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Blow-up criteria for Boussinesq system and MHD system and Landau-Lifshitz equations in a bounded domain

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Abstract

In this paper, we prove some blow-up criteria for the 3D Boussinesq system with zero heat conductivity and MHD system and Landau-Lifshitz equations in a bounded domain.

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Keywords: blow-up criterion; Boussinesq system; MHD system; Landau-Lifshitz equations

1 Introduction

Let Ω be a bounded, simply connected domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, and ν be the unit outward normal vector to $\partial\Omega$. First, we consider the regularity criterion of the Boussinesq system with zero heat conductivity:

$$\operatorname{div} u = 0, \tag{1.1}$$

$$\partial_t u + u \cdot \nabla u + \nabla\pi - \Delta u = \theta e_3, \tag{1.2}$$

$$\partial_t \theta + u \cdot \nabla\theta = 0 \quad \text{in } \Omega \times (0,\infty), \tag{1.3}$$

$$u \cdot \nu = 0, \qquad \operatorname{curl} u \times \nu = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.4}$$

$$(u,\theta)(\cdot,0) = (u_0,\theta_0) \quad \text{in } \Omega \subseteq \mathbb{R}^3, \tag{1.5}$$

where u, π, and θ denote the unknown velocity vector field, pressure scalar, and temperature scalar of the fluid, respectively. $\omega := \operatorname{curl} u$ is the vorticity, and $e_3 := (0,0,1)^t$.

When $\theta = 0$, (1.1) and (1.2) are the well-known Navier-Stokes system. Giga [1], Kim [2], and Kang and Kim [3] have proved some Serrin-type regularity criteria.

The first aim of this paper is to prove a new regularity criterion for problem (1.1)-(1.5).

Theorem 1.1 *Let $u_0 \in H^3$ and $\theta_0 \in W^{1,p}$ with $3 < p \le 6$ and $\operatorname{div} u_0 = 0$ in Ω and $u_0 \cdot \nu = 0$, $\operatorname{curl} u_0 \times \nu = 0$ on $\partial\Omega$. Let (u,θ) be a strong solution of problem* (1.1)-(1.5). *If u satisfies*

$$\nabla u \in L^1\big(0,T;\mathrm{BMO}(\Omega)\big) \tag{1.6}$$

with $0 < T < \infty$, *then the solution* (u, θ) *can be extended beyond* $T > 0$. *Here* BMO *denotes the space of bounded mean oscillation*.

Secondly, we consider the blow-up criterion for the 3D MHD system

$$\operatorname{div} u = \operatorname{div} b = 0, \tag{1.7}$$

$$\partial_t u + u \cdot \nabla u + \nabla\left(\pi + \frac{1}{2}|b|^2\right) - \Delta u = b \cdot \nabla b, \tag{1.8}$$

$$\partial_t b + u \cdot \nabla b - b \cdot \nabla u = \Delta b \quad \text{in } \Omega \times (0, \infty), \tag{1.9}$$

$$u \cdot \nu = 0, \qquad \operatorname{curl} u \times \nu = 0, \qquad b \cdot \nu = 0, \qquad \operatorname{curl} b \times \nu = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{1.10}$$

$$(u, b)(\cdot, 0) = (u_0, b_0) \quad \text{in } \Omega \subseteq \mathbb{R}^3. \tag{1.11}$$

Here b is the magnetic field of the fluid.

It is well known that problem (1.7)-(1.11) has a unique local strong solution [4]. But whether this local solution can exist globally is an outstanding problem. Kang and Kim [3] proved some Serrin-type regularity criteria.

The second aim of this paper is to prove a new regularity criterion for problem (1.7)-(1.11).

Theorem 1.2 *Let* $u_0, b_0 \in H^3$ *with* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$ *in* Ω *and* $u_0 \cdot \nu = b_0 \cdot \nu = 0$, $\operatorname{curl} u_0 \times \nu = \operatorname{curl} b_0 \times \nu = 0$ *on* $\partial\Omega$. *Let* (u, b) *be a strong solution to problem* (1.7)-(1.11). *If* (1.6) *holds, then the solution* (u, b) *can be extended beyond* $T > 0$.

Remark 1.1 When $\Omega := \mathbb{R}^3$, our result gives the well-known regularity criterion

$$\omega := \operatorname{curl} u \in L^1(0, T; \dot{B}^0_{\infty,\infty}),$$

but the method of proof we use is different from that in [5, 6]. Here $\dot{B}^0_{\infty,\infty}$ denotes the homogeneous Besov space [7].

Next, we consider the following 3D density-dependent MHD equations:

$$\operatorname{div} u = \operatorname{div} b = 0, \tag{1.12}$$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.13}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla\left(\pi + \frac{1}{2}|b|^2\right) - \Delta u = b \cdot \nabla b, \tag{1.14}$$

$$\partial_t b + u \cdot \nabla b - b \cdot \nabla u = \Delta b \quad \text{in } \Omega \times (0, \infty), \tag{1.15}$$

$$u = 0, \qquad b \cdot \nu = 0, \qquad \operatorname{curl} b \times \nu = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{1.16}$$

$$(\rho, \rho u, b)(\cdot, 0) = (\rho_0, \rho_0 u_0, b_0) \quad \text{in } \Omega \subset \mathbb{R}^3. \tag{1.17}$$

For this problem, Wu [8] proved that if the initial data ρ_0, u_0, and b_0 satisfy

$$\begin{aligned} &0 \leq \rho_0 \in H^2, \qquad u_0 \in H_0^1 \cap H^2, \qquad b_0 \in H^2, \\ &-\Delta u_0 + \nabla\left(\pi_0 + \frac{1}{2}|b_0|^2\right) = b_0 \cdot \nabla b_0 + \sqrt{\rho_0} g \end{aligned} \tag{1.18}$$

for some $(\pi_0, g) \in H^1 \times L^2$, then there exists a positive time T_* and a unique strong solution (ρ, u, b) to problem (1.12)-(1.17) such that

$$\begin{aligned} &\rho \in C([0,T_*]; H^2), \qquad u \in C([0,T_*]; H_0^1 \cap H^2) \cap L^2(0,T_*; H^2), \\ &u_t \in L^2(0,T_*; H_0^1), \qquad \sqrt{\rho} u_t \in L^\infty(0,T_*; L^2), \\ &b \in L^\infty(0,T_*; H^2) \cap L^2(0,T_*; H^3), \qquad b_t \in L^\infty(0,T_*; L^2) \cap L^2(0,T_*; H^1). \end{aligned} \tag{1.19}$$

When $b = 0$, Kim [2] proved the following regularity criterion:

$$u \in L^{\frac{2s}{s-3}}\big(0,T; L_w^s(\Omega)\big) \quad \text{with } 3 < s \leq \infty. \tag{1.20}$$

Here L_w^s denotes the weak-L^s space, and $L_w^\infty = L^\infty$.

The aim of this paper is to refine (1.20) as follows.

Theorem 1.3 *Let ρ_0, u_0, and b_0 satisfy* (1.18). *Let (ρ, u, b) be a strong solution of problem* (1.12)-(1.17) *in the class* (1.19). *Suppose that u satisfies one of the following two conditions*:

$$\text{(i)} \quad \int_0^T \frac{\|u(t)\|_{L_w^s}^{\frac{2s}{s-3}}}{1 + \log(e + \|u(t)\|_{L_w^s})} \, dt < \infty \quad \textit{with } 3 < s \leq \infty, \tag{1.21}$$

$$\text{(ii)} \quad u \in L^2\big(0,T; \mathrm{BMO}(\Omega)\big) \tag{1.22}$$

with $0 < T < \infty$. *Then the solution (ρ, u, b) can be extended beyond $T > 0$.*

Finally, we consider the 3D Landau-Lifshitz system:

$$\partial_t d - \Delta d = d|\nabla d|^2 + d \times \Delta d, \qquad |d| = 1 \quad \text{in } \Omega \times (0,\infty), \tag{1.23}$$

$$\partial_\nu d = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.24}$$

$$d(\cdot,0) = d_0, \qquad |d_0| = 1 \quad \text{in } \Omega \subseteq \mathbb{R}^3. \tag{1.25}$$

Carbou and Fabrie [9] showed the existence and uniqueness of local smooth solutions. When $\Omega := \mathbb{R}^n$ ($n = 2,3,4$), Fan and Ozawa [10] proved some regularity criteria. The aim of this paper is to prove a logarithmic blow-up criterion for problem (1.23)-(1.25) when Ω is a bounded domain. We will prove the following:

Theorem 1.4 *Let $d_0 \in H^3(\Omega)$ with $|d_0| = 1$ in Ω and $\partial_\nu d_0 = 0$ on $\partial\Omega$. Let d be a local smooth solution to problem* (1.23)-(1.25). *If d satisfies*

$$\int_0^T \frac{\|\nabla d\|_{L^q}^{\frac{2q}{q-3}}}{1 + \log(e + \|\nabla d\|_{L^q})} \, dt < \infty \quad \textit{with } 3 < q \leq \infty \tag{1.26}$$

and $0 < T < \infty$, *then the solution can be extended beyond $T > 0$.*

In Section 2, we give some preliminary lemmas, which will be used in the following sections. The proof of Theorem 1.1 for problem (1.1)-(1.5) will be given in Section 3. The new regularly criterion of Theorem 1.2 for the 3D MHD problem (1.7)-(1.11) will be proved in Section 4. In Section 5, we prove Theorem 1.3, and in Section 6, we give the main proof of final Theorem 1.4.

2 Preliminary lemmas

In the following proofs, we will use the logarithmic Sobolev inequality [11]

$$\|\nabla u\|_{L^\infty} \le C\big(1 + \|\nabla u\|_{\mathrm{BMO}} \log\big(e + \|u\|_{W^{s,p}}\big)\big) \quad \text{with } s > 1 + \frac{3}{p} \tag{2.1}$$

and the following three lemmas.

Lemma 2.1 ([12]) *Let $\Omega \subseteq \mathbb{R}^3$ be a smooth bounded domain, let $b : \Omega \to \mathbb{R}^3$ be a smooth vector field, and let $1 < p < \infty$. Then*

$$\begin{aligned} -\int_\Omega \Delta b \cdot b|b|^{p-2}\,dx = {}& \frac{1}{2}\int_\Omega |b|^{p-2}|\nabla b|^2\,dx + 4\frac{p-2}{p^2}\int_\Omega \big|\nabla |b|^{\frac{p}{2}}\big|^2\,dx \\ & - \int_{\partial\Omega} |b|^{p-2}(b\cdot\nabla)b\cdot\nu\,d\sigma - \int_{\partial\Omega} |b|^{p-2}(\operatorname{curl} b \times \nu)\cdot b\,d\sigma. \end{aligned} \tag{2.2}$$

Lemma 2.2 ([13, 14]) *Let Ω be a smooth and bounded open set, and let $1 < p < \infty$. Then we have the estimate*

$$\|b\|_{L^p(\partial\Omega)} \le C\|b\|_{L^p(\Omega)}^{1-\frac{1}{p}}\|b\|_{W^{1,p}(\Omega)}^{\frac{1}{p}} \tag{2.3}$$

for all $b \in W^{1,p}(\Omega)$.

Lemma 2.3 *We have*

$$\|f\|_{L^\infty(\Omega)} \le C\big(1 + \|f\|_{\mathrm{BMO}(\Omega)} \log^{\frac{1}{2}}\big(e + \|f\|_{W^{1,4}(\Omega)}\big)\big) \tag{2.4}$$

for all $f \in W_0^{1,4}(\Omega)$.

Proof When $\Omega := \mathbb{R}^3$, (2.4) is proved by Ogawa [15]. For a bounded domain Ω in $\mathbb{R}^3$, we define

$$\tilde{f} := \begin{cases} f & \text{in } \Omega, \\ 0 & \text{in } \Omega^c := \mathbb{R}^3 \setminus \Omega. \end{cases}$$

Then we have [13], p.71,

$$\|\tilde{f}\|_{W^{1,4}(\mathbb{R}^3)} = \|f\|_{W^{1,4}(\Omega)},$$

and it is obvious that

$$\|\tilde{f}\|_{L^\infty(\mathbb{R}^3)} = \|f\|_{L^\infty(\Omega)}, \|\tilde{f}\|_{\mathrm{BMO}(\mathbb{R}^3)} = \|f\|_{\mathrm{BMO}(\Omega)}.$$

Thus, (2.4) is proved. □

Finally, when b satisfies $b \cdot \nu = 0$ on $\partial\Omega$, we will also use the identity

$$(b \cdot \nabla)b \cdot \nu = -(b \cdot \nabla)\nu \cdot b \quad \text{on } \partial\Omega \tag{2.5}$$

for any sufficiently smooth vector field b.

3 Proof of Theorem 1.1

Since it is easy to prove that problem (1.1)-(1.5) has a unique local-in-time strong solution, we omit the details. We only need to establish a priori estimates.

First, thanks to the maximum principle, it follows from (1.1) and (1.3) that

$$\|\theta\|_{L^\infty(0,T;L^\infty)} \le C. \tag{3.1}$$

Testing (1.2) by u and using (1.1) and (3.1), we see that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + \int_\Omega |\operatorname{curl} u|^2\,dx \le \int_\Omega \theta e_3 \cdot u\,dx \le \frac{1}{2}\int_\Omega \theta^2\,dx + \frac{1}{2}\int_\Omega u^2\,dx,$$

which gives

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} \le C. \tag{3.2}$$

Applying curl to (1.2) and setting $\omega := \operatorname{curl} u$, we find that

$$\partial_t\omega + u \cdot \nabla\omega - \Delta\omega = \omega \cdot \nabla u + \operatorname{curl}(\theta e_3). \tag{3.3}$$

Testing (3.3) by ω and using (1.1) and (3.1), we infer that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\omega|^2\,dx + \int_\Omega |\operatorname{curl}\omega|^2\,dx &= \int_\Omega (\omega \cdot \nabla)u \cdot \omega\,dx + \int_\Omega \theta e_3 \operatorname{curl}\omega\,dx \\ &\le \|\nabla u\|_{L^\infty}\int_\Omega \omega^2\,dx + \frac{1}{2}\int_\Omega |\operatorname{curl}\omega|^2\,dx + C,\end{aligned}$$

which implies

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\omega|^2\,dx + \int_\Omega |\operatorname{curl}\omega|^2\,dx &\le C\|\nabla u\|_{L^\infty}\int_\Omega |\omega|^2\,dx + C \\ &\le C\big(1 + \|\nabla u\|_{\mathrm{BMO}}\big)\log\big(e + \|u\|_{H^3}\big)\int_\Omega |\omega|^2\,dx + C,\end{aligned}$$

and therefore

$$\int_\Omega |\omega|^2\,dx + \int_{t_0}^t \|\operatorname{curl}\omega\|_{L^2}^2\,d\tau \le C(e + y)^{C_0\epsilon}, \tag{3.4}$$

provided that

$$\int_{t_0}^t \|\nabla u\|_{\mathrm{BMO}}\,d\tau \le \epsilon \ll 1, \tag{3.5}$$

and $y(t) := \sup_{[t_0,t]}\|u\|_{H^3}$ for any $0 < t_0 \le t \le T$, and C_0 is an absolute constant.

Applying ∂_t to (1.2), we deduce that

$$\partial_t^2 u + u \cdot \nabla u_t + \nabla \pi_t - \Delta u_t = -u_t \cdot \nabla u + \theta_t e_3. \tag{3.6}$$

Testing (3.6) by u_t and using (1.1), (1.3), (3.1), and (3.2), we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2\,dx + \int_\Omega |\operatorname{curl} u_t|^2\,dx \\
&\quad = -\int_\Omega u_t \cdot \nabla u \cdot u_t\,dx + \int_\Omega \theta_t e_3 u_t\,dx \\
&\quad = -\int_\Omega u_t \cdot \nabla u \cdot u_t\,dx - \int_\Omega \operatorname{div}(u\theta) e_3 u_t\,dx \\
&\quad = -\int_\Omega u_t \cdot \nabla u \cdot u_t\,dx + \int_\Omega u\theta \nabla(e_3 u_t)\,dx \\
&\quad \le \|\nabla u\|_{L^\infty} \int_\Omega |u_t|^2\,dx + \frac{1}{2}\int_\Omega |\operatorname{curl} u_t|^2\,dx + C \\
&\quad \le C\bigl(1 + \|\nabla u\|_{\mathrm{BMO}}\bigr)\log(e+y)\int_\Omega |u_t|^2\,dx + \frac{1}{2}\int_\Omega |\operatorname{curl} u_t|^2\,dx + C,
\end{aligned}$$

which yields

$$\int_\Omega |u_t|^2\,dx + \int_{t_0}^t \int_\Omega |\operatorname{curl} u_t|^2\,dx\,d\tau \le C(e+y)^{C_0\epsilon}. \tag{3.7}$$

On the other hand, thanks to the H^2-theory of the Stokes system, if follows from (1.2), (3.1), (3.4), and (3.7) that

$$\begin{aligned}
\|u\|_{H^2} &\le C\|-\Delta u + \nabla \pi\|_{L^2} \\
&\le C\|\partial_t u + u \cdot \nabla u - \theta e_3\|_{L^2} \\
&\le C\|u_t\|_{L^2} + C\|u\|_{L^6}\|\nabla u\|_{L^3} + C\|\theta\|_{L^2} \\
&\le C\|u_t\|_{L^2} + C\|\nabla u\|_{L^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}} + C,
\end{aligned}$$

which implies

$$\|u\|_{H^2} \le C\|u_t\|_{L^2} + C\|\nabla u\|_{L^2}^3 + C \le C(e+y)^{C_0\epsilon}. \tag{3.8}$$

Applying ∇ to (1.3), testing by $|\nabla\theta|^{p-2}\nabla\theta$ $(2 \le p < \infty)$, and using (1.1), we get

$$\begin{aligned}
\frac{d}{dt}\|\nabla\theta\|_{L^p} &\le C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^p} \\
&\le C\bigl(1 + \|\nabla u\|_{\mathrm{BMO}}\bigr)\log(e+y)\|\nabla\theta\|_{L^p},
\end{aligned}$$

which leads to

$$\|\nabla\theta\|_{L^\infty(t_0,t;L^p)} \le C(e+y)^{C_0\epsilon} \quad \text{with } 2 \le p < \infty. \tag{3.9}$$

Testing (3.6) by $-\Delta u_t + \nabla\pi_t$ and using (1.1), (1.3), (3.7), (3.8), and (3.9), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\operatorname{curl} u_t|^2\,dx + \int_\Omega |-\Delta u_t + \nabla\pi_t|^2\,dx \\
&\quad= \int_\Omega (-u_t\cdot\nabla u + \theta_t e_3 - u\cdot\nabla u_t)(-\Delta u_t + \nabla\pi_t)\,dx \\
&\quad\le \bigl(\|\nabla u\|_{L^6}\|u_t\|_{L^3} + \|u\|_{L^\infty}\|\nabla\theta\|_{L^2} + \|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\bigr)\|-\Delta u_t + \nabla\pi_t\|_{L^2} \\
&\quad\le \|u\|_{H^2}\bigl(\|u_t\|_{H^1} + \|\nabla\theta\|_{L^2}\bigr)\|-\Delta u_t + \nabla\pi_t\|_{L^2} \\
&\quad\le \frac{1}{2}\|-\Delta u_t + \nabla\pi_t\|_{L^2}^2 + C\|u\|_{H^2}^2\bigl(\|u_t\|_{H^1}^2 + \|\nabla\theta\|_{L^2}^2\bigr),
\end{aligned}$$

which leads to

$$\int_\Omega |\operatorname{curl} u_t|^2\,dx + \int_{t_0}^t \|u_t\|_{H^2}^2\,d\tau \le C(e+y)^{C_0\epsilon}. \tag{3.10}$$

On the other hand, if follows from (3.3), (3.10), (3.9), and (3.8) that

$$\begin{aligned}
\|u\|_{H^3} &\le C\bigl(1 + \|\Delta\omega\|_{L^2}\bigr) \\
&\le C\bigl(1 + \bigl\|\partial_t\omega + u\cdot\nabla\omega - \omega\cdot\nabla u - \operatorname{curl}(\theta e_3)\bigr\|_{L^2}\bigr) \\
&\le C\bigl(1 + \|\partial_t\omega\|_{L^2} + \|u\|_{L^\infty}\|\nabla\omega\|_{L^2} + \|\omega\|_{L^4}\|\nabla u\|_{L^4} + \|\nabla\theta\|_{L^2}\bigr) \\
&\le C(e+y)^{C_0\epsilon},
\end{aligned}$$

which gives

$$\|u\|_{L^\infty(0,T;H^3)} \le C \tag{3.11}$$

and

$$\|\theta\|_{L^\infty(0,T;W^{1,p})} \le C \quad \text{with } 3\le p\le 6. \tag{3.12}$$

This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

We only need to prove a priori estimates.

First, testing (1.8) by u and using (1.7), we see that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + \int_\Omega |\operatorname{curl} u|^2\,dx = \int_\Omega (b\cdot\nabla)b\cdot u\,dx. \tag{4.1}$$

Testing (1.9) by b and using (1.7), we find that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega b^2\,dx + \int_\Omega |\operatorname{curl} b|^2\,dx = \int_\Omega (b\cdot\nabla)u\cdot b\,dx. \tag{4.2}$$

Summing up (4.1) and (4.2), we get the well-known energy inequality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \bigl(u^2 + b^2\bigr)\,dx + \int_\Omega \bigl(|\operatorname{curl} u|^2 + |\operatorname{curl} b|^2\bigr)\,dx \le 0. \tag{4.3}$$

Testing (1.9) by $|b|^{p-2}b$ $(2 \le p \le 6)$ and using (1.7), (2.2), (2.3), and (2.5), we derive

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega |b|^p\,dx + \frac{1}{2}\int_\Omega |b|^{p-2}|\nabla b|^2\,dx + 4\frac{p-2}{p^2}\int_\Omega \bigl|\nabla |b|^{\frac{p}{2}}\bigr|^2\,dx \\
&\quad = -\int_{\partial\Omega} |b|^{p-2}(b\cdot\nabla)\nu\cdot b\,d\sigma + \int_\Omega b\cdot\nabla u\cdot |b|^{p-2}b\,dx \\
&\quad \le C\int_{\partial\Omega} |b|^p\,dx + \|\nabla u\|_{L^\infty}\int_\Omega |b|^p\,dx \\
&\quad \le 2\frac{p-2}{p^2}\int_\Omega \bigl|\nabla |b|^{\frac{p}{2}}\bigr|^2\,dx + C\bigl(1+\|\nabla u\|_{L^\infty}\bigr)\int_\Omega |b|^p\,dx \\
&\quad \le 2\frac{p-2}{p^2}\int_\Omega \bigl|\nabla |b|^{\frac{p}{2}}\bigr|^2\,dx + C\bigl(1+\|\nabla u\|_{\mathrm{BMO}}\bigr)\int_\Omega |b|^p\,dx\log(e+y),
\end{aligned}$$

which implies

$$\|b\|_{L^\infty(t_0,t;L^p)} + \int_{t_0}^t\int_\Omega |b|^2|\nabla b|^2\,dx\,d\tau \le C(e+y)^{C_0\epsilon} \quad \text{with } 2\le p\le 6, \tag{4.4}$$

with the same y and ϵ as in (3.5).

Taking curl to (1.8) and (1.9), respectively, and setting $\omega := \operatorname{curl} u$ and $j := \operatorname{curl} b$, we infer that

$$\partial_t\omega + u\cdot\nabla\omega - \Delta\omega = \omega\cdot\nabla u + b\cdot\nabla j + \sum_i \nabla b_i\times\partial_i b, \tag{4.5}$$

$$\partial_t j + u\cdot\nabla j - \Delta j = b\cdot\nabla\omega + \sum_i \nabla b_i\times\partial_i u - \sum_i \nabla u_i\times\partial_i b. \tag{4.6}$$

Testing (4.5) and (4.6) by ω and j, respectively, summing up the result, and using (1.7), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \bigl(\omega^2+j^2\bigr)\,dx + \int_\Omega \bigl(|\operatorname{curl}\omega|^2 + |\operatorname{curl} j|^2\bigr)\,dx \\
&\quad = \int_\Omega (\omega\cdot\nabla)u\cdot\omega\,dx + \sum_i\int_\Omega (\nabla b_i\times\partial_i b)\omega\,dx \\
&\qquad + \sum_i\int_\Omega (\nabla b_i\times\partial_i u)\cdot j\,dx - \sum_i\int_\Omega (\nabla u_i\times\partial_i b)\cdot j\,dx \\
&\quad \le C\|\nabla u\|_{L^\infty}\int_\Omega \bigl(\omega^2+j^2\bigr)\,dx \\
&\quad \le C\bigl(1+\|\nabla u\|_{\mathrm{BMO}}\bigr)\int_\Omega \bigl(\omega^2+j^2\bigr)\,dx\log(e+y),
\end{aligned}$$

which implies

$$\int_\Omega \bigl(\omega^2+j^2\bigr)\,dx + \int_{t_0}^t\int_\Omega \bigl(|\operatorname{curl}\omega|^2 + |\operatorname{curl} j|^2\bigr)\,dx\,d\tau \le C(e+y)^{C_0\epsilon}. \tag{4.7}$$

Thus, it follows from (1.8), (1.9), and (4.7) that

$$\int_{t_0}^{t}\int_\Omega\big(|u_t|^2+|b_t|^2\big)\,dx\,d\tau\le C(e+y)^{C_0\epsilon}. \tag{4.8}$$

Applying ∂_t to (1.8), we have

$$\partial_t^2u+u\cdot\nabla u_t+\nabla\pi_t-\Delta u_t=\operatorname{div}(b\otimes b)_t-u_t\cdot\nabla u. \tag{4.9}$$

Testing (4.9) by u_t and using (1.7), we get

$$\begin{aligned}
&\frac12\frac{d}{dt}\int_\Omega|u_t|^2\,dx+\int_\Omega|\operatorname{curl}u_t|^2\,dx\\
&\quad=-\sum_{i,j}\int_\Omega\big(b^ib^j\big)_t\partial_ju_t^i\,dx-\int_\Omega u_t\cdot\nabla u\cdot u_t\,dx\\
&\quad\le C\|b_t\|_{L^3}\|b\|_{L^6}\|\nabla u_t\|_{L^2}+\|\nabla u\|_{L^2}\|u_t\|_{L^4}^2\\
&\quad\le C\|b_t\|_{L^2}^{\frac12}\|\operatorname{curl}b_t\|_{L^2}^{\frac12}\|\operatorname{curl}u_t\|_{L^2}\|b\|_{L^6}+C\|\nabla u\|_{L^2}\|u_t\|_{L^2}^{\frac12}\|\operatorname{curl}u_t\|_{L^2}^{\frac32}\\
&\quad\le\delta\|\operatorname{curl}u_t\|_{L^2}^2+\delta\|\operatorname{curl}b_t\|_{L^2}^2+C\|b_t\|_{L^2}^2\|b\|_{L^6}^4+C\|\nabla u\|_{L^2}^4\|u_t\|_{L^2}^2
\end{aligned} \tag{4.10}$$

for any $\delta\in(0,1)$.

Applying ∂_t to (1.9), we have

$$\partial_t^2b+u\cdot\nabla b_t-\Delta b_t=b_t\cdot\nabla u+b\cdot\nabla u_t-u_t\cdot\nabla b. \tag{4.11}$$

Testing (4.11) by b_t and using (1.7), we deduce that

$$\begin{aligned}
&\frac12\frac{d}{dt}\int_\Omega|b_t|^2\,dx+\int_\Omega|\operatorname{curl}b_t|^2\,dx\\
&\quad=\int_\Omega(b_t\cdot\nabla u+b\cdot\nabla u_t-u_t\cdot\nabla b)b_t\,dx\\
&\quad\le\|\nabla u\|_{L^2}\|b_t\|_{L^4}^2+\|b\|_{L^6}\|\nabla u_t\|_{L^2}\|b_t\|_{L^3}+\|\nabla b\|_{L^2}\|u_t\|_{L^4}\|b_t\|_{L^4}\\
&\quad\le\delta\|\operatorname{curl}b_t\|_{L^2}^2+\delta\|\operatorname{curl}u_t\|_{L^2}^2\\
&\qquad+C\|\nabla u\|_{L^2}^4\|b_t\|_{L^2}^2+C\|b\|_{L^6}^4\|b_t\|_{L^2}^2+C\|\nabla b\|_{L^2}^4\big(\|u_t\|_{L^2}^2+\|b_t\|_{L^2}^2\big)
\end{aligned} \tag{4.12}$$

for any $\delta\in(0,1)$.

Combining (4.10) and (4.12), taking δ small enough, and using (4.7) and (4.8), we have

$$\int_\Omega\big(|u_t|^2+|b_t|^2\big)\,dx+\int_{t_0}^{t}\int_\Omega\big(|\operatorname{curl}u_t|^2+|\operatorname{curl}b_t|^2\big)\,dx\,d\tau\le C(e+y)^{C_0\epsilon}. \tag{4.13}$$

It follows from (1.8), (1.9), (4.7), and (4.13) that

$$\|u\|_{L^\infty(t_0,t;H^2)}+\|b\|_{L^\infty(t_0,t;H^2)}\le C(e+y)^{C_0\epsilon}. \tag{4.14}$$

Testing (4.9) by $\nabla(\pi + \frac{1}{2}|b|^2)_t - \Delta u_t$ and using (1.7), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\operatorname{curl} u_t|^2\,dx + \int_\Omega \left|\nabla\left(\pi + \frac{1}{2}|b|^2\right)_t - \Delta u_t\right|^2 dx \\
&\quad = \int_\Omega \big((b\cdot\nabla b)_t - u_t\cdot\nabla u - u\cdot\nabla u_t\big)\left(\nabla\left(\pi + \frac{1}{2}|b|^2\right)_t - \Delta u_t\right) dx \\
&\quad \le C\big(\|b\|_{L^\infty}\|\nabla b_t\|_{L^2} + \|b_t\|_{L^6}\|\nabla b\|_{L^3} + \|u_t\|_{L^6}\|\nabla u\|_{L^3} \\
&\qquad + \|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\big)\left\|\nabla\left(\pi + \frac{1}{2}|b|^2\right)_t - \Delta u_t\right\|_{L^2} \\
&\quad \le \frac{1}{4}\left\|\nabla\left(\pi + \frac{1}{2}|b|^2\right)_t - \Delta u_t\right\|_{L^2}^2 + C\big(\|u\|_{L^\infty}^2 + \|\nabla u\|_{L^3}^2\big)\|\nabla u_t\|_{L^2}^2 \\
&\qquad + C\big(\|b\|_{L^\infty}^2 + \|\nabla b\|_{L^3}^2\big)\|\nabla b_t\|_{L^2}^2.
\end{aligned} \tag{4.15}$$

Similarly, testing (4.11) by $-\Delta b_t$, we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\operatorname{curl} b_t|^2\,dx + \int_\Omega |\Delta b_t|^2\,dx \\
&\quad = \int_\Omega (u_t\cdot\nabla b + u\cdot\nabla b_t - b_t\cdot\nabla u - b\cdot\nabla u_t)\Delta b_t\,dx \\
&\quad \le \big(\|u_t\|_{L^6}\|\nabla b\|_{L^3} + \|u\|_{L^\infty}\|\nabla b_t\|_{L^2} + \|\nabla u\|_{L^3}\|b_t\|_{L^6} + \|b\|_{L^\infty}\|\nabla u_t\|_{L^2}\big)\|\Delta b_t\|_{L^2} \\
&\quad \le \frac{1}{4}\|\Delta b_t\|_{L^2}^2 + C\big(\|u\|_{L^\infty}^2 + \|\nabla u\|_{L^3}^2\big)\|\nabla b_t\|_{L^2}^2 + C\big(\|b\|_{L^\infty}^2 + \|\nabla b\|_{L^3}^2\big)\|\nabla u_t\|_{L^2}^2.
\end{aligned} \tag{4.16}$$

Combining (4.15) and (4.16) and using (4.14) and (4.13), we have

$$\int_\Omega \big(|\operatorname{curl} u_t|^2 + |\operatorname{curl} b_t|^2\big)\,dx + \int_{t_0}^t \int_\Omega \big(|\Delta u_t|^2 + |\Delta b_t|^2\big)\,dx\,d\tau \le C(e+y)^{C_0\epsilon}. \tag{4.17}$$

On the other hand, it follows from (4.5), (4.6), (4.3), (4.17), and (4.14) that

$$\begin{aligned}
\|u(t)\|_{H^3} + \|b(t)\|_{H^3} &\le C\big(1 + \|\Delta\omega\|_{L^2} + \|\Delta j\|_{L^2}\big) \\
&\le C\bigg(1 + \Big\|\partial_t\omega + u\cdot\nabla\omega - \omega\cdot\nabla u - b\cdot\nabla j - \sum_i \nabla b_i \times \partial_i b\Big\|_{L^2} \\
&\qquad + \Big\|\partial_t j + u\cdot\nabla j - b\cdot\nabla\omega + \sum_i \nabla u_i \times \partial_i b - \sum_i \nabla b_i \times \partial_i u\Big\|_{L^2}\bigg) \\
&\le C\big(e + y(t)\big)^{C_0\epsilon},
\end{aligned}$$

which yields

$$\|u\|_{L^\infty(0,T;H^3)} + \|b\|_{L^\infty(0,T;H^3)} \le C,$$

This completes the proof of Theorem 1.2.

5 Proof of Theorem 1.3

We only need to establish a priori estimates.

First, it follows from (1.12) and (1.13) that

$$\|\rho\|_{L^\infty(0,T;L^\infty)} \le C. \tag{5.1}$$

Testing (1.14) by u and using (1.12) and (1.13), we see that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho u^2\,dx + \int_\Omega |\nabla u|^2\,dx = \int_\Omega (b\cdot\nabla)b\cdot u\,dx \tag{5.2}$$

and testing (1.15) by b and using (1.12) and (1.16), we find that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |b|^2\,dx + \int_\Omega |\operatorname{curl} b|^2\,dx = \int_\Omega (b\cdot\nabla)u\cdot b\,dx. \tag{5.3}$$

Summing up (5.2) and (5.3), we get the well-known energy inequality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \bigl(\rho|u|^2 + |b|^2\bigr)\,dx + \int_\Omega \bigl(|\nabla u|^2 + |\operatorname{curl} b|^2\bigr)\,dx \le 0. \tag{5.4}$$

(I) Let (1.21) hold.

Testing (1.15) by $|b|^{p-2}b$ ($2 \le p < \infty$), using (1.12), (2.2), (2.3), and (2.5), setting $\phi = |b|^{\frac{p}{2}}$, and using the Gagliardo-Nirenberg inequality [3]

$$\|\phi\|_{L^{\frac{2s}{s-2},2}} \le C\|\phi\|_{L^2}^{1-\frac{3}{s}}\|\phi\|_{H^1}^{\frac{3}{s}} \quad \text{with } 3 < s \le \infty \tag{5.5}$$

and the generalized Hölder inequality [7]

$$\|fg\|_{L^{p,q}} \le C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}} \tag{5.6}$$

with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, we derive

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega |b|^p\,dx + \frac{1}{2}\int_\Omega |b|^{p-2}|\nabla b|^2\,dx + 4\frac{p-2}{p^2}\int_\Omega \bigl|\nabla|b|^{\frac{p}{2}}\bigr|^2\,dx \\
&\quad = -\int_{\partial\Omega} |b|^{p-2}(b\cdot\nabla)\nu\cdot b\,d\sigma + \int_\Omega (b\cdot\nabla)u\cdot|b|^{p-2}b\,dx \\
&\quad \le \|\nabla\nu\|_{L^\infty}\int_{\partial\Omega}|b|^p\,d\sigma - \sum_i\int_\Omega b_i u\partial_i\bigl(|b|^{p-2}b\bigr)\,dx \\
&\quad \le C\int_{\partial\Omega}\phi^2\,d\sigma + C\int_\Omega |u\phi\nabla\phi|\,dx \\
&\quad \le C\int_{\partial\Omega}\phi^2\,d\sigma + C\|u\|_{L^s_w}\|\phi\|_{L^{\frac{2s}{s-2},2}}\|\nabla\phi\|_{L^2} \\
&\quad \le C\|\phi\|_{L^2}\|\phi\|_{H^1} + C\|u\|_{L^s_w}\|\phi\|_{L^2}^{1-\frac{3}{s}}\|\nabla\phi\|_{L^2}^{1+\frac{3}{s}} \\
&\quad \le 2\frac{p-2}{p^2}\int_\Omega |\nabla\phi|^2\,dx + C\|\phi\|_{L^2}^2 + C\|u\|_{L^s_w}^{\frac{2s}{s-3}}\|\phi\|_{L^2}^2,
\end{aligned}$$

which yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega \phi^2\,dx + C\int_\Omega |\nabla\phi|^2\,dx &\le C\bigl(1+\|u\|_{L^s_w}^{\frac{2s}{s-3}}\bigr)\|\phi\|_{L^2}^2\\ &\le C\biggl(1+\frac{\|u\|_{L^s_w}^{\frac{2s}{s-3}}}{1+\log(e+\|u\|_{L^s_w})}\biggr)\|\phi\|_{L^2}^2\bigl(1+\log\bigl(e+\|u\|_{L^s_w}\bigr)\bigr)\\ &\le C\biggl(1+\frac{\|u\|_{L^s_w}^{\frac{2s}{s-3}}}{1+\log(e+\|u\|_{L^s_w})}\biggr)\bigl(1+\log(e+y)\bigr)\|\phi\|_{L^2}^2,\end{aligned}$$

from which it follows that

$$\|b\|_{L^\infty(t_0,t;L^p)} + \int_{t_0}^t\int_\Omega |b|^2|\nabla b|^2\,dx\,d\tau \le C\bigl(e+y(t)\bigr)^{C_0\epsilon} \tag{5.7}$$

with

$$y(t) := \sup_{[t_0,t]}\|u\|_{W^{1,4}}$$

for any $0 < t_0 \le t \le T$, where C_0 is an absolute constant, provided that

$$\int_{t_0}^T \frac{\|u\|_{L^s_w}^{\frac{2s}{s-3}}}{1+\log(e+\|u\|_{L^s_w})}\,d\tau \le \epsilon \ll 1. \tag{5.8}$$

Testing (1.14) by u_t and using (1.12) and (1.13), we infer that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + \int_\Omega \rho|u_t|^2\,dx &= -\int_\Omega \rho u\cdot\nabla u\cdot u_t\,dx + \int_\Omega b\cdot\nabla b\cdot u_t\,dx\\ &=: I_1 + I_2.\end{aligned} \tag{5.9}$$

We first compute I_2:

$$\begin{aligned}I_2 &= \int_\Omega \operatorname{div}(b\otimes b)\cdot u_t\,dx = -\int_\Omega b\otimes b:\nabla u_t\,dx\\ &= -\frac{d}{dt}\int_\Omega b\otimes b:\nabla u\,dx + 2\int_\Omega b\otimes b_t:\nabla u\,dx\\ &\le -\frac{d}{dt}\int_\Omega b\otimes b:\nabla u\,dx + C\|b_t\|_{L^2}\|b\|_{L^6}\|\nabla u\|_{L^3}\\ &\le -\frac{d}{dt}\int_\Omega b\otimes b:\nabla u\,dx + C\|b_t\|_{L^2}\|b\|_{L^6}\|\nabla u\|_{L^2}^{\frac12}\|u\|_{H^2}^{\frac12}\\ &\le -\frac{d}{dt}\int_\Omega b\otimes b:\nabla u\,dx + \delta\|b_t\|_{L^2}^2 + \delta\|u\|_{H^2}^2 + C\|b\|_{L^6}^4\|\nabla u\|_{L^2}^2\end{aligned} \tag{5.10}$$

for any $0 < \delta < 1$.

We use (5.1), (5.5), and (5.6) to bound I_1 as follows:

$$I_1 \le \|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}\|_{L^\infty}\|u\|_{L^s_w}\|\nabla u\|_{L^{\frac{2s}{s-2},2}}$$

$$
\begin{aligned}
&\le C\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{L^s_w}\|\nabla u\|_{L^2}^{1-\frac{3}{s}}\|u\|_{H^2}^{\frac{3}{s}} \\
&\le \delta\|\sqrt{u}_t\|_{L^2}^2 + \delta\|u\|_{H^2}^2 + C\|u\|_{L^s_w}^{\frac{2s}{s-3}}\|\nabla u\|_{L^2}^2
\end{aligned}
\tag{5.11}
$$

for any $0 < \delta < 1$.

On the other hand, by the H^2-theory of the Stokes system, using (5.1), (5.5), and (5.6), we obtain

$$
\begin{aligned}
\|u\|_{H^2} &\le C\left\|-\Delta u + \nabla\left(\pi + \frac{1}{2}|b|^2\right)\right\|_{L^2} \\
&\le C\|\rho\partial_t u + \rho u\cdot\nabla u - b\cdot\nabla b\|_{L^2} \\
&\le C\|\sqrt{\rho}u_t\|_{L^2} + C\|u\|_{L^s_w}\|\nabla u\|_{L^{\frac{2s}{s-2},2}} + C\|b\cdot\nabla b\|_{L^2} \\
&\le C\|\sqrt{\rho}u_t\|_{L^2} + C\|u\|_{L^s_w}\|\nabla u\|_{L^2}^{1-\frac{3}{s}}\|u\|_{H^2}^{\frac{3}{s}} + C\|b\cdot\nabla b\|_{L^2},
\end{aligned}
$$

which gives

$$
\|u\|_{H^2} \le C\|\sqrt{\rho}u_t\|_{L^2} + C\|b\cdot\nabla b\|_{L^2} + C\|u\|_{L^s_w}^{\frac{s}{s-3}}\|\nabla u\|_{L^2}. \tag{5.12}
$$

Testing (1.15) by $b_t - \Delta b$ and using (5.5) and (5.6), we deduce that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |\operatorname{curl} b|^2\,dx + \int_\Omega \left(|b_t|^2 + |\Delta b|^2\right)dx \\
&= \int_\Omega (b\cdot\nabla u - u\cdot\nabla b)(b_t - \Delta b)\,dx \\
&\le \left(\|u\|_{L^s_w}\|\nabla b\|_{L^{\frac{2s}{s-2},2}} + \|b\|_{L^6}\|\nabla u\|_{L^3}\right)\left(\|b_t\|_{L^2} + \|\Delta b\|_{L^2}\right) \\
&\le C\left(\|u\|_{L^s_w}\|\nabla b\|_{L^2}^{1-\frac{3}{s}}\|b\|_{H^2}^{\frac{3}{s}} + C\|b\|_{L^6}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}\right)\left(\|b_t\|_{L^2} + \|\Delta b\|_{L^2}\right) \\
&\le \frac{1}{2}\left(\|b_t\|_{L^2}^2 + \|\Delta b\|_{L^2}^2\right) + \delta\|u\|_{H^2}^2 + C\|b\|_{L^6}^4\|\nabla u\|_{L^2}^2 + C\|u\|_{L^s_w}^{\frac{2s}{s-3}}\|\nabla b\|_{L^2}^2 + C
\end{aligned}
\tag{5.13}
$$

for any $0 < \delta < 1$.

It is easy to compute that

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega |b|^4\,dx &\le C\int_\Omega |b|^3|b_t|\,dx \\
&\le C\|b\|_{L^6}^3\|b_t\|_{L^2} \le \delta\|b_t\|_{L^2}^2 + C\|b\|_{L^6}^6
\end{aligned}
\tag{5.14}
$$

for any $0 < \delta < 1$.

Combining (5.9), (5.10), (5.11), (5.12), (5.13) and (5.14), and taking δ small enough, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(|\nabla u|^2 + |\operatorname{curl} b|^2 + b\otimes b : \nabla u + C_0|b|^4\right)dx \\
&\quad + \int_\Omega \left(\rho|u_t|^2 + |b_t|^2 + |\Delta b|^2\right)dx + \|u\|_{H^2}^2 \\
&\le C\|b\|_{L^6}^4\|\nabla u\|_{L^2}^2 + C\|u\|_{L^s_w}^{\frac{2s}{s-3}}\left(\|\nabla u\|_{L^2}^2 + \|\operatorname{curl} b\|_{L^2}^2\right) + C\|b\cdot\nabla b\|_{L^2}^2 + C.
\end{aligned}
\tag{5.15}
$$

Using (5.4), (5.7), (5.8), and the Gronwall inequality, we have

$$\begin{aligned}
&\int_\Omega \big(|\nabla u|^2 + |\operatorname{curl} b|^2 + b\otimes b : \nabla u + C_0|b|^4\big)\,dx \\
&\quad\le \bigg[\int_\Omega \big(|\nabla u_0|^2 + |\operatorname{curl} b_0|^2 + b_0\otimes b_0 : \nabla u_0 + C_0|b_0|^4\big)\,dx \\
&\qquad + C\|b\|^4_{L^\infty(t_0,t;L^6)}\int_{t_0}^t \|\nabla u\|^2_{L^2}\,d\tau + C(t-t_0) + C\int_{t_0}^t \|b\cdot\nabla b\|^2_{L^2}\,d\tau\bigg] \\
&\qquad \times \exp\bigg(\int_{t_0}^t \|u\|^{\frac{2s}{s-3}}_{L^s_w}\,d\tau\bigg) \\
&\quad\le C(e+y)^{C_0\epsilon}\exp\bigg[\int_{t_0}^t \frac{\|u\|^{\frac{2s}{s-3}}_{L^s_w}}{1+\log(e+\|u\|_{L^s_w})}\,d\tau\big(1+\log(e+y)\big)\bigg] \\
&\quad\le C(e+y)^{C_0\epsilon}.
\end{aligned} \tag{5.16}$$

Plugging (5.16) into (5.15) and integrating over $[t_0,t]$, we have

$$\int_{t_0}^t\int_\Omega \big(\rho|u_t|^2 + |b_t|^2 + |\Delta b|^2\big)\,dx\,d\tau + \int_{t_0}^t \|u\|^2_{H^2}\,d\tau \le C(e+y)^{C_0\epsilon}. \tag{5.17}$$

Applying ∂_t to (1.15), testing by u_t, and using (1.12) and (1.13), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|u_t|^2\,dx + \int_\Omega |\nabla u_t|^2\,dx \\
&\quad= -\int_\Omega \rho u\cdot\nabla|u_t|^2\,dx - \int_\Omega \rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)\,dx \\
&\qquad - \int_\Omega \rho u_t\cdot\nabla u\cdot u_t\,dx + \int_\Omega b\otimes b_t : \nabla u_t\,dx + \int_\Omega b_t\otimes b : \nabla u_t\,dx \\
&\quad\le C\|u\|_{L^6}\|\sqrt{\rho}u_t\|_{L^3}\|\nabla u_t\|_{L^2} + C\|u\|_{L^6}\|\nabla u\|_{L^6}\|u_t\|_{L^6}\|\nabla u\|_{L^2} \\
&\qquad + C\|u\|^2_{L^6}\|\Delta u\|_{L^2}\|u_t\|_{L^6} + C\|u\|^2_{L^6}\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2} \\
&\qquad + C\|\sqrt{\rho}u_t\|^2_{L^4}\|\nabla u\|_{L^2} + C\|b\|_{L^6}\|b_t\|_{L^3}\|\nabla u_t\|_{L^2} \\
&\quad\le C\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|^{\frac12}_{L^2}\|\sqrt{\rho}u_t\|^{\frac12}_{L^6}\|\nabla u_t\|_{L^2} \\
&\qquad + C\|\nabla u\|^2_{L^2}\|u\|_{H^2}\|\nabla u_t\|_{L^2} + C\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|^{\frac12}_{L^2}\|\sqrt{\rho}u_t\|^{\frac32}_{L^6} \\
&\qquad + C\|b\|_{L^6}\|b_t\|_{L^3}\|\nabla u_t\|_{L^2} \\
&\quad\le C\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|^{\frac12}_{L^2}\|\nabla u_t\|^{\frac32}_{L^2} + C\|\nabla u\|^2_{L^2}\|u\|_{H^2}\|\nabla u_t\|_{L^2} \\
&\qquad + C\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|^{\frac12}_{L^2}\|\nabla u_t\|^{\frac32}_{L^2} + C\|b\|_{L^6}\|b_t\|_{L^3}\|\nabla u_t\|_{L^2} \\
&\quad\le \frac14\|\nabla u_t\|^2_{L^2} + C\|\nabla u\|^4_{L^2}\big(\|\sqrt{\rho}u_t\|^2_{L^2} + \|u\|^2_{H^2}\big) + C\|b\|^2_{L^6}\|b_t\|^2_{L^3} \\
&\quad\le \frac14\|\nabla u_t\|^2_{L^2} + C\|\nabla u\|^4_{L^2}\big(\|\sqrt{\rho}u_t\|^2_{L^2} + \|u\|^2_{H^2}\big) + \frac14\|\operatorname{curl} b_t\|^2_{L^2} \\
&\qquad + C\|b\|^4_{L^6}\|b_t\|^2_{L^2}.
\end{aligned} \tag{5.18}$$

Applying ∂_t to (1.15), testing by b_t, and using (1.12), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |b_t|^2\,dx+\int_\Omega |\operatorname{curl} b_t|^2\,dx \\
&\quad=-\int_\Omega (u_t\cdot\nabla b-b_t\nabla u-b\cdot\nabla u_t)b_t\,dx \\
&\quad\le \|u_t\|_{L^6}\|\nabla b\|_{L^2}\|b_t\|_{L^3}+\|\nabla u\|_{L^2}\|b_t\|_{L^4}^2+\|\nabla u_t\|_{L^2}\|b\|_{L^6}\|b_t\|_{L^3} \\
&\quad\le \frac{1}{4}\|\nabla u_t\|_{L^2}^2+\frac{1}{4}\|\operatorname{curl} b_t\|_{L^2}^2+C\|\nabla b\|_{L^2}^4\|b_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^4\|b_t\|_{L^2}^2. \qquad (5.19)
\end{aligned}$$

Combining (5.18) and (5.19) and integrating over $[t_0,t]$, we have

$$\int_\Omega \bigl(|\rho u_t|^2+|b_t|^2\bigr)\,dx+\int_{t_0}^t\int_\Omega \bigl(|\nabla u_t|^2+|\operatorname{curl} b_t|^2\bigr)\,dx\,d\tau\le C(e+y)^{C_0\epsilon}. \qquad (5.20)$$

Similarly to (5.12), we deduce that

$$\begin{aligned}
\|u\|_{H^2}&\le C\|\sqrt{\rho}u_t\|_{L^2}+C\|u\|_{L^6}\|\nabla u\|_{L^3}+C\|b\|_{L^6}\|\nabla b\|_{L^3} \\
&\le C\|\sqrt{\rho}u_t\|_{L^2}+C\|u\|_{L^6}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}+C\|b\|_{L^6}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}},
\end{aligned}$$

which leads to

$$\|u\|_{H^2}^2\le C\|\sqrt{\rho}u_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^6+C\|\nabla b\|_{L^2}^6+\frac{1}{2}\|b\|_{H^2}^2. \qquad (5.21)$$

Similarly, we have

$$\begin{aligned}
\|b\|_{H^2}&\le C\|b_t+u\cdot\nabla b-b\cdot\nabla u\|_{L^2} \\
&\le C\|b_t\|_{L^2}+C\|u\|_{L^6}\|\nabla b\|_{L^3}+C\|b\|_{L^6}\|\nabla u\|_{L^3} \\
&\le C\|b_t\|_{L^2}+C\|u\|_{L^6}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}}+C\|b\|_{L^6}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}},
\end{aligned}$$

which implies

$$\|b\|_{H^2}^2\le C\|b_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^6+C\|\nabla b\|_{L^2}^6+\frac{1}{2}\|u\|_{H^2}^2. \qquad (5.22)$$

Combining (5.21) and (5.22) and using (5.20) and (5.16), we conclude that

$$\|u\|_{H^2}^2+\|b\|_{H^2}^2\le C(e+y)^{C_0\epsilon}, \qquad (5.23)$$

and thus

$$\|u\|_{L^\infty(0,T;H^2)}+\|b\|_{L^\infty(0,T;H^2)}\le C. \qquad (5.24)$$

Now it is standard to prove that

$$\|u\|_{L^2(0,T;H^3)}+\|b\|_{L^2(0,T;H^3)}\le C, \qquad (5.25)$$

$$\|\rho\|_{L^\infty(0,T;H^2)} \le C. \tag{5.26}$$

(II) Let (1.22) hold.

Similarly to (5.7), we take $s = \infty$ and using (2.4), we still get (5.7), provided that

$$\int_{t_0}^{T} \|u(t)\|_{\mathrm{BMO}}^2 \,dt \le \epsilon \ll 1. \tag{5.27}$$

We still have (5.9), (5.10), (5.11) with $s = \infty$, (5.12) with $s = \infty$, (5.13) with $s = \infty$, and (5.14), (5.15) with $s = \infty$, and then using (5.27) and (2.4), we arrive at (5.16) and (5.17). Then by the same calculations as those in (5.18)-(5.26), we conclude that (5.18)-(5.26) hold.

This completes the proof of Theorem 1.3.

6 Proof of Theorem 1.4

We only need to establish a priori estimates.

First, using the formula $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ and the fact that $|d| = 1$ implies $d\Delta d = -|\nabla d|^2$, we have the following equivalent equation:

$$\frac{1}{2}d_t - \frac{1}{2}d \times d_t = \Delta d + d|\nabla d|^2. \tag{6.1}$$

Testing (6.1) by d_t and using $(a \times b) \cdot b = 0$ and $d \cdot d_t = 0$, we get

$$\frac{d}{dt}\int_\Omega |\nabla d|^2\,dx + \int_\Omega |d_t|^2\,dx \le 0. \tag{6.2}$$

Testing (1.23) by $-\Delta d_t$ and using $|d| = 1$, we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta d|^2\,dx + \int_\Omega |\nabla d_t|^2\,dx \\
&\quad = -\int_\Omega \bigl(d|\nabla d|^2 + d \times \Delta d\bigr) \cdot \Delta d_t\,dx \\
&\quad = \int_\Omega \nabla\bigl(d|\nabla d|^2 + d \times \Delta d\bigr) \cdot \nabla d_t\,dx \\
&\quad \le C\bigl(\|\nabla d\|_{L^q}\|\nabla d\|^2_{L^{\frac{4q}{q-2}}} + \|\nabla d\|_{L^q}\|\Delta d\|_{L^{\frac{2q}{q-2}}} + \|\nabla\Delta d\|_{L^2}\bigr)\|\nabla d_t\|_{L^2} \\
&\quad \le C\bigl(\|\nabla d\|_{L^q}\|\Delta d\|_{L^{\frac{2q}{q-2}}} + \|\nabla\Delta d\|_{L^2}\bigr)\|\nabla d_t\|_{L^2} \\
&\quad \le C\bigl(\|\nabla d\|_{L^q}\|\Delta d\|_{L^2}^{1-\frac{3}{q}}\|d\|_{H^3}^{\frac{3}{q}} + \|d\|_{H^3}\bigr)\|\nabla d_t\|_{L^2} \\
&\quad \le \frac{1}{4}\|\nabla d_t\|_{L^2}^2 + \delta\|d\|_{H^3}^2 + C\|\nabla d\|_{L^q}^{\frac{2q}{q-3}}\|\Delta d\|_{L^2}^2
\end{aligned} \tag{6.3}$$

for any $0 < \delta < 1$. Here we have used the Gagliardo-Nirenberg inequalities

$$\|\nabla d\|^2_{L^{\frac{4q}{q-2}}} \le C\|d\|_{L^\infty}\|\Delta d\|_{L^{\frac{2q}{q-2}}}, \tag{6.4}$$

$$\|\Delta d\|_{L^{\frac{2q}{q-2}}} \le C\|\Delta d\|_{L^2}^{1-\frac{3}{q}}\|d\|_{H^3}^{\frac{3}{q}}. \tag{6.5}$$

Applying ∂_i to (1.23), we get

$$\partial_i d_t - \Delta \partial_i d = \partial_i\big(d|\nabla d|^2\big) + \partial_i d \times \Delta d + d \times \Delta \partial_i d.$$

Testing this equation by $\Delta\partial_i d$, summing over i, and using (6.4) and (6.5) and $|d| = 1$, we obtain

$$\begin{aligned}
\|d\|_{H^3} &\le C\big(\|d\|_{L^2} + \|\nabla\Delta d\|_{L^2}\big) \\
&\le C + C\|\nabla d_t\|_{L^2} + C\big\|\nabla\big(d|\nabla d|^2\big)\big\|_{L^2} + \sum_i C\|\partial_i d \times \Delta d\|_{L^2} \\
&\le C + C\|\nabla d_t\|_{L^2} + C\|\nabla d\|_{L^q}\|\nabla d\|^2_{L^{\frac{4q}{q-2}}} + C\|\nabla d\|_{L^q}\|\Delta d\|_{L^{\frac{2q}{q-2}}} \\
&\le C + C\|\nabla d_t\|_{L^2} + C\|\nabla d\|_{L^q}\|\Delta d\|_{L^{\frac{2q}{q-2}}} \\
&\le C + C\|\nabla d_t\|_{L^2} + C\|\nabla d\|_{L^q}\|\Delta d\|_{L^2}^{1-\frac{3}{q}}\|d\|_{H^3}^{\frac{3}{q}},
\end{aligned}$$

which yields

$$\|d\|_{H^3} \le C + C\|\nabla d_t\|_{L^2} + C\|\nabla d\|_{L^q}^{\frac{q}{q-3}}\|\Delta d\|_{L^2}. \tag{6.6}$$

Plugging (6.6) into (6.3) and taking δ small enough, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\Delta d|^2\,dx + \int_\Omega |\nabla d_t|^2\,dx \\
&\quad\le C + C\|\nabla d\|_{L^q}^{\frac{2q}{q-3}}\|\Delta d\|_{L^2}^2 \\
&\quad\le C + C\frac{\|\nabla d\|_{L^q}^{\frac{2q}{q-3}}}{1+\log(e+\|\nabla d\|_{L^q})}\|\Delta d\|_{L^2}^2\log\big(e+\|\nabla d\|_{L^q}\big) \\
&\quad\le C + C\frac{\|\nabla d\|_{L^q}^{\frac{2q}{q-3}}}{1+\log(e+\|\nabla d\|_{L^q})}\|\Delta d\|_{L^2}^2\log(e+y),
\end{aligned}$$

which implies

$$\int_\Omega |\Delta d|^2\,dx + \int_{t_0}^t\int_\Omega |\nabla d_t|^2\,dx\,d\tau \le C(e+y)^{C_0\epsilon}, \tag{6.7}$$

provided that

$$\int_{t_0}^T \frac{\|\nabla d\|_{L^q}^{\frac{2q}{q-3}}}{1+\log(e+\|\nabla d\|_{L^q})}\,d\tau \le \epsilon \ll 1,$$

with $y(t) := \sup_{[t_0,t]}\|d\|_{H^3}$ for any $0 < t_0 \le t \le T$, where C_0 is an absolute constant.

It follows from (1.23), (6.6), and (6.7) that

$$\int_\Omega |d_t|^2\,dx + \int_{t_0}^t \|d\|_{H^3}^2\,d\tau \le C(e+y)^{C_0\epsilon}. \tag{6.8}$$

Applying ∂_t to (1.23), testing by $-\Delta d_t$, and using $|d| = 1$, (6.7), and (6.8), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla d_t|^2\,dx + \int_\Omega |\Delta d_t|^2\,dx \\
&\quad = -\int_\Omega \bigl[\partial_t\bigl(d|\nabla d|^2\bigr) + d_t \times \Delta d\bigr]\Delta d_t\,dx \\
&\quad \le C\bigl(\|\nabla d\|_{L^6}^2\|d_t\|_{L^6} + \|\nabla d\|_{L^6}\|\nabla d_t\|_{L^3} + \|d_t\|_{L^\infty}\|\Delta d\|_{L^2}\bigr)\|\Delta d_t\|_{L^2} \\
&\quad \le C\bigl(\|\nabla d\|_{L^6}^2\|d_t\|_{L^6} + \|\Delta d\|_{L^2}\|\nabla d_t\|_{L^2}^{\frac{1}{2}}\|\Delta d_t\|_{L^2}^{\frac{1}{2}} + \|\Delta d\|_{L^2}\|d_t\|_{L^2}\bigr)\|\Delta d_t\|_{L^2} \\
&\quad \le \frac{1}{2}\|\Delta d_t\|_{L^2}^2 + C\|d\|_{H^2}^4\|d_t\|_{H^1}^2 + C\|d\|_{H^2}^2\|d_t\|_{L^2}^2,
\end{aligned}$$

which implies

$$\int_\Omega |\nabla d_t|^2\,dx + \int_{t_0}^t \|\Delta d_t\|_{L^2}^2\,d\tau \le C(e+y)^{C_0\epsilon}. \tag{6.9}$$

It follows from (6.6), (6.7), (6.8), and (6.9) that

$$\|d\|_{H^3} \le C + C\|\nabla d_t\|_{L^2} + C\|\nabla d\|_{L^6}^2\|\Delta d\|_{L^2} \le C(e+y)^{C_0\epsilon},$$

which leads to

$$\|d\|_{L^\infty(0,T;H^3)} \le C.$$

This completes the proof of Theorem 1.4.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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