Guan et al. *Boundary Value Problems* (2016) 2016:141
DOI 10.1186/s13661-016-0650-3

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

On the existence of positive solutions and negative solutions of singular fractional differential equations via global bifurcation techniques

Yongliang Guan, Zengqin Zhao* and Xiuli Lin

*Correspondence:
zqzhaoy@163.com
School of Mathematical Sciences, Qufu Normal University, Qufu, 273165, P.R. China

Abstract
We are concerned with a type of fractional differential equations attached to boundary conditions. We investigate the existence of positive solutions and negative solutions via global bifurcation techniques.

Keywords: bifurcation; fractional differential; positive solution

1 Introduction

Boundary value problems of nonlinear fractional differential equations have been studied extensively in recent years (see, for instance, [1–4] and the references therein). In addition, since Rabinowitz established unilateral global bifurcation theorems, there has been much research in global bifurcation theory and it has been applied to obtain the existence and multiplicity for solutions of differential equations (see, for instance, [5–16] and their references). However, little of the previous research is involved with both global bifurcation techniques and fractional differential equations. In this paper, we will deal with fractional differential equations via global bifurcation techniques.

In [8], Ma *et al.* studied the problem

$$\begin{cases} u''''(t) = rg(t)f(u(t)), & t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1.1}$$

where the following conditions have been adopted:

(H1) $f : \mathbb{R} \to \mathbb{R}$ is continuous and $sf(s) > 0$ for $s \neq 0$.

(H2) There exist $f_0, f_\infty \in (0, \infty)$ such that

$$f_0 = \lim_{s\to 0} \frac{f(s)}{s}, \qquad f_\infty = \lim_{s\to\infty} \frac{f(s)}{s}.$$

(H3) $g : [0,1] \to \mathbb{R}$ is a continuous function which attains both positive and negative values.

The main result that Ma *et al.* established in [8] is stated as follows:

Let (H1)-(H3) hold. Assume that

$$r \in \left(\frac{\lambda_+}{f_\infty}, \frac{\lambda_+}{f_0}\right) \cup \left(\frac{\lambda_-}{f_0}, \frac{\lambda_-}{f_\infty}\right)$$

or

$$r \in \left(\frac{\lambda_+}{f_0}, \frac{\lambda_+}{f_\infty}\right) \cup \left(\frac{\lambda_-}{f_\infty}, \frac{\lambda_-}{f_0}\right).$$

Then (1.2) has at least one positive solution. Here λ_+ and λ_- are two simple principal eigenvalues of the following linear eigenvalue problem:

$$\begin{cases} u''''(t) = \lambda g(t)u(t), & t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

Inspired by [8], we will tackle the following fractional differential equation attached to boundary conditions:

$$\begin{cases} D^\alpha_{0+}u(t) + rf(t,u(t)) = 0, & t \in (0,1), \\ t^{n-\alpha}u^{(n-2)}(t)|_{t=0} = t^{n-\alpha}u^{(n-3)}(t)|_{t=0} = \cdots = t^{n-\alpha}u(t)|_{t=0} = u(1) = 0, \end{cases} \tag{1.2}$$

where $r > 0$, $\alpha > 1$, n is always the smallest integer greater than or equal to α in this paper. α can be integer or not. We consider (1.2) under the following assumption.

(C_1) $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ is continuous and $xf(t,x) > 0$ for $x \neq 0$.

(C_2^+) There exists a nonnegative function $a_{0^+} \in C[0,1]$ such that

$$\lim_{x\to 0^+} \frac{t^{n-1}(1-t)^{n-1}f(t,x)}{x} = a_{0^+}(t)$$

for all $t \in (0,1)$ uniformly. Moreover, $a_{0^+}(t)$ does not identically vanish in any subinterval of $(0,1)$.

(C_2^-) There exists a nonnegative function $a_{0^-} \in C[0,1]$ such that

$$\lim_{x\to 0^-} \frac{t^{n-1}(1-t)^{n-1}f(t,x)}{x} = a_{0^-}(t)$$

for all $t \in (0,1)$ uniformly. Moreover, $a_{0^-}(t)$ does not identically vanish in any subinterval of $(0,1)$.

(C_3^+) There exists a nonnegative function $a_{+\infty} \in C[0,1]$ such that

$$\lim_{x\to +\infty} \frac{t^{n-1}(1-t)^{n-1}f(t,x)}{x} = a_{+\infty}(t)$$

for all $t \in (0,1)$ uniformly. Moreover, $a_{+\infty}(t)$ does not identically vanish in any subinterval of $(0,1)$.

(C_3^-) There exists a nonnegative function $a_{-\infty} \in C[0,1]$ such that

$$\lim_{x\to -\infty} \frac{t^{n-1}(1-t)^{n-1}f(t,x)}{x} = a_{-\infty}(t)$$

for all $t \in (0,1)$ uniformly. Moreover, $a_{-\infty}(t)$ does not identically vanish in any subinterval of $(0,1)$.

(C_4) There exists a constant number $b > 0$ such that

$$0 < \frac{t^{n-1}(1-t)^{n-1}f(t,x)}{x} \leq b$$

for all $(t,x) \in (0,1) \times (\mathbb{R} - \{0\})$.

(C_5) There exists a function $\varphi \in C[0,1]$ such that

$$\frac{f(t,x)}{x} \geq \varphi(t) \geq 0$$

for all $(t,x) \in (0,1) \times (\mathbb{R} - \{0\})$. In addition, $\varphi(t)$ does not identically vanish in any subinterval of $(0,1)$.

Remark 1.1 The conditions that we adopt admit the singularity of $f(t,x)$ at $t = 0$ and $t = 1$, while (H1) and (H2) in [8] require the continuity of g there. What is more, the right side of equation (1.1) has been separated into two parts, functions of t and $u(t)$, respectively. However, we do not ask for that separation in this paper. Moreover, we tackle αth-order differential equations in this paper, where $\alpha > 1$ may be integer or not, while [8] only deals with fourth-order differential equations.

The rest of the paper is arranged as follows. In Section 2, some preliminary definitions and results will be presented. In Section 3, we will prove some property of H, L_{0^+}, and $L_{+\infty}$ as preparations. In Section 4, we will give our main results and prove it. In Section 5, we will present two examples to apply our main results.

2 Preliminary

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (0,+\infty) \to \mathbb{R}$ is defined by

$$I_{0+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds,$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$. In addition, I_{0+}^{0} is the identical operator.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $g : (0,+\infty) \to \mathbb{R}$ is defined by

$$D_{0+}^{\alpha} g(t) = \left(\frac{d}{dt}\right)^{n} I_{0+}^{n-\alpha} g(t),$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$.

When $\alpha \in \mathbb{N}^+$, the Riemann-Liouville fractional integral and derivative of order α coincide with the usual integral and derivative of integer order, respectively.

Consider the following problem:

$$\begin{cases} D_{0+}^{\alpha} u(t) + h(t) = 0, \quad t \in (0,1), \\ t^{n-\alpha} u^{(n-2)}(t)|_{t=0} = t^{n-\alpha} u^{(n-3)}(t)|_{t=0} = \cdots = t^{n-\alpha} u(t)|_{t=0} = u(1) = 0. \end{cases} \tag{2.1}$$

If we set

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \leq t \leq s \leq 1, \end{cases}$$

then we will have the following.

Lemma 2.1 *Let $h \in C(0,1)$. If $h(t)$ is absolutely integrable in $(0,a)$ for any number $a \in (0,1)$, we set*

$$u(t) = \int_0^1 G(t,s)h(s)\,ds, \tag{2.2}$$

i.e.,

$$u(t) = \frac{1}{\Gamma(\alpha)}\left[-\int_0^t (t-s)^{\alpha-1}h(s)\,ds + t^{\alpha-1}\int_0^1 (1-s)^{\alpha-1}h(s)\,ds\right], \tag{2.3}$$

then $u(t)$ must be a solution of (2.1).

Proof If (2.3) is satisfied, it is easy to verify that $u(1) = 0$. Since

$$u^{(i)}(t) = \frac{1}{\Gamma(\alpha-i)}\left[-\int_0^t (t-s)^{\alpha-i-1}h(s)\,ds + t^{\alpha-i-1}\int_0^1 (1-s)^{\alpha-1}h(s)\,ds\right],$$
$$i = 0,1,2,\ldots,n-2,$$

we have

$$\begin{aligned} \left|t^{n-\alpha}u^{(i)}(t)\right| &\leq \frac{1}{\Gamma(\alpha-i)}\int_0^t (t-s)^{\alpha-i-1}\left|h(s)\right|ds + \frac{t^{n-i-1}}{\Gamma(\alpha-i)}\left|\int_0^1 (1-s)^{\alpha-1}h(s)\,ds\right| \\ &= \frac{(t-\xi)^{\alpha-i-1}}{\Gamma(\alpha-i)}\int_0^t \left|h(s)\right|ds + \frac{t^{n-i-1}}{\Gamma(\alpha-i)}\left|\int_0^1 (1-s)^{\alpha-1}h(s)\,ds\right|, \\ &\quad i = 0,1,2,\ldots,n-2, \xi \in [0,t]. \end{aligned}$$

Hence

$$t^{n-\alpha}u^{(i)}(t)|_{t=0} = 0, \quad i = 0,1,2,\ldots,n-2. \qquad \square$$

Definition 2.3 Let X be a real Banach space and let K be a subset of X. Then K is called an order cone if:

(i) K is closed, nonempty, and $K \neq \theta$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in K \Rightarrow ax + by \in K$;

(iii) $x \in K$ and $-x \in K \Rightarrow x = \theta$.

On this basis, $u \in K$ is denoted by $u \geq \theta$ while $u > \theta$ means that $u \in K$ and $u \neq \theta$. Moreover, K is called solid if $\mathrm{int}(K) \neq \phi$, *i.e.*, K has interior points. $u \gg \theta$ means that u is an interior point of K.

In this paper, the space we choose is

$$X = \left\{ u \in C(0,1) : \lim_{t \to 0_+} \frac{u(t)}{t^{\alpha-1}(1-t)} \text{ and } \lim_{t \to 1_-} \frac{u(t)}{t^{\alpha-1}(1-t)} \text{ both exist} \right\}.$$

X is a Banach space equipped with the norm

$$\|u\| = \sup_{0<t<1} \left| \frac{u(t)}{t^{\alpha-1}(1-t)} \right|.$$

We set the cone

$$K_+ := \{ u \in X : u(t) \geq 0, \forall t \in (0,1) \}.$$

Clearly K_+ is a solid cone of X.

Now we present some conditions that Lemma 2.2 needs:

(H_1^+) The operators $L, N : X \to X$ are compact on the real Banach space X. $L + N$ is positive. L is linear and $\frac{\|Nu\|}{\|u\|} \to 0$ as $\|u\| \to 0$. Moreover, X has an order cone K with $X = K - K$.

(H_2^+) The spectral radius $r(L)$ of L is positive.

(H_3^+) L is strongly positive, *i.e.*, $Lu \gg \theta$ for $u > \theta$.

Two results which will be very important in this paper are stated as follows.

Lemma 2.2 (Corollary 15.12 in [17]) *We set*

$$S_+ = \{ (\mu, u) \in \mathbb{R} \times X : (\mu, u) \textit{ is a solution of } u = \mu(Lu + Nu) \textit{ with } \mu > 0 \textit{ and } u > \theta \}.$$

If (H_1^+) *and* (H_2^+) *are satisfied, then* $(r(L)^{-1}, \theta)$ *is a bifurcation point of* $u = \mu(Lu + Nu)$ *and* $\overline{S}_+$ *contains an unbounded solution component* $C_+(r(L)^{-1})$ *which passes through* $(r(L)^{-1}, \theta)$.

If additionally (H_3^+) *is satisfied, then* $(\mu, u) \in C_+(r(L)^{-1})$ *and* $\mu \neq r(L)^{-1}$ *always implies* $\mu > 0$ *and* $u > \theta$.

Lemma 2.3 (Theorem 19.3 in [18]) *Let X be a Banach space and $K \subset X$ be a solid cone. $L : X \to X$ is linear, compact, and strongly positive. Then we have*:

(a) *$r(L) > 0$, $r(L)$ is a simple eigenvalue with an eigenvector $v \gg \theta$ and there is no other eigenvalue with positive eigenvector.*

(b) *$|\lambda| < r(L)$ for all eigenvalue $\lambda \neq r(L)$.*

(c) *For $y > \theta$, $\lambda \leq r(L)$, the equation $\lambda u - Lu = y$ has no solution in K.*

(d) *Let $S : X \to X$ be a linear operator. If $Sx - Lx \geq \theta$ on K, then $r(s) \geq r(L)$, while $r(s) > r(L)$ if $Sx - Lx \gg \theta$ for $x > \theta$.*

3 Property of H, L_{0^+}, and $L_{+\infty}$

We set

$$(Hu)(t) = \int_0^1 G(t,s) f\big(s, u(s)\big)\, ds.$$

First of all, we show that H is well defined in X if (C_4) is satisfied.

Recalling that

$$(Hu)(t)=\frac{1}{\Gamma(\alpha)}\left[t^{\alpha-1}\int_0^1(1-s)^{\alpha-1}f\bigl(s,u(s)\bigr)\,ds-\int_0^t(t-s)^{\alpha-1}f\bigl(s,u(s)\bigr)\,ds\right],$$

what we only need to do is to describe the nature of $f(s,u(s))$ when s is close to 0 and to 1. Due to condition (C_4), we have

$$\begin{aligned}\bigl|f\bigl(s,u(s)\bigr)\bigr| &\le \frac{b}{s^{n-1}(1-s)^{n-1}}\bigl|u(s)\bigr| \\ &= bs^{\alpha-n}(1-s)^{2-n}\left|\frac{u(s)}{s^{\alpha-1}(1-s)}\right| \\ &\le b\|u\|s^{\alpha-n}(1-s)^{2-n}. \end{aligned} \tag{3.1}$$

Then we can see that H is well defined on X. Furthermore, we have the following result.

Lemma 3.1 *If* (C_4) *is satisfied, then* $H:X\to X$ *is compact.*

Proof The proof is divided into two parts.

Part 1. We prove that $Hu\in X$ for $u\in X$ in this part.

Suppose $u\in X$. It is obvious that $Hu\in C(0,1)$.

We will prove that $\lim_{t\to 0_+}\frac{(Hu)(t)}{t^{\alpha-1}(1-t)}$ exists in the next. We have

$$\frac{(Hu)(t)}{t^{\alpha-1}}=\frac{1}{\Gamma(\alpha)}\left[\int_0^1(1-s)^{\alpha-1}f\bigl(s,u(s)\bigr)\,ds-\frac{\int_0^t(t-s)^{\alpha-1}f(s,u(s))\,ds}{t^{\alpha-1}}\right]. \tag{3.2}$$

Moreover, due to (3.1), we have

$$\begin{aligned}\left|\frac{\int_0^t(t-s)^{\alpha-1}f(s,u(s))\,ds}{t^{\alpha-1}}\right| &\le \frac{\int_0^t(t-s)^{\alpha-1}|f(s,u(s))|\,ds}{t^{\alpha-1}} \\ &\le \frac{b\|u\|\int_0^t(t-s)^{\alpha-1}s^{\alpha-n}(1-s)^{2-n}\,ds}{t^{\alpha-1}} \\ &= b\|u\|(1-\xi)^{2-n}\frac{\int_0^t(t-s)^{\alpha-1}s^{\alpha-n}\,ds}{t^{\alpha-1}} \\ &= b\|u\|(1-\xi)^{2-n}\frac{\Gamma(\alpha)\Gamma(\alpha-n+1)}{\Gamma(2\alpha-n+1)}t^{\alpha-n+1},\quad \xi\in(0,t). \end{aligned}$$

So

$$\lim_{t\to 0_+}\frac{\int_0^t(t-s)^{\alpha-1}f(s,u(s))\,ds}{t^{\alpha-1}}=0. \tag{3.3}$$

By (3.2) and (3.3), we know that

$$\lim_{t\to 0_+}\frac{(Hu)(t)}{t^{\alpha-1}(1-t)}=\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f\bigl(s,u(s)\bigr)\,ds. \tag{3.4}$$

At last, we prove that $\lim_{t\to 1_-}\frac{(Hu)(t)}{t^{\alpha-1}(1-t)}$ exists:

$$\begin{aligned}\frac{(Hu)(t)}{1-t} &= \frac{t^{\alpha-1}\int_0^1(1-s)^{\alpha-1}f(s,u(s))\,ds - \int_0^t(t-s)^{\alpha-1}f(s,u(s))\,ds}{\Gamma(\alpha)(1-t)} \\ &= \frac{(t^{\alpha-1}-1)\int_0^1(1-s)^{\alpha-1}f(s,u(s))\,ds}{\Gamma(\alpha)(1-t)} \\ &\quad + \frac{\int_0^1(1-s)^{\alpha-1}f(s,u(s))\,ds - \int_0^t(t-s)^{\alpha-1}f(s,u(s))\,ds}{\Gamma(\alpha)(1-t)} \\ &= \frac{(t^{\alpha-1}-1)\int_0^1(1-s)^{\alpha-1}f(s,u(s))\,ds}{\Gamma(\alpha)(1-t)} + \frac{\int_t^1(1-s)^{\alpha-1}f(s,u(s))\,ds}{\Gamma(\alpha)(1-t)} \\ &\quad + \frac{\int_0^t[(1-s)^{\alpha-1}-(t-s)^{\alpha-1}]f(s,u(s))\,ds}{\Gamma(\alpha)(1-t)},\end{aligned} \tag{3.5}$$

where

$$\lim_{t\to 1_-}\frac{(t^{\alpha-1}-1)\int_0^1(1-s)^{\alpha-1}f(s,u(s))\,ds}{\Gamma(\alpha)(1-t)} = -\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-1}f\bigl(s,u(s)\bigr)\,ds. \tag{3.6}$$

Moreover, by (3.1) we have

$$\begin{aligned}\left|\frac{\int_t^1(1-s)^{\alpha-1}f(s,u(s))\,ds}{\Gamma(\alpha)(1-t)}\right| &\le \left|\frac{b\|u\|\int_t^1(1-s)^{\alpha-n+1}s^{\alpha-n}\,ds}{\Gamma(\alpha)(1-t)}\right| \\ &= \left|\frac{b\|u\|\xi^{\alpha-n}}{\Gamma(\alpha)(\alpha-n+2)}(1-t)^{\alpha-n+1}\right|, \quad \xi\in(t,1),\end{aligned}$$

then

$$\lim_{t\to 1_-}\frac{\int_t^1(1-s)^{\alpha-1}f(s,u(s))\,ds}{\Gamma(\alpha)(1-t)} = 0. \tag{3.7}$$

Now we have to deal with

$$\frac{\int_0^t[(1-s)^{\alpha-1}-(t-s)^{\alpha-1}]f(s,u(s))\,ds}{\Gamma(\alpha)(1-t)}.$$

Since

$$\frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{(1-t)} - (\alpha-1)(1-s)^{\alpha-2}$$

does not change sign, by (3.1) we have

$$\begin{aligned}&\left|\frac{\int_0^t[(1-s)^{\alpha-1}-(t-s)^{\alpha-1}]f(s,u(s))\,ds}{\Gamma(\alpha)(1-t)} - \frac{\int_0^t(1-s)^{\alpha-2}f(s,u(s))\,ds}{\Gamma(\alpha-1)}\right| \\ &\quad = \frac{1}{\Gamma(\alpha)}\left|\int_0^t\left[\frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{(1-t)} - (\alpha-1)(1-s)^{\alpha-2}\right]f\bigl(s,u(s)\bigr)\,ds\right| \\ &\quad \le \frac{b\|u\|}{\Gamma(\alpha)}\left|\int_0^t\left[\frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{(1-t)} - (\alpha-1)(1-s)^{\alpha-2}\right]s^{\alpha-n}(1-s)^{2-n}\,ds\right|.\end{aligned} \tag{3.8}$$

After some calculations (more details being omitted), we know that

$$\lim_{t\to 1_-}\int_0^t\left[\frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{(1-t)}-(\alpha-1)(1-s)^{\alpha-2}\right]s^{\alpha-n}(1-s)^{2-n}\,ds=0. \tag{3.9}$$

Moreover,

$$\begin{aligned}&\left|\frac{\int_0^1(1-s)^{\alpha-2}f(s,u(s))\,ds}{\Gamma(\alpha-1)}-\frac{\int_0^t(1-s)^{\alpha-2}f(s,u(s))\,ds}{\Gamma(\alpha-1)}\right|\\&\quad=\left|\frac{\int_t^1(1-s)^{\alpha-2}f(s,u(s))\,ds}{\Gamma(\alpha-1)}\right|\\&\quad\le\left|\frac{b\|u\|\int_t^1(1-s)^{\alpha-n}s^{\alpha-n}\,ds}{\Gamma(\alpha-1)}\right|\\&\quad=\left|\frac{b\|u\|\xi^{\alpha-n}(1-t)^{\alpha-n+1}}{(\alpha-n+1)\Gamma(\alpha-1)}\right|,\quad \xi\in(t,1).\end{aligned} \tag{3.10}$$

Due to (3.5)-(3.10), we have

$$\lim_{t\to 1_-}\frac{(Hu)(t)}{t^{\alpha-1}(1-t)}=\frac{1}{\Gamma(\alpha-1)}\int_0^1\left[(1-s)^{\alpha-2}-(1-s)^{\alpha-1}\right]f\bigl(s,u(s)\bigr)\,ds. \tag{3.11}$$

Up to now, we have proved that for $u\in X$ there must be $Hu\in X$.

Part 2. We will show that $H:X\to X$ is compact in this part.

For a bounded subset of X named D, we have to prove that

$$\left\{g:g(t)=\frac{(Hu)(t)}{t^{\alpha-1}(1-t)}\text{ for some }u\in D\right\}$$

is uniformly bounded and equicontinuous.

Since D is bounded, we can choose a constant number $\overline{M}>0$ such that $\|u\|\le\overline{M}$ for all $u\in D$. By (3.4) and (3.1), we have

$$\begin{aligned}\lim_{t\to 0_+}\left|\frac{(Hu)(t)}{t^{\alpha-1}(1-t)}\right|&=\left|\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f\bigl(s,u(s)\bigr)\,ds\right|\\&\le\frac{b\|u\|}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-n+1}s^{\alpha-n}\,ds\\&\le\frac{b\overline{M}\Gamma(\alpha-n+2)\Gamma(\alpha-n+1)}{\Gamma(\alpha)\Gamma(2\alpha-2n+3)}.\end{aligned} \tag{3.12}$$

Similarly, by (3.11) and (3.1), we have

$$\begin{aligned}\lim_{t\to 1_-}\left|\frac{(Hu)(t)}{t^{\alpha-1}(1-t)}\right|&=\left|\frac{1}{\Gamma(\alpha-1)}\int_0^1\left[(1-s)^{\alpha-2}-(1-s)^{\alpha-1}\right]f\bigl(s,u(s)\bigr)\,ds\right|\\&\le\frac{b\|u\|}{\Gamma(\alpha-1)}\left|\int_0^1\left[(1-s)^{\alpha-2}-(1-s)^{\alpha-1}\right]s^{\alpha-n}(1-s)^{2-n}\,ds\right|\\&=\frac{b\|u\|}{\Gamma(\alpha-1)}\left[\frac{\Gamma(\alpha-n+1)\Gamma(\alpha-n+1)}{\Gamma(2\alpha-2n+2)}-\frac{\Gamma(\alpha-n+1)\Gamma(\alpha-n+2)}{\Gamma(2\alpha-2n+3)}\right]\\&\le\frac{b\overline{M}\Gamma(\alpha-n+1)\Gamma(\alpha-n+2)}{\Gamma(\alpha-1)\Gamma(2\alpha-2n+3)}.\end{aligned} \tag{3.13}$$

Moreover, examining the proof of (3.4) and (3.11), we can find that the convergence of (3.4) and (3.11) are independent of $u \in D$, *i.e.*, for any $\varepsilon > 0$ there must exist a constant number $\delta > 0$ such that for any $u \in D$, $0 < t < \delta$ always implies $|\frac{(Hu)(t)}{t^{\alpha-1}(1-t)} - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,u(s))\,ds| \leq \varepsilon$ and $1-\delta < t < 1$ always implies $|\frac{(Hu)(t)}{t^{\alpha-1}(1-t)} - \frac{1}{\Gamma(\alpha-1)}\int_0^1 [(1-s)^{\alpha-2} - (1-s)^{\alpha-1}] f(s,u(s))\,ds| < \varepsilon$. Furthermore, thanks to $0 < \frac{t^{n-1}(1-t)^{n-1}f(t,x)}{x} \leq b$ and the boundedness of D, recalling that

$$\begin{aligned}
\frac{(Hu)(t)}{t^{\alpha-1}(1-t)} &= \frac{1}{\Gamma(\alpha)(1-t)}\int_0^1 (1-s)^{\alpha-1} f\bigl(s,u(s)\bigr)\,ds \\
&\quad - \frac{1}{\Gamma(\alpha)t^{\alpha-1}(1-t)}\int_0^t (t-s)^{\alpha-1} f\bigl(s,u(s)\bigr)\,ds \\
&= \int_0^1 \frac{(1-s)^{\alpha-n+1}s^{\alpha-n}}{\Gamma(\alpha)(1-t)}\frac{s^{n-1}(1-s)^{n-1}f(s,u(s))}{u(s)}\frac{u(s)}{\|u\|s^{\alpha-1}(1-s)}\,ds \\
&\quad + \int_0^t \frac{(1-s)^{2-n}(t-s)^{\alpha-1}s^{\alpha-n}}{\Gamma(\alpha)t^{\alpha-1}(1-t)}\frac{s^{n-1}(1-s)^{n-1}f(s,u(s))}{u(s)} \\
&\quad \times \frac{u(s)}{\|u\|s^{\alpha-1}(1-s)}\,ds,
\end{aligned}$$

we will know that $\{g : g(t) = \frac{(Hu)(t)}{t^{\alpha-1}(1-t)}$ for some $u \in D\}$ is uniformly bounded and equicontinuous. □

We set

$$\begin{aligned}
(L_{0^+}u)(t) &= \int_0^1 G(t,s)\frac{a_{0^+}(s)}{s^{n-1}(1-s)^{n-1}}u(s)\,ds, \\
(L_{+\infty}u)(t) &= \int_0^1 G(t,s)\frac{a_{+\infty}(s)}{s^{n-1}(1-s)^{n-1}}u(s)\,ds.
\end{aligned}$$

Lemma 3.2 *$L_{0^+}, L_{+\infty} : X \to X$ are linear, compact, and strongly positive operators.*

Proof Similar to Lemma 3.1, we can verify that $L_{0^+}, L_{+\infty} : X \to X$ are compact operators. Moreover, they are obviously linear operators. We only prove that $L_{0^+} : X \to X$ is strongly positive in the following since the proof of the other one is similar.

For $u > \theta$, similar with (3.4) and (3.11), we have

$$\lim_{t\to 0_+}\frac{(L_{0^+}u)(t)}{t^{\alpha-1}(1-t)} = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\frac{a_{0^+}(s)}{s^{n-1}(1-s)^{n-1}}u(s)\,ds > 0$$

and

$$\lim_{t\to 1_-}\frac{(L_{0^+}u)(t)}{t^{\alpha-1}(1-t)} = \frac{1}{\Gamma(\alpha-1)}\int_0^1 \bigl[(1-s)^{\alpha-2} - (1-s)^{\alpha-1}\bigr]\frac{a_{0^+}(s)}{s^{n-1}(1-s)^{n-1}}u(s)\,ds > 0.$$

Noticing that $G(t,s) > 0$ for $0 < t < 1$, $0 < s < 1$ (proof omitted), we know that there must exist a constant number $c > 0$ such that $\frac{(L_{0^+}u)(t)}{t^{\alpha-1}(1-t)} > c$ for all $t \in (0,1)$, which implies that $L_{0^+}u \gg \theta$. □

4 Main result and its proof

Since L_{0^+} and $L_{+\infty}$ satisfy the conditions of Lemma 2.3, we know that $r(L_{0^+}) > 0$ and $r(L_{+\infty}) > 0$. We set

$$\mu_{0^+} = r(L_{0^+})^{-1}, \qquad \mu_{+\infty} = r(L_{+\infty})^{-1}.$$

Then we will have the following conclusion.

Theorem 4.1 *Let* (C_1), (C_2^+), (C_3^+), (C_4), *and* (C_5) *hold. Then if* $\mu_{+\infty} < r < \mu_{0^+}$ *or* $\mu_{0^+} < r < \mu_{+\infty}$, *there must exist at least one positive solution of* (1.2).

Proof The proof is organized as follows.

In *Part* 1, we consider the auxiliary equation (see (4.1)) of which solutions of the kind $\mu = 1$ will be the solutions of (1.1) and we transform it into a functional operator equation (see (4.4)).

In *Part* 2, we intend to verify that L_{0^+}, N_{0^+} satisfy all the conditions required to apply Lemma 2.2 and Lemma 2.3, where N_{0^+} will be defined in *Part* 1.

In *Part* 3, we apply Lemma 2.2 and Lemma 2.3 to get the existence of at least one positive solution of (1.2).

Part 1. Consider the following problem:

$$\begin{cases} D_{0+}^{\alpha} u(t) + \mu r f(t, u(t)) = 0, \quad t \in (0,1), \\ t^{n-\alpha} u^{(n-2)}(t)|_{t=0} = t^{n-\alpha} u^{(n-3)}(t)|_{t=0} = \cdots = t^{n-\alpha} u(t)|_{t=0} = u(1) = 0. \end{cases} \tag{4.1}$$

We call $(\mu, u) \in \mathbb{R} \times X$ a solution of (4.1) if it satisfies (4.1). It is clear that any solution of (4.1) of the form $(1, u)$ yields a solution u of (1.2).

Thanks to (3.1), we can see that $f(s, u(s))$ is absolutely integrable in $(0, a)$ for any $a \in (0,1)$ and $u \in X$. Then due to Lemma 2.1, $(\mu, u) \in \mathbb{R} \times X$ is a solution of (4.1) if

$$u(t) = \mu r \int_0^1 G(t,s) f\bigl(s, u(s)\bigr)\, ds. \tag{4.2}$$

If we decompose $f(t,x)$ to become

$$f(t,x) = \frac{a_{0^+}(t)}{t^{n-1}(1-t)^{n-1}} x + \hat{a}_{0^+}(t,x),$$

then due to (C_2^+), we know that

$$\lim_{x \to 0^+} \frac{t^{n-1}(1-t)^{n-1} \hat{a}_{0^+}(t,x)}{x} = 0 \quad \text{for } t \in (0,1) \text{ uniformly.} \tag{4.3}$$

Moreover, (4.2) is equivalent to

$$\begin{aligned} u(t) &= \mu r \int_0^1 G(t,s) \frac{a_{0^+}(s)}{s^{n-1}(1-s)^{n-1}} u(s)\, ds \\ &\quad + \mu r \int_0^1 G(t,s) \hat{a}_{0^+}\bigl(s, u(s)\bigr)\, ds \\ &:= \mu r (L_{0^+} u)(t) + \mu r (N_{0^+} u)(t), \end{aligned}$$

i.e., u is a solution of (4.1) if and only if

$$u = \mu r(L_{0^+}u) + \mu r(N_{0^+}u). \tag{4.4}$$

Part 2. It is obvious that L_{0^+}, N_{0^+} is positive by (C_1). Moreover, we have confirmed that $L_{0^+} : X \to X$ is linear, compact, and strongly positive in Lemma 3.2. In addition, we also know that $N_{0^+} = H - L_{0^+}$ is compact due to Lemma 3.1. It is left for us to verify that $\frac{\|N_{0^+}u\|}{\|u\|} \to 0$ as $\|u\| \to 0$.

Similar to (3.4) and (3.11), we have

$$\begin{aligned}
&\lim_{t\to 0_+} \frac{|(N_{0^+}u)(t)|}{\|u\|t^{\alpha-1}(1-t)} \\
&\quad= \frac{|\int_0^1 (1-s)^{\alpha-1}\hat{a}_{0^+}(s,u(s))\,ds|}{\|u\|\Gamma(\alpha)} \\
&\quad\le \frac{\int_0^1 (1-s)^{\alpha-1}|\hat{a}_{0^+}(s,u(s))|\,ds}{\|u\|\Gamma(\alpha)} \\
&\quad= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha+1-n}s^{\alpha-n}\left|\frac{s^{n-1}(1-s)^{n-1}\hat{a}_{0^+}(s,u(s))}{u(s)}\frac{u(s)}{\|u\|s^{\alpha-1}(1-s)}\right|ds \\
&\quad\le \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha+1-n}s^{\alpha-n}\left|\frac{s^{n-1}(1-s)^{n-1}\hat{a}_{0^+}(s,u(s))}{u(s)}\right|ds \\
&\quad= \frac{1}{\Gamma(\alpha)}\left|\frac{\xi^{n-1}(1-\xi)^{n-1}\hat{a}_{0^+}(\xi,u(\xi))}{u(\xi)}\right|\int_0^1 (1-s)^{\alpha+1-n}s^{\alpha-n}\,ds \\
&\quad= \frac{\Gamma(\alpha-n+2)\Gamma(\alpha-n+1)}{\Gamma(\alpha)\Gamma(2\alpha-2n+3)}\left|\frac{\xi^{n-1}(1-\xi)^{n-1}\hat{a}_{0^+}(\xi,u(\xi))}{u(\xi)}\right|, \quad \xi\in(0,1),
\end{aligned} \tag{4.5}$$

and similarly

$$\begin{aligned}
&\lim_{t\to 1_-} \frac{|(N_{0^+}u)(t)|}{\|u\|t^{\alpha-1}(1-t)} \\
&\quad= \frac{|\int_0^1 [(1-s)^{\alpha-2}-(1-s)^{\alpha-1}]\hat{a}_{0^+}(s,u(s))\,ds|}{\|u\|\Gamma(\alpha-1)} \\
&\quad\le \frac{1}{\Gamma(\alpha-1)}\int_0^1 \bigl[(1-s)^{\alpha-n}-(1-s)^{\alpha-n+1}\bigr]s^{\alpha-n} \\
&\qquad\times\left|\frac{s^{n-1}(1-s)^{n-1}\hat{a}_{0^+}(s,u(s))}{u(s)}\frac{u(s)}{\|u\|s^{\alpha-1}(1-s)}\right|ds \\
&\quad\le \frac{1}{\Gamma(\alpha-1)}\int_0^1 \bigl[(1-s)^{\alpha-n}-(1-s)^{\alpha-n+1}\bigr]s^{\alpha-n}\left|\frac{s^{n-1}(1-s)^{n-1}\hat{a}_{0^+}(s,u(s))}{u(s)}\right|ds \\
&\quad= \frac{1}{\Gamma(\alpha-1)}\left|\frac{\xi^{n-1}(1-\xi)^{n-1}\hat{a}_{0^+}(\xi,u(\xi))}{u(\xi)}\right|\int_0^1 \bigl[(1-s)^{\alpha-n}-(1-s)^{\alpha-n+1}\bigr]s^{\alpha-n}\,ds \\
&\quad= \frac{1}{\Gamma(\alpha-1)}\left[\frac{\Gamma(\alpha-n+1)\Gamma(\alpha-n+1)}{\Gamma(2\alpha-2n+2)}-\frac{\Gamma(\alpha-n+1)\Gamma(\alpha-n+2)}{\Gamma(2\alpha-2n+3)}\right] \\
&\qquad\times\left|\frac{\xi^{n-1}(1-\xi)^{n-1}\hat{a}_{0^+}(\xi,u(\xi))}{u(\xi)}\right|, \quad \xi\in(0,1).
\end{aligned} \tag{4.6}$$

Moreover,

$$\begin{aligned}
&\frac{|(N_{0^+}u)(t)|}{\|u\|t^{\alpha-1}(1-t)}\\
&\quad=\left|\frac{1}{\Gamma(\alpha)(1-t)}\int_0^1(1-s)^{\alpha-1}\frac{\hat{a}_{0^+}(s,u(s))}{\|u\|}\,ds\right.\\
&\qquad\left.-\frac{1}{\Gamma(\alpha)t^{\alpha-1}(1-t)}\int_0^t(t-s)^{\alpha-1}\frac{\hat{a}_{0^+}(s,u(s))}{\|u\|}\,ds\right|\\
&\quad\le\frac{1}{\Gamma(\alpha)(1-t)}\int_0^1(1-s)^{\alpha-1}\frac{|\hat{a}_{0^+}(s,u(s))|}{\|u\|}\,ds\\
&\qquad+\frac{1}{\Gamma(\alpha)t^{\alpha-1}(1-t)}\int_0^t(t-s)^{\alpha-1}\frac{|\hat{a}_{0^+}(s,u(s))|}{\|u\|}\,ds\\
&\quad=\int_0^1\frac{(1-s)^{\alpha-n+1}s^{\alpha-n}}{\Gamma(\alpha)(1-t)}\left|\frac{s^{n-1}(1-s)^{n-1}\hat{a}_{0^+}(s,u(s))}{u(s)}\right|\left|\frac{u(s)}{\|u\|s^{\alpha-1}(1-s)}\right|ds\\
&\qquad+\int_0^t\frac{(1-s)^{2-n}(t-s)^{\alpha-1}s^{\alpha-n}}{\Gamma(\alpha)t^{\alpha-1}(1-t)}\left|\frac{s^{n-1}(1-s)^{n-1}\hat{a}_{0^+}(s,u(s))}{u(s)}\right|\left|\frac{u(s)}{\|u\|s^{\alpha-1}(1-s)}\right|ds\\
&\quad\le\int_0^1\frac{(1-s)^{\alpha-n+1}s^{\alpha-n}}{\Gamma(\alpha)(1-t)}\left|\frac{s^{n-1}(1-s)^{n-1}\hat{a}_{0^+}(s,u(s))}{u(s)}\right|ds\\
&\qquad+\int_0^t\frac{(1-s)^{2-n}(t-s)^{\alpha-1}s^{\alpha-n}}{\Gamma(\alpha)t^{\alpha-1}(1-t)}\left|\frac{s^{n-1}(1-s)^{n-1}\hat{a}_{0^+}(s,u(s))}{u(s)}\right|ds\\
&\quad=\left|\frac{\xi_1^{n-1}(1-\xi_1)^{n-1}\hat{a}_{0^+}(\xi_1,u(\xi_1))}{u(\xi_1)}\right|\int_0^1\frac{(1-s)^{\alpha-n+1}s^{\alpha-n}}{\Gamma(\alpha)(1-t)}\,ds\\
&\qquad+\left|\frac{\xi_2^{n-1}(1-\xi_2)^{n-1}\hat{a}_{0^+}(\xi_2,u(\xi_2))}{u(\xi_2)}\right|(1-\xi_2)^{2-n}\int_0^t\frac{(t-s)^{\alpha-1}s^{\alpha-n}}{\Gamma(\alpha)t^{\alpha-1}(1-t)}\,ds\\
&\quad=\left|\frac{\xi_1^{n-1}(1-\xi_1)^{n-1}\hat{a}_{0^+}(\xi_1,u(\xi_1))}{u(\xi_1)}\right|\frac{\Gamma(\alpha-n+2)\Gamma(\alpha-n+1)}{\Gamma(\alpha)\Gamma(2\alpha-2n+3)(1-t)}\\
&\qquad+\left|\frac{\xi_2^{n-1}(1-\xi_2)^{n-1}\hat{a}_{0^+}(\xi_2,u(\xi_2))}{u(\xi_2)}\right|(1-\xi_2)^{2-n}\frac{\Gamma(\alpha-n+1)t^{\alpha-n+1}}{\Gamma(2\alpha-n+1)(1-t)},\\
&\qquad\xi_1\in(0,1),\xi_2\in(0,t).
\end{aligned}\tag{4.7}$$

Because $a_{0^+}(s,u(s))$ has the property similar with (3.1) (the proof is easy, so we omit it), we can see that the convergence of (4.5) and (4.6) are independent of $u\in X$ (*i.e.*, for any $\varepsilon>0$ there must exist a constant number $\delta>0$ such that for any $u\in D$, $0<t<\delta$ always implies $|\frac{|(N_{0^+}u)(t)|}{\|u\|t^{\alpha-1}(1-t)}-\frac{|\int_0^1(1-s)^{\alpha-1}\hat{a}_{0^+}(s,u(s))\,ds|}{\|u\|\Gamma(\alpha)}|\le\varepsilon$ and $1-\delta<t<1$ always implies $|\frac{|(N_{0^+}u)(t)|}{\|u\|t^{\alpha-1}(1-t)}-\frac{|\int_0^1[(1-s)^{\alpha-2}-(1-s)^{\alpha-1}]\hat{a}_{0^+}(s,u(s))\,ds|}{\|u\|\Gamma(\alpha-1)}|<\varepsilon$) and applying (4.3) on (4.5)-(4.7), we can finish this part.

Part 3. Applying Lemma 2.2 and Lemma 2.3, we can draw a conclusion as below.

For (4.4), from $(\frac{\mu_{0^+}}{r},\theta)$ there emanates an unbounded continuum of positive solutions $C_+\subset\overline{D}_+$. Here

$$D_+=\bigl\{(\mu,u)\in\mathbb{R}\times X:u=\mu rL_{0^+}u+\mu rN_{0^+}u\text{ with }\mu>0\text{ and }u>\theta\bigr\}.$$

Furthermore, $(\mu,u)\in C_+$ and $\mu\neq\frac{\mu_{0^+}}{r}$ always implies $u>\theta$.

To verify the existence of at least one positive solution of (1.2), we only need to show that C_+ crosses the hyperplane $\{1\} \times X$ in $\mathbb{R} \times X$. To this end, it will be enough to show that C_+ joins $(\frac{\mu_{0^+}}{r}, \theta)$ to $(\frac{\mu_{+\infty}}{r}, +\infty)$.

Suppose $(\mu_n, u_n) \in C_+$ satisfy $\mu_n + \|u_n\| \to \infty$ now, we will show that $\{\mu_n\}$ is bounded first:

If not, choosing a subsequence and relabeling it if necessary, we have $\lim_{n\to\infty} \mu_n \to \infty$. Now defining

$$(L_\varphi u)(t) = \int_0^1 G(t,s)\varphi(s)u(s)\,ds, \quad 0 < t < 1,$$

we can verify that $L_\varphi : X \to X$ is linear, compact, and strongly positive (the proofs are similar to but simpler than those corresponding to L_0 since φ is continuous, so we omit them). Then, due to Lemma 2.3(c), we see that there is a contradiction for sufficiently large n in the following inequality:

$$u_n(t) = \mu_n r \int_0^1 G(t,s) f\bigl(s, u_n(s)\bigr)\,ds > \mu_n r \int_0^1 G(t,s)\varphi(s)u_n(s)\,ds = \mu_n r(L_\varphi u_n)(t).$$

Then $\{\mu_n\}$ is bounded and hence $\lim_{n\to\infty} \|u_n\| \to \infty$.

Second, let $v_n = \frac{u_n}{\|u_n\|}$, we will show that $\{v_n\}$ is relatively compact. Similar to (3.4) and (3.11) and by (3.1), we have

$$\begin{aligned} 0 &\le \lim_{t\to 0_+} \frac{v_n(t)}{t^{\alpha-1}(1-t)} = \frac{\mu_n r}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \frac{f(s,u_n(s))}{\|u_n\|}\,ds \\ &\le \frac{\mu_n r b}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-n+1} s^{\alpha-n}\,ds \\ &= \frac{\mu_n r b \Gamma(\alpha-n+2)\Gamma(\alpha-n+1)}{\Gamma(\alpha)\Gamma(2\alpha-2n+3)} \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} 0 &\le \lim_{t\to 1_-} \frac{v_n(t)}{t^{\alpha-1}(1-t)} = \frac{\mu_n r}{\Gamma(\alpha-1)} \int_0^1 \bigl[(1-s)^{\alpha-2} - (1-s)^{\alpha-1}\bigr] \frac{f(s,u_n(s))}{\|u_n\|}\,ds \\ &\le \frac{\mu_n r b}{\Gamma(\alpha-1)} \int_0^1 \bigl[(1-s)^{\alpha-n} - (1-s)^{\alpha-n+1}\bigr] s^{\alpha-n}\,ds \\ &= \frac{\mu_n r b}{\Gamma(\alpha-1)} \biggl[\frac{\Gamma(\alpha-n+1)\Gamma(\alpha-n+1)}{\Gamma(2\alpha-2n+2)} - \frac{\Gamma(\alpha-n+1)\Gamma(\alpha-n+2)}{\Gamma(2\alpha-2n+3)}\biggr] \\ &= \frac{\mu_n r b \Gamma(\alpha-n+1)\Gamma(\alpha-n+2)}{\Gamma(\alpha-1)\Gamma(2\alpha-2n+3)}. \end{aligned} \tag{4.9}$$

Moreover, recall that

$$\begin{aligned} \frac{v_n(t)}{t^{\alpha-1}(1-t)} &= \frac{1}{\Gamma(\alpha)} \biggl[\frac{1}{1-t} \int_0^1 (1-s)^{\alpha-1} \frac{f(s,u_n(s))}{\|u_n\|}\,ds \\ &\quad - \frac{1}{t^{\alpha-1}(1-t)} \int_0^t (t-s)^{\alpha-1} \frac{f(s,u_n(s))}{\|u_n\|}\,ds\biggr]. \end{aligned} \tag{4.10}$$

Due to (3.1) and the boundedness of $\{\mu_n\}$, we know that the convergence of (4.8) and the convergence of (4.9) are uniform for $t \in (0,1)$, *i.e.*, they are independent of n. Then applying (3.1) on (4.10), we will see that $\{g : g(t) = \frac{\nu_n(t)}{t^{\alpha-1}(1-t)}$ for some $u \in D\}$ is uniformly bounded and equicontinuous, which exactly means that $\{\nu_n\}$ is relatively compact. Hence, we can find a subsequence of $\{(\mu_n, \nu_n)\}$ convergent to (μ_*, ν_*). Obviously $\|\nu_*\| = 1$ and $\nu_* > \theta$. We relabel the subsequence $\{(\mu_n, \nu_n)\}$ for convenience.

Before claiming that C_+ joins $(\frac{\mu_{0^+}}{r}, \theta)$ to $(\frac{\mu_{+\infty}}{r}, +\infty)$, we still have some preparatory work to do. Decomposing $f(t,x)$ to become

$$f(t,x) = \frac{a_{+\infty}(t)}{t^{\alpha-1}(1-t)^{n-1}} x + \hat{a}_{+\infty}(t,x),$$

then (4.2) is equivalent to

$$\begin{aligned} u(t) &= \mu r \int_0^1 G(t,s) \frac{a_{+\infty}(s)}{s^{\alpha-1}(1-s)^{n-1}} u(s)\, ds \\ &\quad + \mu r \int_0^1 G(t,s) \hat{a}_{+\infty}\bigl(s, u(s)\bigr)\, ds \\ &:= \mu r (L_{+\infty} u)(t) + \mu r (N_{+\infty} u)(t). \end{aligned} \tag{4.11}$$

We will show that, for any $t \in (0,1)$, we have

$$\lim_{u \in K, \|u\| \to \infty} \frac{(N_{+\infty} u)(t)}{\|u\|} = 0.$$

Since

$$\begin{aligned} \frac{(N_{+\infty} u)(t)}{\|u\|} &= \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \int_0^1 (1-s)^{\alpha-1} \frac{\hat{a}_{+\infty}(s, u(s))}{\|u\|}\, ds \\ &\quad - \int_0^t (t-s)^{\alpha-1} \frac{\hat{a}_{+\infty}(s, u(s))}{\|u\|}\, ds \end{aligned}$$

we only need to prove that

$$\lim_{u \in K, \|u\| \to \infty} \int_0^1 (1-s)^{\alpha-1} \frac{\hat{a}_{+\infty}(s, u(s))}{\|u\|}\, ds = 0$$

and

$$\lim_{u \in K, \|u\| \to \infty} \int_0^t (t-s)^{\alpha-1} \frac{\hat{a}_{+\infty}(s, u(s))}{\|u\|}\, ds = 0.$$

We only prove the first one since the proof of the second one is similar.

It is not hard to see that there exists a constant number $b_{+\infty} > 0$ such that

$$\left| \frac{t^{n-1}(1-t)^{n-1} \hat{a}_{+\infty}(t,x)}{x} \right| \le b_{+\infty}$$

for all $(t,x) \in (0,1) \times (0,+\infty)$ due to $0 < \frac{t^{n-1}(1-t)^{n-1} f(t,x)}{x} \le b$ and $a_{+\infty} \in C[0,1]$. So for any $\varepsilon > 0$, we can find a sufficiently small $\delta > 0$ satisfying the requirement that, for any $u \in K$, we must have

$$\begin{aligned}\left|\int_0^{\delta}(1-s)^{\alpha-1}\frac{\hat{a}_{+\infty}(s,u(s))}{\|u\|}\,ds\right| &\le \int_0^{\delta}(1-s)^{\alpha-1}\frac{|\hat{a}_{+\infty}(s,u(s))|}{\|u\|}\,ds\\ &\le \int_0^{\delta}(1-s)^{\alpha-n+1}s^{\alpha-n}\frac{b_{+\infty}|u(s)|}{\|u\|s^{\alpha-1}(1-s)}\,ds\\ &\le b_{+\infty}\left|\int_0^{\delta}(1-s)^{\alpha-n+1}s^{\alpha-n}\right|ds<\frac{\varepsilon}{4}\end{aligned}\tag{4.12}$$

and similarly

$$\left|\int_{1-\delta}^{1}(1-s)^{\alpha-1}\frac{\hat{a}_{+\infty}(s,u(s))\,ds}{\|u\|}\right|<\frac{\varepsilon}{4}.\tag{4.13}$$

Now we consider $|\int_{\delta}^{1-\delta}(1-s)^{\alpha-1}\frac{\hat{a}_{+\infty}(s,u(s))\,ds}{\|u\|}|$. Due to (C_3^+), we know that there must exist a constant number $M_1>0$ such that

$$x\ge M_1\quad\text{implies}\quad\left|\frac{t^{n-1}(1-t)^{n-1}\hat{a}_{+\infty}(t,x)}{x}\right|<\frac{\Gamma(2\alpha-2n+3)}{\Gamma(\alpha-n+2)\Gamma(\alpha-n+1)}\frac{\varepsilon}{4}.\tag{4.14}$$

Moreover, because $\hat{a}_{+\infty}(t,x)$ is continuous in $[\delta,1-\delta]\times[0,M_1]$, there must exist a constant number $M_2>0$ such that

$$\left|\hat{a}_{+\infty}(t,x)\right|\le M_2\quad\text{for all } \delta\le t\le 1-\delta, 0\le x\le M_1.\tag{4.15}$$

For any $u\in X$, we set

$$I_u=\{t\in(0,1):u(t)<M_1\},\qquad J_u=\{t\in(0,1):u(t)\ge M_1\}.$$

Then by (4.14), (4.15), we have

$$\begin{aligned}&\left|\int_{\delta}^{1-\delta}(1-s)^{\alpha-1}\frac{\hat{a}_{+\infty}(s,u(s))\,ds}{\|u\|}\right|\\ &\quad\le\int_{\delta}^{1-\delta}(1-s)^{\alpha-1}\frac{|\hat{a}_{+\infty}(s,u(s))|\,ds}{\|u\|}\\ &\quad\le\int_{[\delta,1-\delta]\cap I_u}(1-s)^{\alpha-1}\frac{|\hat{a}_{+\infty}(s,u(s))|\,ds}{\|u\|}+\int_{[\delta,1-\delta]\cap J_u}(1-s)^{\alpha-1}\frac{|\hat{a}_{+\infty}(s,u(s))|\,ds}{\|u\|}\\ &\quad\le\frac{M_2}{\|u\|}\int_{[\delta,1-\delta]\cap I_u}(1-s)^{\alpha-1}\,ds\\ &\qquad+\int_{[\delta,1-\delta]\cap J_u}\left|\frac{s^{n-1}(1-s)^{n-1}\hat{a}_{\infty}(s,u(s))}{u(s)}\right|\left|\frac{u(s)}{\|u\|s^{\alpha-1}(1-s)}\right|(1-s)^{\alpha-n+1}s^{\alpha-n}\,ds\\ &\quad\le\frac{M_2}{\|u\|}\int_0^1(1-s)^{\alpha-1}\,ds+\frac{\Gamma(2\alpha-2n+3)}{\Gamma(\alpha-n+2)\Gamma(\alpha-n+1)}\frac{\varepsilon}{4}\int_0^1(1-s)^{\alpha-n+1}s^{\alpha-n}\,ds\\ &\quad=\frac{M_2}{\alpha\|u\|}+\frac{\varepsilon}{4}.\end{aligned}\tag{4.16}$$

By (4.12), (4.13), (4.16), we have

$$\lim_{u\in K,\|u\|\to\infty}\left|\int_0^1 (1-s)^{\alpha-1}\frac{\hat{a}_{+\infty}(s,u(s))\,ds}{\|u\|}\right| = 0.$$

Up to now, we have proved that for any $t \in (0,1)$, we must have

$$\lim_{u\in K,\|u\|\to\infty}\frac{(N_{+\infty}u)(t)}{\|u\|} = 0.$$

So if we divide

$$u_n(t) = \mu_n r(L_{+\infty}u_n)(t) + \mu_n r(N_{+\infty}u_n)(t), \quad t \in (0,1),$$

by $\|u_n\|$ and let $n \to \infty$, we will have

$$v_*(t) = \mu_* r(L_{+\infty}v_*)(t), \quad t \in (0,1),$$

i.e.,

$$v_* = \mu_* r(L_{+\infty}v_*).$$

Recalling the result of Lemma 2.3, we will know that $\mu_* = \frac{\mu_{+\infty}}{r}$, which implies C_+ joins $(\frac{\mu_{0^+}}{r},\theta)$ to $(\frac{\mu_{+\infty}}{r},+\infty)$. □

We set the cone

$$K_- := \{u \in X : u(t) \leq 0, \forall t \in (0,1)\}.$$

Clearly K_- is a solid cone in X.

We set

$$(L_{0^-}u)(t) = \int_0^1 G(t,s)\frac{a_{0^-}(s)}{s^{n-1}(1-s)^{n-1}}u(s)\,ds,$$
$$(L_{-\infty}u)(t) = \int_0^1 G(t,s)\frac{a_{-\infty}(s)}{s^{n-1}(1-s)^{n-1}}u(s)\,ds.$$

Similar to what we have done with L_{0^+} and $L_{+\infty}$ we can verify that L_{0^-} and $L_{-\infty}$ satisfy the conditions of Lemma 2.3, so we know that $r(L_{0^-}) > 0$ and $r(L_{-\infty}) > 0$. We set

$$\mu_{0^-} = r(L_{0^-})^{-1}, \qquad \mu_{-\infty} = r(L_{-\infty})^{-1}.$$

Then we will have the following conclusion.

Theorem 4.2 *Let* (C_1), (C_2^-), (C_3^-), (C_4), *and* (C_5) *hold. Then if* $\mu_{-\infty} < r < \mu_{0^-}$ *or* $\mu_{0^-} < r < \mu_{-\infty}$, *there must exist at least one negative solution of* (1.2).

Proof The proof is similar to that of Theorem 4.1, so we omit it. □

5 Examples

Example 5.1 Consider the following problem:

$$(\mathrm{E}_1) \quad \begin{cases} D_{0^+}^{\frac{10}{3}}u(t) + r\frac{2u(t)+t\sin(u(t))}{t^3(1-t)^2} = 0, \quad t \in (0,1), \\ t^{\frac{2}{3}}u''(t)|_{t=0} = t^{\frac{2}{3}}u'(t)|_{t=0} = t^{\frac{2}{3}}u(t)|_{t=0} = u(1) = 0. \end{cases}$$

If we set

$$(L_0u)(t) = \int_0^1 G(t,s)\frac{(2+s)}{s^3(1-s)^2}u(s)\,ds, \qquad (L_\infty u)(t) = \int_0^1 G(t,s)\frac{2}{s^3(1-s)^2}u(s)\,ds$$

and

$$\mu_0 = r(L_0)^{-1}, \qquad \mu_\infty = r(L_\infty)^{-1},$$

then there must be at least one positive solution and one negative solution of (E_1) when $\mu_0 < r < \mu_\infty$.

Proof Let $\alpha = \frac{10}{3}$, $n = 4$, $f(t,x) = \frac{2x+t\sin x}{t^3(1-t)^2}$, we can easily verify that (C_1), (C_2^+), (C_2^-), (C_3^+), (C_3^-), (C_4), (C_5) hold with $a_{0^+}(t) = a_{0^-}(t) = (2+t)(1-t)$, $a_{+\infty}(t) = a_{-\infty}(t) = 2(1-t)$, $\varphi(t) \equiv 1$, and $b = 3$. Then applying Theorem 4.1 and Theorem 4.2 we will complete the proof (we can verify that $\mu_0 < \mu_\infty$ by Lemma 2.3(d)). □

Example 5.2 Consider the following problem:

$$(\mathrm{E}_2) \quad \begin{cases} u''(t) + r\frac{u(t)-\arctan\frac{u(t)}{2}}{t(1-t)} = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

If we set

$$(L_0u)(t) = \int_0^1 G(t,s)\frac{1}{2s(1-s)}u(s)\,ds, \qquad (L_\infty u)(t) = \int_0^1 G(t,s)\frac{1}{s(1-s)}u(s)\,ds$$

and

$$\mu_0 = r(L_0)^{-1}, \qquad \mu_\infty = r(L_\infty)^{-1},$$

then there must be at least one positive solution and one negative solution of (E_2) when $\mu_\infty < r < \mu_0$.

Proof Let $\alpha = n = 2$, $f(t,x) = \frac{x-\arctan\frac{x}{2}}{t(1-t)}$, we can easily verify that (C_1), (C_2^+), (C_2^-), (C_3^+), (C_3^-), (C_4), (C_5) hold with $a_{0^+}(t) = a_{0^-}(t) \equiv \frac{1}{2}$, $a_{+\infty}(t) = a_{-\infty}(t) \equiv 1$, $\varphi(t) \equiv \frac{1}{2}$ and $b = 1$. Then applying Theorem 4.1 and Theorem 4.2 we will complete the proof (we can verify that $\mu_\infty < \mu_0$ by Lemma 2.3(d)). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
Research was supported by the National Natural Science Foundation of China (11571197, 11471187), the Doctoral Program Foundation of Education Ministry of China (20133705110003), the Natural Science Foundation of Shandong Province of China (ZR2014AM007, ZR2015AL002) and a Project of Shandong Province Higher Educational Science and Technology Program (J15LI16).

Received: 19 April 2016 Accepted: 26 July 2016 Published online: 04 August 2016

References
1. Cui, Y: Uniqueness of solution for boundary value problems for fractional differential equations. Appl. Math. Lett. **51**, 48-54 (2016)
2. Teng, K: Existence and multiplicity results for fractional differential inclusions with Dirichlet boundary conditions. Appl. Math. Comput. **220**, 792-801 (2013)
3. Sun, Y, Zhao, M: Positive solutions for a class of fractional differential equations with integral boundary conditions. Appl. Math. Lett. **34**, 17-21 (2014)
4. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
5. Chen, R, Ma, R: Positive solutions of the second-order differential systems in reactor dynamics. Appl. Math. Comput. **219**, 3882-3892 (2012)
6. Dai, G, Ma, R: Unilateral global bifurcation phenomena and nodal solutions for *p*-Laplacian. J. Differ. Equ. **252**, 2448-2468 (2012)
7. Dai, G, Wang, H, Yang, B: Global bifurcation and positive solution for a class of fully nonlinear problems. Comput. Math. Appl. **69**, 771-776 (2015)
8. Ma, R, Gao, C, Han, X: On linear and nonlinear fourth-order eigenvalue problems with indefinite weight. Nonlinear Anal. **74**, 6965-6969 (2011)
9. Liu, X, Sun, J: Asymptotic bifurcation points, and global bifurcation of nonlinear operators and its applications. Nonlinear Anal. **75**, 7-21 (2012)
10. Rynne, BP: Infinitely many solutions of superlinear fourth order boundary value problems. Topol. Methods Nonlinear Anal. **19**, 303-312 (2002)
11. Ma, R, Gao, C, Chen, X: Infinitely many solutions to superlinear second order m-point boundary value problems. Bound. Value Probl. **2011**, 14 (2011)
12. Ma, R, Thompson, B: Multiplicity results for second-order two-point boundary value problems with nonlinearities across several eigenvalues. Appl. Math. Lett. **18**, 587-595 (2005)
13. Rabinowitz, PH: Some global results for nonlinear eigenvalue problems. J. Funct. Anal. **7**, 487-513 (1971)
14. Dancer, EN: On the structure of solutions of non-linear eigenvalue problems. Indiana Univ. Math. J. **23**, 1069-1076 (1974)
15. Dancer, EN: Global solutions branches for positive maps. Arch. Ration. Mech. Anal. **55**, 207-213 (1974)
16. Rynne, BP: Spectral properties and nodal solutions for second-order, *m*-point, boundary value problems. Nonlinear Anal. **67**, 3318-3327 (2007)
17. Zeidler, E: Nonlinear Functional Analysis and Its Applications I: Fixed-Point Theorems. Springer, New York (1986)
18. Deimling, K: Nonlinear Functional Analysis. Springer, Berlin (1985)