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On a parabolic equation related to the p-Laplacian

Huashui Zhan*

*Correspondence:
huashuizhan@163.com
School of Applied Mathematics,
Xiamen University of Technology,
Xiamen, 361024, China

Abstract
Consider a parabolic equation related to the p-Laplacian. If the diffusion coefficient of the equation is degenerate on the boundary, no matter we can define the trace of the solution on the boundary or not, by choosing a suitable test function, the stability of the solutions always can be established without a boundary condition.

MSC: 35K55; 46E35; 35R35

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1 Introduction and the main results

Consider an equation related to the p-Laplacian,

$$u_t = \operatorname{div}\left(\rho^\alpha |\nabla u|^{p-2}\nabla u\right) + f(u,x,t), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with appropriately smooth boundary, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$, $p > 1$, $\alpha > 0$. Yin and Wang [1] first studied the equation

$$u_t = \operatorname{div}\left(\rho^\alpha |\nabla u|^{p-2}\nabla u\right) \tag{1.2}$$

and showed that, when $\alpha > p-1$, the solution of the equation is completely controlled by the initial value.

The author studied in cooperation with Zhan and Yuan [2] the following equation:

$$u_t = \operatorname{div}\left(\rho^\alpha |\nabla u|^{p-2}\nabla u\right) + \sum_{i=1}^{N} \frac{\partial b_i(u)}{\partial x_i}, \quad (x,t) \in Q_T, \tag{1.3}$$

and had shown that, to consider the well-posedness of equation (1.3), instead of the whole boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.4}$$

only the partial boundary condition

$$u(x,t) = 0, \quad (x,t) \in \Sigma_p \times (0,T) \tag{1.5}$$

is necessary. Here, $\Sigma_p \subseteq \partial\Omega$ is just a portion of $\partial\Omega$, which is determined by the first order derivative term $\frac{\partial b_i(u)}{\partial x_i}$, $i = 1, 2, \ldots, N$. Certainly, the initial value is always necessary,

$$u(x,0) = u_0(x). \tag{1.6}$$

In our paper, we will consider the well-posedness of the solutions of equation (1.1). First of all, we give some basic functional spaces. For every fixed $t \in [0,T]$, we introduce the Banach space

$$V_t(\Omega) = \left\{u(x) : u(x) \in L^2(\Omega) \cap W_0^{1,1}(\Omega), \left|\nabla u(x)\right|^p \in L^1(\Omega)\right\},$$
$$\|u\|_{V_t(\Omega)} = \|u\|_{2,\Omega} + \|\nabla u\|_{p,\Omega},$$

and denote by $V_t'(\Omega)$ its dual. By $\mathbf{W}(Q_T)$ we denote the Banach space

$$\begin{cases} \mathbf{W}(Q_T) = \{u : [0,T] \to V_t(\Omega) | u \in L^2(Q_T), |\nabla u|^p \in L^1(Q_T), u = 0 \text{ on } \Gamma_T\}, \\ \|u\|_{\mathbf{W}(Q_T)} = \|\nabla u\|_{p,Q_T} + \|u\|_{2,Q_T}. \end{cases}$$

$\mathbf{W}'(Q_T)$ is the dual of $\mathbf{W}(Q_T)$ (the space of linear functionals over $\mathbf{W}(Q_T)$):

$$w \in \mathbf{W}'(Q_T) \quad \Longleftrightarrow \quad \begin{cases} w = w_0 + \sum_{i=1}^n D_i w_i, \quad w_0 \in L^2(Q_T), w_i \in L^{p'}(Q_T), \\ \forall\phi \in \mathbf{W}(Q_T), \quad \ll w, \phi \gg = \int_{Q_T} (w_0\phi + \sum_i w_i D_i\phi)\, dz. \end{cases}$$

The norm in $\mathbf{W}'(Q_T)$ is defined by

$$\|v\|_{\mathbf{W}'(Q_T)} = \sup\left\{\ll v, \phi \gg | \phi \in \mathbf{W}(\mathbf{Q_T}), \|\phi\|_{\mathbf{W}(Q_T)} \leq 1\right\}.$$

Definition 1.1 A function $u(x,t)$ is said to be a weak solution of equation (1.1) with the initial value (1.6), if

$$u \in L^\infty(Q_T), \qquad u_t \in \mathbf{W}'(Q_T), \qquad \rho^\alpha |\nabla u|^p \in L^1(Q_T), \tag{1.7}$$

and for any function $\varphi \in C_0^\infty(Q_T)$, we have

$$\iint_{Q_T} \left(-u\varphi_t + \rho^\alpha |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi - f(u,x,t)\varphi\right) dx\, dt = 0. \tag{1.8}$$

The initial value, as usual, is satisfied in the sense that

$$\lim_{t\to 0} \int_\Omega u(x,t)\phi(x)\, dx = \int_\Omega u_0(x)\phi(x)\, dx, \quad \forall\phi(x) \in C_0^\infty(\Omega). \tag{1.9}$$

We can easily obtain the existence of the weak solution.

Theorem 1.2 *Let us suppose* $1 < p$, $0 < \alpha$, $f(s,x,t)$ *is a Lipschitz function. If*

$$u_0(x) \in L^\infty(\Omega), \quad \rho^\alpha \|\nabla u_0\|^p \in L^1(\Omega), \tag{1.10}$$

then equation (1.1) *with initial value* (1.6) *has a weak solution u in the sense of Definition* 1.1.

We mainly are concerned with the stability of the solutions. As in [1–3], due to the fact that the weak solution defined in our paper satisfies (1.7), when $\alpha < p-1$, we can define the trace of u on the boundary, while for $\alpha \geq p-1$, the obtained weak solution lacks the regularity to define the trace on the boundary. However, in the short paper, by choosing a suitable test function, we can obtain the stability of the weak solutions without the boundary condition only if $\alpha > 0$. In other words, whether the weak solution is regular enough to define the trace on the boundary is not so important.

The main result of our paper is the following theorem.

Theorem 1.3 *Let u and v be two weak solutions of equation* (1.1) *with the different initial values $u(x,0) = v(x,0)$, respectively. If $\alpha > 0$ and $f(s,x,t)$ is a Lipschitz function, then*

$$\int_\Omega \left|u(x,t) - v(x,t)\right| dx \leq \int_\Omega \left|u_0(x) - v_0(x)\right| dx. \tag{1.11}$$

Comparing with [1], the greatest improvement lies in that we do not require any boundary condition, no matter that $\alpha < p-1$ or $\alpha \geq p-1$. At the same time, the nonlinear source term $f(u,x,t)$ adds the difficulty when we use the compact convergence theorem. The proof of the existence of the weak solution is quite different from that in [1]. Moreover, we consider the following equation, which seems similar to our equation (1.1):

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + f(u,x,t), \quad (x,t) \in Q_T, \tag{1.12}$$

which has been studied thoroughly for a long time, one may refer to [4–7]. Generally, to the growth order of u in $f(u,x,t)$ should be added some restrictions. Very recently, Benedikt *et al.* [8] have studied the equation

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + q(x)u^\gamma, \quad (x,t) \in Q_T, \tag{1.13}$$

with $0 < \gamma < 1$, and shown that the uniqueness of the solutions of equation (1.13) is not true. From the short comment, one can see that the degeneracy of the coefficient ρ^α plays an important role in the well-posedness of the solutions, it even can eliminate the action from the source term $f(u,x,t)$. By the way, the author has been interested in the boundary value condition of a degenerate parabolic equation for some time, one may refer to [9].

2 The proof of Theorem 1.2

By [10] and [11], we have the following lemma.

Lemma 2.1 *Let $q \geq 1$. If $u_\varepsilon \in L^\infty(0,T;L^2(\Omega)) \cap \mathbf{W}(Q_T)$, $\|u_{\varepsilon t}\|_{\mathbf{W}'(Q_T)} \leq c$, $\|\nabla(|u_\varepsilon|^{q-1} \times u_\varepsilon)\|_{p,Q_T} \leq c$, then there is a subsequence of $\{u_\varepsilon\}$ which is relatively compact in $L^s(Q_T)$ with $s \in (1,\infty)$.*

To study equation (1.1), since f is Lipschitz, without loss the generality, we may assume that $f \in C^1$. We consider, as *e.g.* in [12] by Ragusa, the associated regularized problem

$$u_{\varepsilon t} - \operatorname{div}\left(\rho_\varepsilon^\alpha\left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p-2}{2}}\nabla u_\varepsilon\right) - f(u_\varepsilon,x,t) = 0, \quad (x,t) \in Q_T, \tag{2.1}$$

$$u_\varepsilon(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{2.2}$$

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x), \quad x \in \Omega, \tag{2.3}$$

where $\rho_\varepsilon = \rho * \delta_\varepsilon + \varepsilon$, $\varepsilon > 0$, δ_ε is the mollifier as usual, $u_{\varepsilon,0} \in C_0^\infty(\Omega)$ and $\rho_\varepsilon^\alpha |\nabla u_{\varepsilon,0}|^p \in L^1(\Omega)$ is uniformly bounded, and $u_{\varepsilon,0}$ converges to u_0 in $W_0^{1,p}(\Omega)$. It is well known that the above problem has a unique classical solution [12–14].

Lemma 2.2 *There is a subsequence of u_ε (we still denote it as u_ε), which converges to a weak solution u of equation* (1.1) *with the initial value* (1.5).

Proof By the maximum principle, there is a constant c only dependent on $\|u_0\|_{L^\infty(\Omega)}$ but independent on ε, such that

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \le c. \tag{2.4}$$

Multiplying (2.1) by u_ε and integrating it over Q_T, we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega u_\varepsilon^2\,dx + \iint_{Q_T} \rho_\varepsilon^\alpha \left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2\,dx\,dt + \iint_{Q_T} u_\varepsilon f(u_\varepsilon,x,t)\,dx\,dt \\ &\quad = \frac{1}{2}\int_\Omega u_0^2\,dx. \end{aligned} \tag{2.5}$$

By (2.4), (2.5), and the assumption that f is C^1, we have

$$\frac{1}{2}\int_\Omega u_\varepsilon^2\,dx + \iint_{Q_T} \rho_\varepsilon^\alpha \left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p-2}{2}} |\nabla u_\varepsilon|^2\,dx\,dt \le c. \tag{2.6}$$

For small enough $\lambda > 0$, let $\Omega_\lambda = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \lambda\}$. Since $p > 1$, by (2.6),

$$\int_0^T \int_{\Omega_\lambda} |\nabla u_\varepsilon|\,dx\,dt \le c\left(\int_0^T \int_{\Omega_\lambda} |\nabla u_\varepsilon|^p\,dx\,dt\right)^{\frac{1}{p}} \le c(\lambda). \tag{2.7}$$

Now, for any $v \in \mathbf{W}(Q_T)$, $\|v\|_{W(Q_T)} = 1$,

$$\langle u_{\varepsilon t}, v\rangle = -\iint_{Q_T} \rho_\varepsilon^\alpha \left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p(x)-2}{2}} \nabla u_\varepsilon \cdot \nabla v\,dx\,dt + \iint_{Q_T} v f(u_\varepsilon,x,t)\,dx\,dt,$$

by the Young inequality, we can show that

$$\left|\langle u_{\varepsilon t}, v\rangle\right| \le c\left[\iint_{Q_T} \rho_\varepsilon^\alpha |\nabla u_\varepsilon|^p\,dx\,dt + \iint_{Q_T} \left(|v|^p + |\nabla v|^p\right)dx\,dt\right] \le c,$$

then

$$\|u_{\varepsilon t}\|_{\mathbf{W}'(Q_T)} \le c. \tag{2.8}$$

Now, let $\varphi \in C_0^1(\Omega)$, $0 \le \varphi \le 1$, such that

$$\varphi|_{\Omega_{2\lambda}} = 1, \qquad \varphi|_{\Omega\setminus\Omega_\lambda} = 0.$$

Then

$$\left|\langle(\varphi u_\varepsilon)_t, v\rangle\right| = \left|\langle\varphi u_{\varepsilon t}, v\rangle\right| \leq \left|\langle u_{\varepsilon t}, v\rangle\right|,$$

we have

$$\left\|\left(\varphi(x)u\right)_{\varepsilon t}\right\|_{\mathbf{W}'(Q_T)} \leq \|u_{\varepsilon t}\|_{\mathbf{W}'(Q_T)} \leq c, \tag{2.9}$$

$$\iint_{Q_T} \left|\nabla(\varphi u_\varepsilon)\right|^p dx\,dt \leq c(\lambda)\left(1 + \int_0^T \int_{\Omega_\lambda} |\nabla u_\varepsilon|^p\,dx\,dt\right) \leq c(\lambda), \tag{2.10}$$

and so

$$\left\|\nabla\left(\varphi u_\varepsilon\right)\right\|_{p,Q_T} \leq c(\lambda). \tag{2.11}$$

By Lemma 2.1, φu_ε is relatively compact in $L^s(Q_T)$ with $s \in (1,\infty)$. Then $\varphi u_\varepsilon \to \varphi u$ a.e. in Q_T. In particular, due to the arbitrariness of λ, $u_\varepsilon \to u$ a.e. in Q_T.

Hence, by (2.4), (2.6), (2.8), there exist a function u and an n-dimensional vector function $\overrightarrow{\zeta} = (\zeta_1, \ldots, \zeta_n)$ satisfying

$$u \in L^\infty(Q_T), \qquad u_t \in \mathbf{W}'(Q_T), \qquad |\overrightarrow{\zeta}| \in L^{\frac{p}{p-1}}(Q_T),$$

and

$$\begin{aligned}
&u_\varepsilon \rightharpoonup *u, \quad \text{in } L^\infty(Q_T),\\
&\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^p_{\text{loc}}(Q_T),\\
&\rho_\varepsilon^\alpha |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \rightharpoonup \overrightarrow{\zeta} \quad \text{in } L^{\frac{p}{p-1}}(Q_T).
\end{aligned}$$

In order to prove u satisfies equation (1.1), we notice that, for any function $\varphi \in C_0^\infty(Q_T)$,

$$\iint_{Q_T} \left[-u_\varepsilon\varphi_t + \rho_\varepsilon^\alpha\left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p-2}{2}} \nabla u_\varepsilon \cdot \nabla\varphi - f(u_\varepsilon, x, t)\varphi\right] dx\,dt = 0 \tag{2.12}$$

and $u_\varepsilon \to u$ is almost everywhere convergent, so $f(u_\varepsilon, x, t) \to f(u, x, t)$ is true. Then

$$\iint_{Q_T} \left(\frac{\partial u}{\partial t}\varphi + \vec{\zeta} \cdot \nabla\varphi + f(u, x, t)\varphi\right) dx\,dt = 0. \tag{2.13}$$

Now, similar to [4] or [15], we can prove

$$\iint_{Q_T} \rho^\alpha |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,dx\,dt = \iint_{Q_T} \overrightarrow{\zeta} \cdot \nabla\varphi\,dx\,dt \tag{2.14}$$

for any function $\varphi \in C_0^\infty(Q_T)$, we omit the details here. Thus u satisfies equation (1.1).

Finally, we can prove (1.9) as in [11], we also omit the details here. Then Lemma 2.2 is proved. □

Theorem 1.2 is a direct corollary of Lemma 2.2.

3 The stability of the solutions

As we have said before, by choosing a suitable test function, we can prove the stability of the solutions without any boundary value condition only if $\alpha > 0$.

Proof of Theorem 1.3 For any given positive integer n, let $g_n(s)$ be an odd function, and

$$g_n(s) = \begin{cases} 1, & s > \frac{1}{n}, \\ n^2 s^2 e^{1-n^2 s^2}, & s \le \frac{1}{n}. \end{cases} \tag{3.1}$$

Clearly,

$$\lim_{n\to 0} g_n(s) = \operatorname{sgn}(s), \quad s \in (-\infty, +\infty), \tag{3.2}$$

and

$$0 \le g_n'(s) \le \frac{c}{s}, \quad 0 < s < \frac{1}{n}, \tag{3.3}$$

where c is independent of n.

Let $\beta \le \frac{\alpha}{p}$ and

$$\phi(x) = \rho^\beta(x). \tag{3.4}$$

By taking the limit, we can choose $g_n(\phi(u-v))$ as the test function, then

$$\begin{aligned} &\int_\Omega g_n\bigl(\phi(u-v)\bigr)\frac{\partial(u-v)}{\partial t}\,dx \\ &\quad + \int_\Omega \rho^\alpha\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr)\cdot \phi\nabla(u-v)g_n'\,dx \\ &\quad + \int_\Omega \rho^\alpha\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr)\cdot \nabla\phi(u-v)g_n'\,dx \\ &\quad + \int_\Omega \bigl(f(u,x,t) - f(v,x,t)\bigr)g_n\bigl(\phi(u-v)\bigr) = 0. \end{aligned} \tag{3.5}$$

Thus

$$\begin{aligned} &\lim_{n\to\infty}\int_\Omega g_n\bigl(\phi(u-v)\bigr)\frac{\partial(u-v)}{\partial t}\,dx \\ &\quad = \int_\Omega \operatorname{sgn}\bigl(\phi(u-v)\bigr)\frac{\partial(u-v)}{\partial t}\,dx \\ &\quad = \int_\Omega \operatorname{sgn}(u-v)\frac{\partial(u-v)}{\partial t}\,dx = \frac{d}{dt}\|u-v\|_1, \end{aligned} \tag{3.6}$$

$$\int_\Omega \rho^\alpha\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr)\cdot\nabla(u-v)g_n'\phi(x)\,dx \ge 0, \tag{3.7}$$

$$\begin{aligned} &\left|\int_\Omega \rho^\alpha\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr)\cdot\nabla\phi(u-v)g_n'\bigl(\phi(u-v)\bigr)\,dx\right| \\ &\quad \le c\int_{\{x:\rho^\beta(u-v)<\frac{1}{n}\}} \left|\rho^{\frac{\alpha}{p}}\rho^{\frac{\alpha(p-1)}{p}}\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr)\right|\frac{1}{\phi}\,dx \end{aligned}$$

$$\leq c\left(\int_{\{x:\rho^\beta(u-v)<\frac{1}{n}\}} \frac{\rho^\alpha}{\rho^{p\beta}}\,dx\right)^{\frac{1}{p}}$$
$$\times\left(\int_{\{x:\rho^\beta(u-v)<\frac{1}{n}\}} \left|\rho^{\frac{\alpha(p-1)}{p}}\left(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right)\right|^{\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}. \tag{3.8}$$

Let $n\to\infty$. If $\{x\in\Omega:\rho^\beta|u-v|=0\}$ is a set with 0 measure, then

$$\left(\int_{\{x:\rho^\beta(u-v)=0\}} \frac{\rho^\alpha}{\rho^{p\beta}}\,dx\right)^{\frac{1}{p}}=0.$$

If $\{x\in\Omega:\rho^\beta|u-v|=0\}$ is a set with positive measure, then

$$\left(\int_{\{x:\rho^\beta(u-v)=0\}} \left|\rho^{\frac{\alpha(p-1)}{p}}\left(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right)\right|^{\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}=0.$$

Both cases lead to the right hand side of (3.8) going to 0 as $n\to\infty$. Meanwhile,

$$\begin{aligned}
&\lim_{n\to\infty}\left|\int_\Omega \big(f(u,x,t)-f(v,x,t)\big)g_n\big(\phi(u-v)\big)\,dx\right| \\
&=\left|\int_\Omega \big(f(u,x,t)-f(v,x,t)\big)\operatorname{sgn}\big(\phi(u-v)\big)\,dx\right| \\
&=\left|\int_\Omega \big(f(u,x,t)-f(v,x,t)\big)\operatorname{sgn}(u-v)\,dx\right| \\
&\leq c\int_\Omega |u-v|\,dx.
\end{aligned}$$

Now, let $n\to\infty$ in (3.5). Then

$$\frac{d}{dt}\|u-v\|_1\leq c\|u-v\|_1.$$

It implies that

$$\int_\Omega \big|u(x,t)-v(x,t)\big|dx\leq c(T)\int_\Omega |u_0-v_0|\,dx.$$

Theorem 1.3 is proved. □

Competing interests
The author declares to have no competing interests.

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