left problem, whose proofs are similar to that of Lemma 2.1, and therefore they are omitted here.

Lemma 2.2 *Under the assumption* (H_1), *for sufficiently small* $\varepsilon > 0$, *problem* (2.9) *has a solution* $Q_0^-(\xi_2^-, y)$ *and satisfies the estimate*

$$\begin{aligned} &\bigl(\gamma_0(c,y) - \varphi(c,y)\bigr)\exp\bigl(-\sqrt{\tilde{\sigma}_{l_2}}\xi_2^-\bigr) \le Q_0^-\bigl(\xi_2^-, y\bigr) \\ &\qquad \le \bigl(\gamma_0(c,y) - \varphi(c,y)\bigr)\exp\bigl(-\sqrt{\bar{\sigma}_{l_2}}\xi_2^-\bigr), \end{aligned}$$

where

$$\tilde{\sigma}_{l_2} = \begin{cases} \sigma_3, & \gamma_0(c,y) - \varphi(c,y) > 0, \\ \sigma_1, & \gamma_0(c,y) - \varphi(c,y) < 0, \end{cases} \qquad \bar{\sigma}_{l_2} = \begin{cases} \sigma_1, & \gamma_0(c,y) - \varphi(c,y) > 0, \\ \sigma_3, & \gamma_0(c,y) - \varphi(c,y) < 0. \end{cases}$$

Lemma 2.3 *Under the assumption* (H_1), *for sufficiently small* $\varepsilon > 0$, *problem* (2.10) *has a solution* $W_0^-(x, \xi_3^-)$ *and satisfies the estimate*

$$\begin{aligned} &\bigl(g(x,0) - \varphi(x,0)\bigr)\exp\bigl(-\sqrt{\tilde{\sigma}_{l_3}}\xi_3^-\bigr) \le W_0^-\bigl(x, \xi_3^-\bigr) \\ &\qquad \le \bigl(g(x,0) - \varphi(x,0)\bigr)\exp\bigl(-\sqrt{\bar{\sigma}_{l_3}}\xi_3^-\bigr), \end{aligned}$$

where

$$\tilde{\sigma}_{l_3} = \begin{cases} \sigma_3, & g(x,0) - \varphi(x,0) > 0, \\ \sigma_1, & g(x,0) - \varphi(x,0) < 0, \end{cases} \qquad \bar{\sigma}_{l_3} = \begin{cases} \sigma_1, & g(x,0) - \varphi(x,0) > 0, \\ \sigma_3, & g(x,0) - \varphi(x,0) < 0. \end{cases}$$

Lemma 2.4 *Under the assumption* (H_1), *for sufficiently small* $\varepsilon > 0$, *problem* (2.11) *has a solution* $\Pi_0^-(x, \xi_4^-)$ *and satisfies the estimate*

$$\bigl(g(x,1) - \varphi(x,1)\bigr)\exp\bigl(-\sqrt{\tilde{\sigma}_{l_4}}\xi_4^-\bigr) \le \Pi_0^-\bigl(x, \xi_4^-\bigr) \le \bigl(g(x,1) - \varphi(x,1)\bigr)\exp\bigl(-\sqrt{\bar{\sigma}_{l_4}}\xi_4^-\bigr),$$

where

$$\tilde{\sigma}_{l_4} = \begin{cases} \sigma_3, & g(x,1) - \varphi(x,1) > 0, \\ \sigma_1, & g(x,1) - \varphi(x,1) < 0, \end{cases} \qquad \bar{\sigma}_{l_4} = \begin{cases} \sigma_1, & g(x,1) - \varphi(x,1) > 0, \\ \sigma_3, & g(x,1) - \varphi(x,1) < 0. \end{cases}$$

Regarding to the problem (2.2), we find that the analyses of $\bar{U}^+$, L^+, W^+, and Π^+ are similar to the left problem (2.1). Consequently, we will not calculate them in detail. In order to determine the parameters $\gamma_n(y)$ ($n > 0$), we would concentrate our attention on Q^+. For the zeroth-order approximation $Q_0^+(\xi_2^+, y)$, we get the following boundary value problem:

$$\begin{cases} \frac{\partial^2 Q_0^+}{\partial \xi_2^{+,2}} = f_2(\bar{U}_0^+(c,y) + Q_0^+(\xi_2^+, y), c, y), \\ Q_0^+(0,y) = \gamma_0(c,y) - \psi(c,y), \qquad Q_0^+(+\infty, y) = 0. \end{cases} \tag{2.12}$$

Due to the continuity of $f_2(u,x,y)$, we define

$$\sigma_4 = \max\left\{ \frac{\partial f_2}{\partial u}\big(\psi(x,y) + \zeta_2, x, y\big) \Big| |\zeta_2| \leq N_2^+ \right\},$$

where $\zeta_4 = \kappa_1 L^+ + \kappa_2 Q^+ + \kappa_3 W^+ + \kappa_2 \Pi^+$ for $0 \leq \kappa_i \leq 1, i = 1,2$, and N_2^+ is large enough. In a completely similar way for the left problem, we obtain the following result.

Lemma 2.5 *Under the assumption* (H$_1$), *for sufficiently small* $\varepsilon > 0$, *problem* (2.12) *has a solution* $Q_0^+(\xi_2^+, y)$ *and satisfies the estimate*

$$\begin{aligned} \big(\gamma_0(c,y) - \psi(c,y)\big) \exp\big(-\sqrt{\tilde{\sigma}_{l_5}}\xi_2^+\big) &\leq Q_0^+\big(\xi_2^+, y\big) \\ &\leq \big(\gamma_0(c,y) - \psi(c,y)\big) \exp\big(-\sqrt{\bar{\sigma}_{l_5}}\xi_2^+\big), \end{aligned}$$

where

$$\tilde{\sigma}_{l_5} = \begin{cases} \sigma_4, & \gamma_0(c,y) - \psi(c,y) > 0, \\ \sigma_2, & \gamma_0(c,y) - \psi(c,y) < 0, \end{cases} \qquad \bar{\sigma}_{l_5} = \begin{cases} \sigma_2, & \gamma_0(c,y) - \psi(c,y) > 0, \\ \sigma_4, & \gamma_0(c,y) - \psi(c,y) < 0. \end{cases}$$

In order that the solutions of the two problems are smoothly connected at $x = c$. The smoothness is guaranteed under an extra condition,

$$\frac{\partial U^-}{\partial x}(c,y,\varepsilon) = \frac{\partial U^+}{\partial x}(c,y,\varepsilon). \tag{2.13}$$

Substituting the formal solutions into (2.13) and comparing the same power of ε, we have

$$\begin{aligned} &\frac{\partial Q_0^-}{\partial \xi_2^-}\bigg|_{\xi_2^-=0} = \frac{\partial Q_0^+}{\partial \xi_2^+}\bigg|_{\xi_2^+=0}, \\ &\frac{\partial \varphi}{\partial x}\bigg|_{x=c} + \frac{\partial Q_1^-}{\partial \xi_2^-}\bigg|_{\xi_2^-=0} = \frac{\partial \psi}{\partial x}\bigg|_{x=c} + \frac{\partial Q_1^+}{\partial \xi_2^+}\bigg|_{\xi_2^+=0}, \\ &\dots \end{aligned} \tag{2.14}$$

Let

$$w = \frac{\partial Q_0^-}{\partial \xi_2^-},$$

then the boundary value problem (2.9) becomes

$$\frac{\partial w}{\partial \xi_2^-} = f_1\big(\varphi(c,y) + Q_0^-(\xi_2^-,y), c, y\big). \tag{2.15}$$

Multiplying (2.15) by $2\frac{\partial Q_0^-}{\partial \xi_2^-}$ and integrating on $[0,\infty)$, we obtain

$$\begin{aligned}\left(\frac{\partial Q_0^-}{\partial \xi_2^-}\bigg|_{\xi_2^-=0}\right)^2 &= 2f_1\big(\varphi(c,y) + Q_0^-(0,y), c, y\big)Q_0^-(0,y)\\ &\quad + 2\int_0^\infty \frac{\partial f_1}{\partial u}\big(\varphi(c,y) + Q_0^-(\xi_2^-,y), c, y\big)Q_0^- \frac{\partial Q_0^-}{\partial \xi_2^-}\,\mathrm{d}\xi_2^-. \end{aligned}\tag{2.16}$$

Analogously, we have

$$\begin{aligned}\left(\frac{\partial Q_0^+}{\partial \xi_2^+}\bigg|_{\xi_2^+=0}\right)^2 &= 2f_2\big(\psi(c,y) + Q_0^+(0,y), c, y\big)Q_0^+(0,y)\\ &\quad + 2\int_0^\infty \frac{\partial f_2}{\partial u}\big(\psi(c,y) + Q_0^+(\xi_2^+,y), c, y\big)Q_0^+ \frac{\partial Q_0^+}{\partial \xi_2^+}\,\mathrm{d}\xi_2^+. \end{aligned}\tag{2.17}$$

By substituting (2.16) and (2.17) into (2.14), we get

$$\begin{aligned}&f_2\big(\psi(c,y) + Q_0^+(0,y), c, y\big)Q_0^+(0,y) - f_1\big(\varphi(c,y) + Q_0^-(0,y), c, y\big)Q_0^-(0,y)\\ &\quad= \int_0^\infty \frac{\partial f_1}{\partial u}\big(\varphi(c,y) + Q_0^-(\xi_2^-,y), c, y\big)Q_0^- \frac{\partial Q_0^-}{\partial \xi_2^-}\,\mathrm{d}\xi_2^-\\ &\qquad - \int_0^\infty \frac{\partial f_2}{\partial u}\big(\psi(c,y) + Q_0^+(\xi_2^+,y), c, y\big)Q_0^+ \frac{\partial Q_0^+}{\partial \xi_2^+}\,\mathrm{d}\xi_2^+\\ &\quad\triangleq F(\gamma_0, y).\end{aligned}$$

(H_2) Assume

$$f_2\big(\psi(c,y) + Q_0^+(0,y), c, y\big)Q_0^+(0,y) - f_1\big(\varphi(c,y) + Q_0^-(0,y), c, y\big)Q_0^-(0,y) = F(\gamma_0, y)$$

has a unique solution $\gamma_0(c,y)$.

Likewise, $\gamma_i(c,y), i = 1,2,\ldots$, can be determined recursively.

Therefore, we have thus constructed the zeroth-order asymptotic solution

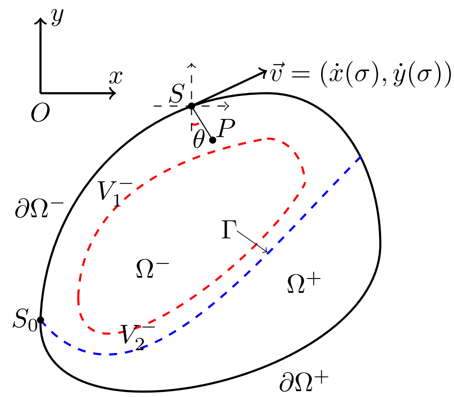
$$\widetilde{U}(x,y) = \begin{cases} U^-, & (x,y) \in \Omega^-,\\ U^+, & (x,y) \in \Omega^+, \end{cases} \tag{2.18}$$

where

$$\begin{aligned} U^- &= \varphi(x,y) + L_0^-(\xi_1^-,y) + Q_0^-(\xi_2^-,y) + W_0^-(x,\xi_3^-) + \Pi_0^-(x,\xi_4^-), \quad (x,y) \in \Omega^-,\\ U^+ &= \psi(x,y) + L_0^+(\xi_1^+,y) + Q_0^+(\xi_2^+,y) + W_0^+(x,\xi_3^+) + \Pi_0^+(x,\xi_4^+), \quad (x,y) \in \Omega^+. \end{aligned}$$

Case 2 *Ω is a bounded domain with $\partial\Omega$ of class C^∞.*

Figure 2 The domain Ω.



We study the problem (1.1) on the bounded domain Ω with ∂Ω of class C^∞. In order to construct the formal asymptotic solution we need to propose suitable parameter transformation. We first introduce local coordinates $(\rho_1^\pm, \sigma_1^\pm)$ in an interior neighborhood $V_1^\pm$ of $\partial\Omega^\pm$and $(\rho_2^\pm, \sigma_2^\pm)$ in a neighborhood $V_2^\pm$ of $\Gamma^\pm$, where $\rho_1^\pm$ denotes the distance PS from a point $P \in V_1^\pm$ to the boundary $\partial\Omega^\pm$, and $\sigma_1^\pm$ is the arc length from a given point $S_0 \in \partial\Omega^\pm$ to the point $S \in \partial\Omega^\pm$. $\rho_2^\pm$ denotes the distance PS from a point $P \in V_2^\pm$ to the boundary $\Gamma^\pm$, and $\sigma_2^\pm$ is the arc length from a given point $S_0 \in \Gamma^\pm$ to the point $S \in \Gamma^\pm$ (see Figure 2).

Since $\partial\Omega^\pm$ and $\Gamma^\pm$ are sufficiently smooth, the neighborhood $V_i^\pm$ can be expressed as

$$0 \le \rho_i^\pm \le \alpha_i^\pm, \qquad 0 \le \sigma_i^\pm \le \beta_i^\pm, \quad i = 1, 2.$$

Thus there is a 1-1 correspondence between the coordinates (x, y) and $(\rho_i^\pm, \sigma_i^\pm)$ in $V_i^\pm$.

Let the parameter representation of $\partial\Omega^\pm$ be given by

$$x = x(\sigma_1^\pm), \qquad y = y(\sigma_1^\pm),$$

and the parameter representation of $\Gamma^\pm$ be given by

$$x = x(\sigma_2^\pm), \qquad y = y(\sigma_2^\pm).$$

By the arc length formula, we have

$$\sigma = \int_0^\sigma \sqrt{\dot{x}(\tau)^2 + \dot{y}(\tau)^2}\, d\tau.$$

Taking a derivative with respect to σ in both sides of the above equation, we get

$$\dot{x}(\sigma)^2 + \dot{y}(\sigma)^2 = 1.$$

On the other hand, the unit tangent vector is $\vec{v} = (\dot{x}(\sigma_i^\pm), \dot{y}(\sigma_i^\pm))$ at the point S. Using the method of differential triangles, we have

$$x - x(\sigma_i^\pm) = \rho_i^\pm \sin\theta = \rho_i^\pm \dot{y}(\sigma_i^\pm),$$

$$y(\sigma_i^{\pm}) - y = \rho_i^{\pm} \cos\theta = \rho_i^{\pm} \dot{x}(\sigma_i^{\pm}).$$

Consequently, we have in $V_i^{\pm}$

$$x = x(\sigma_i^{\pm}) + \rho_i^{\pm} \dot{y}(\sigma_i^{\pm}), \qquad y = y(\sigma_i^{\pm}) - \rho_i^{\pm} \dot{x}(\sigma_i^{\pm}).$$

We look for the formal asymptotic solutions $U^{\mp}(x,y,\varepsilon)$ of the problem (2.1) and (2.2) in the following form:

$$\begin{aligned} U^-(x,y,\varepsilon) &= \bar{U}^-(x,y,\varepsilon) + V^-(\tau_1^-,\sigma_1^-,\varepsilon) + W^-(\tau_2^-,\sigma_2^-,\varepsilon), \\ U^+(x,y,\varepsilon) &= \bar{U}^+(x,y,\varepsilon) + V^+(\tau_1^+,\sigma_1^+,\varepsilon) + W^+(\tau_2^+,\sigma_2^+,\varepsilon), \end{aligned} \tag{2.19}$$

with

$$\begin{aligned} \bar{U}^{(\mp)}(x,y,\varepsilon) &= \bar{U}_0^{(\mp)}(x,y) + \varepsilon \bar{U}_1^{(\mp)}(x,y) + \cdots, \\ V^{(\mp)}(\tau_1^{\mp},\sigma_1^{\mp},\varepsilon) &= V_0^{(\mp)}(\tau_1^{\mp},\sigma_1^{\mp}) + \varepsilon V_1^{(\mp)}(\tau_1^{\mp},\sigma_1^{\mp}) + \cdots, \\ W^{(\mp)}(\tau_2^{\mp},\sigma_2^{\mp},\varepsilon) &= W_0^{(\mp)}(\tau_2^{\mp},\sigma_2^{\mp}) + \varepsilon W_1^{(\mp)}(\tau_2^{\mp},\sigma_2^{\mp}) + \cdots, \end{aligned}$$

where $\tau_i^{\mp} = \frac{\rho_i^{\mp}}{\varepsilon}$, and ρ_2^- and ρ_2^+ are symmetrical with respect to Γ.

Remark Note that the formal solution U^- (or U^+) in (2.19) consists of a regular part and two boundary layer parts (near the curve Γ and the boundary $\partial\Omega^+$ (or $\partial\Omega^-$)), which is different from that in the first case that Ω is a rectangular region. In the first case, the piecewise smooth boundary Ω comprises four line segments. Therefore, we have four boundary layer parts in (2.3), besides a regular part.

Expressing our dependent variables in $(\rho_i^{\mp},\sigma_i^{\mp})$, we retain the notation of our function symbols. For example, $f(u,x,y)$ with $(x,y) \in V_i^{\pm}$ is written as $f(u,\rho_i^{\mp},\sigma_i^{\mp})$. The problem (2.1) and (2.2) become in $(\rho_i^{\mp},\sigma_i^{\mp})$ coordinates, respectively,

$$\begin{aligned} &\varepsilon^2 \left\{ \frac{\partial^2 U^{\mp}}{\partial \rho_i^{\mp,2}} + \frac{1}{J_i^{\mp,2}} \frac{\partial^2 U^{\mp}}{\partial \sigma_i^{\mp,2}} + \left(\frac{\partial^2 \rho_i^{\mp}}{\partial x^2} + \frac{\partial^2 \rho_i^{\mp}}{\partial y^2} \right) \frac{\partial U^{\mp}}{\partial \rho_i^{\mp}} + \left(\frac{\partial^2 \sigma_i^{\mp}}{\partial x^2} + \frac{\partial^2 \sigma_i^{\mp}}{\partial y^2} \right) \frac{\partial U^{\mp}}{\partial \sigma_i^{\mp}} \right\} \\ &\quad = f_1(U^{\mp},\rho_i^{\mp},\sigma_i^{\mp}), \end{aligned} \tag{2.20}$$

subject to the boundary conditions

$$U^{\mp}|_{\partial\Omega^{\mp}} = \tilde{g}(\sigma_1^{\mp}), \qquad U^{\mp}|_{\Gamma^{\mp}} = \gamma(\varepsilon,\sigma_2^{\mp}),$$

where

$$\begin{aligned} J_i^{\mp} &= \begin{vmatrix} \frac{\partial x}{\partial \sigma_i^{\mp}} & \frac{\partial x}{\partial \rho_i^{\mp}} \\ \frac{\partial y}{\partial \sigma_i^{\mp}} & \frac{\partial y}{\partial \rho_i^{\mp}} \end{vmatrix}, \\ \tilde{g}(\sigma_1^{\mp}) &= g(x(\sigma_1^{\mp}), y(\sigma_1^{\mp})), \\ \gamma(\varepsilon,\sigma_2^{\mp}) &= \gamma(\varepsilon, x(\sigma_2^{\mp}), y(\sigma_2^{\mp})). \end{aligned}$$

By the method of the boundary layer function [4, 13],

$$\begin{aligned} &f_1(U^{\mp}, \rho_i^{\mp}, \sigma_i^{\mp}) \\ &\quad = f_1(\bar{U}^{\mp} + V^{\mp} + W^{\mp}, \rho_i^{\mp}, \sigma_i^{\mp}) \\ &\quad = f_1(\bar{U}^{\mp}, \rho_i^{\mp}, \sigma_i^{\mp}) + f_1(\bar{U}^{\mp} + V^{\mp}, \rho_i^{\mp}, \sigma_i^{\mp}) - f_1(\bar{U}^{\mp}, \rho_i^{\mp}, \sigma_i^{\mp}) \\ &\qquad + f_1(\bar{U}^{\mp} + V^{\mp} + W^{\mp}, \rho_i^{\mp}, \sigma_i^{\mp}) - f_1(\bar{U}^{\mp} + V^{\mp}, \rho_i^{\mp}, \sigma_i^{\mp}). \end{aligned}$$

For simplicity, we only consider the approximation of zeroth order. For the left problem, considering the boundary layer term of $\partial\Omega^-$, we can get the boundary value problem

$$\begin{cases} \frac{\partial^2 V_0^-}{\partial \tau_1^{-,2}} = f_1(\bar{U}_0^-(x(\sigma_1^-), y(\sigma_1^-)) + V_0^-, 0, \sigma_1^-), \\ V_0^-(0, \sigma_1^-) = \tilde{g}(\sigma_1^-) - \bar{U}_0^-(x(\sigma_1^-), y(\sigma_1^-)), \qquad V_0^-(\infty, 0) = 0. \end{cases} \tag{2.21}$$

For $\theta \in (0, 1)$, we have

$$f_1(\bar{U}_0^-(x(\sigma_1^-), y(\sigma_1^-)) + V_0^-, 0, \sigma_1^-) = \frac{\partial f_1}{\partial u}(\bar{U}_0^-(x(\sigma_1^-), y(\sigma_1^-)) + \theta V_0^-, 0, \sigma_1^-) V_0^-.$$

Rewrite the first equation of (2.21) as the equivalent equation

$$\frac{\partial^2 V_0^-}{\partial \tau_1^{-,2}} = \frac{\partial f_1}{\partial u}(\bar{U}_0^-(x(\sigma_1^-), y(\sigma_1^-)) + \theta V_0^-, 0, \sigma_1^-) V_0^-.$$

By the assumption (H_1), we know $\frac{\partial f_1}{\partial u}(u, x, y) \geq \sigma_1 > 0$. It follows that $V_0^-(\tau_1^-, \sigma_1^-)$ as the function of τ_1^- is concave for $V_0^-(0, \sigma_1^-) > 0$ and convex for $V_0^-(0, \sigma_1^-) < 0$. Because of $V_0^-(\infty, 0) = 0$, as long as $V_0^-(0, \sigma_1^-) \neq 0$, it is impossible that the function $V_0^-(\tau_1^-, \sigma_1^-)$ can change its sign for $\tau_1^- > 0$. Hence we have the result that $V_0^-(\tau_1^-, \sigma_1^-)$ decreases monotonously for $V_0^-(0, \sigma_1^-) > 0$, and increases monotonously for $V_0^-(0, \sigma_1^-) < 0$.

If $V_0^-(0, \sigma_1^-) > 0$, we have

$$\frac{\partial^2 V_0^-}{\partial \tau_1^{-,2}} \frac{\partial V_0^-}{\partial \tau_1^-} < \sigma_1 V_0^- \frac{\partial V_0^-}{\partial \tau_1^-}.$$

It follows that

$$\left(\frac{\partial V_0^-}{\partial \tau_1^-}\right)^2 \Big|_{\tau_1^-}^{\infty} < \sigma_1 (V_0^-)^2 \Big|_{\tau_1^-}^{\infty},$$

from which we obtain

$$-\frac{\partial V_0^-}{\partial \tau_1^-} > \sigma_1 V_0^-.$$

Integrating once again, we get

$$V_0^-(\tau_1^-, \sigma_1^-) = O(e^{-\sigma_1 \tau_1^-}). \tag{2.22}$$

Of course, the same reason holds for the case $V_0^-(0, \sigma_1^-) < 0$.

Consider the boundary layer term of Γ. By equation (2.22), it follows that

$$\begin{aligned} &f_1\bigl(\bar{U}^{(\mp)} + V^{(\mp)} + W^{(\mp)}, \rho_i^{\mp}, \sigma_i^{\mp}\bigr) - f_1\bigl(\bar{U}^{(\mp)} + V^{(\mp)}, \rho_i^{\mp}, \sigma_i^{\mp}\bigr) \\ &\quad = f_1\bigl(\bar{U}^{(\mp)} + W^{(\mp)}, \rho_i^{\mp}, \sigma_i^{\mp}\bigr) - f_1\bigl(\bar{U}^{(\mp)}, \rho_i^{\mp}, \sigma_i^{\mp}\bigr) + \Delta F_5, \end{aligned}$$

where ΔF_5 can be negligible. For the zeroth-order approximation of $W_0^-(0,\sigma_2^-)$, we get the boundary value problem

$$\begin{cases} \frac{\partial^2 W_0^-}{\partial \tau_2^{-,2}} = f_1(\bar{U}_0^-(x(\sigma_2^-), y(\sigma_2^-)) + W_0^-, 0, \sigma_2^-), \\ W_0^-(0,\sigma_2^-) = \gamma_0(\sigma_2^-) - \bar{U}_0^-(x(\sigma_2^-), y(\sigma_2^-)), \qquad W_0^-(\infty, 0) = 0. \end{cases} \tag{2.23}$$

For $\theta \in (0,1)$, we have

$$f_1\bigl(\bar{U}_0^-\bigl(x(\sigma_2^-), y(\sigma_2^-)\bigr) + W_0^-, 0, \sigma_2^-\bigr) = \frac{\partial f_1}{\partial u}\bigl(\bar{U}_0^-\bigl(x(\sigma_2^-), y(\sigma_2^-)\bigr) + \theta W_0^-, 0, \sigma_2^-\bigr) W_0^-.$$

Rewrite the first equation of (2.23) into the equivalent equation

$$\frac{\partial^2 W_0^-}{\partial \tau_2^{-,2}} = \frac{\partial f_1}{\partial u}\bigl(\bar{U}_0^-\bigl(x(\sigma_2^-), y(\sigma_2^-)\bigr) + \theta W_0^-, 0, \sigma_2^-\bigr) W_0^-. \tag{2.24}$$

Similar to the analysis to the boundary value problem (2.21), for the boundary value problem (2.23), we obtain the same result,

$$W_0^-\bigl(\tau_2^-, \sigma_2^-\bigr) = O\bigl(e^{-\sigma_1 \tau_2^-}\bigr).$$

Consider the function of the left problem (2.1)

$$U_0^-(x,y) = \bar{U}_0^-(x,y) + V_0^-\left(\frac{\rho_1^-}{\varepsilon}, \sigma_1^-\right)\chi_1\bigl(\rho_1^-\bigr) + W_0^-\left(\frac{\rho_2^-}{\varepsilon}, \sigma_2^-\right)\chi_2\bigl(\rho_2^-\bigr), \tag{2.25}$$

where

$$\chi_i\bigl(\rho_i^-\bigr) = \begin{cases} 1, & 0 \le \rho_i^- \le \frac{1}{2}\alpha_i^-, \\ 0, & \frac{3}{4}\alpha_i^- \le \rho_i^- \le \alpha_i^-, \end{cases} \qquad i = 1,2.$$

Analogously, the right problem (2.2) have the same form

$$U_0^+(x,y) = \bar{U}_0^+(x,y) + V_0^+\left(\frac{\rho_1^+}{\varepsilon}, \sigma_1^+\right)\nu_1\bigl(\rho_1^+\bigr) + W_0^+\left(\frac{\rho_2^+}{\varepsilon}, \sigma_2^+\right)\nu_2\bigl(\rho_2^+\bigr), \tag{2.26}$$

where

$$\nu_i\bigl(\rho_i^+\bigr) = \begin{cases} 1, & 0 \le \rho_i^+ \le \frac{1}{2}\alpha_i^+, \\ 0, & \frac{3}{4}\alpha_i^+ \le \rho_i^+ \le \alpha_i^+, \end{cases} \qquad i = 1,2.$$

In order that the solutions of the two problems are smoothly connected at Γ, considering the approximation of zeroth order for the boundary layer terms, we have

$$\left.\frac{\partial W_0^-}{\partial \tau_2^-}\right|_{\tau_2^-=0} = \left.\frac{\partial W_0^+}{\partial \tau_2^+}\right|_{\tau_2^+=0}. \tag{2.27}$$

Let

$$z=\frac{\partial W_0^-}{\partial \tau_2^-},$$

then the boundary value problem (2.23) becomes

$$\frac{\partial z}{\partial \tau_2^-}=f_1\big(\bar{U}_0^-\big(x(\sigma_2^-),y(\sigma_2^-)\big)+W_0^-,0,\sigma_2^-\big). \tag{2.28}$$

Multiplying (2.28) by $2\frac{\partial W_0^-}{\partial \tau_2^-}$ and integrating on $[0,\infty)$, we obtain

$$\begin{aligned}\left(\frac{\partial W_0^-}{\partial \tau_2^-}\bigg|_{\tau_2^-=0}\right)^2 &= 2f_1\big(\bar{U}_0^-\big(x(\sigma_2^-),y(\sigma_2^-)\big)+W_0^-(0),0,\sigma_2^-\big)\\ &\quad+2\int_0^\infty \frac{\partial f_1}{\partial u}\big(\bar{U}_0^-\big(x(\sigma_2^-),y(\sigma_2^-)\big)+W_0^-,0,\sigma_2^-\big)\frac{\partial W_0^-}{\partial \tau_2^-}W_0^-\,\mathrm{d}\tau_2^-.\end{aligned} \tag{2.29}$$

In the same way,

$$\begin{aligned}\left(\frac{\partial W_0^+}{\partial \tau_2^+}\bigg|_{\tau_2^+=0}\right)^2 &= 2f_2\big(\bar{U}_0^+\big(x(\sigma_2^+),y(\sigma_2^+)\big)+W_0^+(0),0,\sigma_2^+\big)\\ &\quad+2\int_0^\infty \frac{\partial f_1}{\partial u}\big(\bar{U}_0^+\big(x(\sigma_2^+),y(\sigma_2^+)\big)+W_0^+,0,\sigma_2^+\big)\frac{\partial W_0^+}{\partial \tau_2^+}W_0^+\,\mathrm{d}\tau_2^+.\end{aligned} \tag{2.30}$$

By substituting (2.29) and (2.30) into (2.27), we obtain

$$\begin{aligned}&f_2\big(\bar{U}_0^+\big(x(\sigma_2^+),y(\sigma_2^+)\big)+W_0^+(0),0,\sigma_2^+\big)-f_1\big(\bar{U}_0^-\big(x(\sigma_2^-),y(\sigma_2^-)\big)+W_0^-(0),0,\sigma_2^-\big)\\ &\quad=\int_0^\infty \frac{\partial f_1}{\partial u}\big(\bar{U}_0^-\big(x(\sigma_2^-),y(\sigma_2^-)\big)+W_0^-,0,\sigma_2^-\big)\frac{\partial W_0^-}{\partial \tau_2^-}W_0^-\,\mathrm{d}\tau_2^-\\ &\qquad-\int_0^\infty \frac{\partial f_1}{\partial u}\big(\bar{U}_0^+\big(x(\sigma_2^+),y(\sigma_2^+)\big)+W_0^+,0,\sigma_2^+\big)\frac{\partial W_0^+}{\partial \tau_2^+}W_0^+\,\mathrm{d}\tau_2^+\\ &\quad\triangleq \tilde{F}\big(\gamma_0,\sigma_2^-,\sigma_2^+\big).\end{aligned}$$

(H_3) Assume that

$$\begin{aligned}&f_2\big(\bar{U}_0^+\big(x(\sigma_2^+),y(\sigma_2^+)\big)+W_0^+(0),0,\sigma_2^+\big)-f_1\big(\bar{U}_0^-\big(x(\sigma_2^-),y(\sigma_2^-)\big)+W_0^-(0),0,\sigma_2^-\big)\\ &\quad\triangleq \tilde{F}\big(\gamma_0,\sigma_2^-,\sigma_2^+\big)\end{aligned}$$

has a unique solution $\gamma_0(\sigma_2^-,\sigma_2^+)$.

Likewise, $\gamma_i(\sigma_2^-,\sigma_2^+), i=1,2,\ldots$ can be determined recursively.

As mentioned above, we have constructed the zeroth-order asymptotic solution of the problems (1.1) in the form

$$\widetilde{U}(x,y)=\begin{cases}U_0^-(x,y), & (x,y)\in\Omega^-,\\ \gamma_0(x,y), & (x,y)\in\Gamma,\\ U_0^+(x,y), & (x,y)\in\Omega^+.\end{cases} \tag{2.31}$$

3 Main result

In this section, we will consider the uniform validity of the solution for the problem (1.1). Before doing this, we need to present a technical lemma whose proof can be found in [4].

Lemma 3.1 *Let the twice continuously differentiable functions* $\Phi(x,y)$, $\Psi_1(x,y)$, *and* $\Psi_2(x,y)$ *satisfy the inequalities*

$$L[\Psi_1] < L[\Phi] < L[\Psi_2], \quad (x,y) \in D,$$

where L is the differential operator defined in (1.1), *elliptic in bounded domain* $D \in \mathbb{R}$ *with respect to the functions* $\Phi + \theta_1(\Psi_1 - \Phi)$ *and* $\Phi + \theta_2(\Psi_2 - \Phi)$, *and* $\frac{\partial f}{\partial u} \geq 0$ *for all* $(x,y) \in D$ *and for all functions* $u(x,y) \in C^2(D)$; $0 < \theta_i = \theta_i(x,y) < 1, i = 1,2$. *If*

$$\Psi_2(x,y) \leq \Phi(x,y) \leq \Psi_1(x,y), \quad (x,y) \in \partial D,$$

then this relation holds also for all points $(x,y) \in \bar{D}$.

Now we state the existence and asymptotical results for the problem (1.1).

Theorem 3.1 *Suppose that* Ω *is a bounded rectangular domain as in Figure* 1. *If the conditions* (H_1)-(H_2) *hold, then for sufficiently small* $\varepsilon > 0$, *the problem* (1.1) *has a solution* $U(x,y)$ *with*

$$U(x,y) = \widetilde{U}(x,y) + O(\varepsilon),$$

where $\widetilde{U}(x,y)$ *is defined in* (2.18).

Proof Note that $\widetilde{U}(x,y)$ satisfies the boundary value problem

$$\begin{cases} \varepsilon^2 \Delta \widetilde{U}(x,y) - f_i(\widetilde{U},x,y) = O(\varepsilon), & (x,y) \in \Omega, i = 1,2, \\ \widetilde{U}(x,y)|_{\partial\Omega \cup \Gamma} = 0. \end{cases} \tag{3.1}$$

Set

$$U(x,y) = \widetilde{U}(x,y) + R(x,y),$$

with

$$R(x,y) = \begin{cases} R_1(x,y), & (x,y) \in \Omega^-, \\ R_2(x,y), & (x,y) \in \Omega^+. \end{cases}$$

Thus we have the problem for the remainder term $R(x,y)$

$$\begin{cases} \varepsilon^2 \Delta \widetilde{U}(x,y) + \varepsilon^2 \Delta R(x,y) - f_i(\widetilde{U} + R,x,y) = O(\varepsilon), & (x,y) \in \Omega, i = 1,2, \\ R(x,y)|_{\partial\Omega \cup \Gamma} = 0. \end{cases} \tag{3.2}$$

It follows from equations (3.1) and (3.2) that

$$\begin{cases} L_1(R) \equiv \varepsilon^2 \Delta R(x,y) - f_i(\widetilde{U} + R, x, y) + f_i(\widetilde{U}, x, y) = O(\varepsilon), & (x,y) \in \Omega, i = 1,2, \\ R(x,y)|_{\partial\Omega\cup\Gamma} = 0. \end{cases}$$

Barrier functions are provided by taking β, where

$$\beta = \sum_{i=1}^{2} \sum_{k=1}^{2} \beta_{i,k},$$

and $\beta_{i,k} = (-1)^{k-1}\lambda\varepsilon, i = 1,2$. λ is sufficiently large.

For $k = 1$, and $\beta_{i,1} = \lambda\varepsilon, i = 1,2$, we have

$$\begin{aligned} L_1[\beta_{i,1}] &\equiv \varepsilon^2 \Delta\beta_{i,1} - f_i(\widetilde{U} + \beta_{i,1}, x, y) + f_i(\widetilde{U}, x, y) \\ &= -\frac{\partial f}{\partial u}(\widetilde{U} + \theta\lambda\varepsilon, x, y)\lambda\varepsilon \\ &\le -\sigma_i \lambda\varepsilon. \end{aligned}$$

Analogously, for $k = 2$, and $\beta_{i,2} = -\lambda\varepsilon, i = 1,2$,

$$\begin{aligned} L_1[\beta_{i,2}] &\equiv \varepsilon^2 \Delta\beta_{i,2} - f_i(\widetilde{U} + \beta_{i,1}, x, y) + f_i(\widetilde{U}, x, y) \\ &= \frac{\partial f}{\partial u}(\widetilde{U} + \theta\lambda\varepsilon, x, y)\lambda\varepsilon \\ &\ge \sigma_i \lambda\varepsilon. \end{aligned}$$

Accordingly, $R = O(\varepsilon)$ is uniform in $\bar{\Omega}$ by Lemma 3.1. □

Theorem 3.2 *Suppose that* Ω *is a bounded domain with* $\partial\Omega$ *of class* C^∞. *If the conditions* (H_1) *and* (H_3) *hold, then for sufficiently small* $\varepsilon > 0$, *the problem* (1.1) *has a solution* $U(x,y)$ *with*

$$U(x,y) = \widetilde{U}(x,y) + O(\varepsilon),$$

where $\widetilde{U}(x,y)$ *is defined as in* (2.31).

The proof of this result is quite similar to that given earlier for the Theorem 3.1 and thereby is omitted.

4 Example

In this section, we present an example on a rectangular domain $\Omega = \{0 < x < 1, 0 < y < 1\}$ to illustrate our results. Consider the following boundary value problem

$$\begin{cases} \varepsilon^2(\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y)) = f(u,x,y), \\ u(0,y) = y, \qquad u(1,y) = y, \\ u(x,0) = 2, \qquad u(x,1) = x, \end{cases} \tag{4.1}$$

where

$$f(u,x,y) = \begin{cases} u(x,y), & (x,y) \in \Omega^-, \\ u(x,y) - 1, & (x,y) \in \Omega^+, \end{cases}$$

with $\Omega^- = \{0 < x < \frac{1}{2}, 0 < y < 1\}$, $\Omega^+ = \{\frac{1}{2} < x < 1, 0 < y < 1\}$. Obviously, $f_i(u,x,y) \in C^2(R \times \Omega^\mp)$, $i = 1,2$, and $f_1(u, \frac{1}{2}, y) \neq f_2(u, \frac{1}{2}, y)$.

The reduced equation $f_1(u,x,y) = 0$ has a solution $\varphi(x,y) = 0$, and $f_2(u,x,y) = 0$ has a solution $\psi(x,y) = 1$, satisfying the condition

$$\frac{\partial f_i}{\partial u}(u,x,y) = 1 > 0, \quad (u,x,y) \in \mathbb{R} \times \Omega^\mp, i = 1,2.$$

It follows that the assumption (H_1) is verified.

Moreover, we can find that $\gamma_0(y) = \frac{1}{4}$ satisfies the assumption (H_2). Therefore the zeroth-order asymptotic solution of the problem (4.1) is obtained as follows:

$$U(x,y) = \begin{cases} ye^{-\frac{x}{\varepsilon}} + \frac{1}{4}e^{-\frac{\frac{1}{2}-x}{\varepsilon}} + 2e^{-\frac{y}{\varepsilon}} + xe^{-\frac{1-y}{\varepsilon}}, & (x,y) \in \Omega^-, \\ \frac{1}{4}, & (x,y) \in \Gamma, \\ 1 + (y-1)e^{-\frac{1-x}{\varepsilon}} - \frac{3}{4}e^{-\frac{x-\frac{1}{2}}{\varepsilon}} + e^{-\frac{y}{\varepsilon}} + (x-1)e^{-\frac{1-y}{\varepsilon}}, & (x,y) \in \Omega^+. \end{cases}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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