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On the decay and blow-up of solution for a system of nonlinear viscoelastic plate equations with dissipative terms

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Abstract

In this paper, we consider the initial-boundary value problem of nonlinear viscoelastic plate equations with dissipative terms. We prove that, for certain initial data in the stable set, the decay rate estimate of the energy function is exponential or polynomial depending on the exponents of the damping terms in both equations by using Nakao's method. Conversely, for certain initial data in the unstable set, we use the perturbed energy method to show that the solution blows up in finite time when the initial energy is not larger than some positive number. This improves earlier results in the literature.

MSC: 35L70; 35L75; 93D20

Keywords: nonlinear plate equations; viscoelastic terms; dissipative terms; decay; blow-up

1 Introduction

In this paper, we consider the following initial-boundary value problem of the nonlinear viscoelastic plate equations with dissipative terms:

$$\begin{cases} u_{tt} - \gamma\Delta u - \Delta u_t + \Delta^2 u - \Delta u_{tt} + \int_0^t g_1(t-s)\Delta u(s)\,ds + |u_t|^{p-1}u_t \\ \quad = f_1(u,v), \quad (x,t) \in \Omega \times (0,T), \\ v_{tt} - \delta\Delta v - \Delta v_t + \Delta^2 v - \Delta v_{tt} + \int_0^t g_2(t-s)\Delta v(s)\,ds + |v_t|^{q-1}v_t \\ \quad = f_2(u,v), \quad (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \quad x \in \Omega, \\ u(x,t) = \partial_\nu u(x,t) = 0, \qquad v(x,t) = \partial_\nu v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n $(n = 1,2,3)$ with smooth boundary $\partial\Omega$, γ and δ are positive constants, $g_i : R^+ \to R^+$, $f_i : R^2 \to R$, $i = 1,2$, are given functions to be specified later.

The motivation of our work is due to the initial boundary problem of the plate equation

$$\begin{cases} u_{tt} - \Delta u - \Delta u_t + \Delta^2 u - \Delta u_{tt} + a|u_t|^{m-2}u_t = b|u|^{p-2}u, \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = \partial_\nu u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{1.2}$$

which has been discussed by Di and Shang [1] by considering the existence of global solutions and the asymptotic behavior of global solutions with $m \geq p$. Here, we understand Δu_t, $-\Delta u_{tt}$, $a|u_t|^{m-2}u_t$, and $b|u|^{p-2}u$ to be the strong dissipation term, the dispersive term, the nonlinear damping term, and the source term, respectively.

In the absence of the dispersive term and the nonlinear damping term, model (1.2) reduces to the following wave equation ($n \geq 1$)

$$u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} = f(u). \tag{1.3}$$

In 2000, Shang [2] studied the well-posedness, asymptotic behavior, and the finite time blow-up of the solutions under some suitable conditions on f and for $n = 1, 2, 3$. In 2004, Zhang and Hu [3] showed the existence and the stability of global weak solutions. In 2007, Xie and Zhong [4] obtained the existence of global attractors in $H_0^1(\Omega) \times H_0^1(\Omega)$, where the nonlinear term f satisfies a critical exponential growth assumption. In 2008, Xu *et al.* [5] used the multiplier method to investigate the asymptotic behavior of solutions for (1.3). Kafini and Messaoudi [6] considered a nonlinear wave equation and obtained a finite-time blow-up result with arbitrary positive initial energy. For more related results, the reader is referred to [7–10].

In the absence of the dispersive term and $m = 0$, model (1.2) reduces to the wave equation

$$u_{tt} - \Delta u - \Delta u_t - \Delta u_{tt} + u_t = |u|^{p-2}u. \tag{1.4}$$

Xu and Yang [11] established a blow-up result for certain solutions of (1.4) with arbitrary positive initial energy, where $1 < p < \infty$ if $n = 1, 2$ and $1 < p \leq \frac{n+2}{n-2}$ if $n \geq 3$.

Messaoudi and Mukiawa [12] studied the fourth-order viscoelastic plate equation

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)\,ds = 0$$

in the bounded domain $\Omega = (0, \pi) \times (-l, l) \subset R^2$ with nontraditional boundary conditions. The authors established the well-posedness of the solution and a decay result.

Another model related to (1.1) is

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)\,ds + h_1(u_t) = f_1(u,v), & (x,t) \in \Omega \times (0,T), \\ v_{tt} - \Delta v + \int_0^t g_2(t-s)\Delta v(s)\,ds + h_2(v_t) = f_2(u,v), & (x,t) \in \Omega \times (0,T), \end{cases} \tag{1.5}$$

where Ω is a bounded domain in R^n ($n = 1, 2, 3$) with smooth boundary $\partial\Omega$. For problem (1.5) with $h_1(u_t) = -\Delta u_t$ and $h_2(v_t) = -\Delta v_t$, Liang and Gao [13] obtained that the decay estimate of the energy function is exponential with certain initial data in the stable set. On the contrary, a solution with positive initial energy blows up in finite time when the initial data is inside the unstable set. For $h_1(u_t) = |u_t|^{m-1}u_t$ and $h_2(v_t) = |v_t|^{r-1}v_t$, Han and Wang [14] showed several results concerned with local existence, global existence, and finite-time blow-up with negative initial energy. The latter blow-up result has been improved by Messaoudi, Said-Houari, and Guesmia [15, 16] by studying a larger class of initial data for which the initial energy can take positive values and obtained that the rate of decay of the total energy depends on those of the relaxation functions. Wu [17] considered the following problem for $(x,t) \in \Omega \times (0,T)$:

$$\begin{cases} u_{tt} - M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + |u_t|^{p-1}u_t = f_1(u,v), \\ v_{tt} - M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + |v_t|^{p-1}v_t = f_2(u,v), \end{cases} \tag{1.6}$$

where Ω is a bounded domain in R^n $(n = 1,2,3)$ with smooth boundary $\partial\Omega$. He obtained that the decay estimate of the energy function is exponential or polynomial depending on the exponents of the damping terms in both equations, and the blow-up of solution with nonnegative initial energy was established.

Motivated by previous works, it is interesting to study the global existence, uniform decay, and finite time blow-up of solution to problem (1.1). Firstly, we establish that the solution is global in time under certain initial data in the stable set. After that, we show the decay estimate of solutions by Nakao's method [18]. Precisely, we establish that the decay estimate of energy function is exponential or polynomial depending on the parameters p and q. Secondly, we study the finite time blow-up of problem (1.1) with $\gamma = \delta = 1$. By adopting and modifying the methods used in [15] we prove the blow-up of solutions when the energy is negative or nonnegative and less than the critical value E_1 (given in (4.3)). In this way, our results allow a wider region for the blow-up results.

The paper is organized as follow. In Section 2, we present preliminaries and some lemmas. In Section 3, the global existence and decay property are derived. Finally, the blow-up results of (1.1) with $\gamma = \delta = 1$ are obtained in the case of initial energy being nonnegative.

2 Preliminaries

In this section, we give some lemmas and assumptions. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^1(\Omega)$ with their usual products and norms. We use the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ for $2 \le p \le \frac{2n}{n-2}$ if $n \ge 3$ or $2 \le p$ if $n = 1,2$. In this case, the embedding constant is denoted by c_*, that is,

$$\|u\|_p \le c_* \|\nabla u\|_2. \tag{2.1}$$

Next, we give the assumptions for problem (1.1).

(A1) The relaxation functions $g_1(s)$ and $g_2(s)$ are of class C^1, nonnegative and nonincreasing for $s \ge 0$, and satisfy

$$\gamma - \int_0^\infty g_1(s)\,ds = l > 0, \qquad \delta - \int_0^\infty g_2(s)\,ds = k > 0.$$

Concerning the functions $f_1(u,v)$ and $f_2(u,v)$, we take (see [15])

$$f_1(u,v) = (m+1)\bigl(a|u+v|^{m-1}(u+v) + b|u|^{\frac{m-3}{2}}|v|^{\frac{m+1}{2}}u\bigr) \tag{2.2}$$

and

$$f_2(u,v) = (m+1)\bigl(a|u+v|^{m-1}(u+v) + b|v|^{\frac{m-3}{2}}|u|^{\frac{m+1}{2}}v\bigr) \tag{2.3}$$

with constants $a, b > 0$. We can easily verify that

$$uf_1(u,v) + vf_2(u,v) = (m+1)F(u,v), \quad (u,v) \in R^2,$$

where

$$F(u,v) = a|u+v|^{m+1} + 2b|uv|^{\frac{m+1}{2}}.$$

(A2) For the nonlinearity, we suppose that

$$\begin{cases} m > 1, & n = 1,2, \\ 1 < m \leq 3, & n = 3, \end{cases} \tag{2.4}$$

and

$$\begin{cases} p,q \geq 1, & n = 1,2, \\ 1 \leq p,q \leq 5, & n = 3. \end{cases} \tag{2.5}$$

As in [15], we still have the following results.

Lemma 2.1 (Sobolev-Poincaré inequality) *Let $2 \leq k < +\infty$ and $n \leq 3$. Then there is a constant $\tilde{c} = c(\Omega,k)$ such that*

$$\|u\|_k \leq \tilde{c}\|\Delta u\|_2, \quad u \in H_0^2(\Omega). \tag{2.6}$$

Lemma 2.2 *There exist two positive constants c_0 and c_1 such that*

$$c_0\big(|u|^{m+1} + |v|^{m+1}\big) \leq F(u,v) \leq c_1\big(|u|^{m+1} + |v|^{m+1}\big), \quad (u,v) \in R^2.$$

Lemma 2.3 *Suppose that* (2.4) *holds. Then there exists $\eta > 0$ such that, for any $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$, we have*

$$\|u+v\|_{m+1}^{m+1} + 2\|uv\|_{\frac{m+1}{2}}^{\frac{m+1}{2}} \leq \eta\big(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\big)^{\frac{m+1}{2}}.$$

We also need the following technical lemma.

Lemma 2.4 ([15]) *For any $g \in C^1$ and $\phi \in H^1(0,T)$, we have*

$$-2\int_0^t \int_\Omega g(t-s)\phi\phi_t\,dx\,ds = \frac{d}{dt}\left(g \circ \phi - \int_0^t g(s)\,ds\|\phi\|_2^2\right) + g(t)\|\phi\|_2^2 - g' \circ \phi,$$

where

$$g \circ \phi := \int_0^t g(t-s)\int_\Omega \big|\phi(s) - \phi(t)\big|^2\,dx\,ds.$$

Now, we are in a position to state the local existence result to problem (1.1), which can be established by using arguments similar to those in [14]. We omit the proof.

Theorem 2.5 *Let $u_0, v_0 \in H_0^2(\Omega)$ and $u_1, v_1 \in H_0^1(\Omega)$. Assume that* (A1) *and* (A2) *are satisfied. Then there exists a couple solution (u,v) of problem* (1.1) *such that, for some $T > 0$,*

$$u \in L^\infty\big(0,T;H_0^2(\Omega) \cap L^{p+1}(\Omega)\big), \qquad v \in L^\infty\big(0,T;H_0^2(\Omega) \cap L^{q+1}(\Omega)\big),$$
$$u_t \in L^\infty\big(0,T;H_0^1(\Omega)\big) \cap L^{p+1}(\Omega), \qquad v_t \in L^\infty\big(0,T;H_0^1(\Omega)\big) \cap L^{q+1}(\Omega).$$

We conclude this section by stating Nakao's lemma, which will be used in establishing the decay rate of solutions to problem (1.1).

Lemma 2.6 ([18]) *Let $\phi(t)$ be a nonincreasing and nonnegative function on $[0,T]$, $T>1$, such that*

$$\phi^{1+r}(t) \le \omega_0\bigl(\phi(t)-\phi(t+1)\bigr), \quad t\in[0,T],$$

where $\omega_0>1$ and $r\ge 0$. Then we have, for all $t\in[0,T]$,

(i) *if $r=0$, then*

$$\phi(t)\le \phi(0)e^{-\omega_1[t-1]^+},$$

(ii) *if $r>0$, then*

$$\phi(t)\le \bigl(\phi^{-r}(0)+\omega_0^{-1}r[t-1]^+\bigr)^{-\frac{1}{r}},$$

where $\omega_1 := \ln(\frac{\omega_0}{\omega_0-1})$ and $[t-1]^+ := \max\{t-1,0\}$.

Remark 2.7 For simplicity, we take $a=b=1$ in (2.2) and (2.3) throughout this paper.

3 Global existence and energy decay

In this section, we focus our attention on the global existence and decay rate of the solution to problem (1.1). We first define

$$\begin{aligned} I(t) &:= \|\Delta u\|_2^2+\|\Delta v\|_2^2+\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2+\left(\gamma-\int_0^t g_1(s)\,ds\right)\|\nabla u\|_2^2 \\ &\quad +\left(\delta-\int_0^t g_2(s)\,ds\right)\|\nabla v\|_2^2+g_1\circ\nabla u+g_2\circ\nabla v-(m+1)\int_\Omega F(u,v)\,dx, \end{aligned} \tag{3.1}$$

$$\begin{aligned} J(t) &:= \frac{1}{2}\left(\|\Delta u\|_2^2+\|\Delta v\|_2^2+\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2+\left(\gamma-\int_0^t g_1(s)\,ds\right)\|\nabla u\|_2^2\right. \\ &\quad \left.+\left(\delta-\int_0^t g_2(s)\,ds\right)\|\nabla v\|_2^2+g_1\circ\nabla u+g_2\circ\nabla v\right)-\int_\Omega F(u,v)\,dx, \end{aligned} \tag{3.2}$$

and define the energy function as

$$E(t):=\frac{1}{2}\bigl(\|u_t\|_2^2+\|v_t\|_2^2\bigr)+J(t). \tag{3.3}$$

Lemma 3.1 *Suppose that* (A1) *and* (2.4) *hold. Let (u,v) be the solution of problem* (1.1). *Then $E(t)$ is a nonincreasing function, that is, for $t\ge 0$,*

$$\begin{aligned} \frac{d}{dt}E(t) &= -\|\nabla u_t\|_2^2-\|\nabla v_t\|_2^2+\frac{1}{2}\bigl(g_1'\circ\nabla u\bigr)+\frac{1}{2}\bigl(g_2'\circ\nabla v\bigr)-\frac{1}{2}g_1(t)\|\nabla u\|_2^2 \\ &\quad -\frac{1}{2}g_2(t)\|\nabla v\|_2^2-\|u_t\|_{p+1}^{p+1}-\|v_t\|_{q+1}^{q+1}. \end{aligned} \tag{3.4}$$

Proof Multiplying $(1.1)_1$ by u_t and $(1.2)_2$ by v_t, integrating over Ω, summing up, and then using integration by parts, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\left(\|u_t\|_2^2+\|v_t\|_2^2+\gamma\|\nabla u\|_2^2+\delta\|\nabla v\|_2^2+\|\Delta u\|_2^2+\|\Delta v\|_2^2+\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\right)\right.\\
&\qquad\left.-\int_\Omega F(u,v)\,dx\right]\\
&\quad=-\|\nabla u_t\|_2^2-\|\nabla v_t\|_2^2+\int_0^t\int_\Omega g_1(t-s)\nabla u(s)\cdot\nabla u_t\,dx\,ds\\
&\qquad+\int_0^t\int_\Omega g_2(t-s)\nabla v(s)\cdot\nabla v_t\,dx\,ds\\
&\qquad-\|u_t\|_{p+1}^{p+1}-\|v_t\|_{q+1}^{q+1}.
\end{aligned}$$

Applying Lemma 2.4 to the third and fourth terms on the right-hand side of this equality, we get (3.4). □

Lemma 3.2 *Suppose that* (A1) *and* (2.4) *hold. Let* (u,v) *be the solution of problem* (1.1). *Assume further that* $I(0)>0$ *and*

$$\alpha_1:=(m+1)c_1\tilde{c}^{m+1}\left(\frac{2(m+1)}{m-1}E(0)\right)^{\frac{m-1}{2}}<1. \tag{3.5}$$

Then

$$I(t)>0,\quad t\geq 0. \tag{3.6}$$

Proof Since $I(0)>0$, by continuity there exists a maximal time $t_{\max}>0$ (possibly $t_{\max}=T$) such that

$$I(t)\geq 0,\quad t\in[0,t_{\max}],$$

which implies that, for $t\in[0,t_{\max}]$,

$$\begin{aligned}
J(t)&=\frac{m-1}{2(m+1)}\left(\|\Delta u\|_2^2+\|\Delta v\|_2^2+\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2+\left(\gamma-\int_0^t g_1(s)\,ds\right)\|\nabla u\|_2^2\right.\\
&\quad\left.+\left(\delta-\int_0^t g_2(s)\,ds\right)\|\nabla v\|_2^2+g_1\circ\nabla u+g_2\circ\nabla v\right)+\frac{1}{m+1}I(t)\\
&\geq\frac{m-1}{2(m+1)}\left(\|\Delta u\|_2^2+\|\Delta v\|_2^2+l\|\nabla u\|_2^2+k\|\nabla v\|_2^2\right).
\end{aligned}\tag{3.7}$$

Since $E(t)$ is nonincreasing by (3.4), using (3.7) and (3.3), we get, for $t\in[0,t_{\max}]$,

$$\begin{aligned}
&\|\Delta u\|_2^2+\|\Delta v\|_2^2+l\|\nabla u\|_2^2+k\|\nabla v\|_2^2\\
&\quad\leq\frac{2(m+1)}{m-1}J(t)\leq\frac{2(m+1)}{m-1}E(t)\leq\frac{2(m+1)}{m-1}E(0).
\end{aligned}\tag{3.8}$$

Using Lemma 2.2, (2.6), (3.8), and (3.5), we obtain, for $t \in [0, t_{\max}]$,

$$\begin{aligned}(m+1)\int_\Omega F(u,v)\,dx &\le (m+1)c_1\bigl(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\bigr)\\ &\le (m+1)c_1\tilde{c}^{m+1}\bigl(\|\Delta u\|_2^{m+1} + \|\Delta v\|_2^{m+1}\bigr)\\ &\le \alpha_1\bigl(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\bigr)\\ &< \|\Delta u\|_2^2 + \|\Delta v\|_2^2. \end{aligned} \tag{3.9}$$

Thus,

$$I(t) > 0, \quad t \in [0, t_{\max}].$$

By repeating these steps and using the fact that

$$\lim_{t\to t_{\max}} (m+1)c_1\tilde{c}^{m+1}\left(\frac{2(m+1)}{m-1}E(t)\right)^{\frac{m-1}{2}} \le \alpha_1 < 1,$$

this implies that we can take $t_{\max} = T$. □

Lemma 3.3 *Let the assumptions of Lemma* 3.2 *hold. Then there exists* $\eta_1 > 1$ *such that*

$$\|\Delta u\|_2^2 + \|\Delta v\|_2^2 \le \eta_1 I(t), \quad t \in [0, T), \tag{3.10}$$

where $\eta_1 = \frac{1}{1-\alpha_1}$.

Proof From (3.9) we have

$$(m+1)\int_\Omega F(u,v)\,dx \le \alpha_1\bigl(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\bigr).$$

Letting $\eta_1 = \frac{1}{1-\alpha_1}$ and using (3.1), we obtain (3.10). □

Theorem 3.4 *Suppose that* (A1) *and* (A2) *hold. Let* $u_0, v_0 \in H_0^2(\Omega)$ *and* $u_1, v_1 \in H_0^1(\Omega)$ *satisfy* $I(0) > 0$ *and* (3.5). *Then the solution* (u,v) *of problem* (1.1) *is global and bounded. Furthermore, if*

$$\gamma + \delta > \frac{7+5\eta_1}{2}\max\left\{\int_0^\infty g_1(s)\,ds, \int_0^\infty g_2(s)\,ds\right\}, \tag{3.11}$$

then we have the following decay estimates:

(i) *if* $p = q = 1$, *then, for* $t \ge 0$,

$$E(t) \le E(0)e^{-\tau_1 t},$$

(ii) *if* $\max\{p,q\} > 1$, *then, for* $t \ge 0$,

$$E(t) \le \left(E^{-\max\{\frac{p-1}{2},\frac{q-1}{2}\}}(0) + \tau_2\max\left\{\frac{p-1}{2}, \frac{q-1}{2}\right\}[t-1]^+\right)^{-\frac{2}{\max\{p,q\}-1}},$$

where τ_1 *and* τ_2 *are some positive constants.*

Proof First, to prove that $T = \infty$, it suffices to show that $\|\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2$ is bounded independently of t. Thanks to (3.3), (3.4), and (3.6), we have

$$\begin{aligned} E(0) \geq E(t) &\geq J(t) \\ &= \frac{m-1}{2(m+1)}\biggl(\|\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 \\ &\quad + \biggl(\gamma - \int_0^t g_1(s)\,ds\biggr)\|\nabla u\|_2^2 \\ &\quad + \biggl(\delta - \int_0^t g_2(s)\,ds\biggr)\|\nabla v\|_2^2 + g_1 \circ \nabla u + g_2 \circ \nabla v\biggr) + \frac{1}{m+1}I(t) \\ &> \frac{m-1}{2(m+1)}\bigl(\|\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\bigr). \end{aligned}$$

Therefore,

$$\|\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 \leq \alpha_2 E(0),$$

where α_2 is a positive constant depending only on m. Thus, we obtain the global existence.

We further derive the decay rate of the energy function for problem (1.1) by Nakao's method [18]. For this purpose, we have to show that the energy function defined by (3.3) satisfies the hypothesis of Lemma 2.6. Integrating (3.4) over $[t, t+1]$, we have

$$E(t) - E(t+1) = D_1^{p+1}(t) + D_2^{q+1}(t), \tag{3.12}$$

where

$$\begin{aligned} D_1^{p+1}(t) &:= \int_t^{t+1} \|\nabla u_t\|_2^2\,ds - \frac{1}{2}\int_t^{t+1} g_1' \circ \nabla u\,ds \\ &\quad + \frac{1}{2}\int_t^{t+1} g_1\|\nabla u\|_2^2\,ds + \int_t^{t+1} \|u_t\|_{p+1}^{p+1}\,ds, \end{aligned} \tag{3.13}$$

$$\begin{aligned} D_2^{q+1}(t) &:= \int_t^{t+1} \|\nabla v_t\|_2^2\,ds - \frac{1}{2}\int_t^{t+1} g_2' \circ \nabla v\,ds \\ &\quad + \frac{1}{2}\int_t^{t+1} g_2\|\nabla v\|_2^2\,ds + \int_t^{t+1} \|v_t\|_{q+1}^{q+1}\,ds. \end{aligned} \tag{3.14}$$

By (3.13), (3.14), and the Hölder inequality, we observe that

$$\int_t^{t+1}\int_\Omega |u_t|^2\,dx\,ds + \int_t^{t+1}\int_\Omega |v_t|^2\,dx\,ds \leq c_1(\Omega)D_1^2(t) + c_2(\Omega)D_2^2(t), \tag{3.15}$$

where $c_1(\Omega) = \mathrm{vol}(\Omega)^{\frac{p-1}{p+1}}$ and $c_2(\Omega) = \mathrm{vol}(\Omega)^{\frac{q-1}{q+1}}$.

By the mean value theorem there exist $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\bigl\|u_t(t_i)\bigr\|_2^2 + \bigl\|v_t(t_i)\bigr\|_2^2 \leq 4c_1(\Omega)D_1^2(t) + 4c_2(\Omega)D_2^2(t), \quad i = 1, 2. \tag{3.16}$$

Next, multiplying Eq. $(1.1)_1$ by u and Eq. $(1.1)_2$ by v, integrating over $\Omega \times [t_1, t_2]$, and using integration by parts, we obtain

$$\begin{aligned}\int_{t_1}^{t_2} I(t)\,dt = {} & -\int_{t_1}^{t_2}\int_\Omega uu_{tt}\,dx\,dt - \int_{t_1}^{t_2}\int_\Omega vv_{tt}\,dx\,dt - \int_{t_1}^{t_2}\int_\Omega \nabla u\cdot\nabla u_t\,dx\,dt \\ & - \int_{t_1}^{t_2}\int_\Omega \nabla v\cdot\nabla v_t\,dx\,dt - \int_{t_1}^{t_2}\int_\Omega \nabla u\cdot\nabla u_{tt}\,dx\,dt \\ & - \int_{t_1}^{t_2}\int_\Omega \nabla v\cdot\nabla v_{tt}\,dx\,dt + \int_{t_1}^{t_2}\|\nabla u_t\|_2^2\,dt \\ & + \int_{t_1}^{t_2}\|\nabla v_t\|_2^2\,dt + \int_{t_1}^{t_2} g_1\circ\nabla u\,dt + \int_{t_1}^{t_2} g_2\circ\nabla v\,dt \\ & + \int_{t_1}^{t_2}\int_\Omega\int_0^t g_1(t-s)\nabla u(t)\cdot\big(\nabla u(s)-\nabla u(t)\big)\,ds\,dx\,dt \\ & + \int_{t_1}^{t_2}\int_\Omega\int_0^t g_2(t-s)\nabla v(t)\cdot\big(\nabla v(s)-\nabla v(t)\big)\,ds\,dx\,dt \\ & - \int_{t_1}^{t_2}\int_\Omega |u_t|^{p-1}u_t u\,dx\,dt - \int_{t_1}^{t_2}\int_\Omega |v_t|^{q-1}v_t v\,dx\,dt. \end{aligned} \tag{3.17}$$

Integrating by parts and applying the Cauchy-Schwarz inequality in the first term of the right-hand side of (3.17), we obtain

$$\left|\int_{t_1}^{t_2}\int_\Omega uu_{tt}\,dx\,dt\right| \le \sum_{i=1}^{2}\big\|u_t(t_i)\big\|_2\big\|u(t_i)\big\|_2 + \int_{t_1}^{t_2}\big\|u_t(t)\big\|_2^2\,dt, \tag{3.18}$$

$$\left|\int_{t_1}^{t_2}\int_\Omega vv_{tt}\,dx\,dt\right| \le \sum_{i=1}^{2}\big\|v_t(t_i)\big\|_2\big\|v(t_i)\big\|_2 + \int_{t_1}^{t_2}\big\|v_t(t)\big\|_2^2\,dt, \tag{3.19}$$

$$\left|\int_{t_1}^{t_2}\int_\Omega \nabla u\cdot\nabla u_{tt}\,dx\,dt\right| \le \sum_{i=1}^{2}\big\|\nabla u_t(t_i)\big\|_2\big\|\nabla u(t_i)\big\|_2 + \int_{t_1}^{t_2}\big\|\nabla u_t(t)\big\|_2^2\,dt, \tag{3.20}$$

and

$$\left|\int_{t_1}^{t_2}\int_\Omega \nabla v\cdot\nabla v_{tt}\,dx\,dt\right| \le \sum_{i=1}^{2}\big\|\nabla v_t(t_i)\big\|_2\big\|\nabla v(t_i)\big\|_2 + \int_{t_1}^{t_2}\big\|\nabla v_t(t)\big\|_2^2\,dt. \tag{3.21}$$

Now, we estimate the third term of the right-hand side of inequality (3.17). By the Cauchy-Schwarz inequality we have

$$\left|\int_{t_1}^{t_2}\int_\Omega \nabla u\cdot\nabla u_t\,dx\,dt\right| \le \int_{t_1}^{t_2}\|\nabla u\|_2\|\nabla u_t\|_2\,dt \tag{3.22}$$

and

$$\left|\int_{t_1}^{t_2}\int_\Omega \nabla v\cdot\nabla v_t\,dx\,dt\right| \le \int_{t_1}^{t_2}\|\nabla v\|_2\|\nabla v_t\|_2\,dt. \tag{3.23}$$

Since

$$\begin{aligned}
&\int_\Omega \int_0^t g_1(t-s)\nabla u(t)\cdot\big(\nabla u(s)-\nabla u(t)\big)\,ds\,dx \\
&\quad= \frac{1}{2}\bigg[\int_0^t g_1(t-s)\big(\big\|\nabla u(t)\big\|_2^2+\big\|\nabla u(s)\big\|_2^2\big)\,ds - \int_0^t g_1(t-s)\big(\big\|\nabla u(t)-\nabla u(s)\big\|_2^2\big)\,ds\bigg] \\
&\qquad - \int_\Omega \int_0^t g_1(s)\big|\nabla u(t)\big|^2\,ds\,dx \\
&\quad= -\frac{1}{2}\int_\Omega \int_0^t g_1(s)\big|\nabla u(t)\big|^2\,ds\,dx + \frac{1}{2}\int_0^t g_1(t-s)\big\|\nabla u(s)\big\|_2^2\,ds - \frac{1}{2}(g_1\circ\nabla u) \\
&\quad\le \frac{1}{2}\int_0^t g_1(t-s)\big\|\nabla u(s)\big\|_2^2\,ds - \frac{1}{2}(g_1\circ\nabla u)
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
&\int_\Omega \int_0^t g_2(t-s)\nabla v(t)\cdot\big(\nabla v(s)-\nabla v(t)\big)\,ds\,dx \\
&\quad\le \frac{1}{2}\int_0^t g_2(t-s)\big\|\nabla v(s)\big\|_2^2\,ds - \frac{1}{2}(g_2\circ\nabla v),
\end{aligned} \tag{3.25}$$

by (3.18)-(3.25) we have

$$\begin{aligned}
\int_{t_1}^{t_2} I(t)\,dt \le{}& \sum_{i=1}^{2}\big\|u_t(t_i)\big\|_2\big\|u(t_i)\big\|_2 + \sum_{i=1}^{2}\big\|v_t(t_i)\big\|_2\big\|v(t_i)\big\|_2 \\
&+ \int_{t_1}^{t_2}\big(\big\|u_t(t)\big\|_2^2+\big\|v_t(t)\big\|_2^2\big)\,dt + \int_{t_1}^{t_2}\|\nabla u\|_2\|\nabla u_t\|_2\,dt \\
&+ \int_{t_1}^{t_2}\|\nabla v\|_2\|\nabla v_t\|_2\,dt + \sum_{i=1}^{2}\big\|\nabla u_t(t_i)\big\|_2\big\|\nabla u(t_i)\big\|_2 \\
&+ \sum_{i=1}^{2}\big\|\nabla v_t(t_i)\big\|_2\big\|\nabla v(t_i)\big\|_2 + 2\int_{t_1}^{t_2}\big(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\big)\,dt \\
&+ \frac{1}{2}\int_{t_1}^{t_2}\int_0^t g_1(t-s)\big\|\nabla u(s)\big\|_2^2\,ds\,dt + \frac{1}{2}\int_{t_1}^{t_2}\int_0^t g_2(t-s)\big\|\nabla v(s)\big\|_2^2\,ds\,dt \\
&+ \frac{1}{2}\int_{t_1}^{t_2} g_1\circ\nabla u\,dt + \frac{1}{2}\int_{t_1}^{t_2} g_2\circ\nabla v\,dt - \int_{t_1}^{t_2}\int_\Omega |u_t|^{p-1}u_t u\,dx\,dt \\
&- \int_{t_1}^{t_2}\int_\Omega |v_t|^{q-1}v_t v\,dx\,dt.
\end{aligned} \tag{3.26}$$

Now, we will estimate the right-hand side of (3.26). First, by (3.16), (2.1), and (3.8), letting $\beta=\min\{l,k\}$, we have

$$\begin{aligned}
\big\|u_t(t_i)\big\|_2\big\|u(t_i)\big\|_2 &\le c_*\sqrt{4c_1(\Omega)D_1^2(t)+4c_2(\Omega)D_2^2(t)}\sup_{t_1\le s\le t_2}\big\|\nabla u(s)\big\|_2 \\
&\le c_*\left(\frac{2(m+1)}{\beta(m-1)}\right)^{\frac{1}{2}}\sqrt{4c_1(\Omega)D_1^2(t)+4c_2(\Omega)D_2^2(t)}\sup_{t_1\le s\le t_2}E^{\frac{1}{2}}(s)
\end{aligned} \tag{3.27}$$

and

$$\|v_t(t_i)\|_2 \|v(t_i)\|_2 \le c_* \left(\frac{2(m+1)}{\beta(m-1)}\right)^{\frac{1}{2}} \sqrt{4c_1(\Omega)D_1^2(t) + 4c_2(\Omega)D_2^2(t)} \sup_{t_1 \le s \le t_2} E^{\frac{1}{2}}(s). \tag{3.28}$$

By the Hölder inequality and (3.13) we find

$$\int_{t_1}^{t_2} \|\nabla u_t\|_2 \, dt \le \left(\int_{t_1}^{t_2} 1^2 \, dt\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \|\nabla u_t\|_2^2 \, dt\right)^{\frac{1}{2}} \le D^{\frac{p+1}{2}}(t). \tag{3.29}$$

Then we have

$$\int_{t_1}^{t_2} \|\nabla u\|_2 \|\nabla u_t\|_2 \, dt \le \left(\frac{2(m+1)}{\beta(m-1)}\right)^{\frac{1}{2}} D_1^{\frac{p+1}{2}}(t) \sup_{t_1 \le s \le t_2} E^{\frac{1}{2}}(s), \tag{3.30}$$

and similarly we obtain

$$\int_{t_1}^{t_2} \|\nabla v\|_2 \|\nabla v_t\|_2 \, dt \le \left(\frac{2(m+1)}{\beta(m-1)}\right)^{\frac{1}{2}} D_2^{\frac{q+1}{2}}(t) \sup_{t_1 \le s \le t_2} E^{\frac{1}{2}}(s). \tag{3.31}$$

By (3.13) and (3.14) we observe that

$$\int_t^{t+1} \|\nabla u_t\|_2^2 \, ds + \int_t^{t+1} \|\nabla v_t\|_2^2 \, ds \le D_1^{p+1}(t) + D_2^{q+1}(t).$$

By the mean value theorem there exist $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|\nabla u_t(t_i)\|_2^2 + \|\nabla v_t(t_i)\|_2^2 \le 4D_1^{p+1}(t) + 4D_2^{q+1}(t). \tag{3.32}$$

From (3.8) and (3.32) we have

$$\|\nabla u_t(t_i)\|_2 \|\nabla u(t_i)\|_2 \le \left(\frac{2(m+1)}{\beta(m-1)}\right)^{\frac{1}{2}} \sqrt{4D_1^{p+1}(t) + 4D_2^{q+1}(t)} \sup_{t_1 \le s \le t_2} E^{\frac{1}{2}}(s) \tag{3.33}$$

and

$$\|\nabla v_t(t_i)\|_2 \|\nabla v(t_i)\|_2 \le \left(\frac{2(m+1)}{\beta(m-1)}\right)^{\frac{1}{2}} \sqrt{4D_1^{p+1}(t) + 4D_2^{q+1}(t)} \sup_{t_1 \le s \le t_2} E^{\frac{1}{2}}(s). \tag{3.34}$$

Applying Young's inequality to convolution $\|\phi * \psi\|_q \le \|\phi\|_r \|\psi\|_s$ with

$$\frac{1}{q} = \frac{1}{r} + \frac{1}{s} - 1, \quad 1 \le q, r, s \le \infty,$$

and noting that if $q = 1$, then $r = 1$ and $s = 1$, we get

$$\begin{aligned} \int_{t_1}^{t_2} \int_0^t g_1(t-s) \|\nabla u(s)\|_2^2 \, ds \, dt &\le \int_{t_1}^{t_2} g_1(t) \, dt \int_{t_1}^{t_2} \|\nabla u(t)\|_2^2 \, dt \\ &\le (\gamma - \beta) \int_{t_1}^{t_2} \|\nabla u(t)\|_2^2 \, dt \end{aligned} \tag{3.35}$$

and

$$\int_{t_1}^{t_2}\int_0^t g_2(t-s)\|\nabla v(s)\|_2^2\,ds\,dt \le (\delta-\beta)\int_{t_1}^{t_2}\|\nabla v(t)\|_2^2\,dt. \tag{3.36}$$

From (3.1), (3.9), (3.10), (3.35), (3.36), and (A1) we have

$$\begin{aligned}
&\frac{1}{2}\left(\int_{t_1}^{t_2}\int_0^t g_1(t-s)\|\nabla u(s)\|_2^2\,ds\,dt + \int_{t_1}^{t_2}\int_0^t g_2(t-s)\|\nabla v(s)\|_2^2\,ds\,dt\right)\\
&\quad\le \frac{\gamma+\delta-\beta}{2\beta}\int_{t_1}^{t_2}\left(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\right)dt\\
&\quad\le \frac{\gamma+\delta-\beta}{2\beta}\int_{t_1}^{t_2}\left(I(t)+(m+1)\int_\Omega F(u,v)\,dx\right)dt\\
&\quad\le \frac{\gamma+\delta-\beta}{2\beta}(1+\eta_1)\int_{t_1}^{t_2} I(t)\,dt.
\end{aligned} \tag{3.37}$$

To estimate the eleventh and twelfth terms on the right-hand side of (3.26), we use (3.37) to obtain

$$\begin{aligned}
\frac{1}{2}\int_{t_1}^{t_2}(g_1\circ\nabla u + g_2\circ\nabla v)\,dt &= \frac{1}{2}\int_{t_1}^{t_2}\int_0^t g_1(t-s)\left(\|\nabla u(s)-\nabla u(t)\|_2^2\right)ds\,dt\\
&\quad + \frac{1}{2}\int_{t_1}^{t_2}\int_0^t g_2(t-s)\left(\|\nabla v(s)-\nabla v(t)\|_2^2\right)ds\,dt\\
&\le \int_{t_1}^{t_2}\int_0^t g_1(t-s)\left(\|\nabla u(s)\|_2^2 + \|\nabla u(t)\|_2^2\right)ds\,dt\\
&\quad + \int_{t_1}^{t_2}\int_0^t g_2(t-s)\left(\|\nabla v(s)\|_2^2 + \|\nabla v(t)\|_2^2\right)ds\,dt\\
&\le \frac{2(\gamma+\delta-\beta)}{\beta}(1+\eta_1)\int_{t_1}^{t_2} I(t)\,dt.
\end{aligned} \tag{3.38}$$

Using Hölder inequality, (2.1), (3.8), and (3.13), we have

$$\begin{aligned}
\left|\int_{t_1}^{t_2}\int_\Omega |u_t|^{p-1}u_t u\,dx\,dt\right| &\le \int_{t_1}^{t_2}\|u_t\|_{p+1}^p\|u\|_{p+1}\,dt\\
&\le c_*\int_{t_1}^{t_2}\|u_t\|_{p+1}^p\|\nabla u\|_2\,dt\\
&\le c_*\left(\frac{2(m+1)}{\beta(m-1)}\right)^{\frac{1}{2}}\int_{t_1}^{t_2}\|u_t\|_{p+1}^p\,dt\sup_{t_1\le s\le t_2}E^{\frac{1}{2}}(s)\\
&\le c_*\left(\frac{2(m+1)}{\beta(m-1)}\right)^{\frac{1}{2}}D_1^p(t)\sup_{t_1\le s\le t_2}E^{\frac{1}{2}}(s)
\end{aligned} \tag{3.39}$$

and

$$\left|\int_{t_1}^{t_2}\int_\Omega |v_t|^{q-1}v_t v\,dx\,dt\right| \le c_*\left(\frac{2(m+1)}{\beta(m-1)}\right)^{\frac{1}{2}}D_2^q(t)\sup_{t_1\le s\le t_2}E^{\frac{1}{2}}(s). \tag{3.40}$$

Therefore, applying (3.13)-(3.15), (3.27)-(3.31), (3.33)-(3.34), and (3.37)-(3.40) to (3.26) we obtain

$$\begin{aligned}\beta_2\int_{t_1}^{t_2} I(t)\,dt \le{}& 4c_*\beta_1\sqrt{4c_1(\Omega)D_1^2(t)+4c_2(\Omega)D_2^2(t)}\sup_{t_1\le s\le t_2}E^{\frac{1}{2}}(s)\\ &+c_1(\Omega)D_1^2(t)+c_2(\Omega)D_2^2(t)+\beta_1\bigl(D_1^{\frac{p+1}{2}}(t)+D_2^{\frac{q+1}{2}}(t)\bigr)\sup_{t_1\le s\le t_2}E^{\frac{1}{2}}(s)\\ &+2\bigl(D_1^{p+1}(t)+D_2^{q+1}(t)\bigr)+4\beta_1\sqrt{4D_1^{p+1}(t)+4D_2^{q+1}(t)}\sup_{t_1\le s\le t_2}E^{\frac{1}{2}}(s)\\ &+c_*\beta_1\bigl(D_1^p(t)+D_2^q(t)\bigr)\sup_{t_1\le s\le t_2}E^{\frac{1}{2}}(s),\end{aligned}\tag{3.41}$$

where

$$\beta_1:=\left(\frac{2(m+1)}{\beta(m-1)}\right)^{\frac{1}{2}},\qquad \beta_2:=1-\frac{5(\gamma+\delta-\beta)}{2\beta}(1+\eta_1).$$

Note that the assumption

$$\gamma+\delta>\frac{7+5\eta_1}{2}\max\left\{\int_0^\infty g_1(s)\,ds,\int_0^\infty g_2(s)\,ds\right\}$$

gives $\beta_2>0$. Thus,

$$\begin{aligned}\int_{t_1}^{t_2} I(t)\,dt \le{}& c_3\Bigl[\sqrt{4c_1(\Omega)D_1^2(t)+4c_2(\Omega)D_2^2(t)}\sup_{t_1\le s\le t_2}E^{\frac{1}{2}}(s)+D_1^2(t)+D_2^2(t)\\ &+\bigl(D_1^{\frac{p+1}{2}}(t)+D_2^{\frac{q+1}{2}}(t)\bigr)\sup_{t_1\le s\le t_2}E^{\frac{1}{2}}(s)+D_1^{p+1}(t)+D_2^{q+1}(t)\\ &+\sqrt{4D_1^{p+1}(t)+4D_2^{q+1}(t)}\sup_{t_1\le s\le t_2}E^{\frac{1}{2}}(s)\\ &+\bigl(D_1^p(t)+D_2^q(t)\bigr)\sup_{t_1\le s\le t_2}E^{\frac{1}{2}}(s)\Bigr],\end{aligned}\tag{3.42}$$

where

$$c_3:=\frac{\max\{4c_*\beta_1,c_1(\Omega),c_2(\Omega),4\beta_1\}}{\beta_2}.$$

On the other hand, from the definition of $J(t)$ and $I(t)$, (3.9), and (3.10) we have

$$\begin{aligned}J(t)={}&\frac{m-1}{2(m+1)}\biggl(\|\Delta u\|_2^2+\|\Delta v\|_2^2+\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2+\Bigl(\gamma-\int_0^t g_1(s)\,ds\Bigr)\|\nabla u\|_2^2\\ &+\Bigl(\delta-\int_0^t g_2(s)\,ds\Bigr)\|\nabla v\|_2^2+g_1\circ\nabla u+g_2\circ\nabla v\biggr)+\frac{1}{m+1}I(t)\\ ={}&\frac{m-1}{2(m+1)}\Bigl(I(t)+(m+1)\int_\Omega F(u,v)\,dx\Bigr)+\frac{1}{m+1}I(t)\\ \le{}&\Bigl(1+\frac{m-1}{2(m+1)}\eta_1\Bigr)I(t)\\ :={}&c_4I(t).\end{aligned}\tag{3.43}$$

Hence, integrating (3.3) over (t_1, t_2) and then using (3.43), (3.42), and (3.15), we deduce that

$$\begin{aligned}\int_{t_1}^{t_2} E(t)\,dt &= \frac{1}{2}\int_{t_1}^{t_2}\bigl(\|u_t\|_2^2 + \|v_t\|_2^2\bigr)\,dt + \int_{t_1}^{t_2} J(t)\,dt \\ &\le \frac{1}{2}\int_{t_1}^{t_2}\bigl(\|u_t\|_2^2 + \|v_t\|_2^2\bigr)\,dt + c_4\int_{t_1}^{t_2} I(t)\,dt \\ &\le c_5\Bigl[D_1^2(t) + D_2^2(t) + \sqrt{4c_1(\Omega)D_1^2(t) + 4c_2(\Omega)D_2^2(t)}\sup_{t_1\le s\le t_2} E^{\frac{1}{2}}(s) \\ &\quad + \bigl(D_1^{\frac{p+1}{2}}(t) + D_2^{\frac{q+1}{2}}(t)\bigr)\sup_{t_1\le s\le t_2} E^{\frac{1}{2}}(s) + D_1^{p+1}(t) + D_2^{q+1}(t) \\ &\quad + \sqrt{4D_1^{p+1}(t) + 4D_2^{q+1}(t)}\sup_{t_1\le s\le t_2} E^{\frac{1}{2}}(s) \\ &\quad + \bigl(D_1^p(t) + D_2^q(t)\bigr)\sup_{t_1\le s\le t_2} E^{\frac{1}{2}}(s)\Bigr], \end{aligned} \tag{3.44}$$

where $c_5 := c_3c_4$. By integrating (3.4) over $[t, t_2]$ we obtain

$$\begin{aligned} E(t) = E(t_2) + \int_t^{t_2}\Bigl(&\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 - \frac{1}{2}\bigl(g_1' \circ \nabla u\bigr) - \frac{1}{2}\bigl(g_2' \circ \nabla v\bigr) + \frac{1}{2}g_1(s)\|\nabla u\|_2^2 \\ &+ \frac{1}{2}g_2(s)\|\nabla v\|_2^2 + \|u_t\|_{p+1}^{p+1} + \|v_t\|_{q+1}^{q+1}\Bigr)\,ds. \end{aligned} \tag{3.45}$$

Since $t_2 - t_1 \ge \frac{1}{2}$ and $E(t)$ is nonincreasing in t, it follows that

$$\int_{t_1}^{t_2} E(t)\,dt \ge \int_{t_1}^{t_2} E(t_2)\,dt \ge \frac{1}{2}E(t_2).$$

Then, we have

$$\begin{aligned} E(t) &\le 2\int_{t_1}^{t_2} E(t)\,dt + \int_t^{t_2}\Bigl(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 \\ &\quad - \frac{1}{2}\bigl(g_1' \circ \nabla u\bigr) - \frac{1}{2}\bigl(g_2' \circ \nabla v\bigr) + \frac{1}{2}g_1(s)\|\nabla u\|_2^2 \\ &\quad + \frac{1}{2}g_2(s)\|\nabla v\|_2^2 + \|u_t\|_{p+1}^{p+1} + \|v_t\|_{q+1}^{q+1}\Bigr)\,ds \\ &\le 2\int_{t_1}^{t_2} E(t)\,dt + D_1^{p+1}(t) + D_2^{q+1}(t). \end{aligned} \tag{3.46}$$

Consequently, since $E(t)$ is nonincreasing, combining (3.44) with (3.46), we obtain

$$\begin{aligned} E(t) &\le c_6\Bigl(D_1^2(t) + D_2^2(t) + \sqrt{4c_1(\Omega)D_1^2(t) + 4c_2(\Omega)D_2^2(t)}E^{\frac{1}{2}}(t) \\ &\quad + \bigl(D_1^{\frac{p+1}{2}}(t) + D_2^{\frac{q+1}{2}}(t)\bigr)E^{\frac{1}{2}}(t) + D_1^{p+1}(t) + D_2^{q+1}(t) \\ &\quad + \sqrt{4D_1^{p+1}(t) + 4D_2^{q+1}(t)}E^{\frac{1}{2}}(t) + \bigl(D_1^p(t) + D_2^q(t)\bigr)E^{\frac{1}{2}}(t)\Bigr). \end{aligned}$$

Then a simple application of Young's inequality gives, for $t \geq 0$,

$$E(t) \leq c_7\big[D_1^2(t) + D_2^2(t) + D_1^{p+1}(t) + D_2^{q+1}(t) + D_1^{2p}(t) + D_2^{2q}(t)\big], \tag{3.47}$$

where c_6 and c_7 are positive constants.

Therefore, we have the following decay estimate.

(i) For $p = q = 1$, from (3.47) and (3.12) we get

$$E(t) \leq c_8\big[E(t) - E(t+1)\big],$$

where we choose $c_8 > 1$. Thus, by Lemma 2.6 we obtain

$$E(t) \leq E(0)e^{-\tau_1 t}, \quad t \geq 0,$$

where $\tau_1 := \ln \frac{c_8}{c_8 - 1}$.

(ii) For $\max\{p, q\} > 1$, it follows from (3.47) that, for $t \geq 0$,

$$E(t) \leq c_7\big[\big(1 + D_1^{p-1}(t) + D_1^{2p-2}(t)\big)D_1^2(t) + \big(1 + D_2^{q-1}(t) + D_2^{2q-2}(t)\big)D_2^2(t)\big].$$

Since $D_1(t) \leq E^{\frac{1}{p+1}}(t) \leq E^{\frac{1}{p+1}}(0)$ and $D_2(t) \leq E^{\frac{1}{q+1}}(t) \leq E^{\frac{1}{q+1}}(0)$ by (3.12) and (3.4), we have, for $t \geq 0$,

$$\begin{aligned} E(t) &\leq c_7\big[\big(1 + E^{\frac{p-1}{p+1}}(0) + E^{\frac{2p-2}{p+1}}(0)\big)D_1^2(t) + \big(1 + E^{\frac{q-1}{q+1}}(0) + E^{\frac{2q-2}{q+1}}(0)\big)D_2^2(t)\big] \\ &\leq c_9\big(D_1^2(t) + D_2^2(t)\big). \end{aligned}$$

Then, setting $\rho := \max\{\frac{p-1}{2}, \frac{q-1}{2}\}$, we obtain

$$\begin{aligned} E^{1+\rho}(t) &\leq \big[c_9\big(D_1^2(t) + D_2^2(t)\big)\big]^{1+\rho} \\ &\leq c_{10}\big(D_1^{2\rho+2}(t) + D_2^{2\rho+2}(t)\big) \\ &= c_{10}\big(D_1^{2\rho-p+1}(t)D_1^{p+1}(t) + D_2^{2\rho-q+1}(t)D_2^{q+1}(t)\big) \\ &\leq c_{10}\big(E^{\frac{2\rho-p+1}{p+1}}(0)D_1^{p+1}(t) + E^{\frac{2\rho-q+1}{q+1}}(0)D_2^{q+1}(t)\big) \\ &\leq c_{11}\big(D_1^{p+1}(t) + D_2^{q+1}(t)\big) \\ &= c_{11}\big(E(t) - E(t+1)\big), \end{aligned} \tag{3.48}$$

where $c_{10} := 2^\rho \cdot c_9^{1+\rho}$ and $c_{11} := c_{10} \max(E^{\frac{2\rho-p+1}{p+1}}(0), E^{\frac{2\rho-q+1}{q+1}}(0))$. Application of Lemma 2.6 to (3.48) yields

$$E(t) \leq \big(E^{-\rho}(0) + \tau_2\rho[t-1]^+\big)^{-\frac{1}{\rho}}, \quad t \geq 0,$$

with $\tau_2 := c_{11}^{-1}$.

The proof of Theorem 3.4 is completed. □

4 Blow-up result

In this section, we deal with the blow-up of solution to problem (1.1) with $\gamma = \delta = 1$. In order to state our result, we make an extra assumption on g_1 and g_2,

$$\max\left\{\int_0^\infty g_1(s)\,ds, \int_0^\infty g_2(s)\,ds\right\} < \min\left\{\frac{2(m-1)}{2m-1}, \frac{2(m+1)(E_1 - E(0))}{(2m-1)\lambda_1^2}\right\}, \tag{4.1}$$

where λ_1 and E_1 are given in (4.3) and (4.4), respectively.

Next, we define the functional G that helps in establishing desired results by

$$G(\chi) := \frac{1}{2}\chi^2 - \eta\chi^{m+1}, \quad \chi > 0, \tag{4.2}$$

where η is the constant from Lemma 2.3.

Remark 4.1

(i) We can verify that the functional G is increasing in $(0, \lambda_1)$, decreasing in $(\lambda_1, +\infty)$, $G(\lambda) \to -\infty$ as $\lambda \to +\infty$, and G has attains the maximum

$$E_1 := G(\lambda_1) = \frac{m-1}{2(m+1)}\lambda_1^2 \tag{4.3}$$

at

$$\lambda_1 := \left(\frac{1}{\eta(m+1)}\right)^{\frac{1}{m-1}}. \tag{4.4}$$

(ii) We observe from (3.3), Lemma 2.3, and (4.3) that

$$\begin{aligned} E(t) \ge J(t) &\ge \frac{1}{2}w^2(t) - \int_\Omega F(u,v)\,dx \\ &\ge \frac{1}{2}w^2(t) - \eta\bigl(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\bigr)^{\frac{m+1}{2}} \\ &\ge \frac{1}{2}w^2(t) - \eta w^{m+1}(t) = G\bigl(w(t)\bigr), \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} w(t) := \bigl(&\|\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 + l\|\nabla u\|_2^2 \\ &+ k\|\nabla v\|_2^2 + g_1 \circ \nabla u + g_2 \circ \nabla v\bigr)^{\frac{1}{2}}. \end{aligned} \tag{4.6}$$

Before we state and prove our main result, we need the following lemma, which is similar to a lemma from [17] to study some classes of the coupled equations.

Lemma 4.2 *Assume that* (A1) *and* (2.4) *hold*, $u_0, v_0 \in H_0^2(\Omega)$, *and* $u_1, v_1 \in H_0^1(\Omega)$. *Let* (u,v) *be a solution of* (1.1) *with initial data satisfying* $E(0) < E_1$ *and* $w(0) > \lambda_1$, *that is*,

$$\bigl(\|\Delta u_0\|_2^2 + \|\Delta v_0\|_2^2 + \|\nabla u_1\|_2^2 + \|\nabla v_1\|_2^2 + l\|\nabla u_0\|_2^2 + k\|\nabla v_0\|_2^2\bigr)^{\frac{1}{2}} > \lambda_1. \tag{4.7}$$

Then there exists $\lambda_2 > \lambda_1$ *such that, for all* $t \geq 0$,

$$w(t) \geq \lambda_2. \tag{4.8}$$

Proof The proof is similar to that of Lemma 4.2 in [17]. □

Theorem 4.3 *Suppose that* (A1), (2.4), *and* (4.1) *hold*, $u_0, v_0 \in H_0^2(\Omega)$, *and* $u_1, v_1 \in H_0^1(\Omega)$. *Assume further that* $m > \max(p, q)$ *and* $I(0) < 0$. *Suppose that one of the following is satisfied*:

(i) $E(0) < 0$,

(ii) $0 \leq E(0) < E_1$ *and* $w(0) > \lambda_1$.

Then the solution of problem (1.1) *blows up at a finite time, that is, there exists* $T < +\infty$ *such that*

$$\begin{aligned} \lim_{t\to T^-} \bigl(&\|u_t\|_2^2 + \|v_t\|_2^2 + \|\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \\ &+ \|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1} \bigr) = +\infty. \end{aligned} \tag{4.9}$$

Proof For case (ii), $0 \leq E(0) < E_1$, set

$$H(t) := E_2 - E(t), \quad t \geq 0, \tag{4.10}$$

where $E_2 := \frac{E_1 + E(0)}{2}$. By (3.4) we see that $H'(t) \geq 0$. Thus, we obtain

$$H(t) \geq H(0) = E_2 - E(0), \quad t \geq 0. \tag{4.11}$$

Moreover, from (4.5), (4.8), and (4.3) we see that

$$\begin{aligned} H(t) &= E_2 - E(t) \\ &\leq E_1 - \frac{1}{2} w^2(t) + \int_\Omega F(u, v)\,dx \\ &\leq E_1 - \frac{1}{2} \lambda_1^2 + \int_\Omega F(u, v)\,dx \\ &= -\frac{\lambda_1^2}{m+1} + \int_\Omega F(u, v)\,dx. \end{aligned} \tag{4.12}$$

Then, by (4.11), (4.12), and Lemma 2.2 we have

$$0 < H(0) \leq H(t) \leq \int_\Omega F(u, v)\,dx \leq c_1 \bigl(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1} \bigr). \tag{4.13}$$

Let

$$\begin{aligned} A(t) := {} & H^{1-\sigma}(t) + \epsilon \int_\Omega (u u_t + v v_t)\,dx + \frac{\epsilon}{2} \int_\Omega \bigl(|\nabla u|^2 + |\nabla v|^2 \bigr)\,dx \\ &+ \epsilon \int_\Omega (\nabla u \cdot \nabla u_t + \nabla v \cdot \nabla v_t)\,dx, \end{aligned} \tag{4.14}$$

where ϵ and σ are positive constants to be specified latter. By taking the derivative of (4.14) and using Eq. (1.1) with $\gamma = \delta = 1$ we get

$$\begin{aligned}
A'(t) = {} & (1-\sigma)H^{-\sigma}(t)H'(t) + \epsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \epsilon\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) \\
& + \epsilon\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right) - \epsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\
& + \epsilon \int_\Omega \int_0^t g_1(t-s)\nabla u(s)\cdot\nabla u(t)\,ds\,dx \\
& + \epsilon \int_\Omega \int_0^t g_2(t-s)\nabla v(s)\cdot\nabla v(t)\,ds\,dx - \epsilon\int_\Omega \left(u|u_t|^{p-1}u_t + v|v_t|^{q-1}v_t\right)dx \\
& + (m+1)\epsilon\int_\Omega F(u,v)\,dx.
\end{aligned} \tag{4.15}$$

Using the Hölder and Young inequalities, we observe that

$$\begin{aligned}
& \int_\Omega \int_0^t g_1(t-s)\nabla u(s)\cdot\nabla u(t)\,ds\,dx \\
& \quad = \int_\Omega \int_0^t g_1(t-s)\nabla u(t)\cdot\left(\nabla u(s) - \nabla u(t)\right)ds\,dx + \int_0^t g_1(t-s)\,ds\left\|\nabla u(t)\right\|_2^2 \\
& \quad \geq -(g_1 \circ \nabla u) + \frac{3}{4}\int_0^t g_1(s)\,ds\left\|\nabla u(t)\right\|_2^2
\end{aligned} \tag{4.16}$$

and

$$\int_\Omega \int_0^t g_2(t-s)\nabla v(s)\cdot\nabla v(t)\,ds\,dx \geq -(g_2\circ\nabla v) + \frac{3}{4}\int_0^t g_2(s)\,ds\left\|\nabla v(t)\right\|_2^2. \tag{4.17}$$

Taking (4.16) and (4.17) into account, using (4.10) and the definition of $E(t)$ by (3.3) to substitute for $\int_\Omega F(u,v)\,dx$, (4.15) becomes

$$\begin{aligned}
A'(t) \geq {} & (1-\sigma)H^{-\sigma}(t)H'(t) + \epsilon a_1\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \epsilon a_1\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right) \\
& + \epsilon a_2\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) + \epsilon a_2(g_1\circ\nabla u + g_2\circ\nabla v) + \epsilon a_3\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\
& - \epsilon\int_\Omega \left(u|u_t|^{p-1}u_t + v|v_t|^{q-1}v_t\right)dx + (m+1)\epsilon H(t) - (m+1)\epsilon E_2,
\end{aligned}$$

where

$$\begin{aligned}
& a_1 = \frac{m+3}{2}, \qquad a_2 = \frac{m-1}{2}, \\
& a_3 = \frac{m-1}{2} - \frac{2m-1}{4}\max\left\{\int_0^\infty g_1(s)\,ds, \int_0^\infty g_2(s)\,ds\right\}.
\end{aligned}$$

By (4.1) we observe that $a_3 > 0$, and then by the definition of $w(t)$ by (4.6) we have

$$\begin{aligned}
A'(t) \geq {} & (1-\sigma)H^{-\sigma}(t)H'(t) + \epsilon a_1\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \epsilon a_3\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) \\
& + \epsilon a_3\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right) + \epsilon a_3\left(l\|\nabla u\|_2^2 + k\|\nabla v\|_2^2\right) + \epsilon a_3(g_1\circ\nabla u + g_2\circ\nabla v) \\
& - \epsilon\int_\Omega \left(u|u_t|^{p-1}u_t + v|v_t|^{q-1}v_t\right)dx + (m+1)\epsilon H(t) - (m+1)\epsilon E_2
\end{aligned}$$

$$
\begin{aligned}
&= (1-\sigma)H^{-\sigma}(t)H'(t) + \epsilon a_1\big(\|u_t\|_2^2 + \|v_t\|_2^2\big) + \epsilon a_3 w^2(t) \\
&\quad - \epsilon \int_\Omega \big(u|u_t|^{p-1}u_t + v|v_t|^{q-1}v_t\big)\,dx \\
&\quad + (m+1)\epsilon H(t) - (m+1)\epsilon E_2.
\end{aligned}
$$

Since $w(t) \geq \lambda_2$ by (4.8) and $\lambda_2 > \lambda_1$ by Lemma 4.2, we note that

$$
\begin{aligned}
a_3 w^2(t) - (m+1)E_2 &= a_3 \frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2} w^2(t) + a_3\lambda_1^2 \frac{w^2(t)}{\lambda_2^2} - (m+1)E_2 \\
&\geq c_2 w^2(t) + c_3,
\end{aligned}
$$

where $c_2 = a_3\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2} > 0$ and $c_3 = a_3\lambda_1^2 - (m+1)E_2$.

Furthermore, by the definition of E_1, $E_2 = \frac{E_1+E(0)}{2}$ and assumption (4.1) we see that

$$
\begin{aligned}
c_3 &= a_3\lambda_1^2 - (m+1)E_2 \\
&= \left(\frac{m-1}{2} - \frac{2m-1}{4}\max\left\{\int_0^\infty g_1(s)\,ds, \int_0^\infty g_2(s)\,ds\right\}\right)\lambda_1^2 - (m+1)E_2 \\
&= \frac{(m+1)(E_1 - E(0))}{2} - \frac{(2m-1)\lambda_1^2}{4}\max\left\{\int_0^\infty g_1(s)\,ds, \int_0^\infty g_2(s)\,ds\right\} > 0.
\end{aligned}
$$

Therefore, based on the above arguments, we conclude that

$$
\begin{aligned}
A'(t) &\geq (1-\sigma)H^{-\sigma}(t)H'(t) + \epsilon a_1\big(\|u_t\|_2^2 + \|v_t\|_2^2\big) + \epsilon c_2 w^2(t) \\
&\quad - \epsilon \int_\Omega \big(u|u_t|^{p-1}u_t + v|v_t|^{q-1}v_t\big)\,dx + (m+1)\epsilon H(t).
\end{aligned}
\tag{4.18}
$$

To proceed further, by the Hölder and Young inequalities we have

$$
\left|\int_\Omega |u_t|^{p-1}u_t u\,dx\right| \leq \frac{\delta_1^{p+1}}{p+1}\|u\|_{p+1}^{p+1} + \frac{p\delta_1^{-\frac{p+1}{p}}}{p+1}\|u_t\|_{p+1}^{p+1}
$$

and

$$
\left|\int_\Omega |v_t|^{q-1}v_t v\,dx\right| \leq \frac{\delta_2^{q+1}}{q+1}\|v\|_{q+1}^{q+1} + \frac{q\delta_2^{-\frac{q+1}{q}}}{q+1}\|v_t\|_{q+1}^{q+1},
$$

where δ_1 and δ_2 are positive constants depending on t and will be specified later.

Then, inserting the last two inequalities into (4.18), we obtain

$$
\begin{aligned}
A'(t) &\geq (1-\sigma)H^{-\sigma}(t)H'(t) + \epsilon a_1\big(\|u_t\|_2^2 + \|v_t\|_2^2\big) + \epsilon c_2 w^2(t) \\
&\quad - \epsilon\left(\frac{\delta_1^{p+1}}{p+1}\|u\|_{p+1}^{p+1} + \frac{p\delta_1^{-\frac{p+1}{p}}}{p+1}\|u_t\|_{p+1}^{p+1} + \frac{\delta_2^{q+1}}{q+1}\|v\|_{q+1}^{q+1} + \frac{q\delta_2^{-\frac{q+1}{q}}}{q+1}\|v_t\|_{q+1}^{q+1}\right) \\
&\quad + (m+1)\epsilon H(t).
\end{aligned}
$$

At this point, choosing δ_1 and δ_2 such that

$$\delta_1^{-\frac{p+1}{p}} = M_1 H^{-\sigma}(t), \qquad \delta_2^{-\frac{q+1}{q}} = M_2 H^{-\sigma}(t),$$

we get that

$$\begin{aligned} A'(t) &\geq (1-\sigma - M\epsilon)H^{-\sigma}(t)H'(t) + \epsilon a_1\big(\|u_t\|_2^2 + \|v_t\|_2^2\big) + \epsilon c_2 w^2(t) \\ &\quad - \epsilon M_1^{-p} H^{\sigma p}(t)\|u\|_{p+1}^{p+1} - \epsilon M_2^{-q} H^{\sigma q}(t)\|v\|_{q+1}^{q+1} + (m+1)\epsilon H(t), \end{aligned} \tag{4.19}$$

where M_1, M_2 are positive constants, and $M = \frac{pM_1}{p+1} + \frac{qM_2}{q+1}$. It follows from (4.13) that

$$M_1^{-p} H^{\sigma p}(t) \leq M_1^{-p} c_1^{\sigma p}\big(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\big)^{\sigma p}$$

and

$$M_2^{-q} H^{\sigma q}(t) \leq M_2^{-q} c_1^{\sigma q}\big(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\big)^{\sigma q}.$$

Substitution of these two inequalities into (4.19) yields

$$\begin{aligned} A'(t) &\geq (1-\sigma - M\epsilon)H^{-\sigma}(t)H'(t) + \epsilon a_1\big(\|u_t\|_2^2 + \|v_t\|_2^2\big) \\ &\quad + \epsilon c_2 w^2(t) - \epsilon M_1^{-p} c_1^{\sigma p}\big(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\big)^{\sigma p}\|u\|_{p+1}^{p+1} \\ &\quad - \epsilon M_2^{-q} c_1^{\sigma q}\big(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\big)^{\sigma q}\|v\|_{q+1}^{q+1} + (m+1)\epsilon H(t). \end{aligned}$$

Since $p < m$ and $q < m$, we note that

$$\begin{aligned} \|u\|_{p+1}^{p+1} &\leq c_4\|u\|_{m+1}^{p+1} \leq c_4(\|u\|_{m+1} + \|v\|_{m+1})^{p+1}, \\ \|v\|_{q+1}^{q+1} &\leq c_5\|v\|_{m+1}^{q+1} \leq c_5(\|u\|_{m+1} + \|v\|_{m+1})^{q+1}, \end{aligned}$$

where $c_4 = \operatorname{vol}(\Omega)^{\frac{m-p}{m+1}}$ and $c_5 = \operatorname{vol}(\Omega)^{\frac{m-q}{m+1}}$. Thus,

$$\begin{aligned} A'(t) &\geq (1-\sigma - M\epsilon)H^{-\sigma}(t)H'(t) + \epsilon a_1\big(\|u_t\|_2^2 + \|v_t\|_2^2\big) + \epsilon c_2 w^2(t) \\ &\quad - \epsilon M_1^{-p} c_1^{\sigma p} c_4\big(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\big)^{\sigma p}(\|u\|_{m+1} + \|v\|_{m+1})^{p+1} \\ &\quad - \epsilon M_2^{-q} c_1^{\sigma q} c_5\big(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\big)^{\sigma q}(\|u\|_{m+1} + \|v\|_{m+1})^{q+1} + (m+1)\epsilon H(t) \\ &\geq (1-\sigma - M\epsilon)H^{-\sigma}(t)H'(t) + \epsilon a_1\big(\|u_t\|_2^2 + \|v_t\|_2^2\big) + \epsilon c_2 w^2(t) \\ &\quad - \epsilon M_1^{-p} c_1^{\sigma p} c_7(\|u\|_{m+1} + \|v\|_{m+1})^{\sigma p(m+1)+p+1} + (m+1)\epsilon H(t) \\ &\quad - \epsilon M_2^{-q} c_1^{\sigma q} c_8(\|u\|_{m+1} + \|v\|_{m+1})^{\sigma q(m+1)+q+1}, \end{aligned} \tag{4.20}$$

where the last inequality is derived from

$$\begin{aligned} \big(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\big)^{\sigma p} &\leq c_6(\|u\|_{m+1} + \|v\|_{m+1})^{\sigma p(m+1)}, \\ \big(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\big)^{\sigma q} &\leq c_6(\|u\|_{m+1} + \|v\|_{m+1})^{\sigma q(m+1)} \end{aligned}$$

because of

$$(x+y)^{\lambda} \le c_6\big(x^{\lambda}+y^{\lambda}\big), \quad x,y \ge 0, \lambda > 0, c_6 > 0, \tag{4.21}$$

and the constants $c_7 = c_6c_4$ and $c_8 = c_6c_5$.

Now, letting

$$0 < \sigma < \min\left\{\frac{m-p}{p(m+1)}, \frac{m-q}{q(m+1)}, \frac{m-1}{2(m+1)}\right\}, \tag{4.22}$$

we have

$$2 \le \sigma p(m+1)+p+1 \le m+1, \qquad 2 \le \sigma q(m+1)+q+1 \le m+1.$$

Hence, by the inequality

$$\|v\|_{m+1}^{s} \le c\big(\mathrm{vol}(\Omega), m\big)\big(\|\nabla v\|_2^2 + \|v\|_{m+1}^{m+1}\big), \quad v \in H_0^1(\Omega), 2 \le s \le m+1, \tag{4.23}$$

we have

$$\|u\|_{m+1}^{\sigma p(m+1)+p+1} \le c_9\big(\|\nabla u\|_2^2 + \|u\|_{m+1}^{m+1}\big) \tag{4.24}$$

and

$$\|v\|_{m+1}^{\sigma q(m+1)+q+1} \le c_{10}\big(\|\nabla v\|_2^2 + \|v\|_{m+1}^{m+1}\big), \tag{4.25}$$

where c_9 and c_{10} are positive constants. Taking (4.24) and (4.25) into consideration and using the definition of $w(t)$, (4.20) takes the form

$$\begin{aligned} A'(t) \ge{}& (1-\sigma-M\epsilon)H^{-\sigma}(t)H'(t) + \epsilon a_1\big(\|u_t\|_2^2 + \|v_t\|_2^2\big) + \epsilon c_2(g_1 \circ \nabla u + g_2 \circ \nabla v) \\ &+ \epsilon\big(c_2\beta - M_1^{-p}c_1^{\sigma p}c_{11} - M_2^{-q}c_1^{\sigma q}c_{12}\big)\big(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\big) + (m+1)\epsilon H(t) \\ &+ \epsilon c_2\big(\|\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\big) \\ &- \epsilon\big(M_1^{-p}c_1^{\sigma p}c_{11} + M_2^{-q}c_1^{\sigma q}c_{12}\big)\big(\|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\big), \end{aligned} \tag{4.26}$$

where $c_{11} = c_6 \cdot c_7 \cdot c_9$ and $c_{12} = c_6 \cdot c_8 \cdot c_{10}$.

At this moment, setting $a_4 = \min\{c_2\beta, \frac{m+1}{2}\}$, decomposing $\epsilon(m+1)H(t)$ in (4.26) by

$$\epsilon(m+1)H(t) = 2a_4\epsilon H(t) + (m+1-2a_4)\epsilon H(t),$$

and using (4.10) and Lemma 2.2, we obtain

$$\begin{aligned} A'(t) \ge{}& (1-\sigma-M\epsilon)H^{-\sigma}(t)H'(t) + \epsilon(a_1-a_4)\big(\|u_t\|_2^2 + \|v_t\|_2^2\big) \\ &+ \epsilon(c_2-a_4)\big(\|\Delta u\|_2^2 + \|\Delta v\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\big) \\ &+ \epsilon\big(c_2\beta - M_1^{-p}c_1^{\sigma p}c_{11} - M_2^{-q}c_1^{\sigma q}c_{12} - a_4\big)\big(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\big) \end{aligned}$$

$$+\epsilon\bigl(2a_4c_0-\bigl(M_1^{-p}c_1^{\sigma p}c_{11}+M_2^{-q}c_1^{\sigma q}c_{12}\bigr)\bigr)\bigl(\|u\|_{m+1}^{m+1}+\|v\|_{m+1}^{m+1}\bigr)$$
$$+\epsilon(c_2-a_4)(g_1\circ\nabla u+g_2\circ\nabla v)+(m+1-2a_4)\epsilon H(t).$$

Choosing M_1 and M_2 large enough such that

$$c_2\beta-M_1^{-p}c_1^{\sigma p}c_{11}-M_2^{-q}c_1^{\sigma q}c_{12}-a_4>\frac{c_2\beta-a_4}{2},$$
$$2a_4c_0-\bigl(M_1^{-p}c_1^{\sigma p}c_{11}+M_2^{-q}c_1^{\sigma q}c_{12}\bigr)>a_4c_0,$$

we get

$$\begin{aligned}A'(t)&\geq(1-\sigma-M\epsilon)H^{-\sigma}(t)H'(t)+\epsilon c_{13}\bigl(\|u_t\|_2^2+\|v_t\|_2^2\bigr)+\epsilon c_{14}(g_1\circ\nabla u+g_2\circ\nabla v)\\&\quad+\epsilon c_{14}\bigl(\|\Delta u\|_2^2+\|\Delta v\|_2^2+\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\bigr)+\epsilon c_{15}H(t)\\&\quad+\epsilon c_{16}\bigl(\|\nabla u\|_2^2+\|\nabla v\|_2^2\bigr)+\epsilon c_{17}\bigl(\|u\|_{m+1}^{m+1}+\|v\|_{m+1}^{m+1}\bigr)\end{aligned}$$

for some positive constants c_i, $i=13,14,\dots,17$. Once M_1 and M_2 are fixed, we pick $\epsilon>0$ small enough such that

$$1-\sigma-M\epsilon\geq0$$

and

$$\begin{aligned}A(0)&=H^{1-\sigma}(0)+\epsilon\int_\Omega(u_0u_1+v_0v_1)\,dx+\frac{\epsilon}{2}\bigl(\|\nabla u_0\|_2^2+\|\nabla v\|_2^2\bigr)\\&\quad+\epsilon\int_\Omega(\nabla u_0\cdot\nabla u_1+\nabla v_0\cdot\nabla v_1)\,dx>0.\end{aligned}\tag{4.27}$$

Thus, there exists $K>0$ such that

$$\begin{aligned}A'(t)&\geq\epsilon K\bigl(\|u_t\|_2^2+\|v_t\|_2^2+\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2+\|\nabla u\|_2^2\\&\quad+\|\nabla v\|_2^2+\|u\|_{m+1}^{m+1}+\|v\|_{m+1}^{m+1}+H(t)\bigr),\end{aligned}\tag{4.28}$$

which, together with (4.27), implies that

$$A(t)\geq A(0)>0,\quad t\geq0.$$

On the other hand, by the Hölder and Young inequalities, (4.22), and (4.23) we have that

$$\begin{aligned}&\left(\int_\Omega(u_tu+v_tv)\,dx\right)^{\frac{1}{1-\sigma}}\\&\quad\leq2^{\frac{\sigma}{1-\sigma}}\bigl(\|u_t\|_2^{\frac{1}{1-\sigma}}\|u\|_2^{\frac{1}{1-\sigma}}+\|v_t\|_2^{\frac{1}{1-\sigma}}\|v\|_2^{\frac{1}{1-\sigma}}\bigr)\\&\quad\leq c_{18}\bigl(\|u_t\|_2^{\frac{1}{1-\sigma}}\|u\|_{m+1}^{\frac{1}{1-\sigma}}+\|v_t\|_2^{\frac{1}{1-\sigma}}\|v\|_{m+1}^{\frac{1}{1-\sigma}}\bigr)\\&\quad\leq c_{19}\bigl(\|u_t\|_2^2+\|v_t\|_2^2+\|u\|_{m+1}^{\frac{2}{1-2\sigma}}+\|v\|_{m+1}^{\frac{2}{1-2\sigma}}\bigr)\\&\quad\leq c_{20}\bigl(\|u_t\|_2^2+\|v_t\|_2^2+\|\nabla u\|_2^2+\|\nabla v\|_2^2+\|u\|_{m+1}^{m+1}+\|v\|_{m+1}^{m+1}\bigr),\end{aligned}\tag{4.29}$$

and, similarly,

$$\begin{aligned}
&\left(\int_\Omega (\nabla u_t \cdot \nabla u + \nabla v_t \cdot \nabla v)\,dx\right)^{\frac{1}{1-\sigma}} \\
&\quad \le 2^{\frac{\sigma}{1-\sigma}} \left(\|\nabla u_t\|_2^{\frac{1}{1-\sigma}} \|\nabla u\|_2^{\frac{1}{1-\sigma}} + \|\nabla v_t\|_2^{\frac{1}{1-\sigma}} \|\nabla v\|_2^{\frac{1}{1-\sigma}}\right) \\
&\quad \le c_{21}\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 + \|\nabla u\|_2^{\frac{2}{1-2\sigma}} + \|\nabla v\|_2^{\frac{2}{1-2\sigma}}\right).
\end{aligned} \tag{4.30}$$

By using (4.9) we get

$$\|\nabla u\|_2^{\frac{2}{1-2\sigma}} \le c^{\frac{1}{1-2\sigma}} \le \frac{c^{\frac{1}{1-2\sigma}}}{H(0)} H(t) \tag{4.31}$$

and

$$\|\nabla v\|_2^{\frac{2}{1-2\sigma}} \le c^{\frac{1}{1-2\sigma}} \le \frac{c^{\frac{1}{1-2\sigma}}}{H(0)} H(t). \tag{4.32}$$

Substitution of these two inequalities into (4.30) yields

$$\left(\int_\Omega (\nabla u_t \cdot \nabla u + \nabla v_t \cdot \nabla v)\,dx\right)^{\frac{1}{1-\sigma}} \le c_{22}\left(H(t) + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right). \tag{4.33}$$

Similarly, we obtain

$$\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\frac{1}{1-\sigma}} \le 2^{\frac{\sigma}{1-\sigma}} \left(\|\nabla u\|_2^{\frac{2}{1-\sigma}} + \|\nabla v\|_2^{\frac{2}{1-\sigma}}\right) \le c_{23} H(t). \tag{4.34}$$

By using (4.29), (4.33)-(4.34), and (4.13) we get, for $t \ge 0$,

$$\begin{aligned}
A^{\frac{1}{1-\sigma}}(t) &\le 2^{\frac{\sigma}{1-\sigma}} \left(H(t) + \left(\int_\Omega (uu_t + vv_t)\,dx\right)^{\frac{1}{1-\sigma}} + \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\frac{1}{1-\sigma}} \right. \\
&\qquad \left. + \left(\int_\Omega (\nabla u \cdot \nabla u_t + \nabla v \cdot \nabla v_t)\,dx\right)^{\frac{1}{1-\sigma}}\right) \\
&\le c_{24}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right. \\
&\qquad \left. + \|u\|_{m+1}^{m+1} + \|v\|_{m+1}^{m+1}\right),
\end{aligned} \tag{4.35}$$

where c_i, $i = 18, 19, \dots, 24$, are positive constants. Combining (4.28) and (4.35), we get

$$A'(t) \ge c_{25} A^{\frac{1}{1-\sigma}}(t), \quad t \ge 0, \tag{4.36}$$

where $c_{25} = \frac{\epsilon K}{c_{24}}$. Integration of (4.36) over $(0, t)$ then yields

$$A^{\frac{\sigma}{1-\sigma}}(t) \ge \frac{1}{A^{\frac{-\sigma}{1-\sigma}}(0) - \frac{\sigma c_{25}}{1-\sigma} t}, \quad t \ge 0. \tag{4.37}$$

This shows that $A(t)$ blows up in finite time T and

$$T \le \frac{1-\sigma}{\sigma c_{25} A^{\frac{\sigma}{1-\sigma}}(0)}. \tag{4.38}$$

Furthermore, we get from (4.35) that

$$\lim_{t\to T^-}\left(\|u_t\|_2^2+\|v_t\|_2^2++\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2+\|\nabla u\|_2^2+\|\nabla v\|_2^2+\|u\|_{m+1}^{m+1}+\|v\|_{m+1}^{m+1}\right)=+\infty.$$

Thus, the solution of problem (1.1) blows up in finite time.

For case (i), $E(0)<0$, we set $H(t)=-E(t)$ instead of (4.10). Then, applying the same arguments as in case (ii), we have the desired result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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References

1. Di, HF, Shang, YD: Global existence and asymptotic behavior of solutions for the double dispersive-dissipative wave equation with nonlinear damping and source terms. Bound. Value Probl. **2015**, 29 (2015)
2. Shang, YD: Initial boundary value problem of equation $u_{tt}-\Delta u-\Delta u_t-\Delta u_{tt}=f(u)$. Acta Math. Appl. Sin. **23**, 385-393 (2000)
3. Zhang, HW, Hu, QY: Existence of global weak solution and stability of a class nonlinear evolution equation. Acta Math. Sci. Ser. A **24**, 329-336 (2004)
4. Xie, YQ, Zhong, CK: The existence of global attractors for a class nonlinear equation. Anal. Appl. **336**, 54-69 (2007)
5. Xu, RZ, Zhao, XR, Shen, JH: Asymptotic behavior of solution for fourth order wave equation with dispersive and dissipative terms. Appl. Math. Mech. **29**, 259-262 (2008)
6. Kafini, M, Messaoudi, SA: A blow-up result in a nonlinear viscoelastic problem with arbitrary positive initial energy. Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal. **20**(6), 657-665 (2013)
7. Xie, YQ, Zhong, CK: Asymptotic behavior of a class of nonlinear evolution equation. Nonlinear Anal. **71**(11), 5095-5105 (2009)
8. Carvalho, AN, Cholewa, JW: Local well posedness, asymptotic behavior and asymptotic bootstrapping for a class of semilinear evolution equation equations of the second order in time. Trans. Am. Math. Soc. **361**(5), 2567-2586 (2009)
9. Sun, CY, Yang, L, Duan, JQ: Asymptotic behavior for a semilinear second order evolution equation. Trans. Am. Math. Soc. **363**(11), 6085-6109 (2011)
10. Chen, CS, Wang, H, Zhu, SL: Global attractor and decay estimates of solutions to a class of nonlinear evolution equations. Math. Methods Appl. Sci. **34**(5), 497-508 (2011)
11. Xu, RZ, Yang, YB: Finite time blow up for the nonlinear fourth-order dispersive-dissipative wave equation at high energy level. Int. J. Math. **23**, 1250060 (2012)
12. Messaoudi, SA, Mukiawa, SE: Existence and general decay of a viscoelastic plate equation. Electron. J. Differ. Equ. **2016**, 22 (2016)
13. Liang, F, Gao, HJ: Exponential energy decay and blow-up of solutions for a system of nonlinear viscoelastic wave equations with strong damping. Bound. Value Probl. **2011**, 22 (2011)
14. Han, XS, Wang, MX: Global existence and blow-up of solutions for a system of nonlinear viscoelastic wave equations with damping and source. Nonlinear Anal. TMA **71**, 5427-5450 (2009)
15. Messaoudi, SA, Said-Houari, B: Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms. J. Math. Anal. Appl. **365**, 277-287 (2010)
16. Said-Houari, B, Messaoudi, SA, Guesmia, A: General decay of solutions of a nonlinear system of viscoelastic wave equations. Nonlinear Differ. Equ. Appl. **18**, 659-684 (2011)
17. Wu, ST: On decay and blow-up of solutions for a system of nonlinear wave equations. J. Math. Anal. Appl. **394**, 360-377 (2012)
18. Nakao, M: Asymptotic stability of the bounded or almost periodic solution of the wave equation with nonlinear dissipative term. J. Math. Anal. Appl. **58**(2), 336-343 (1977)