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# Twin iterative solutions for a fractional differential turbulent flow model

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**Abstract**
We investigate the existence of twin iterative solutions for a fractional $p$-Laplacian equation with nonlocal boundary conditions. Using the monotone iterative technique, we establish a new existence result on the maximal and minimal solutions under suitable nonlinear growth conditions. We also consider some interesting particular cases and give an example to illustrate our main results.

**Keywords:** extremal solutions; monotone iterative technique; $p$-Laplacian operator; nonlocal boundary value problem; fractional differential equation

## 1 Introduction

In this paper, we are concerned with the existence of twin iterative solutions for the following nonlocal fractional differential equation with a $p$-Laplacian operator:

$$\begin{cases} -\mathscr{D}_{\mathbf{t}}^{\beta}(\varphi_p(-\mathscr{D}_{\mathbf{t}}^{\alpha}x))(t) = h(t)f(x(t)), \quad 0 < t < 1, \\ x(0) = 0, \qquad \mathscr{D}_{\mathbf{t}}^{\alpha}x(0) = \mathscr{D}_{\mathbf{t}}^{\alpha}x(1) = 0, \qquad x(1) = \int_0^1 x(t)\,dA(t), \end{cases} \tag{1.1}$$

where $\mathscr{D}_{\mathbf{t}}^{\alpha}$, $\mathscr{D}_{\mathbf{t}}^{\beta}$ are the standard Riemann-Liouville derivatives with $1 < \alpha, \beta \leq 2$, $\int_0^1 x(s)\,dA(s)$ is the Riemann-Stieltjes integral with respect to a function $A$ of bounded variation, and $\varphi_p$ is the $p$-Laplacian operator defined by $\varphi_p(s) = |s|^{p-2}s$, $p > 1$. Obviously, $\varphi_p(s)$ is invertible, and its inverse operator is $\varphi_q(s)$, where $q > 1$ is the constant such that $\frac{1}{p} + \frac{1}{q} = 1$.

It is well known that the $p$-Laplacian equation can describe a fundamental mechanics problem arising from turbulent flow in a porous medium; see [1]. Based on this background, some interesting results relative to the equation $(\varphi_p(x'(t)))' = f(t, x(t))$ subject to certain boundary value conditions have been obtained in [2–6] and references therein. On the other hand, fractional calculus has been greatly developed in recent years. In particular, fractional-order models have been proved to be more accurate than integer-order models for the description of many physical phenomena with long memory, such as viscoelasticity, electrochemistry control, porous media, electromagnetic, polymer rheology, and some hereditary properties of various materials and processes (for the reseach of fractional models and relative problems, we refer readers to [7–27]). Thus, fractional-order differential equations with $p$-Laplacian operator have attracted great interest from the mathematical research community.

Recently, Wang *et al.* [22] investigated the existence of multiple positive solutions for nonlocal fractional $p$-Laplacian equation

$$\begin{cases} \mathscr{D}_{\mathbf{t}}^{\beta}(\varphi_p(\mathscr{D}_{\mathbf{t}}^{\alpha}x))(t)+f(t,x(t))=0, \quad t\in(0,1), \\ x(0)=0, \qquad x(1)=ax(\xi), \qquad \mathscr{D}_{\mathbf{t}}^{\alpha}x(0)=0, \end{cases} \tag{1.2}$$

where $0<\alpha\leq 2$, $0<\beta\leq 1$, $0\leq a\leq 1$, $0<\xi<1$. By using Krasnosel'skii's fixed point theorem and the Leggett-Williams theorem, some results on the existence of positive solutions are obtained. And then, by means of the upper and lower solutions method, Wang *et al.* [28] studied the existence of positive solutions for the following nonlocal fractional $p$-Laplacian equation:

$$\begin{cases} \mathscr{D}_{\mathbf{t}}^{\beta}(\varphi_p(\mathscr{D}_{\mathbf{t}}^{\alpha}x))(t)+f(t,x(t))=0, \quad t\in(0,1), \\ x(0)=0, \qquad x(1)=ax(\xi), \qquad \mathscr{D}_{\mathbf{t}}^{\alpha}x(0)=0, \qquad \mathscr{D}_{\mathbf{t}}^{\alpha}x(1)=b\mathscr{D}_{\mathbf{t}}^{\alpha}x(\eta), \end{cases} \tag{1.3}$$

where $1<\alpha,\beta\leq 2$, $0\leq a,b\leq 1$, $0<\xi,\eta<1$. More recently, Zhang *et al.* [29] considered the following fractional-order model for turbulent flow in a porous medium:

$$\begin{cases} -\mathscr{D}_{\mathbf{t}}^{\beta}(\varphi_p(-\mathscr{D}_{\mathbf{t}}^{\alpha}x))(t)=f(x(t),\mathscr{D}_{\mathbf{t}}^{\gamma}x(t)), \quad t\in(0,1), \\ \mathscr{D}_{\mathbf{t}}^{\alpha}x(0)=\mathscr{D}_{\mathbf{t}}^{\alpha+1}x(0)=\mathscr{D}_{\mathbf{t}}^{\alpha}x(1)=0, \qquad \mathscr{D}_{\mathbf{t}}^{\gamma}x(0)=0, \qquad \mathscr{D}_{\mathbf{t}}^{\gamma}x(1)=\int_0^1\mathscr{D}_{\mathbf{t}}^{\gamma}x(s)\,dA(s), \end{cases}$$

where $\mathscr{D}_{\mathbf{t}}^{\alpha}$, $\mathscr{D}_{\mathbf{t}}^{\beta}$, $\mathscr{D}_{\mathbf{t}}^{\gamma}$ are the standard Riemann-Liouville derivatives, $\int_0^1 x(s)\,dA(s)$ is the Riemann-Stieltjes integral, $0<\gamma\leq 1<\alpha\leq 2<\beta<3$, $\alpha-\gamma>1$, $A$ is a function of bounded variation, and $dA$ can be a signed measure. In the case where the nonlinearity $f(u,v)$ may be singular at both $u=0$ and $v=0$, the uniqueness of a positive solution for a fractional model of turbulent flow in a porous medium was established via the fixed point theorem of the mixed monotone operator.

Motivated by the mentioned works, in this paper, we consider the twin iterative solutions of fractional-order model for turbulent flow in a porous medium. Differently from the mentioned works, we not only obtain the minimal and maximal solutions of the nonlocal boundary value problem of the fractional $p$-Laplacian equation (1.1), but we also derive estimates of the lower and upper bounds of the extremal solutions and construct a convergent iterative scheme for finding these solutions. In addition, we consider some particular cases and give an example to illustrate our main results.

## 2 Preliminaries and lemmas

Our work is carried out based on various definitions and semigroup properties of the Riemann-Liouville fractional calculus. We give some preliminaries and lemmas for convenience of the reader.

**Definition 2.1** (see [30–32]) The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $x:(0,+\infty)\to\mathbb{R}$ is given by

$$I^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}x(s)\,ds,$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$.

**Definition 2.2** (see [30–32]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$\mathscr{D}_{\mathbf{t}}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_0^t (t-s)^{n-\alpha-1} x(s)\, ds,$$

where $n = [\alpha] + 1$ with $[\alpha]$ denoting the integer part of a number $\alpha$, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

**Proposition 2.1** (see [30–32])

(1) *If* $x \in L^1(0,1)$ *and* $\alpha > \beta > 0$, *then*

$$I^{\alpha} I^{\beta} x(t) = I^{\alpha+\beta} x(t), \qquad \mathscr{D}_{\mathbf{t}}^{\beta} I^{\alpha} x(t) = I^{\alpha-\beta} x(t), \qquad \mathscr{D}_{\mathbf{t}}^{\beta} I^{\beta} x(t) = x(t).$$

(2) *If* $\alpha > 0$, $\beta > 0$, *then*

$$\mathscr{D}_{\mathbf{t}}^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}.$$

**Proposition 2.2** (see [30–32]) *For* $\alpha > 0$, *if* $f(x)$ *is integrable, then*

$$I^{\alpha} \mathscr{D}_{\mathbf{t}}^{\alpha} x(t) = f(x) + c_1 x^{\alpha-1} + c_2 x^{\alpha-2} + \cdots + c_n x^{\alpha-n},$$

*where* $c_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$, *and* $n = [\alpha]$.

Now consider the following linear fractional differential equation with nonlocal boundary condition:

$$\begin{cases} -\mathscr{D}_{\mathbf{t}}^{\alpha} x(t) = r(t), & t \in (0,1), \\ x(0) = 0, & x(1) = \int_0^1 x(t)\, dA(t). \end{cases} \tag{2.1}$$

Let

$$G_{\alpha}(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} [t(1-s)]^{\alpha-1}, & 0 \le t \le s \le 1, \\ [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1. \end{cases} \tag{2.2}$$

**Lemma 2.1** (see [29]) *Given* $r \in L^1(0,1)$, *the boundary value problem*

$$\begin{cases} -\mathscr{D}_{\mathbf{t}}^{\alpha} x(t) = r(t), & 0 < t < 1, \\ x(0) = x(1) = 0, \end{cases} \tag{2.3}$$

*has the unique solution*

$$x(t) = \int_0^1 G_{\alpha}(t,s) r(s)\, ds.$$

On the other hand, it follows from Proposition 2.1 that the unique solution of the problem

$$\begin{cases} -\mathscr{D}_{\mathbf{t}}^{\alpha} x(t) = 0, & 0 < t < 1, \\ x(0) = 0, & x(1) = 1, \end{cases} \tag{2.4}$$

is $t^{\alpha-1}$. Let

$$\mathcal{A} = \int_0^1 t^{\alpha-1}\,dA(t), \qquad \mathcal{G}_A(s) = \int_0^1 G_\alpha(t,s)\,dA(t).$$

According to the strategy of [31] and [32], we have the following lemmas.

**Lemma 2.2** (see [33]) *If* $1 < \alpha \le 2$ *and* $r \in L^1(0,1)$, *then the boundary value problem* (2.1) *has the unique solution*

$$x(t) = \int_0^1 H(t,s)r(s)\,ds, \tag{2.5}$$

*where*

$$H(t,s) = \frac{t^{\alpha-1}}{1-\mathcal{A}}\mathcal{G}_A(s) + G_\alpha(t,s). \tag{2.6}$$

**Lemma 2.3** (see [31]) *Let* $0 \le \mathcal{A} < 1$ *and* $\mathcal{G}_A(s) \ge 0$ *for* $s \in [0,1]$. *Then* $G_\alpha(t,s)$ *and* $H(t,s)$ *have the following properties*:

(1) $G_\alpha(t,s)$ *and* $H(t,s)$ *are nonnegative and continuous for* $(t,s) \in [0,1] \times [0,1]$.

(2) $G_\alpha(t,s)$ *satisfies*

$$\frac{t^{\alpha-1}(1-t)s(1-s)^{\alpha-1}}{\Gamma(\alpha)} \le G_\alpha(t,s) \le \frac{\alpha-1}{\Gamma(\alpha)}s(1-s)^{\alpha-1} \quad \text{for } t,s \in [0,1]. \tag{2.7}$$

(3) *There exist two constants* $a$, $b$ *such that*

$$at^{\alpha-1}\mathcal{G}_A(s) \le H(t,s) \le bt^{\alpha-1}, \quad s,t \in [0,1]. \tag{2.8}$$

Let $\frac{1}{q} + \frac{1}{p} = 1$, where $p$ is given by (1.1). We consider the associated linear boundary value problem

$$\begin{cases} -\mathscr{D}_{\mathbf{t}}^{\beta}(\varphi_p(-\mathscr{D}_{\mathbf{t}}^{\alpha}x))(t) = r(t), & t \in (0,1), \\ x(0) = 0, \qquad \mathscr{D}_{\mathbf{t}}^{\alpha}x(0) = \mathscr{D}_{\mathbf{t}}^{\alpha}x(1) = 0, \qquad x(1) = \int_0^1 x(t)\,dA(t) \end{cases} \tag{2.9}$$

for $r \in L^1(0,1)$ and $r \ge 0$.

**Lemma 2.4** *The associated linear BVP* (2.9) *has the unique positive solution*

$$x(t) = \int_0^1 H(t,s)\left(\int_0^1 G_\beta(s,\tau)r(\tau)\,d\tau\right)^{q-1} ds.$$

*Proof* Let $w = -\mathscr{D}_{\mathbf{t}}^{\alpha}x$ and $v = \varphi_p(w)$. Consider the boundary value problem

$$\begin{cases} -\mathscr{D}_{\mathbf{t}}^{\beta}v(t) = r(t), & t \in (0,1), \\ v(0) = v(1) = 0. \end{cases}$$

By Lemma 2.1 we have

$$v(t) = \int_0^1 G_\beta(t,s)r(s)\,ds, \quad t \in [0,1]. \tag{2.10}$$

Noting that $-\mathscr{D}_{\mathbf{t}}^{\alpha}x = w$, $w = \varphi_p^{-1}(v)$, we get from (2.9) and (2.10) that the solution of (2.9) satisfies

$$\begin{cases} -\mathscr{D}_{\mathbf{t}}^{\alpha}x(t) = \varphi_p^{-1}(\int_0^1 G_\beta(t,s)r(s)\,ds), & t \in (0,1), \\ x(0) = 0, \qquad x(1) = \int_0^1 x(t)\,dA(t). \end{cases}$$

By Lemma 2.2 the solution of the BVP (2.9) can be written as

$$x(t) = \int_0^1 H(t,s)\varphi_p^{-1}\left(\int_0^1 G_\beta(s,\tau)r(\tau)\,d\tau\right)ds, \quad t \in [0,1].$$

Since $r(s) \geq 0$, $s \in [0,1]$, the solution of equation (2.9) is

$$x(t) = \int_0^1 H(t,s)\left(\int_0^1 G_\beta(s,\tau)r(\tau)\,d\tau\right)^{q-1}ds, \quad t \in [0,1]. \qquad \square$$

For convenience of presentation, we list some assumptions to be used throughout the rest of the paper.

(H0) $A$ is a function of bounded variation satisfying $\mathcal{G}_A(s) \geq 0$ for $s \in [0,1]$ and $0 \leq \mathcal{A} < 1$.

(H1) $f : [0,+\infty) \to [0,+\infty)$ is continuous and nondecreasing, and there exists a constant $\epsilon > 0$ such that, for any $x \in [0,+\infty)$,

$$f(\mu x) \geq \mu^{\epsilon} f(x), \quad \forall 0 < \mu \leq 1. \tag{2.11}$$

(H2) $h \geq 0$ satisfies

$$0 < \int_0^1 s(1-s)^{\beta-1}h(s)\,ds < +\infty. \tag{2.12}$$

**Remark 2.1** By (H1), for any $\nu > 1$, it is easy to get

$$f(\nu x) \leq \nu^{\epsilon} f(x). \tag{2.13}$$

**Remark 2.2** There are a large number of functions that satisfy (H1). In particular, (H1) can cover mixed cases of the superlinear and sublinear cases. Some basic examples of $f$ that satisfy (H1) are:

(i) $f(s) = \sum_{i=1}^m a_i s^{\gamma_i}$, where $a_i, \gamma_i > 0$, $i = 1,2,\ldots,m$.

(ii) If $0 < \gamma_i, d_i < +\infty$ $(i = 1,2,\ldots,m)$ and $\delta, c > 0$, then

$$f(s) = \left[c + \sum_{i=1}^m d_i(t)x^{\gamma_i}\right]^{\delta}.$$

(iii) $f(s) = \frac{s^\gamma}{1+s^\delta} + s^l$, $\gamma, \delta, l > 0$, $\gamma > \delta$.

(iv) $f(s) = \frac{(a+s^\gamma)s^l}{b+s^\delta}$, $a, b, \gamma, \delta, l > 0$, $l > \delta$.

*Proof* (i) and (ii) are obvious. For (iii) and (iv), obviously, $f : [0,+\infty) \to [0,+\infty)$ is continuous and nondecreasing, and for any $0 < \mu \leq 1$, noticing that $\gamma, \delta, l > 0$, we have

$$f(\mu s) = \frac{\mu^\alpha s^\gamma}{1+\mu^\delta s^\delta} + \mu^l s^l \geq \frac{\mu^\alpha s^\gamma}{1+s^\delta} + \mu^l s^l \geq \mu^{\max\{\gamma,l\}}\left(\frac{s^\gamma}{1+s^\delta} + s^l\right) = \mu^{\max\{\gamma,l\}}f(s)$$

and

$$f(\mu s) = \frac{(a + \mu^\gamma s^\gamma)\mu^l s^l}{b + \mu^\delta s^\delta} \geq \frac{(a + \mu^\gamma s^\gamma)\mu^l s^l}{b + s^\delta} \geq \mu^{\gamma+l}\frac{(a + s^\gamma)s^l}{b + s^\delta} = \mu^{\gamma+l}f(s). \qquad \square$$

Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{R}$ be the set of all real numbers, and $\mathbb{R}_+$ be the set of all nonnegative real numbers. Let $C([0,1],\mathbb{R})$ be the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm

$$\|x\| = \max\bigl\{x(t) : t \in [0,1]\bigr\}.$$

Define the cone $P$ in $C([0,1],\mathbb{R}_+)$ by

$$P = \bigl\{x \in C\bigl([0,1],\mathbb{R}_+\bigr) : \text{there exist two nonnegative numbers } l_x < 1 < L_x \text{ such that } l_x t^{\alpha-1} \leq x(t) \leq L_x t^{\alpha-1}, t \in [0,1]\bigr\}$$

and the operator $T$ by

$$(Tx)(t) = \int_0^1 H(t,s)\biggl(\int_0^1 G_\beta(s,\tau)h(\tau)f\bigl(x(\tau)\bigr)\,d\tau\biggr)^{q-1}\,ds.$$

Then each fixed point of the operator $T$ on $P$ is a positive solution of the BVP (1.1).

**Lemma 2.5** *Assume that* (H0)-(H2) *hold. Then* $T : P \to P$ *is continuous, compact, and nondecreasing.*

*Proof* For any $x \in P$, we can find two positive numbers $L_x > l_x \geq 0$ such that

$$l_x t^{\alpha-1} \leq x(t) \leq L_x t^{\alpha-1}, \quad t \in [0,1]. \tag{2.14}$$

It follows from (H1) that $T$ is increasing with respect to $x$, and thus by (2.8) we have

$$\begin{aligned}
(Tx)(t) &\leq bt^{\alpha-1}\int_0^1\biggl(\int_0^1 G_\beta(s,\tau)h(\tau)f\bigl(x(\tau)\bigr)\,d\tau\biggr)^{q-1}\,ds \\
&\leq bt^{\alpha-1}\int_0^1\biggl(\int_0^1 \frac{\beta-1}{\Gamma(\beta)}\tau(1-\tau)^{\beta-1}h(\tau)f\bigl(L_x\tau^{\alpha-1}\bigr)\,d\tau\biggr)^{q_1-1}\,ds \\
&\leq bt^{\alpha-1}\int_0^1\biggl(\int_0^1 \frac{\beta-1}{\Gamma(\beta)}\tau(1-\tau)^{\beta-1}h(\tau)L_x^\epsilon f\bigl(\tau^{\alpha-1}\bigr)\,d\tau\biggr)^{q_1-1}\,ds \\
&\leq b\biggl(\frac{\beta-1}{\Gamma(\beta)}L_x^\epsilon f(1)\int_0^1 \tau(1-\tau)^{\beta-1}h(\tau)\,d\tau\biggr)^{q-1}t^{\alpha-1} \\
&= L_x^* t^{\alpha-1}
\end{aligned}$$

and

$$\begin{aligned}
(Tx)(t) &\geq at^{\alpha-1}\int_0^1 \mathcal{G}_A(s)\biggl(\int_0^1 G_\beta(s,\tau)h(\tau)f\bigl(x(\tau)\bigr)\,d\tau\biggr)^{q-1}\,ds \\
&\geq at^{\alpha-1}\int_0^1 \mathcal{G}_A(s)\biggl(\int_0^1 \frac{s^{\beta-1}(1-s)\tau(1-\tau)^{\beta-1}}{\Gamma(\beta)}h(\tau)f\bigl(l_x\tau^{\alpha-1}\bigr)\,d\tau\biggr)^{q-1}\,ds
\end{aligned}$$

$$
\begin{aligned}
&\geq at^{\alpha-1}\int_0^1 \mathcal{G}_A(s)\left(\int_0^1 \frac{s^{\beta-1}(1-s)\tau(1-\tau)^{\beta-1}}{\Gamma(\beta)}h(\tau)l_x^{\epsilon}\tau^{\epsilon(\alpha-1)}f(1)\,d\tau\right)^{q-1}ds \\
&= a\left(\frac{l_x^{\epsilon}}{\Gamma(\beta)}\right)^{q-1}\int_0^1 \mathcal{G}_A(s)s^{(\beta-1)(q-1)}(1-s)^{q-1}\,ds \\
&\quad\times\left(\int_0^1 \tau^{(\alpha-1)\epsilon+1}(1-\tau)^{\beta-1}h(\tau)\,d\tau\right)^{q-1}t^{\alpha-1},
\end{aligned}
$$

where

$$
\begin{aligned}
&l_x^* = a\left(\frac{l_x^{\epsilon}}{\Gamma(\beta)}\right)^{q-1}\int_0^1 \mathcal{G}_A(s)s^{(\beta-1)(q-1)}(1-s)^{q-1}\,ds\left(\int_0^1 \tau^{(\alpha-1)\epsilon+1}(1-\tau)^{\beta-1}h(\tau)\,d\tau\right)^{q-1}, \\
&L_x^* = b\left(\frac{\beta-1}{\Gamma(\beta)}L_x^{\epsilon}f(1)\int_0^1 \tau(1-\tau)^{\beta-1}h(\tau)\,d\tau\right)^{q-1},
\end{aligned}
$$

which implies that $T$ is well defined and $T(P)\subset P$. Moreover, $T$ is also uniformly bounded for any bounded set of $P$. In fact, let $D\subset P$ be any bounded set. Then there exists a constant $L>0$ such that $\|x\|\leq L$ for any $x\in D$. Moreover, for any $x\in D$, $s\in[0,1]$, from $\|x\|\leq L<L+1$ and (2.13) we have

$$
f\bigl(x(s)\bigr)\leq f(L+1)\leq (L+1)^{\epsilon}f(1).
$$

Consequently,

$$
\begin{aligned}
\bigl|(Tx)(t)\bigr| &\leq bt^{\alpha-1}\int_0^1\left(\int_0^1 G_{\beta}(s,\tau)h(\tau)f\bigl(x(\tau)\bigr)\,d\tau\right)^{q-1}ds \\
&\leq bt^{\alpha-1}\int_0^1\left(\int_0^1 \frac{\beta-1}{\Gamma(\beta)}\tau(1-\tau)^{\beta-1}h(\tau)f(L+1)\,d\tau\right)^{q_1-1}ds \\
&\leq bt^{\alpha-1}\int_0^1\left(\int_0^1 \frac{\beta-1}{\Gamma(\beta)}\tau(1-\tau)^{\beta-1}h(\tau)(L+1)^{\epsilon}f(1)\,d\tau\right)^{q_1-1}ds \\
&\leq b\left(\frac{\beta-1}{\Gamma(\beta)}(L+1)^{\epsilon}f(1)\int_0^1 \tau(1-\tau)^{\beta-1}h(\tau)\,d\tau\right)^{q-1}<+\infty.
\end{aligned}
$$

Therefore, $T(D)$ is uniformly bounded.

On the other hand, according to the Arezelà-Ascoli theorem and the Lebesgue dominated convergence theorem, it is easy to get that $T:P\to P$ is completely continuous. It follows from (H1) that the operator $T$ is nondecreasing. □

## 3 Main results

Define the constant

$$
A = b\left[\frac{(\beta-1)f(1)}{\Gamma(\beta)}\int_0^1 \tau(1-\tau)^{\beta-1}h(\tau)\,d\tau\right]^{q-1}. \tag{3.1}
$$

**Theorem 3.1** *Suppose that conditions* (H0)-(H2) *hold. If there exists a constant* $c>0$ *such that*

$$
(c+1)^{\epsilon(q-1)-1}\leq A^{-1}, \tag{3.2}
$$

*where $A$ is defined by* (3.1), *then the BVP* (1.1) *has the minimal and maximal solutions $x^*$ and $y^*$, which are positive, and there exist some nonnegative constants $m_i \leq n_i$, $i = 1,2$, such that*

$$m_1 t^{\alpha-1} \leq x^*(t) \leq n_1 t^{\alpha-1}, \qquad m_2 t^{\alpha-1} \leq y^*(t) \leq n_2 t^{\alpha-1}, \quad t \in [0,1]. \tag{3.3}$$

*Moreover, for initial values $x_0 = 0$, $y_0 = c + 1$, let $\{x_n\}$ and $\{y_n\}$ be the iterative sequences generated by*

$$x_n(t) = (Tx_{n-1})(t) = T^n x_0(t), \qquad y_n(t) = (Ty_{n-1})(t) = T^n y_0(t). \tag{3.4}$$

*Then*

$$\lim_{n\to+\infty} x_n = x^*, \qquad \lim_{n\to+\infty} y_n = y^*$$

*uniformly for $t \in [0,1]$.*

*Proof* Let $P[0,c] = \{x \in P : 0 \leq \|x\| \leq c + 1\}$. We first prove that $T(P[0,c]) \subset P[0,c]$.

In fact, for any $x \in P[0,c]$, since

$$0 \leq x(t) \leq \max_{t\in[0,1]} x(t) = \|x\| \leq c + 1, \tag{3.5}$$

from (H1) and (3.5) we have

$$\begin{aligned} \|Tx\| &= \max_{t\in[0,1]} \left\{ \int_0^1 H(t,s) \left( \int_0^1 G_\beta(s,\tau) h(\tau) f\bigl(x(\tau)\bigr)\, d\tau \right)^{q-1} ds \right\} \\ &\leq b \int_0^1 \left( \int_0^1 \frac{\beta - 1}{\Gamma(\beta)} \tau(1-\tau)^{\beta-1} h(\tau) f(c+1)\, d\tau \right)^{q_1 - 1} ds \\ &\leq b \left[ \frac{(\beta - 1) f(1)}{\Gamma(\beta)} \int_0^1 \tau(1-\tau)^{\beta-1} h(\tau)\, d\tau \right]^{q-1} (c+1)^{\epsilon(q-1)} \\ &= A(c+1)^{\epsilon(q-1)} \leq c + 1, \end{aligned}$$

which implies that $T(P[0,c]) \subset P[0,c]$.

Let $x_0(t) = 0$ and $x_1(t) = (Tx_0)(t)$, $t \in [0,1]$. Since $x_0(t) \in P[0,c]$, we have $x_1 \in P[0,c]$. Denote

$$x_{n+1} = Tx_n = T^{n+1} x_0, \quad n = 1, 2, \ldots.$$

It follows from $T(P[0,c]) \subset P[0,c]$ that $x_n \in P[0,c]$. Noticing that $T$ is compact, we get that $\{x_n\}$ is a sequentially compact set.

Since $x_1 = Tx_0 = T0 \in P[0,c]$, we have

$$x_1(t) = (Tx_0)(t) = (T0)(t) \geq 0 = x_0(t), \quad t \in [0,1].$$

By induction we get

$$x_{n+1} \geq x_n, \quad n = 0, 1, 2, \ldots.$$

Consequently, there exists $x^* \in P[0,c]$ such that $x_n \to x^*$. Letting $n \to +\infty$, from the continuity of $T$ and $Tx_n = x_{n-1}$ we obtain $Tx^* = x^*$, which implies that $x^*$ is a nonnegative solution of the nonlinear integral equation (1.1). Since $x^* \in P$, there exist constants $0 \leq m_1 < n_1$ such that

$$0 < m_1 t^{\alpha-1} \leq x^*(t) \leq n_1 t^{\alpha-1}, \quad t \in (0,1),$$

and consequently $x^*$ is a positive solution of the boundary value problem (1.1), and (3.3) holds.

On the other hand, let $y_0(t) = c+1, y_1 = Ty_0, t \in [0,1]$. Then $y_0(t) \in P[0,c]$ and $y_1 \in P[0,c]$. Let

$$y_{n+1} = Ty_n = T^{n+1}y_0, \quad n = 1,2,\ldots.$$

It follows from $T(P[0,c]) \subset P[0,c]$ that

$$y_n \in P[0,c], \quad n = 0,1,2,\ldots.$$

By Lemma 2.3, $T$ is compact, and consequently $\{y_n\}$ is a sequentially compact set.

Now, since $y_1 \in P[0,c]$, we get

$$0 \leq y_1(t) \leq \|y_1\| \leq c+1 = y_0(t).$$

It follows from Lemma 2.3 that $y_2 = Ty_1 \leq Ty_0 = y_1$. By induction we obtain

$$y_{n+1} \leq y_n, \quad n = 0,1,2,\ldots.$$

Consequently, there exists $y^* \in P[0,c]$ such that $y_n \to y^*$. Letting $n \to +\infty$, from the continuity of $T$ and $Ty_n = y_{n-1}$ we have $Ty^* = y^*$, which implies that $y^*$ is another nonnegative solution of the boundary value problem (1.1) and $y^*$ also satisfies (3.3) since $y^* \in P$.

Now we prove that $x^*$ and $y^*$ are extremal solutions for a fractional differential equation (1.1). Let $\tilde{x}$ be any positive solution of the boundary value problem (1.1). Then $x_0 = 0 \leq \tilde{x} \leq c+1 = y_0$, and $x_1 = Tx_0 \leq T\tilde{x} = \tilde{x} \leq T(c+1) = w_1$. By induction we have $x_n \leq \tilde{x} \leq y_n$, $n = 1,2,3,\ldots$. Taking the limit, we have $x^* \leq \tilde{x} \leq y^*$. This implies that $x^*$ and $y^*$ are the maximal and minimal solutions of the BVP (1.1), respectively. The proof is completed. □

**Corollary 3.1** *Suppose that conditions* (H0)-(H2) *hold. If*

$$0 < \epsilon < \frac{1}{q-1}, \tag{3.6}$$

*then the BVP* (1.1) *has the minimal and maximal solutions $x^*$ and $y^*$, which are positive, and there exist some constants $0 \leq m_i \leq n_i$, $i = 1,2$, such that*

$$m_1 t^{\alpha-1} \leq x^*(t) \leq n_1 t^{\alpha-1}, \qquad m_2 t^{\alpha-1} \leq y^*(t) \leq n_2 t^{\alpha-1}, \quad t \in [0,1]. \tag{3.7}$$

*Moreover, there exists a positive constant c such that for initial values* $x_0 = 0$, $y_0 = c + 1$, *the iterative sequences generated by*

$$x_n(t) = (Tx_{n-1})(t) = T^n x_0(t), \qquad y_n(t) = (Ty_{n-1})(t) = T^n y_0(t) \tag{3.8}$$

*converge uniformly to* $x^*$ *and* $y^*$ *for* $t \in [0,1]$, *namely,*

$$\lim_{n\to+\infty} x_n = x^*, \qquad \lim_{n\to+\infty} y_n = y^*.$$

*Proof* From (3.6) we have

$$\lim_{x\to+\infty} \frac{x^{\epsilon(q-1)}}{x} = 0,$$

which implies that we can find a constant $c > 0$ large enough such that

$$(c+1)^{\epsilon(q-1)-1} < A^{-1}.$$

By Theorem 3.1 the conclusion of Corollary 3.1 holds. □

**Remark 3.1** Corollary 3.1 is an interesting case of the boundary value problem (1.1). Because of the independence of $\epsilon$ and $q$, condition (3.6) is easy to be satisfied. For example, for $q = 4$ and $\epsilon = \frac{1}{4}$, the BVP (1.1) has the minimal and maximal solutions if (H0)-(H2) are satisfied.

In addition, note that, when $p = 2$, the nonlinear operator $\mathscr{D}_{\mathbf{t}}^{\beta}(\varphi_p(\mathscr{D}_{\mathbf{t}}^{\alpha}))$ reduces to the linear operator $\mathscr{D}_{\mathbf{t}}^{\beta}(\mathscr{D}_{\mathbf{t}}^{\alpha})$, and if $0 < \epsilon < \frac{1}{2}$, then (3.6) naturally holds, and so we have the following corollary.

**Corollary 3.2** *Suppose that* $p = 2$ *and* (H0), (H2) *hold. Moreover, suppose that* $f$ *satisfies*

(h1) $f : [0,+\infty) \to (0,+\infty)$ *is continuous and nondecreasing, and there exists a constant* $0 < \epsilon < \frac{1}{2}$ *such that, for any* $x \in [0,+\infty)$,

$$f(cx) \geq c^{\epsilon} f(x), \quad \forall 0 < c \leq 1.$$

*Then the BVP* (1.1) *has the minimal and maximal solutions* $x^*$ *and* $y^*$, *which are positive, and there exist constants* $0 \leq m_i \leq n_i$, $i = 1,2$, *such that*

$$m_1 t^{\alpha-1} \leq x^*(t) \leq n_1 t^{\alpha-1}, \qquad m_2 t^{\alpha-1} \leq y^*(t) \leq n_2 t^{\alpha-1}, \quad t \in [0,1]. \tag{3.9}$$

*Moreover, there exists a positive constant c such that for initial values* $x_0 = 0$, $y_0 = c + 1$, *the iterative sequences generated by*

$$x_n(t) = (Tx_{n-1})(t) = T^n x_0(t), \qquad y_n(t) = (Ty_{n-1})(t) = T^n y_0(t) \tag{3.10}$$

*converge uniformly to* $x^*$ *and* $y^*$ *for* $t \in [0,1]$, *namely,*

$$\lim_{n\to+\infty} x_n = x^*, \qquad \lim_{n\to+\infty} y_n = y^*.$$

## 4 Example

Consider the following nonlocal boundary value problem of the fractional $p$-Laplacian equation:

$$\begin{cases} -\mathscr{D}_{\mathbf{t}}^{\frac{4}{3}}(\varphi_3(-\mathscr{D}_{\mathbf{t}}^{\frac{3}{2}}x))(t) = \frac{(3+x^\gamma)x^l}{t(1-t)^{\frac{1}{3}}(2+x^\delta)}, & t \in (0,1), \\ x(0) = 0, \quad \mathscr{D}_{\mathbf{t}}^{\frac{3}{2}}x(0) = \mathscr{D}_{\mathbf{t}}^{\frac{3}{2}}x(1) = 0, \quad x(1) = \int_0^1 x(s)\,dA(s), \end{cases} \tag{4.1}$$

where $\gamma$, $\delta$, $l$ are positive constants satisfying $l > \delta$, $\gamma + l < 2$, $A$ is a bounded-variation function satisfying

$$A(t) = \begin{cases} 0, & t \in [0, \frac{1}{2}), \\ 2, & t \in [\frac{1}{2}, \frac{3}{4}), \\ 1, & t \in [\frac{3}{4}, 1]. \end{cases}$$

Then the BVP (4.1) has the minimal and maximal solutions $x^*$ and $y^*$, which are positive, and there exist constants $0 \leq m_i < n_i$, $i = 1, 2$, such that

$$m_1 t^{\frac{1}{2}} \leq x^*(t) \leq n_1 t^{\frac{1}{2}}, \qquad m_2 t^{\frac{1}{2}} \leq y^*(t) \leq n_2 t^{\frac{1}{2}}, \quad t \in [0,1]. \tag{4.2}$$

Moreover, there exists a positive constant $c$ such that for initial values $x_0 = 0$, $y_0 = c + 1$, the iterative sequences generated by

$$x_n(t) = (Tx_{n-1})(t) = T^n x_0(t), \qquad y_n(t) = (Ty_{n-1})(t) = T^n y_0(t) \tag{4.3}$$

converge uniformly to $x^*$ and $y^*$ for $t \in [0,1]$, namely,

$$\lim_{n\to+\infty} x_n = x^*, \qquad \lim_{n\to+\infty} y_n = y^*.$$

Let

$$\alpha = \frac{3}{2}, \qquad \beta = \frac{4}{3}, \qquad p = 3, \qquad f(x) = \frac{(3+x^\gamma)x^l}{(2+x^\delta)}, \qquad h(t) = \frac{1}{t}(1-t)^{\frac{1}{3}}.$$

Then by simple computation problem (4.1) is equivalent to the following multipoint boundary value problem:

$$\begin{cases} -\mathscr{D}_{\mathbf{t}}^{\frac{4}{3}}(\varphi_3(-\mathscr{D}_{\mathbf{t}}^{\frac{3}{2}}x))(t) = \frac{(3+x^\gamma)x^l}{t(1-t)^{\frac{1}{3}}(2+x^\delta)}, & t \in (0,1), \\ x(0) = 0, \quad \mathscr{D}_{\mathbf{t}}^{\frac{3}{2}}x(0) = \mathscr{D}_{\mathbf{t}}^{\frac{3}{2}}x(1) = 0, \quad x(1) = 2x(\frac{1}{2}) - x(\frac{3}{4}). \end{cases}$$

First, we have

$$\mathcal{A} = \int_0^1 t^{\alpha-1}\,dA(t) = 2 \times \left(\frac{1}{2}\right)^{\frac{1}{2}} - \left(\frac{3}{4}\right)^{\frac{1}{2}} = 0.5482 < 1,$$

and by simple computation we have $\mathcal{G}_A(s) \geq 0$, and so (H0) holds.

Next, from Remark 2.2, for any $0 < \mu \leq 1$, we have

$$f(\mu x) \leq \mu^{\gamma+l} f(x),$$

and thus (H1) holds. Now we compute

$$\int_0^1 s(1-s)^{\beta-1} h(s)\, ds = \int_0^1 s(1-s)^{\frac{1}{3}} h(s)\, ds = 1,$$

so (H2) is satisfied.

Thus, by Corollary 3.1 the BVP (4.1) has maximal and minimal solutions that satisfy (4.2) and (4.3).

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors typed, read, and approved the final manuscript.

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