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Existence of solutions for a sequential fractional differential system with coupled boundary conditions

Jiqiang Jiang[1,2*] and Lishan Liu[2]

*Correspondence: qfjjq@163.com
[1]School of Statistics, Qufu Normal University, Qufu, Shandong 273165, P.R. China
[2]School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, P.R. China

Abstract

This paper is concerned with the existence and uniqueness of solutions for a sequential fractional differential system with coupled boundary conditions. The existence of solutions is derived by applying Leray-Schauder's alternative, while the uniqueness of the solution is established via Banach's contraction principle. Two examples are then given to demonstrate the validity of our main results.

MSC: 26A33; 34B10; 34B15; 34B18

Keywords: fractional differential system; sequential fractional derivative; coupled boundary conditions; fixed point theorem

1 Introduction

The human immunodeficiency virus (HIV) causes acquired immune deficiency syndrome (AIDS). The viral infection is characterized by a severe impairment of the immune system and related opportunistic infections. HIV is a retrovirus that targets the $CD4^+$ T lymphocytes, which are the most abundant white blood cells of the immune system. Until now, there are several countries, particularly in Africa, with up to 35% of their populations between the ages of 15 and 50 years infected by HIV, and throughout the world, already over 16 million deaths died of AIDS. Mathematical models have been proven valuable in understanding the dynamics of HIV infection [1–4]. Perelson [5, 6] developed a simple model for the primary infection with HIV. In this model, four categories of cells were defined: uninfected $CD4^+$ T-cells, latently infected $CD4^+$ T-cells, productively infected $CD4^+$ T-cells, and the virus population. In [4], two equations were proposed to describe the evolution of the system for HIV-1 population dynamics:

$$\begin{cases} \frac{dx}{dt} = s - \mu x - \beta xy, \\ \frac{dy}{dt} = \beta xy - \alpha y, \end{cases}$$

where all parameters and variables are non-negative, and x denotes the number of uninfected $CD4^+$ T-cells and y denotes the number of infected cells, s is the assumed constant rate of production of $CD4^+$ T-cells, μ is their per capita death rate, β is the rate of infection of $CD4^+$ T-cells by virus, and α is the rate of disappearance of infected cells.

Recently, Arafal *et al.* [7] introduced fractional-order model of infection of CD4$^+$ T-cells which is described by the following set of FODEs of order $\alpha_1, \alpha_2, \alpha_3 > 0$:

$$\begin{cases} \mathcal{D}^{\alpha_1}(T) = s - KVT - dT + bI, \\ \mathcal{D}^{\alpha_2}(I) = KVT - (b+\delta)I, \\ \mathcal{D}^{\alpha_3}(V) = N\delta I - cV, \end{cases}$$

where T, I, and V denote the concentration of uninfected CD4$^+$ T-cells, infected CD4$^+$ T-cells, and free HIV virus particles in the blood, respectively. δ represents the death rate of infected T-cells and includes the possibility of death by bursting of infected T-cells, hence $\delta \geq d$. The parameter b is the rate at which infected cells return to the uninfected class, while c is the death rate of the virus and N is the average number of viral particles produced by an infected cell.

Motivated by the HIV infection model and its application background, in this paper, we consider the existence of solutions for the nonlinear sequential fractional differential system with coupled boundary conditions (BCs) of the type:

$$\begin{cases} ({}^{c}\mathcal{D}^{p} + \lambda_1 {}^{c}\mathcal{D}^{p-1})u(t) = f_1(t, u(t), v(t)), \quad 0 < t < 1, \\ ({}^{c}\mathcal{D}^{q} + \lambda_2 {}^{c}\mathcal{D}^{q-1})v(t) = f_2(t, u(t), v(t)), \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \qquad u(1) = av(\xi), \\ v(0) = v'(0) = 0, \qquad v(1) = bu(\eta), \end{cases} \tag{1.1}$$

where $\lambda_i > 0$ $(i = 1,2)$ is a parameter, $2 < p, q \leq 3$, ${}^{c}\mathcal{D}^{p}$, ${}^{c}\mathcal{D}^{q}$ are the Caputo fractional derivatives, ξ, η satisfy $\xi, \eta \in (0,1)$ and $ab(\lambda_1\eta - 1 + e^{-\lambda_1\eta})(\lambda_2\xi - 1 + e^{-\lambda_2\xi}) - (\lambda_1 - 1 + e^{-\lambda_1})(\lambda_2 - 1 + e^{-\lambda_2}) \triangleq \Lambda \neq 0$, the nonlinearities $f_1, f_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

In the past decades, fractional calculus has been extensively applied in many fields such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, economics, control theory, signal and image processing, biophysics, and blood flow phenomena. Many mathematicians and applied researchers have tried to model real processes using the fractional calculus. In biology, it has been deduced that the membranes of cells of biological organism have fractional-order electrical conductance [8] and thus are classified in groups of non-integer-order models. Fractional derivatives embody essential features of cell rheological behavior and have enjoyed greatest success in the field of rheology [9]. Fractional-order ordinary differential equations are naturally related to systems with long time memory which exists in most biological systems such as HIV infection, hepatitis C virus (HCV) infection, and cancer pervasion. Also, they are closely related to fractals, which are abundant in biological systems. Thus fractional-order differential equations are also regarded as a better tool for the description of hereditary properties of various materials and processes than the corresponding integer-order differential equations. With this advantage, fractional-order models have become more realistic and practical than the corresponding classical integer-order models, moreover, the dynamics behavior of fractional-order models are also as stable as their integer-order counterpart. Since theoretical results can help to get an in-depth understanding for the dynamic behavior in biological process, the study of abstract fractional dynamic models nowadays is quite relevant and important. On the other hand, BCs in (1.1) are referred to as coupled BCs; they arise in the study of reaction-diffusion equations, Sturm-Liouville problems, mathematical biology and so on; see [10–20]. In [16], Leung studied the following reaction-diffusion system for a prey-predator interaction:

$$\begin{cases} u_t(t,x) = \sigma_1 \Delta u + u(a + f(u,v)), & t \geq 0, x \in \Omega \subset \mathbb{R}^n, \\ v_t(t,x) = \sigma_2 \Delta v + v(-r + g(u,v)), & t \geq 0, x \in \Omega \subset \mathbb{R}^n, \end{cases}$$

subject to the coupled BCs

$$\frac{\partial u}{\partial \eta} = 0, \qquad \frac{\partial v}{\partial \eta} - p(u) - q(v) = 0 \quad \text{on } \partial\Omega,$$

where the functions $u(t,x)$, $v(t,x)$, respectively, represent the density of prey and predator at time $t \geq 0$ and at position $x = (x_1, \ldots, x_n)$. Similar coupled BCs are also studied in [11] for a biochemical system. As far as we know, the nonlinear fractional differential system coupled at equations have been studied extensively. For details, see [21–31] and the references therein. However, there have been a few papers which deal with coupling at boundary conditions for fractional differential systems. Wang *et al.* [32] obtained the existence and uniqueness of positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations,

$$\begin{cases} \mathcal{D}_{0+}^{\alpha} u(t) = f(t, v(t)), \qquad \mathcal{D}_{0+}^{\beta} v(t) = g(t, u(t)), \quad 0 < t < 1, \\ u(0) = v(0) = 0, \qquad u(1) = au(\xi), \qquad v(1) = bv(\xi), \end{cases}$$

where $1 < \alpha, \beta < 2$, $0 \leq a, b < 1$, $0 < \xi < 1$, $f, g : [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous functions, and $\mathcal{D}_{0+}^{\alpha}$, $\mathcal{D}_{0+}^{\beta}$ are the standard Riemann-Liouville fractional derivative.

The paper is organized as follows. In Section 2, we recall some basic concepts of fractional calculus and present some auxiliary lemmas. The main results are presented in Section 3. We give two results: the first one dealing with the existence of solutions is derived by applying Leray-Schauder's alternative; the second one concerning the uniqueness of solutions, established by applying Banach's contraction mapping principle. In Section 4, two examples are given to demonstrate the validity of our main results. Some interesting observations are presented in the conclusions section.

2 Preliminaries and lemmas

In this section, we will present some preliminaries and lemmas that will be used in the proof of our main results.

Definition 2.1 ([33, 34]) The Riemann-Liouville fractional integral of order $\alpha > 0$ is given by

$$I^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds,$$

where $n - 1 < \alpha < n$, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 ([33, 34]) For an $(n-1)$-times absolutely continuous function $u : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined as

$${}^{\mathrm{c}}\mathcal{D}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\, ds, \quad n - 1 < \alpha < n, n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.1 *For any $h, g \in L(0,1) \cap C(0,1)$, the system consisting of the equations*

$$\big({}^{\mathrm{c}}\mathcal{D}^{p} + \lambda_1 {}^{\mathrm{c}}\mathcal{D}^{p-1}\big)u(t) = h(t), \qquad \big({}^{\mathrm{c}}\mathcal{D}^{q} + \lambda_2 {}^{\mathrm{c}}\mathcal{D}^{q-1}\big)v(t) = g(t), \quad t \in (0,1) \tag{2.1}$$

and the BCs

$$\begin{cases} u(0) = u'(0) = 0, & u(1) = av(\xi), \\ v(0) = v'(0) = 0, & v(1) = bu(\eta), \end{cases} \tag{2.2}$$

has a unique integral representation

$$\begin{aligned} u(t) = {} & A_1(t)\bigg\{a\big(\lambda_2\xi - 1 + e^{-\lambda_2\xi}\big)\bigg[\int_0^1 e^{-\lambda_2(1-s)}(Qg)(s)\,ds - b\int_0^\eta e^{-\lambda_1(\eta-s)}(Ph)(s)\,ds\bigg] \\ & - \big(\lambda_2 - 1 + e^{-\lambda_2}\big)\bigg[a\int_0^\xi e^{-\lambda_2(\xi-s)}(Qg)(s)\,ds - \int_0^1 e^{-\lambda_1(1-s)}(Ph)(s)\,ds\bigg]\bigg\} \\ & + \int_0^t e^{-\lambda_1(t-s)}(Ph)(s)\,ds, \end{aligned} \tag{2.3}$$

$$\begin{aligned} v(t) = {} & A_2(t)\bigg\{\big(\lambda_1 - 1 + e^{-\lambda_1}\big)\bigg[\int_0^1 e^{-\lambda_2(1-s)}(Qg)(s)\,ds - b\int_0^\eta e^{-\lambda_1(\eta-s)}(Ph)(s)\,ds\bigg] \\ & - b\big(\lambda_1\eta - 1 + e^{-\lambda_1\eta}\big)\bigg[a\int_0^\xi e^{-\lambda_2(\xi-s)}(Qg)(s)\,ds - \int_0^1 e^{-\lambda_1(1-s)}(Ph)(s)\,ds\bigg]\bigg\} \\ & + \int_0^t e^{-\lambda_2(t-s)}(Qg)(s)\,ds, \end{aligned} \tag{2.4}$$

where

$$A_1(t) = \frac{1}{\Lambda}\big(\lambda_1 t - 1 + e^{-\lambda_1 t}\big), \qquad A_2(t) = \frac{1}{\Lambda}\big(\lambda_2 t - 1 + e^{-\lambda_2 t}\big), \tag{2.5}$$

$$(Ph)(s) = \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)}h(\tau)\,d\tau, \qquad (Qg)(s) = \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}g(\tau)\,d\tau. \tag{2.6}$$

Proof Solving (2.1), we obtain

$$u(t) = c_{11}e^{-\lambda_1 t} + \frac{c_{12}}{\lambda_1}\big(1 - e^{-\lambda_1 t}\big) + \frac{c_{13}}{\lambda_1^2}\big(\lambda_1 t - 1 + e^{-\lambda_1 t}\big) + \int_0^t e^{-\lambda_1(t-s)}(Ph)(s)\,ds, \tag{2.7}$$

$$v(t) = c_{21}e^{-\lambda_2 t} + \frac{c_{22}}{\lambda_2}\big(1 - e^{-\lambda_2 t}\big) + \frac{c_{23}}{\lambda_2^2}\big(\lambda_2 t - 1 + e^{-\lambda_2 t}\big) + \int_0^t e^{-\lambda_2(t-s)}(Qg)(s)\,ds, \tag{2.8}$$

where c_{ij} $(1 \le i \le 2, 1 \le j \le 3)$ are constants to be determined. In the following, we determine c_{ij} $(1 \le i \le 2, 1 \le j \le 3)$, so that $u(t)$ and $v(t)$ satisfy (2.2). By BCs (2.2), we obtain

$$c_{11} = c_{12} = 0, \qquad c_{21} = c_{22} = 0, \tag{2.9}$$

and

$$\begin{aligned} & \frac{\lambda_1 - 1 + e^{-\lambda_1}}{\lambda_1^2}c_{13} - \frac{a(\lambda_2\xi - 1 + e^{-\lambda_2\xi})}{\lambda_2^2}c_{23} \\ & \quad = a\int_0^\xi e^{-\lambda_2(\xi-s)}(Qg)(s)\,ds - \int_0^1 e^{-\lambda_1(1-s)}(Ph)(s)\,ds, \end{aligned} \tag{2.10}$$

$$\begin{aligned}&\frac{b(\lambda_1\eta-1+e^{-\lambda_1\eta})}{\lambda_1^2}c_{13}-\frac{\lambda_2-1+e^{-\lambda_2}}{\lambda_2^2}c_{23}\\&\quad=\int_0^1 e^{-\lambda_2(1-s)}(Qg)(s)\,ds-b\int_0^\eta e^{-\lambda_1(\eta-s)}(Ph)(s)\,ds.\end{aligned}\tag{2.11}$$

Note that

$$\begin{aligned}\begin{vmatrix}\frac{\lambda_1-1+e^{-\lambda_1}}{\lambda_1^2} & -\frac{a(\lambda_2\xi-1+e^{-\lambda_2\xi})}{\lambda_2^2}\\ \frac{b(\lambda_1\eta-1+e^{-\lambda_1\eta})}{\lambda_1^2} & -\frac{\lambda_2-1+e^{-\lambda_2}}{\lambda_2^2}\end{vmatrix}&=\frac{ab(\lambda_1\eta-1+e^{-\lambda_1\eta})(\lambda_2\xi-1+e^{-\lambda_2\xi})}{\lambda_1^2\lambda_2^2}\\&\quad-\frac{(\lambda_1-1+e^{-\lambda_1})(\lambda_2-1+e^{-\lambda_2})}{\lambda_1^2\lambda_2^2}=\frac{\Lambda}{\lambda_1^2\lambda_2^2}\triangleq\mathfrak{B}\neq 0.\end{aligned}$$

Thus, the system (2.10)-(2.11) has a unique solution for c_{13} and c_{23}. By Cramer's rule and simple calculations, it follows that

$$\begin{aligned}c_{13}&=\frac{1}{\mathfrak{B}}\frac{a(\lambda_2\xi-1+e^{-\lambda_2\xi})}{\lambda_2^2}\biggl[\int_0^1 e^{-\lambda_2(1-s)}(Qg)(s)\,ds-b\int_0^\eta e^{-\lambda_1(\eta-s)}(Ph)(s)\,ds\biggr]\\&\quad-\frac{1}{\mathfrak{B}}\frac{\lambda_2-1+e^{-\lambda_2}}{\lambda_2^2}\biggl[a\int_0^\xi e^{-\lambda_2(\xi-s)}(Qg)(s)\,ds-\int_0^1 e^{-\lambda_1(1-s)}(Ph)(s)\,ds\biggr],\end{aligned}\tag{2.12}$$

$$\begin{aligned}c_{23}&=\frac{1}{\mathfrak{B}}\frac{\lambda_1-1+e^{-\lambda_1}}{\lambda_1^2}\biggl[\int_0^1 e^{-\lambda_2(1-s)}(Qg)(s)\,ds-b\int_0^\eta e^{-\lambda_1(\eta-s)}(Ph)(s)\,ds\biggr]\\&\quad-\frac{1}{\mathfrak{B}}\frac{b(\lambda_1\eta-1+e^{-\lambda_1\eta})}{\lambda_1^2}\biggl[a\int_0^\xi e^{-\lambda_2(\xi-s)}(Qg)(s)\,ds-\int_0^1 e^{-\lambda_1(1-s)}(Ph)(s)\,ds\biggr].\end{aligned}\tag{2.13}$$

Substituting (2.9) and (2.12) in (2.7), one has

$$\begin{aligned}u(t)&=A_1(t)\biggl\{a\bigl(\lambda_2\xi-1+e^{-\lambda_2\xi}\bigr)\biggl[\int_0^1 e^{-\lambda_2(1-s)}(Qg)(s)\,ds-b\int_0^\eta e^{-\lambda_1(\eta-s)}(Ph)(s)\,ds\biggr]\\&\quad-\bigl(\lambda_2-1+e^{-\lambda_2}\bigr)\biggl[a\int_0^\xi e^{-\lambda_2(\xi-s)}(Qg)(s)\,ds-\int_0^1 e^{-\lambda_1(1-s)}(Ph)(s)\,ds\biggr]\biggr\}\\&\quad+\int_0^t e^{-\lambda_1(t-s)}(Ph)(s)\,ds.\end{aligned}$$

So (2.3) holds. Similarly, substituting (2.9) and (2.13) in (2.8) we can get (2.4). This completes the proof of the lemma. □

Lemma 2.2 ([35]) *For any $h,g\in L(0,1)\cap C(0,1)$, we have*

$$\begin{aligned}&\biggl|\int_0^\eta e^{-\lambda_1(\eta-s)}(Ph)(s)\,ds\biggr|\le\frac{\eta^p}{\lambda_1\Gamma(p)}\bigl(1-e^{-\lambda_1\eta}\bigr)\|h\|,\\&\biggl|\int_0^t e^{-\lambda_1(t-s)}(Ph)(s)\,ds\biggr|\le\frac{1}{\lambda_1\Gamma(p)}\bigl(1-e^{-\lambda_1}\bigr)\|h\|,\end{aligned}\tag{2.14}$$

$$\begin{aligned}&\biggl|\int_0^\xi e^{-\lambda_2(\xi-s)}(Qg)(s)\,ds\biggr|\le\frac{\xi^q}{\lambda_2\Gamma(q)}\bigl(1-e^{-\lambda_2\xi}\bigr)\|g\|,\\&\biggl|\int_0^t e^{-\lambda_2(t-s)}(Qg)(s)\,ds\biggr|\le\frac{1}{\lambda_2\Gamma(q)}\bigl(1-e^{-\lambda_2}\bigr)\|g\|.\end{aligned}\tag{2.15}$$

Let $X = C[0,1]$, then $X \times X$ is a Banach space with the norm

$$\|(u,v)\|_1 := \|u\| + \|v\|, \qquad \|u\| = \max_{0\le t\le 1}|u(t)|, \qquad \|v\| = \max_{0\le t\le 1}|v(t)|,$$

for any $(u,v) \in X \times X$.

In view of Lemma 2.1, we define the operator $T : X \times X \to X \times X$ by

$$T(u,v) = \big(T_1(u,v), T_2(u,v)\big),$$

where operators $T_i : X \times X \to X$ ($i = 1,2$) are defined by

$$\begin{aligned} &T_1(u,v)(t) \\ &\quad = A_1(t)\biggl\{a\bigl(\lambda_2\xi - 1 + e^{-\lambda_2\xi}\bigr)\biggl[\int_0^1 e^{-\lambda_2(1-s)}Q(u,v)(s)\,ds - b\int_0^\eta e^{-\lambda_1(\eta-s)}P(u,v)(s)\,ds\biggr] \\ &\qquad - \bigl(\lambda_2 - 1 + e^{-\lambda_2}\bigr)\biggl[a\int_0^\xi e^{-\lambda_2(\xi-s)}Q(u,v)(s)\,ds - \int_0^1 e^{-\lambda_1(1-s)}P(u,v)(s)\,ds\biggr]\biggr\} \\ &\qquad + \int_0^t e^{-\lambda_1(t-s)}P(u,v)(s)\,ds \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} &T_2(u,v)(t) \\ &\quad = A_2(t)\biggl\{\bigl(\lambda_1 - 1 + e^{-\lambda_1}\bigr)\biggl[\int_0^1 e^{-\lambda_2(1-s)}Q(u,v)(s)\,ds - b\int_0^\eta e^{-\lambda_1(\eta-s)}P(u,v)(s)\,ds\biggr] \\ &\qquad - b\bigl(\lambda_1\eta - 1 + e^{-\lambda_1\eta}\bigr)\biggl[a\int_0^\xi e^{-\lambda_2(\xi-s)}Q(u,v)(s)\,ds - \int_0^1 e^{-\lambda_1(1-s)}P(u,v)(s)\,ds\biggr]\biggr\} \\ &\qquad + \int_0^t e^{-\lambda_2(t-s)}Q(u,v)(s)\,ds, \end{aligned} \tag{2.17}$$

with

$$P(u,v)(s) = \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} f_1\bigl(\tau, u(\tau), v(\tau)\bigr)\,d\tau, \tag{2.18}$$

$$Q(u,v)(s) = \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} f_2\bigl(\tau, u(\tau), v(\tau)\bigr)\,d\tau. \tag{2.19}$$

For the sake of convenience, we set

$$A_1 = \sup_{t\in[0,1]}|A_1(t)|, \qquad A_2 = \sup_{t\in[0,1]}|A_2(t)|, \tag{2.20}$$

$$\begin{aligned} M_1 &= \frac{A_1[|ab|(\lambda_2\xi - 1 + e^{-\lambda_2\xi})(1 - e^{-\lambda_1\eta})\eta^{p-1} + (\lambda_2 - 1 + e^{-\lambda_2})(1 - e^{-\lambda_1})] + (1 - e^{-\lambda_1})}{\lambda_1\Gamma(p)}, \\ M_2 &= \frac{A_1[|a|(\lambda_2\xi - 1 + e^{-\lambda_2\xi})(1 - e^{-\lambda_2}) + (\lambda_2 - 1 + e^{-\lambda_2})|a|(1 - e^{-\lambda_2\xi})\xi^{q-1}]}{\lambda_2\Gamma(q)}, \\ M_1' &= \frac{A_2[(\lambda_1 - 1 + e^{-\lambda_1})|b|(1 - e^{-\lambda_1\eta})\eta^{p-1} + |b|(\lambda_1\eta - 1 + e^{-\lambda_1\eta})(1 - e^{-\lambda_1})]}{\lambda_1\Gamma(p)}, \end{aligned} \tag{2.21}$$

$$M_2' = \frac{A_2[(\lambda_1 - 1 + e^{-\lambda_1})(1 - e^{-\lambda_2}) + |ab|(\lambda_1\eta - 1 + e^{-\lambda_1\eta})(1 - e^{-\lambda_2\xi})\xi^{q-1}] + (1 - e^{-\lambda_2})}{\lambda_2\Gamma(q)},$$

and

$$(P\mathbf{1})(s) = \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)}\,d\tau, \qquad (Q\mathbf{1})(s) = \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}\,d\tau. \tag{2.22}$$

Lemma 2.3 *The operator $T: X \times X \to X \times X$ is a completely continuous.*

Proof By continuity of the functions f_i $(i = 1, 2)$, the operator T is continuous.

Let $\Omega \subset X \times X$ be bounded. Then there exist constants $L_i > 0$ $(i = 1, 2)$ such that

$$\bigl|f_i\bigl(t, u(t), v(t)\bigr)\bigr| \le L_i, \quad \forall (u, v) \in \Omega, i = 1, 2.$$

Then for any $(u, v) \in \Omega$, it follows from (2.16), (2.14), (2.15), (2.20), and (2.22) that

$$\begin{aligned}
&\bigl|T_1(u,v)(t)\bigr| \\
&\quad\le A_1\biggl\{|a|\bigl(\lambda_2\xi - 1 + e^{-\lambda_2\xi}\bigr)\biggl[L_2\int_0^1 e^{-\lambda_2(1-s)}(Q\mathbf{1})(s)\,ds + |b|L_1\int_0^\eta e^{-\lambda_1(\eta-s)}(P\mathbf{1})(s)\,ds\biggr] \\
&\qquad + \bigl(\lambda_2 - 1 + e^{-\lambda_2}\bigr)\biggl[|a|L_2\int_0^\xi e^{-\lambda_2(\xi-s)}(Q\mathbf{1})(s)\,ds + L_1\int_0^1 e^{-\lambda_1(1-s)}(P\mathbf{1})(s)\,ds\biggr]\biggr\} \\
&\qquad + L_1\int_0^t e^{-\lambda_1(t-s)}(P\mathbf{1})(s)\,ds \\
&\quad\le A_1\biggl[|a|\bigl(\lambda_2\xi - 1 + e^{-\lambda_2\xi}\bigr)\biggl(\frac{L_2(1-e^{-\lambda_2})}{\lambda_2\Gamma(q)} + \frac{|b|L_1(1-e^{-\lambda_1\eta})\eta^{p-1}}{\lambda_1\Gamma(p)}\biggr) \\
&\qquad + \bigl(\lambda_2 - 1 + e^{-\lambda_2}\bigr)\biggl(\frac{|a|L_2(1-e^{-\lambda_2\xi})\xi^{q-1}}{\lambda_2\Gamma(q)} + \frac{L_1(1-e^{-\lambda_1})}{\lambda_1\Gamma(p)}\biggr)\biggr] \\
&\qquad + \frac{L_1(1-e^{-\lambda_1})}{\lambda_1\Gamma(p)} \\
&\quad\le L_1\frac{A_1[|ab|(\lambda_2\xi - 1 + e^{-\lambda_2\xi})(1-e^{-\lambda_1\eta})\eta^{p-1} + (\lambda_2 - 1 + e^{-\lambda_2})(1-e^{-\lambda_1})] + (1-e^{-\lambda_1})}{\lambda_1\Gamma(p)} \\
&\qquad + L_2\frac{A_1[|a|(\lambda_2\xi - 1 + e^{-\lambda_2\xi})(1-e^{-\lambda_2}) + (\lambda_2 - 1 + e^{-\lambda_2})|a|(1-e^{-\lambda_2\xi})\xi^{q-1}]}{\lambda_2\Gamma(q)},
\end{aligned} \tag{2.23}$$

which implies that

$$\bigl\|T_1(u,v)\bigr\| \le L_1M_1 + L_2M_2, \tag{2.24}$$

where M_1, M_2 are given by (2.21).

By (2.17), (2.14), (2.15), (2.20), (2.22), and proceeding as in (2.23), we can obtain

$$\bigl\|T_2(u,v)\bigr\| \le L_1M_1' + L_2M_2', \tag{2.25}$$

where M_1', M_2' are given by (2.21).

Combining (2.24) with(2.25), we obtain

$$\left\| T(u,v)\right\|_1 = \left\| T_1(u,v)\right\| + \left\| T_2(u,v)\right\| \leq (L_1M_1 + L_2M_2) + \left(L_1M_1' + L_2M_2'\right) = M,$$

which implies that the operator T is uniformly bounded.

Next, we show that T is equicontinuous. For any $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$, noticing (2.22) then we have

$$\begin{aligned}
&\left|T_1(u,v)(t_2) - T_1(u,v)(t_1)\right| \\
&\quad\leq \Bigg|\left[A_1(t_2) - A_1(t_1)\right]\Bigg\{a\left(\lambda_2\xi - 1 + e^{-\lambda_2\xi}\right) \\
&\qquad\times \left[L_2\int_0^1 e^{-\lambda_2(1-s)}(Q\mathbf{1})(s)\,ds - bL_1\int_0^\eta e^{-\lambda_1(\eta-s)}(P\mathbf{1})(s)\,ds\right] \\
&\qquad - \left(\lambda_2 - 1 + e^{-\lambda_2}\right)\left[aL_2\int_0^\xi e^{-\lambda_2(\xi-s)}(Q\mathbf{1})(s)\,ds - L_1\int_0^1 e^{-\lambda_1(1-s)}(P\mathbf{1})(s)\,ds\right]\Bigg\}\Bigg| \\
&\qquad + L_1\left|\int_0^{t_1}\left(e^{-\lambda_1(t_2-s)} - e^{-\lambda_1(t_1-s)}\right)(P\mathbf{1})(s)\,ds + \int_{t_1}^{t_2} e^{-\lambda_1(t_2-s)}(P\mathbf{1})(s)\,ds\right|.
\end{aligned}$$

Analogously, we can obtain the following inequalities:

$$\begin{aligned}
&\left|T_2(u,v)(t_2) - T_2(u,v)(t_1)\right| \\
&\quad\leq \Bigg|\left[A_2(t_2) - A_2(t_1)\right]\Bigg\{\left(\lambda_1 - 1 + e^{-\lambda_1}\right) \\
&\qquad\times \left[L_2\int_0^1 e^{-\lambda_2(1-s)}(Q\mathbf{1})(s)\,ds - bL_1\int_0^\eta e^{-\lambda_1(\eta-s)}(P\mathbf{1})(s)\,ds\right] \\
&\qquad - b\left(\lambda_1\eta - 1 + e^{-\lambda_1\eta}\right)\left[aL_2\int_0^\xi e^{-\lambda_2(\xi-s)}(Q\mathbf{1})(s)\,ds - L_1\int_0^1 e^{-\lambda_1(1-s)}(P\mathbf{1})(s)\,ds\right]\Bigg\}\Bigg| \\
&\qquad + L_2\left|\int_0^{t_1}\left(e^{-\lambda_2(t_2-s)} - e^{-\lambda_2(t_1-s)}\right)(Q\mathbf{1})(s)\,ds + \int_{t_1}^{t_2} e^{-\lambda_2(t_2-s)}(Q\mathbf{1})(s)\,ds\right|.
\end{aligned}$$

Since for any fixed $s \in [0,1]$, the functions $e^{-\lambda_i(t-s)}$, $A_i(t)$ $(i = 1,2)$ are uniformly continuous on the interval on [0,1], we can conclude that the operator $T(u,v)$ is equicontinuous. Thus the operator $T(u,v)$ is completely continuous. The proof is completed. □

Now we state a well-known fixed point theorem, which is needed to prove the existence of solutions for system (1.1).

Lemma 2.4 (Leray-Schauder alternative [36]) *Let E be a Banach space. Assume that $T: E \to E$ be a completely continuous operator. Let*

$$V = \{x \in E | x = \mu Tx \text{ for some } 0 < \mu < 1\}.$$

Then either the set V is unbounded, or T has at least one fixed point.

3 Main results

Theorem 3.1 *Assume that there exist real constants $\rho_i, \delta_i \geq 0$ and $k_i > 0$ $(i = 1, 2)$ such that $\forall t \in [0,1]$, $x, y \in \mathbb{R}$,*

$$\left|f_i(t,x,y)\right| \leq k_i + \rho_i|x| + \delta_i|y|, \quad i = 1, 2. \tag{3.1}$$

In addition, assume that

$$\left(M_1 + M_1'\right)\rho_1 + \left(M_2 + M_2'\right)\rho_2 < 1 \quad \text{and} \quad \left(M_1 + M_1'\right)\delta_1 + \left(M_2 + M_2'\right)\delta_2 < 1,$$

where M_i, M_i' $(i = 1, 2)$ are defined by (2.21). *Then the system* (1.1) *has at least one solution.*

Proof Let us verify that the set $V = \{(u,v) \in X \times X : (u,v) = \mu T(u,v), 0 \leq \mu \leq 1\}$ is bounded. Let $(u,v) \in V$, then $(u,v) = \mu T(u,v)$. For any $t \in [0,1]$, we have

$$u(t) = \mu T_1(u,v)(t), \qquad v(t) = \mu T_2(u,v)(t). \tag{3.2}$$

Then, by (3.1), (3.2), (2.14), and (2.15), for any $t \in [0,1]$, we have

$$\begin{aligned}
\left|u(t)\right| \leq{}& \bigg| A_1(t)\bigg\{ a\left(\lambda_2\xi - 1 + e^{-\lambda_2\xi}\right) \\
&\times \left[\int_0^1 e^{-\lambda_2(1-s)}Q(u,v)(s)\,ds - b\int_0^\eta e^{-\lambda_1(\eta-s)}P(u,v)(s)\,ds\right] \\
&- \left(\lambda_2 - 1 + e^{-\lambda_2}\right)\left[a\int_0^\xi e^{-\lambda_2(\xi-s)}Q(u,v)(s)\,ds - \int_0^1 e^{-\lambda_1(1-s)}P(u,v)(s)\,ds\right]\bigg\} \\
&+ \int_0^t e^{-\lambda_1(t-s)}P(u,v)(s)\,ds\bigg| \\
\leq{}& A_1\bigg\{|a|\left(\lambda_2\xi - 1 + e^{-\lambda_2\xi}\right)\bigg[\left(k_2 + \rho_2\|u\| + \delta_2\|v\|\right)\int_0^1 e^{-\lambda_2(1-s)}(Q\mathbf{1})(s)\,ds \\
&+ |b|\left(k_1 + \rho_1\|u\| + \delta_1\|v\|\right)\int_0^\eta e^{-\lambda_1(\eta-s)}(P\mathbf{1})(s)\,ds\bigg] \\
&+ \left(\lambda_2 - 1 + e^{-\lambda_2}\right)\bigg[|a|\left(k_2 + \rho_2\|u\| + \delta_2\|v\|\right)\int_0^\xi e^{-\lambda_2(\xi-s)}(Q\mathbf{1})(s)\,ds \\
&+ \left(k_1 + \rho_1\|u\| + \delta_1\|v\|\right)\int_0^1 e^{-\lambda_1(1-s)}(P\mathbf{1})(s)\,ds\bigg]\bigg\} \\
&+ \left(k_1 + \rho_1\|u\| + \delta_1\|v\|\right)\int_0^t e^{-\lambda_1(t-s)}(P\mathbf{1})(s)\,ds \\
\leq{}& \frac{A_1[|ab|(\lambda_2\xi - 1 + e^{-\lambda_2\xi})(1 - e^{-\lambda_1\eta})\eta^{p-1} + (\lambda_2 - 1 + e^{-\lambda_2})(1 - e^{-\lambda_1})] + (1 - e^{-\lambda_1})}{\lambda_1\Gamma(p)} \\
&\times \left(k_1 + \rho_1\|u\| + \delta_1\|v\|\right) + \left(k_2 + \rho_2\|u\| + \delta_2\|v\|\right) \\
&\times \frac{A_1[|a|(\lambda_2\xi - 1 + e^{-\lambda_2\xi})(1 - e^{-\lambda_2}) + (\lambda_2 - 1 + e^{-\lambda_2})|a|(1 - e^{-\lambda_2\xi})\xi^{q-1}]}{\lambda_2\Gamma(q)} \\
\leq{}& M_1\left(k_1 + \rho_1\|u\| + \delta_1\|v\|\right) + M_2\left(k_2 + \rho_2\|u\| + \delta_2\|v\|\right).
\end{aligned} \tag{3.3}$$

Hence we have

$$\|u\| \leq M_1\big(k_1 + \rho_1\|u\| + \delta_1\|v\|\big) + M_2\big(k_2 + \rho_2\|u\| + \delta_2\|v\|\big). \tag{3.4}$$

Similarly, proceeding as in (3.3), we can obtain

$$\|v\| \leq M_1'\big(k_1 + \rho_1\|u\| + \delta_1\|v\|\big) + M_2'\big(k_2 + \rho_2\|u\| + \delta_2\|v\|\big). \tag{3.5}$$

Combining (3.4) with (3.5), we obtain

$$\begin{aligned}\|u\| + \|v\| &\leq \big[\big(M_1 + M_1'\big)k_1 + \big(M_2 + M_2'\big)k_2\big] + \big[\big(M_1 + M_1'\big)\rho_1 + \big(M_2 + M_2'\big)\rho_2\big]\|u\| \\ &\quad + \big[\big(M_1 + M_1'\big)\delta_1 + \big(M_2 + M_2'\big)\delta_2\big]\|v\|.\end{aligned}$$

Consequently,

$$\big\|(u,v)\big\|_1 = \|u\| + \|v\| \leq \frac{(M_1 + M_1')k_1 + (M_2 + M_2')k_2}{M_0},$$

where $M_0 = \min\{1 - [(M_1 + M_1')\rho_1 + (M_2 + M_2')\rho_2], 1 - [(M_1 + M_1')\delta_1 + (M_2 + M_2')\delta_2]\}$, which proves that the set V is bounded. Thus, by Lemma 2.4, the operator T has at least one fixed point. Hence the system (1.1) has at least one solution. The proof is complete. □

Theorem 3.2 *Assume that* $f_i : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous functions and there exist constants* $m_i, n_i \geq 0$ $(i = 1,2)$, *such that* $\forall t \in [0,1]$, $x_i, y_i \in \mathbb{R}$,

$$\big|f_i(t,x_1,y_1) - f_i(t,x_2,y_2)\big| \leq m_i|x_1 - x_2| + n_i|y_1 - y_2|, \quad i = 1,2.$$

In addition, assume that

$$\big(M_1 + M_1'\big)(m_1 + n_1) + \big(M_2 + M_2'\big)(m_2 + n_2) < 1,$$

where M_i, M_i' $(i = 1,2)$ *are defined by* (2.21). *Then the system* (1.1) *has a unique solution*.

Proof Define $\sup_{t\in[0,1]} |f_i(t,0,0)| = N_i < \infty$ $(i = 1,2)$ such that

$$r \geq \frac{(M_1 + M_1')N_1 + (M_2 + M_2')N_2}{1 - (M_1 + M_1')(m_1 + n_1) - (M_2 + M_2')(m_2 + n_2)}.$$

We show that $TB_r \subset B_r$, where $B_r = \{(u,v) \in X \times X : \|(u,v)\|_1 < r\}$. For any $(u,v) \in B_r$, we have

$$\begin{aligned}&\big|T_1(u,v)(t)\big| \\ &\quad \leq \max_{t\in[0,1]}\bigg|A_1(t)\bigg\{a\big(\lambda_2\xi - 1 + e^{-\lambda_2\xi}\big) \\ &\qquad \times \bigg[\int_0^1 e^{-\lambda_2(1-s)}Q(u,v)(s)\,ds - b\int_0^\eta e^{-\lambda_1(\eta-s)}P(u,v)(s)\,ds\bigg]\end{aligned}$$

$$
\begin{aligned}
&- \left(\lambda_2 - 1 + e^{-\lambda_2}\right)\left[a \int_0^{\xi} e^{-\lambda_2(\xi-s)} Q(u,v)(s)\,ds - \int_0^1 e^{-\lambda_1(1-s)} P(u,v)(s)\,ds\right]\Bigg\} \\
&+ \int_0^t e^{-\lambda_1(t-s)} P(u,v)(s)\,ds \Bigg| \\
\le{} & A_1 \Bigg\{ |a|\left(\lambda_2\xi - 1 + e^{-\lambda_2\xi}\right)\left[\left(m_2\|u\| + n_2\|v\| + N_2\right) \int_0^1 e^{-\lambda_2(1-s)} (Q\mathbf{1})(s)\,ds \right.\\
&\left. + |b|\left(m_1\|u\| + n_1\|v\| + N_1\right) \int_0^{\eta} e^{-\lambda_1(\eta-s)} (P\mathbf{1})(s)\,ds\right] \\
&+ \left(\lambda_2 - 1 + e^{-\lambda_2}\right)\left[|a|\left(m_2\|u\| + n_2\|v\| + N_2\right) \int_0^{\xi} e^{-\lambda_2(\xi-s)} (Q\mathbf{1})(s)\,ds \right.\\
&\left. + \left(m_1\|u\| + n_1\|v\| + N_1\right) \int_0^1 e^{-\lambda_1(1-s)} (P\mathbf{1})(s)\,ds\right]\Bigg\} \\
&+ \left(m_1\|u\| + n_1\|v\| + N_1\right) \int_0^t e^{-\lambda_1(t-s)} (P\mathbf{1})(s)\,ds \\
\le{} & \frac{A_1[|ab|(\lambda_2\xi - 1 + e^{-\lambda_2\xi})(1 - e^{-\lambda_1\eta})\eta^{p-1} + |(\lambda_2 - 1 + e^{-\lambda_2})|(1 - e^{-\lambda_1})] + (1 - e^{-\lambda_1})}{\lambda_1\Gamma(p)} \\
&\times \left(m_1\|u\| + n_1\|v\| + N_1\right) + \left(m_2\|u\| + n_2\|v\| + N_2\right) \\
&\times \frac{A_1[|a|(\lambda_2\xi - 1 + e^{-\lambda_2\xi})(1 - e^{-\lambda_2}) + (\lambda_2 - 1 + e^{-\lambda_2})|a|(1 - e^{-\lambda_2\xi})\xi^{q-1}]}{\lambda_2\Gamma(q)} \\
\le{} & M_1\left[(m_1 + n_1)r + N_1\right] + M_2\left[(m_2 + n_2)r + N_2\right].
\end{aligned}
\tag{3.6}
$$

Hence

$$\left\|T_1(u,v)\right\| \le M_1\left[(m_1 + n_1)r + N_1\right] + M_2\left[(m_2 + n_2)r + N_2\right].$$

Similarly, for any $(u,v) \in B_r$, proceeding as in (3.6), we can get

$$\left\|T_2(u,v)\right\| \le M_1'\left[(m_1 + n_1)r + N_1\right] + M_2'\left[(m_2 + n_2)r + N_2\right].$$

Consequently,

$$
\begin{aligned}
\left\|T(u,v)\right\|_1 ={} & \left\|T_1(u,v)\right\| + \left\|T_2(u,v)\right\| \\
\le{} & M_1\left[(m_1 + n_1)r + N_1\right] + M_2\left[(m_2 + n_2)r + N_2\right] \\
& + M_1'\left[(m_1 + n_1)r + N_1\right] + M_2'\left[(m_2 + n_2)r + N_2\right] \\
\le{} & r.
\end{aligned}
$$

Now for $(u_1,v_1),(u_2,v_2) \in X \times X$, and for any $t \in [0,1]$, we have

$$
\begin{aligned}
&\left|T_1(u_2,v_2)(t) - T_1(u_1,v_1)(t)\right| \\
&\le \Bigg| A_1(t)\Bigg\{ a\left(\lambda_2\xi - 1 + e^{-\lambda_2\xi}\right)\left[\int_0^1 e^{-\lambda_2(1-s)}\left[Q(u_2,v_2)(s) - Q(u_1,v_1)(s)\right] ds \right.\\
&\qquad \left. - b\int_0^{\eta} e^{-\lambda_1(\eta-s)}\left[P(u_2,v_2)(s) - P(u_1,v_1)(s)\right] ds\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\lambda_2-1+e^{-\lambda_2}\right)\left[a\int_0^{\xi} e^{-\lambda_2(\xi-s)}\left[Q(u_2,v_2)(s)-Q(u_1,v_1)(s)\right]ds\right.\\
&\left.\left.-\int_0^1 e^{-\lambda_1(1-s)}\left[P(u_2,v_2)(s)-P(u_1,v_1)(s)\right]ds\right]\right\}\\
&\left.+\int_0^t e^{-\lambda_1(t-s)}\left[P(u_2,v_2)(s)-P(u_1,v_1)(s)\right]ds\right|\\
\le{}& A_1\left\{|a|\left(\lambda_2\xi-1+e^{-\lambda_2\xi}\right)\left[\left(m_2\|u_2-u_1\|+n_2\|v_2-v_1\|\right)\int_0^1 e^{-\lambda_2(1-s)}(Q\mathbf{1})(s)\,ds\right.\right.\\
&\left.+|b|\left(m_1\|u_2-u_1\|+n_1\|v_2-v_1\|\right)\int_0^{\eta} e^{-\lambda_1(\eta-s)}(P\mathbf{1})(s)\,ds\right]\\
&+\left(\lambda_2-1+e^{-\lambda_2}\right)\left[|a|\left(m_2\|u_2-u_1\|+n_2\|v_2-v_1\|\right)\int_0^{\xi} e^{-\lambda_2(\xi-s)}(Q\mathbf{1})(s)\,ds\right.\\
&\left.\left.+\left(m_1\|u_2-u_1\|+n_1\|v_2-v_1\|\right)\int_0^1 e^{-\lambda_1(1-s)}(P\mathbf{1})(s)\,ds\right]\right\}\\
&+\left(m_1\|u_2-u_1\|+n_1\|v_2-v_1\|\right)\int_0^t e^{-\lambda_1(t-s)}(P\mathbf{1})(s)\,ds\\
\le{}& \frac{A_1[|ab|(\lambda_2\xi-1+e^{-\lambda_2\xi})(1-e^{-\lambda_1\eta})\eta^{p-1}+(\lambda_2-1+e^{-\lambda_2})(1-e^{-\lambda_1})]+(1-e^{-\lambda_1})}{\lambda_1\Gamma(p)}\\
&\times\left(m_1\|u_2-u_1\|+n_1\|v_2-v_1\|\right)+\left(m_2\|u_2-u_1\|+n_2\|v_2-v_1\|\right)\\
&\times\frac{A_1[|a|(\lambda_2\xi-1+e^{-\lambda_2\xi})(1-e^{-\lambda_2})+(\lambda_2-1+e^{-\lambda_2})|a|(1-e^{-\lambda_2\xi})\xi^{q-1}]}{\lambda_2\Gamma(q)}\\
\le{}& M_1\left(m_1\|u_2-u_1\|+n_1\|v_2-v_1\|\right)+M_2\left(m_2\|u_2-u_1\|+n_2\|v_2-v_1\|\right)\\
\le{}& \left[M_1(m_1+n_1)+M_2(m_2+n_2)\right]\left(\|u_2-u_1\|+\|v_2-v_1\|\right).
\end{aligned}
$$

Consequently, for $(u_1,v_1),(u_2,v_2)\in X\times X$, we obtain

$$
\begin{aligned}
&\left\|T_1(u_2,v_2)-T_1(u_1,v_1)\right\|\\
&\quad\le\left[M_1(m_1+n_1)+M_2(m_2+n_2)\right]\left(\|u_2-u_1\|+\|v_2-v_1\|\right).
\end{aligned}\tag{3.7}
$$

Similarly, for $(u_1,v_1),(u_2,v_2)\in X\times X$, we can obtain

$$
\begin{aligned}
&\left\|T_2(u_2,v_2)-T_2(u_1,v_1)\right\|\\
&\quad\le\left[M_1'(m_1+n_1)+M_2'(m_2+n_2)\right]\left(\|u_2-u_1\|+\|v_2-v_1\|\right).
\end{aligned}\tag{3.8}
$$

It follows from (3.7) and (3.8) that

$$
\begin{aligned}
&\left\|T(u_2,v_2)-T(u_1,v_1)\right\|_1\\
&\quad\le\left[\left(M_1+M_1'\right)(m_1+n_1)+\left(M_2+M_2'\right)(m_2+n_2)\right]\left(\|u_2-u_1\|+\|v_2-v_1\|\right).
\end{aligned}
$$

Since $(M_1+M_1')(m_1+n_1)+(M_2+M_2')(m_2+n_2)<1$, T is a contraction operator. So, by the fixed point theorem of the contraction mapping principle, the operator T has a unique fixed point, which is the unique solution of the system (1.1). The proof is complete. □

4 Applications

Example 4.1 Consider the fractional differential system

$$\begin{cases} {}^{c}D^{\frac{5}{2}}(D+2)u(t) = f_1(t, u(t), v(t)), & 0 < t < 1, \\ {}^{c}D^{\frac{9}{4}}(D+3)v(t) = f_2(t, u(t), v(t)), & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = av(\xi), \\ v(0) = v'(0) = 0, & v(1) = bu(\eta), \end{cases} \tag{4.1}$$

where $\lambda_1 = 2$, $\lambda_2 = 3$ are two positive parameters. We take $a = \frac{2}{3}$, $b = 3e$, $\xi = \frac{1}{3}$, $\eta = \frac{1}{2}$.

By direct calculation, we have

$$\begin{aligned} &\Lambda \approx -1.5914367 \neq 0, \\ &A_1 \approx 0.71340273, \qquad A_2 \approx 1.28801043, \\ &M_1 \approx 0.92074193, \qquad M_2 \approx 0.09483186, \\ &M_1' \approx 2.25908866, \qquad M_2' \approx 0.80969458. \end{aligned} \tag{4.2}$$

Let

$$\begin{aligned} f_1(t,x,y) &= \frac{t}{t^2+1}\left(2 + \frac{1}{10}\sin x + \frac{1}{8}\cos^2 y\right), \quad t \in [0,1], x, y \in \mathbb{R}, \\ f_2(t,x,y) &= \frac{1}{(t+2)^2}(2 + x + 2\sin y), \quad t \in [0,1], x, y \in \mathbb{R}. \end{aligned}$$

Notice that

$$\begin{aligned} \left|f_1(t,x,y)\right| &= \left|\frac{t}{t^2+1}\left(2 + \frac{1}{10}\sin x + \frac{1}{8}\cos^2 y\right)\right| \le 2 + \frac{1}{10}|x| + \frac{1}{8}|y|, \\ \left|f_2(t,x,y)\right| &= \left|\frac{1}{(t+2)^2}(2 + x + 2\sin y)\right| \le \frac{1}{2} + \frac{1}{4}|x| + \frac{1}{2}|y|, \end{aligned}$$

and

$$\begin{aligned} &\left(M_1 + M_1'\right)\rho_1 + \left(M_2 + M_2'\right)\rho_2 \approx 0.54411467 < 1, \\ &\left(M_1 + M_1'\right)\delta_1 + \left(M_2 + M_2'\right)\delta_2 \approx 0.84974204 < 1. \end{aligned}$$

Therefore, all conditions of Theorem 3.1 are satisfied, and hence by Theorem 3.1 the system (4.1) has at least one solution.

Example 4.2 In Example 4.1, we only change f_1, f_2, and keep the other conditions unchanged. Λ, A_1, A_2, M_1, M_2, M_1', M_2' are as in (4.2). Let

$$\begin{aligned} f_1(t,x,y) &= \frac{t}{18(t+1)^2}x + \frac{1}{9}\arctan y, \quad t \in [0,1], x, y \in \mathbb{R}, \\ f_2(t,x,y) &= -\frac{1}{8}\cos x + \frac{1}{12}\sin y, \quad t \in [0,1], x, y \in \mathbb{R}. \end{aligned} \tag{4.3}$$

Noticing

$$\left|f_1(t,x_1,y_1)-f_1(t,x_2,y_2)\right| \le \frac{1}{18}|x_1-x_2| + \frac{1}{9}|y_1-y_2|,$$
$$\left|f_2(t,x_1,y_1)-f_2(t,x_2,y_2)\right| \le \frac{1}{8}|x_1-x_2| + \frac{1}{12}|y_1-y_2|,$$

and

$$\left(M_1+M_1'\right)(m_1+n_1)+\left(M_2+M_2'\right)(m_2+n_2)\approx 0.71841477<1.$$

Thus all conditions of Theorem 3.2 are satisfied and, consequently, the system (4.1) has a unique solution (with f_1, f_2 as in (4.3)).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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