Lu et al. *Boundary Value Problems* (2016) 2016:96
DOI 10.1186/s13661-016-0605-8

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Periodic solutions for p-Laplacian Rayleigh equations with singularities

Shipin Lu*, Tao Zhong and Lijuan Chen

*Correspondence:
lushiping88@sohu.com
College of Mathematics & Statistics, Nanjing University of Information Science & Technology, Nanjing, 210044, China

Abstract
In this paper, the problem of existence of periodic solutions is studied for p-Laplacian Rayleigh equations with a singularity at $x = 0$. By using the topological degree theory, some new results are obtained.

Keywords: Rayleigh equation; topological degree; singularity; periodic solution

1 Introduction

In recent years, the periodic problem for some types of singular equations has attracted much attention of many researchers because the singular nonlinearity possesses a significant role in many practical situations. For example, the differential equation

$$x''(t) + cx'(t) - \frac{1}{x(t)} = e(t)$$

described the motion of a piston in a cylinder closed at one extremity. The singular term $-\frac{1}{x}$ in the equation models the restoring force which is caused by a compressed perfect gas (see [1] and the references therein). The interest in studying the equations with a singularity began with some work of Forbat and Huaux [2]. Later, the interest in such problem was renewed by Gordon in [3, 4], and Lazer and Solimini in [5]. For the recent developments on the study of this problem, here, we refer the reader to [6–16], and we notice that the equations studied previously were either of the type of Duffing equations [7–11, 14–16] or of the type of Liénard equations [11, 12, 17, 18]. For example, Jebelean and Mawhin in [1] considered the problem of the existence of positive periodic solutions for the following p-Laplacian Liénard equations with a singularity:

$$\left(|x'|^{p-2}x'\right)' + f(x)x' + g(x) = h(t) \tag{1.1}$$

and

$$\left(|x'|^{p-2}x'\right)' + f(x)x' - g(x) = h(t), \tag{1.2}$$

where $p > 1$ is a constant, $f : [0, +\infty) \to \mathbb{R}$ is an arbitrary continuous function, $h : \mathbb{R} \to \mathbb{R}$ is a T-periodic function with $h \in L^\infty[0, T]$, $g : (0, +\infty) \to (0, +\infty)$ is continuous, and singular at $x = 0$, this means that $g(x)$ is unbounded as $x \to 0^+$. The crucial condition imposed on

$g(x)$ is that $g(x) \to +\infty$ as $x \to 0^+$, *i.e.*, equation (1.1) is of attractive type, and equation (1.2) is of repulsive type. Zhang in [17] studied the problem of periodic solutions of the Liénard equation with a repulsive singularity at $x = 0$,

$$x'' + f(x)x' + g(t, x) = 0. \tag{1.3}$$

In [18], Wang further studied the existence of positive periodic solutions for a delay Liénard equation with a repulsive singularity at $x = 0$,

$$x'' + f(x)x' + g\big(t, x(t-\tau)\big) = 0. \tag{1.4}$$

The methods in [1] for equation (1.2), in [17] for equation (1.3) and in [18] for equation (1.4) were all based on topological degree theory, and the upper and lower solutions techniques was used in [1] for equation (1.1). But as far as we are aware of, few results appeared on the existence of periodic solutions for p-Laplacian Rayleigh equation with a singularity.

Motivated by this, in this paper, we study the existence of positive T-periodic solutions for p-Laplacian Rayleigh equation with a singularity of the form

$$\big(|x'|^{p-2}x'\big)' + f\big(x'\big) - g_1(x) + g_2(x) = h(t) \tag{1.5}$$

and

$$\big(|x'|^{p-2}x'\big)' + f\big(x'\big) + g_1(x) - g_2(x) = h(t), \tag{1.6}$$

where $p > 1$ is a constant, $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary continuous function, $g_1, g_2 : (0, +\infty) \to \mathbb{R}$ are all continuous and $g_1(x)$ is unbounded as $x \to 0^+$, $h : \mathbb{R} \to \mathbb{R}$ is a T-periodic continuous function. Clearly, equation (1.5) and equation (1.6) are all singular at $x = 0$. By using Manásevich-Mawhin's continuation theorem, some new results are obtained.

The interesting thing is that the singular term in equation (1.6) (or in equation (1.5)) is not required to have $g_1(x) \to +\infty$ (or $-g_1(x) \to -\infty$) as $x \to 0^+$. For example, let

$$g_1(u) = \frac{1}{u^\mu}\left|\sin\frac{1}{u}\right|,$$

where $\mu \geq 1$ is a constant. It is easy to verify that $g_1(x)$ does not approach $+\infty$ as $x \to 0^+$. Furthermore, if $x \in C^1(\mathbb{R}, \mathbb{R})$ with T-periodic, then the first order derivative term $f(x)x'$ in equations (1.1)-(1.4) satisfies $\int_0^T f(x(t))x'(t)\,dt = 0$, which is crucial for obtaining an *a priori bounds* of all the possible T-periodic solutions for equation (1.1)-equation (1.4). However, the first order derivative term in equation (1.5) and equation (1.6) is $f(x')$; generally, $\int_0^T f(x'(t))\,dt = 0$ does not hold. This means that the method for estimating an *a priori bounds* of all the possible T-periodic solutions to equation (1.5) and equation (1.6) is different from the corresponding ones in [1, 17, 18].

2 Preliminary lemmas

The following two lemmas (Lemma 2.1 and Lemma 2.2) are all consequences of Theorem 3.1 in [19].

Lemma 2.1 *Assume that there exist constants* $0 < \eta_0 < \eta_1$, $M_2 > 0$, *such that the following conditions hold*.

(1) *For each* $\lambda \in (0,1]$, *each possible positive* T*-periodic solution* x *to the equation*

$$\left(\left|u'\right|^{p-2}u'\right)' + \lambda f\left(u'\right) - \lambda g_1(u) + \lambda g_2(t,u) = \lambda h(t)$$

satisfies the inequalities $\eta_0 < x(t) < \eta_1$ *and* $|x'(t)| < M_2$ *for all* $t \in [0,T]$.

(2) *Each possible solution* c *to the equation*

$$g_1(c) - g_2(c) + \bar{h} = 0$$

satisfies the inequality $\eta_0 < c < \eta_1$.

(3) *It holds*

$$\left(g_1(\eta_0) - g_2(\eta_0) + \bar{h}\right)\left(g_1(\eta_1) - g_2(\eta_1) + \bar{h}\right) < 0.$$

Then equation (1.5) *has at least one* T*-periodic solution* u *such that* $\eta_0 < u(t) < \eta_1$ *for all* $t \in [0,T]$.

Lemma 2.2 *Assume that there exist constants* $0 < \eta_0 < \eta_1$, $M_2 > 0$, *such that the following conditions hold*:

(1) *For each* $\lambda \in (0,1]$, *each possible positive* T*-periodic solution* x *to the equation*

$$\left(\left|u'\right|^{p-2}u'\right)' + \lambda f\left(u'\right) + \lambda g_1(u) - \lambda g_2(u) = \lambda h(t)$$

satisfies the inequalities $\eta_0 < x(t) < \eta_1$ *and* $|x'(t)| < M_2$ *for all* $t \in [0,T]$.

(2) *Each possible solution* c *to the equation*

$$g_1(c) - g_2(c) - \bar{h} = 0$$

satisfies the inequality $\eta_0 < c < \eta_1$.

(3) *It holds*

$$\left(g_1(\eta_0) - g_2(\eta_0) - \bar{h}\right)\left(g_1(\eta_1) - g_2(\eta_1) - \bar{h}\right) < 0.$$

Then equation (1.6) *has at least one* T*-periodic solution* u *such that* $\eta_0 < u(t) < \eta_1$ *for all* $t \in [0,T]$.

In order to study the existence of positive periodic solutions to equation (1.5) and equation (1.6), we list the following assumptions:

(H_1) there are positive constants m_0 and m_1 with $m_0 < m_1$ such that

$$g_1(x) - g_2(x) - |h|_\infty > 0 \quad \text{for all } x \in (0, m_0] \tag{2.1}$$

and

$$g_1(x) - g_2(x) + |h|_\infty < 0 \quad \text{for all } x \in [m_1, \infty), \tag{2.2}$$

where $|h|_\infty = \max_{t\in[0,T]} |h(t)|$;

(H_2) $g_1(x) \geq 0$ for all $x \in (0, +\infty)$, and $\int_0^1 g_1(s)\,ds = +\infty$;

(H_3) $f(0) = 0$;

(H_4) there are constants n, σ_0 and σ_1 with $n > 1$, $0 < \sigma_0 \leq \sigma_1$ such that

$$xf(x) \geq \sigma_0|x|^n \quad \text{or} \quad xf(x) \leq -\sigma_0|x|^n \quad \text{for all } x \in \mathbb{R}$$

and

$$\left|f(x)\right| \leq \sigma_1|x|^{n-1} \quad \text{for all } x \in \mathbb{R}.$$

Now, we embed equation (1.5) and equation (1.6) into the following two equations family with a parameter $\lambda \in (0,1]$, respectively:

$$\left(\left|x'\right|^{p-2}x'\right)' + \lambda f\left(x'\right) - \lambda g_1(x) + \lambda g_2(x) = \lambda h(t), \quad \lambda \in (0,1], \tag{2.3}$$

and

$$\left(\left|x'\right|^{p-2}x'\right)' + \lambda f\left(x'\right) + \lambda g_1(x) - \lambda g_2(x) = \lambda h(t), \quad \lambda \in (0,1]. \tag{2.4}$$

Lemma 2.3 *Assume that assumptions* (H_1) *and* (H_3) *hold, let* m_0 *and* m_1 *be positive constants determined in assumption* (H_1). *Then the following conclusions hold*:

(1) *for each possible positive* T*-periodic solution* $u(t)$ *of equation* (2.3) *there exists* $\tau \in [0, T]$ *such that*

$$m_0 < u(\tau) < m_1; \tag{2.5}$$

(2) *each possible solution* c *to the equation*

$$g_1(c) - g_2(c) + \bar{h} = 0$$

satisfies the inequality $m_0 < c < m_1$;

(3) $g_1(u) - g_2(u) + \bar{h} > 0$ *for all* $u \in (0, m_0]$, *and* $g_1(u) - g_2(u) + \bar{h} < 0$ *for all* $u \in [m_1, +\infty)$.

Proof (1) Suppose that $u(t)$ be an arbitrary positive T-periodic solution to equation (2.3), then

$$\left(\left|u'(t)\right|^{p-2}u'(t)\right)' + \lambda f\left(u'(t)\right) - \lambda g_1\left(u(t)\right) + \lambda g_2\left(u(t)\right) = \lambda h(t), \quad \lambda \in (0,1]. \tag{2.6}$$

Let t_0 and t_1 be the maximum point and the minimum point of $u(t)$ on $[0, T]$, respectively, then $u'(t_0) = 0$ and $u'(t_1) = 0$. We can prove that

$$\left(\left|u'(t)\right|^{p-2}u'(t)\right)'|_{t=t_0} \leq 0. \tag{2.7}$$

In fact, if (2.7) does not hold, then $(|u'|^{p-2}u')'|_{t=t_0} > 0$. By the continuity of $(|u'(t)|^{p-2}u'(t))'$ for $t \in [t_0, t_0 + T]$, we see that there is a constant $\delta \in (0, T)$ such that $(|u'(t)|^{p-2}u'(t))' > 0$

for $t \in (t_0, t_0 + \delta)$, and then $|u'(t)|^{p-2}u'(t) > |u'(t)|^{p-2}u'(t)|_{t=t_0} = 0$ for $t \in (t_0, t_0 + \delta)$, *i.e.*, $u'(t) > 0$ for $t \in (t_0, t_0 + \delta)$, which results in $u(t) > u(t_0)$ for $t \in (t_0, t_0 + \delta)$. So $u(t_0) < \max_{t\in[t_0,t_0+T]} u(t) = \max_{t\in[0,T]} u(t)$, which contradicts the fact that t_0 is the maximum point of $u(t)$ on $[0, T]$. This contradiction implies that (2.7) holds. Similarly, we have

$$\bigl(|u'|^{p-2}u'\bigr)'|_{t=t_1} \geq 0. \tag{2.8}$$

It follows from (2.6) and (2.7) that

$$f\bigl(u'(t_0)\bigr) - g_1\bigl(u(t_0)\bigr) + g_2\bigl(u(t_0)\bigr) \geq -|h|_\infty. \tag{2.9}$$

By using assumption (H_3), we have $f(u'(t_0)) = f(0) = 0$, which together with (2.9) yields

$$g_1\bigl(u(t_0)\bigr) - g_2\bigl(u(t_0)\bigr) \leq |h|_\infty,$$

and, by using condition (2.1) in assumption (H_1), we have

$$u(t_0) > m_0. \tag{2.10}$$

Similarly, condition (2.2) in assumption (H_1), together (2.6) and (2.8), implies that

$$u(t_1) < m_1. \tag{2.11}$$

Without loss of generality, suppose $u(t_0) > m_1$, then by (2.10) and (2.11), we obtain from the intermediate value property of the continuous function $u(t)$ that (2.5) holds.

(2) Conclusion (2), as well as conclusion (3), follows directly from assumption (H_1). □

Similar to the proof of Lemma 2.3, we obtain the following result.

Lemma 2.4 *Assume that assumptions* (H_1) *and* (H_3) *hold, let* m_0 *and* m_1 *be positive constants determined in assumption* (H_1). *Then the following conclusions hold*:

(1) *each possible positive* T*-periodic solution* $u(t)$ *to equation* (2.4) *satisfies*

$$m_0 < u(t) < m_1 \quad \text{for all } t \in [0, T];$$

(2) *each possible solution* c *to the equation*

$$g_1(c) - g_2(c) - \bar{h} = 0$$

satisfies the inequality $m_0 < c < m_1$;

(3) $g_1(u) - g_2(u) - \bar{h} > 0$ *for all* $u \in (0, m_0]$, *and* $g_1(u) - g_2(u) - \bar{h} < 0$ *for all* $u \in [m_1, +\infty)$.

3 Main results

Theorem 3.1 *Assume that assumptions* (H_1), (H_2), *and* (H_4) *hold, then equation* (1.5) *has at least one positive* T*-periodic solution.*

Proof First of all, we will show that there exist M_1, M_2 with $M_1 > m_1$ and $M_2 > 0$ such that each positive T-periodic solution $u(t)$ of equation (2.3) satisfies the inequalities

$$u(t) < M_1, \qquad \left|u'(t)\right| < M_2. \tag{3.1}$$

In fact, if u is a positive T-periodic solution of equation (2.3), then

$$\left(\left|u'(t)\right|^{p-2}u'(t)\right)' + \lambda f\left(u'(t)\right) - \lambda g_1\left(u(t)\right) + \lambda g_2\left(u(t)\right) = \lambda h(t), \quad \lambda \in (0,1]. \tag{3.2}$$

It is easy to see that assumption (H_3) can be deduced from (H_4), so by using Lemma 2.3, we see that there is a point $\tau \in [0, T]$ such that

$$m_0 < u(\tau) < m_1. \tag{3.3}$$

Multiplying (3.2) with $u'(t)$ and integrating over the interval $[0, T]$, we have

$$\int_0^T u'(t)f\left(u'(t)\right)dt = \int_0^T u'(t)h(t)\,dt. \tag{3.4}$$

It follows from assumption (H_4) that

$$\sigma_0 \int_0^T \left|u'(t)\right|^n dt \le \int_0^T \left|u'(t)f\left(u'(t)\right)\right| dt = \left|\int_0^T u'(t)f\left(u'(t)\right)dt\right|,$$

which together with (3.4) yields

$$\sigma_0 \int_0^T \left|u'(t)\right|^n dt \le \left(\int_0^T \left|h(t)\right|^{\frac{n}{n-1}} dt\right)^{\frac{n-1}{n}} \left(\int_0^T \left|u'(t)\right|^n dt\right)^{\frac{1}{n}},$$

i.e.,

$$\int_0^T \left|u'(t)\right|^n dt \le \sigma_0^{\frac{n}{1-n}} \int_0^T \left|h(t)\right|^{\frac{n}{n-1}} dt, \tag{3.5}$$

and then by (3.4), we get

$$\begin{aligned} u(t) &\le u(\tau) + T^{\frac{n-1}{n}} \left(\int_0^T \left|u'(t)\right|^n dt\right)^{\frac{1}{n}} \\ &< m_1 + T^{\frac{n-1}{n}} \sigma_0^{\frac{1}{1-n}} \left(\int_0^T \left|h(t)\right|^{\frac{n}{n-1}} dt\right)^{\frac{n-1}{n}} := M_1. \end{aligned} \tag{3.6}$$

Let $G = \max_{x\in[0,M_1]} |g_2(x)|$, then it follows from (3.2) and the fact of $g_1 \in C((0,\infty), [0,+\infty))$ that

$$\begin{aligned} \int_0^T \left|\left(\left|u'(t)\right|^{p-2}u'(t)\right)'\right| dt &\le \lambda \int_0^T \left|f\left(u'(t)\right)\right| dt + \lambda \int_0^T g_1\left(u(t)\right) dt + \lambda TG \\ &\quad + \lambda \int_0^T \left|h(t)\right| dt. \end{aligned} \tag{3.7}$$

Furthermore, by integrating (3.2) over the interval $[0, T]$, we have

$$\begin{aligned}\int_0^T g_1(u(t))\,dt &= \int_0^T g_2(u(t))\,dt + \int_0^T f(u'(t))\,dt - \int_0^T h(t)\,dt \\ &\le \int_0^T \left|f(u'(t))\right|dt + TG + \int_0^T \left|h(t)\right|dt.\end{aligned}$$

Substituting it into (3.7), and by using assumption (H_3), we have

$$\begin{aligned}\int_0^T \left|\left(\left|u'(t)\right|^{p-2}u'(t)\right)'\right|dt &\le 2\lambda \int_0^T \left|f(u'(t))\right|dt + 2\lambda TG + 2\lambda \int_0^T \left|h(t)\right|dt \\ &\le 2\lambda\sigma_1 \int_0^T \left|u'(t)\right|^{n-1}dt + 2\lambda TG + 2\lambda \int_0^T \left|h(t)\right|dt \\ &\le 2\lambda\sigma_1 T^{\frac{1}{n}}\left(\int_0^T \left|u'(t)\right|^{n}dt\right)^{\frac{n-1}{n}} + 2\lambda TG + 2\lambda \int_0^T \left|h(t)\right|dt,\end{aligned}$$

which together with (3.5) yields

$$\begin{aligned}\int_0^T \left|\left(\left|u'(t)\right|^{p-2}u'(t)\right)'\right|dt \le 2\lambda\Bigg[&\sigma_1\sigma_0^{-1}T^{\frac{1}{n}}\left(\int_0^T \left|h(t)\right|^{\frac{n}{n-1}}dt\right)^{\frac{n-1}{n}} + TG \\ &+ \int_0^T \left|h(t)\right|dt\Bigg].\end{aligned} \tag{3.8}$$

Since $\max_{t\in[0,T]} |u'(t)|^{p-1} \le \int_0^T |(|u'(t)|^{p-2}u'(t))'|\,dt$, it follows from (3.8) that

$$\max_{t\in[0,T]}\left|u'(t)\right|^{p-1} < \lambda M_2, \tag{3.9}$$

i.e.,

$$\max_{t\in[0,T]}\left|u'(t)\right| < M_2, \tag{3.10}$$

where $M_2 = 1 + 2^{\frac{1}{p-1}}[\sigma_1\sigma_0^{-1}T^{\frac{1}{n}}(\int_0^T |h(t)|^{\frac{n}{n-1}}\,dt)^{\frac{n-1}{n}} + TG + \int_0^T |h(t)|\,dt]^{\frac{1}{p-1}}$.

Below, we will show that there exists a constant $M_0 \in (0, m_0)$, such that

$$u(t) > M_0 \quad \text{for all } t \in [0, T]. \tag{3.11}$$

Let τ be determined as in Lemma 2.3. Multiplying (3.2) by $u'(t)$ and integrating over the interval $[\tau, t]$ (or $[t, \tau]$), we get

$$\begin{aligned}&\int_\tau^t \left(\left|u'(s)\right|^{p-2}u'(s)\right)'u'(s)\,ds \\ &\quad + \lambda\int_{t_0}^t f(u'(s))u'(s)\,ds - \lambda\int_\tau^t g_1(u(s))u'(s)\,ds + \lambda\int_\tau^t g_2(u(s))u'(s)\,ds \\ &= \lambda\int_\tau^t h(s)u'(s)\,ds, \quad \lambda \in (0,1).\end{aligned} \tag{3.12}$$

Set $y(t) = |u'(t)|^{p-2}u'(t)$, then $y(t)$ is absolutely continuous and $u'(t) = |y(t)|^{q-2}y(t)$, where $q \in (1, +\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. So

$$\begin{aligned}\int_\tau^t \bigl(\bigl|u'(s)\bigr|^{p-2}u'(s)\bigr)'u'(s)\,ds &= \int_\tau^t \bigl|y(s)\bigr|^{q-2}y(s)y'(s)\,ds \\ &= \frac{|y(t)|^q}{q} - \frac{|y(\tau)|^q}{q} = \frac{|u'(t)|^p}{q} - \frac{|u'(\tau)|^p}{q}.\end{aligned}$$

Substituting it into (3.12), we get

$$\frac{|u'(t)|^p}{q} - \frac{|u'(\tau)|^p}{q} + \lambda\int_\tau^t f(u')u'\,dt = \lambda\int_\tau^t g_1(u)u'\,dt - \lambda\int_\tau^t g_2(u)u'\,dt + \lambda\int_\tau^t h(t)u'\,dt,$$

which yields the estimate

$$\begin{aligned}\lambda\int_{u(t)}^{u(\tau)} g_1(s)\,ds \le{}& \frac{|u'(t)|^p}{q} + \frac{|u'(\tau)|^p}{q} \\ &+ \lambda\int_0^T \bigl|f(u')\bigr|\bigl|u'\bigr|\,dt + \lambda\int_0^T \bigl|g_2(u)\bigr|\bigl|u'\bigr|\,dt + \lambda\int_0^T \bigl|h(t)u'\bigr|\,dt.\end{aligned}$$

From (3.9), we get

$$\lambda\int_{u(t)}^{u(\tau)} g_1(s)\,ds \le \frac{2\lambda M_2^{\frac{p}{p-1}}}{q} + \lambda\Bigl(\max_{0\le u\le M_2}\bigl|f(u)\bigr|\Bigr)TM_2^{\frac{1}{p-1}} + \lambda GTM_2^{\frac{1}{p-1}} + \lambda\|h\|_{L_1}M_2^{\frac{1}{p-1}},$$

which gives

$$\int_{u(t)}^{u(\tau)} g_1(s)\,ds \le M_3 \quad \text{for all } t \in [\tau, \tau + T] \tag{3.13}$$

with

$$M_3 = \frac{2M_2^{\frac{p}{p-1}}}{q} + \Bigl(\max_{0\le u\le M_2}\bigl|f(u)\bigr|\Bigr)TM_2^{\frac{1}{p-1}} + TGM_2^{\frac{1}{p-1}} + \|h\|_{L_1}M_2^{\frac{1}{p-1}}.$$

From (H_2) there exists $M_0 \in (0, m_0)$ such that

$$\int_{M_0}^{m_0} g_1(s)\,ds > M_3. \tag{3.14}$$

Therefore, if there is a $t^* \in [\tau, \tau + T]$ such that $u(t^*) \le M_0$, then from (3.14) we get

$$\int_{u(t^*)}^{u(\tau)} g_1(s)\,ds \ge \int_{M_0}^{m_0} g_1(s)\,ds > M_3,$$

which contradicts (3.13). This contradiction shows that $u(t) > M_0$ for all $t \in [0, T]$. So (3.11) holds. Let $\eta_0 \in (0, M_0)$ and $\eta_1 \in (M_1, +\infty)$ be two constants, then from (3.6), (3.10), and (3.11), we see that each possible positive T-periodic solution u to equation (2.3) satisfies

$$\eta_0 < u(t) < \eta_1, \qquad \bigl|u'(t)\bigr| < M_2.$$

This implies that condition (1) of Lemma 2.1 is satisfied. We can deduce from conclusion (2) of Lemma 2.3 that each possible solution c to the equation

$$g_1(c) - g_2(c) + \bar{h} = 0$$

satisfies the inequality $\eta_0 < c < \eta_1$, and from conclusion (3) of Lemma 2.3, we obtain

$$g_1(c) - g_2(c) + \bar{h} > 0 \quad \text{for } c \in (0, \eta_0]$$

and

$$g_1(c) - g_2(c) + \bar{h} < 0 \quad \text{for } c \in [\eta_1, +\infty),$$

which results in

$$\bigl(g_1(\eta_0) - g_2(\eta_0) + \bar{h}\bigr)\bigl(g_1(\eta_1) - g_2(\eta_1) + \bar{h}\bigr) < 0.$$

So condition (3) of Lemma 2.1 holds. By using Lemma 2.1, we see that equation (1.5) has at least one positive T-periodic solution. The proof is complete. □

Theorem 3.2 *Assume that assumptions* (H_1) *and* (H_3) *hold, then equation* (1.6) *has at least one positive* T*-periodic solution.*

Proof Suppose that $u(t)$ be an arbitrary T-periodic solution to equation (2.4), then

$$\bigl(|u'|^{p-2}u'\bigr)' + \lambda f\bigl(u'\bigr) + \lambda g_1(u) - \lambda g_2(u) = \lambda h(t), \quad \lambda \in (0,1]. \tag{3.15}$$

By using Lemma 2.4, we see that

$$m_0 < u(t) < m_1 \quad \text{for all } t \in [0,T], \tag{3.16}$$

where m_0 and m_1 are constants determined in assumption (H_1). Multiplying equation (3.15) with $(|u'|^{p-2}u')'$ and integrating over the interval $[0,T]$, we have

$$\begin{aligned}&\int_0^T \bigl|\bigl(|u'|^{p-2}u'\bigr)'\bigr|^2\,dt + \lambda\int_0^T \bigl(|u'|^{p-2}u'\bigr)' f\bigl(u'(t)\bigr)\,dt\\ &\quad + \lambda\int_0^T \bigl(|u'|^{p-2}u'\bigr)' g_1(u)\,dt - \lambda\int_0^T \bigl(|u'|^{p-2}u'\bigr)' g_2(u)\,dt\\ &= \lambda\int_0^T \bigl(|u'|^{p-2}u'\bigr)' h(t)\,dt.\end{aligned} \tag{3.17}$$

Let $G_1 = \max_{0\le x\le m_1} |g_2(x)|$, $G_2 = \max_{m_0\le x\le m_1} |g_1(x)|$. Take $y(t) = |u'(t)|^{p-2}u'(t)$, then $u'(t) = |y(t)|^{q-2}y(t)$, where $q \in (1,+\infty)$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_0^T \bigl(|u'|^{p-2}u'\bigr)' f\bigl(u'(t)\bigr)\,dt = \int_0^T f\bigl(|y(t)|^{q-2}y(t)\bigr)y'(t)\,dt = 0.$$

So, it follows from (3.16) and (3.17) that

$$\begin{aligned}
&\int_0^T \bigl|\bigl(|u'|^{p-2}u'\bigr)'\bigr|^2\,dt \\
&\quad\le G_1\int_0^T \bigl|\bigl(|u'|^{p-2}u'\bigr)'\bigr|\,dt + G_2\int_0^T \bigl|\bigl(|u'|^{p-2}u'\bigr)'\bigr|\,dt + |h|_\infty\int_0^T \bigl|\bigl(|u'|^{p-2}u'\bigr)'\bigr|\,dt \\
&\quad\le T^{1/2}\bigl(G_1+G_2+|h|_\infty\bigr)\biggl(\int_0^T \bigl|\bigl(|u'|^{p-2}u'\bigr)'\bigr|^2\,dt\biggr)^{1/2},
\end{aligned}$$

which results in

$$\int_0^T \bigl|\bigl(|u'|^{p-2}u'\bigr)'\bigr|^2\,dt \le T\bigl(G_1+G_2+|h|_\infty\bigr)^2. \tag{3.18}$$

Since

$$\begin{aligned}
\max_{t\in[0,T]}\bigl|u'(t)\bigr|^{p-1} = \max_{t\in[0,T]}\bigl|u'(t)\bigr|^{p-2}\bigl|u'(t)\bigr| &\le \int_0^T \bigl|\bigl(|u'|^{p-2}u'\bigr)'\bigr|\,dt \\
&\le T^{1/2}\biggl(\int_0^T \bigl|\bigl(|u'|^{p-2}u'\bigr)'\bigr|^2\,dt\biggr)^{1/2},
\end{aligned}$$

it follows from (3.18) that

$$\max_{t\in[0,T]}\bigl|u'(t)\bigr|^{p-1} \le T\bigl(G_1+G_2+|h|_\infty\bigr)^{1/2},$$

i.e.,

$$\max_{t\in[0,T]}\bigl|u'\bigr| \le T^{\frac{1}{p-1}}\bigl(G_1+G_2+|h|_\infty\bigr)^{\frac{1}{2(p-1)}}. \tag{3.19}$$

Let $M_2 = T^{\frac{1}{p-1}}(G_1+G_2+|h|_\infty)^{\frac{1}{2(p-1)}} + 1$, and $u(t)$ be an arbitrary T-periodic solution to equation (2.4). Then from (3.16) and (3.19), we see that

$$m_0 < u(t) < m_1, \qquad \bigl|u'(t)\bigr| < M_2 \quad \text{for all } t\in[0,T].$$

This implies that condition (1) of Lemma 2.2 is satisfied, and it is easy to see from assumption (H_1) that conditions (2)-(3) in Lemma 2.2 also hold. By using Lemma 2.2, we see that equation (1.6) possesses a T-periodic solution $u(t)$ such that $m_0 \le u(t) \le m_1$ for all $t\in[0,T]$. The proof is complete. □

Now, if the singular restoring force term $g_1(x)$ in equation (1.5) (or in equation (1.6)) satisfies

$$\lim_{x\to 0^+} g_1(x) = +\infty \tag{3.20}$$

and $g_2(x)$ satisfies

$$\lim_{x\to +\infty} g_2(x) = +\infty, \tag{3.21}$$

then assumption (H_1) holds. Thus, by applying Theorem 3.1 and Theorem 3.2, respectively, we can obtain the following results.

Corollary 3.1 *Assume that* (3.20), (3.21), *and assumptions* (H_2) *and* (H_4) *hold*, *then equation* (1.5) *has at least one positive* T*-periodic solution*.

Corollary 3.2 *Assume that* (3.20), (3.21), *and assumption* (H_3) *hold*, *then equation* (1.6) *has at least one positive* T*-periodic solution*.

Example 3.1 Considering the following equation:

$$x''(t) + \left(x'(t)\right)^3 - \frac{1}{x^2(t)}\left|\sin\frac{1}{x(t)}\right| + x^2(t) = \cos t + 2. \tag{3.22}$$

Corresponding to equation (1.5), $f(u) = u^3$, $g_1(u)$ can be regarded as $g_1(u) = \frac{1}{u^2}|\sin\frac{1}{u}|$, $g_2(u) = u^2 - 2$, and $h(t) = \cos t$. Since

$$\int_0^1 \left(\frac{1}{u^2}\left|\sin\frac{1}{u}\right|\right)du = +\infty,$$

it follows that assumption (H_2) holds. By simple calculating, we can chose $\sigma_0 = \sigma_1 = 1$, $n = 4$, $m_0 = \frac{1}{2}$, and $m_1 = 2$ such that verifying assumptions (H_1) and (H_4). Thus, by using Theorem 3.1, we see that equation (3.22) has at least one positive 2π-periodic solution.

Example 3.2 Consider the following equation:

$$x''(t) + \left(x'(t)\right)^4 + \frac{1}{x^{\frac{1}{2}}(t)} - \frac{1}{x^{\frac{1}{4}}(t)} - x^3(t) = \cos t. \tag{3.23}$$

Corresponding to equation (1.6), $f(u) = u^4$, $g_1(u) = \frac{1}{u^{\frac{1}{2}}} - \frac{1}{u^{\frac{1}{4}}}$, $g_2(u) = u^3$, and $h(t) = \cos t$. It is easy to see that conditions (3.20) and (3.21), and assumption (H_3) are all satisfied. By using Corollary 3.2, we see that equation (3.23) has at least one positive 2π-periodic solution.

Remark 3.1 The first order derivative term in equation (3.23) is $(x'(t))^4$, and then $\int_0^T (x'(t))^4\,dt \neq 0$ for all T-periodic continuous function x, generally. This implies that the methods used for studying equation (1.1) in [1] is not valid for equation (3.23).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

Acknowledgements
The work is sponsored by the National Natural Science Foundation of China (11271197). The authors are grateful to anonymous referees for their constructive comments and suggestions, which have greatly improved this paper.

Received: 23 January 2016 Accepted: 6 May 2016 Published online: 11 May 2016

References

1. Jebelean, P, Mawhin, J: Periodic solutions of singular nonlinear perturbations of the ordinary *p*-Laplacian. Adv. Nonlinear Stud. **2**(3), 299-312 (2002)
2. Forbat, F, Huaux, A: Détermination approchée et stabilité locale de la solution périodique d'une équation différentielle non lineáire. Mém. Publ. Soc. Sci. Arts Lett. Hainaut **76**, 3-13 (1962)
3. Gordon, WB: Conservative dynamical systems involving strong forces. Trans. Am. Math. Soc. **204**, 113-135 (1975)
4. Gordon, WB: A minimizing property of Keplerian orbits. Am. J. Math. **99**, 961-971 (1977)
5. Solimini, S: On forced dynamical systems with a singularity of repulsive type. Nonlinear Anal. **14**, 489-500 (1990)
6. Terracini, S: Remarks on periodic orbits of dynamical systems with repulsive singularities. J. Funct. Anal. **111**, 213-238 (1993)
7. Torres, PJ: Existence of one-signed periodic solutions of some second order differential equations via a Krasnoselskii fixed point theorem. J. Differ. Equ. **190**, 643-662 (2003)
8. Hakl, R, Torres, PJ: On periodic solutions of second-order differential equations with attractive-repulsive singularities. J. Differ. Equ. **248**, 111-126 (2010)
9. Jiang, D, Chu, J, Zhang, M: Multiplicity of positive periodic solutions to superlinear repulsive singular equations. J. Differ. Equ. **211**, 282-302 (2005)
10. Chu, J, Torres, PJ, Zhang, M: Periodic solutions of second order non-autonomous singular dynamical systems. J. Differ. Equ. **239**, 196-212 (2007)
11. Chu, J, Fan, N, Torres, PJ: Periodic solutions for second order singular damped differential equations. J. Math. Anal. Appl. **388**, 665-675 (2012)
12. Martins, R: Existence of periodic solutions for second-order differential equations with singularities and the strong force condition. J. Math. Anal. Appl. **317**, 1-13 (2006)
13. Lei, J, Zhang, MR: Twist property of periodic motion of an atom near a charged wire. Lett. Math. Phys. **60**(1), 9-17 (2002)
14. Torres, PJ: Existence and stability of periodic solutions for second order semilinear differential equations with a singular nonlinearity. Proc. R. Soc. Edinb., Sect. A **137**, 195-201 (2007)
15. Fonda, A, Toader, R: Periodic orbits of radially symmetric Keplerian-like systems: a topological degree approach. J. Differ. Equ. **244**, 3235-3264 (2008)
16. Li, X, Zhang, Z: Periodic solutions for second-order differential equations with a singular nonlinearity. Nonlinear Anal. **69**, 3866-3876 (2008)
17. Zhang, M: Nonuniform nonresonance at the first eigenvalue of the *p*-Laplacian. Nonlinear Anal. **29**(1), 41-51 (1997)
18. Wang, Z: Periodic solutions of Lienard equations with a singularity and a deviating argument. Nonlinear Anal. **16**, 227-234 (2014)
19. Manaśevich, R, Mawhin, J: Periodic solutions for nonlinear systems with *p*-Laplacian-like operators. J. Differ. Equ. **145**, 367-393 (1998)