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Traveling wave solution for a reaction-diffusion competitive-cooperative system with delays

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Dedicated to Professor Weigao Ge

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Abstract
This paper investigates the existence of traveling wave solution to a three species reaction-diffusion system with delays, which includes competitive relationship, cooperative relationship and predator-prey relationship. By using the method of upper-lower solutions, the cross iteration method and Schauder's fixed point theorem, the existence of a traveling wave solution is obtained.

MSC: 92D25; 35K57

Keywords: reaction-diffusion system; traveling wave solution; upper-lower solution

1 Introduction

In population dynamics, Lotka-Volterra competitive, cooperative, and competitive-cooperative systems with diffusion have received great attention and have been studied extensively [1–7]. To illustrate and predict some ecological phenomena, various types of predator-prey model described by differential systems were proposed [8–10]. In studying the dynamics of predator-prey systems, one of the important topics is the existence of traveling wave solutions [11–19].

In this paper, we are concerned with the existence of traveling wave of the following competitive-cooperative system:

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} = d_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1 u_1(x,t)(1 - a_{11}u_1(x,t-\tau_{11}) - a_{12}u_2(x,t-\tau_{12}) \\ \qquad - a_{13}u_3(x,t-\tau_{13})), \\ \frac{\partial u_2(x,t)}{\partial t} = d_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2 u_2(x,t)(1 - a_{21}u_1(x,t-\tau_{21}) - a_{22}u_2(x,t-\tau_{22}) \\ \qquad + a_{23}u_3(x,t-\tau_{23})), \\ \frac{\partial u_3(x,t)}{\partial t} = d_3 \frac{\partial^2 u_3(x,t)}{\partial x^2} + r_3 u_3(x,t)(1 + a_{31}u_1(x,t-\tau_{31}) + a_{32}u_2(x,t-\tau_{32}) \\ \qquad - a_{33}u_3(x,t-\tau_{33})), \end{cases} \tag{1}$$

where all parameters d_i, r_i, a_{ij} are positive constants, $\tau_{ij} \geq 0$, $i,j = 1,2,3$, and the quantities $u_1(x,t)$, $u_2(x,t)$, $u_3(x,t)$ can be interpreted as the population densities of the three species at space x and time t.

It is necessary to point out that, when any one of the quantities $u_1(x,t)$, $u_2(x,t)$, and $u_3(x,t)$ are taken as zero, some cooperative system or competitive system can be derived from system (1), such as, when $u_1 = 0$, system (1) becomes the two species cooperative system

$$\begin{cases} \frac{\partial u_2}{\partial t} = d_2\frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2u_2(x,t)(1 - a_{22}u_2(x,t-\tau_{22}) + a_{23}u_3(x,t-\tau_{23})), \\ \frac{\partial u_3}{\partial t} = d_3\frac{\partial^2 u_3(x,t)}{\partial x^2} + r_3u_3(x,t)(1 + a_{32}u_2(x,t-\tau_{32}) - a_{33}u_3(x,t-\tau_{33})), \end{cases} \tag{2}$$

considered by Huang and Zou [2]. When $u_2 = 0$, system (1) is reduced to the two species predator-prey system

$$\begin{cases} \frac{\partial u_1}{\partial t} = d_1\frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1u_1(x,t)(1 - a_{11}u_1(x,t-\tau_{11}) - a_{13}u_3(x,t-\tau_{13})), \\ \frac{\partial u_3}{\partial t} = d_3\frac{\partial^2 u_3(x,t)}{\partial x^2} + r_3u_3(x,t)(1 + a_{31}u_1(x,t-\tau_{31}) - a_{33}u_3(x,t-\tau_{33})), \end{cases} \tag{3}$$

studied by Zhang and Li [17]. When $u_3 = 0$ system (1) is reduced to the two species competing system

$$\begin{cases} \frac{\partial u_1}{\partial t} = d_1\frac{\partial^2 u_1(x,t)}{\partial x^2} + r_1u_1(x,t)(1 - a_{11}u_1(x,t-\tau_{11}) - a_{12}u_2(x,t-\tau_{12})), \\ \frac{\partial u_2}{\partial t} = d_2\frac{\partial^2 u_2(x,t)}{\partial x^2} + r_2u_2(x,t)(1 - a_{21}u_1(x,t-\tau_{21}) - a_{22}u_2(x,t-\tau_{22})), \end{cases} \tag{4}$$

discussed by Lv and Wang [12].

This paper is organized as follows. In Section 2, we introduce some notations and lemmas which will be essential to our proofs. By applying the cross iteration method and Schauder's fixed point theorem, we establish the existence result of traveling wave solutions for a general delayed reaction-diffusion system. In Section 3, by using the results given in Section 2 and constructing a pair of upper-lower solution, we obtain the existence of traveling wave solutions to the system (1).

2 Preliminaries

For convenience, we first give some notations and definitions of traveling wave solutions.

In this paper, we shall use the standard partial ordering in R^3, namely, for $u = (u_1,u_2,u_3)^T$, $v = (v_1,v_2,v_3)^T$, we denote $u \le v$ if $u_i \le v_i$, $i = 1,2,3$; $u < v$ if $u \le v$ but $u \ne v$; and $u \ll v$ if $u \le v$ but $u_i \ne v_i$, $i = 1,2,3$. If $u \ne v$, we denote $(u,v] = \{w \in R^3 : u < w \le v\}$, $[u,v) = \{w \in R^3 : u \le w < v\}$, and $[u,v] = \{w \in R^3 : u \le w \le v\}$. We use $|\cdot|$ to denote the Euclidean in R^3 and $\|\cdot\|$ to denote the supremum norm in $C([-\tau,0],R^3)$.

Definition 1 ([15, 18]) A traveling wave solution of system (1) is a special solution of the form $u(t,x) = \phi(x+ct)$, $v(t,x) = \varphi(x+ct)$, $w(t,x) = \psi(x+ct)$, where $\phi,\varphi,\psi \in C^2(R,R)$ are the wave profiles that propagate through the one-dimensional spatial domain at a constant velocity $c > 0$.

To show the existence of a traveling wave solution to system (1), we first discuss the following general reaction-diffusion system:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1\frac{\partial^2 u(x,t)}{\partial x^2} + f_1(u(x,t),v(x,t),w(x,t)), \\ \frac{\partial v(x,t)}{\partial t} = d_2\frac{\partial^2 v(x,t)}{\partial x^2} + f_2(u(x,t),v(x,t),w(x,t)), \\ \frac{\partial w(x,t)}{\partial t} = d_3\frac{\partial^2 w(x,t)}{\partial x^2} + f_3(u(x,t),v(x,t),w(x,t)). \end{cases} \tag{5}$$

Substituting $u(x,t) = \phi(x+ct)$, $v(x,t) = \varphi(x+ct)$, $w(x,t) = \psi(x+ct)$ into (5) and denote the traveling wave coordinate $x + ct$ still by t, then (5) has a traveling wave solution if and only if the following system:

$$\begin{cases} d_1\phi''(t) - c\phi'(t) + f_{c1}(\phi_t, \varphi_t, \psi_t) = 0, \\ d_2\varphi''(t) - c\varphi'(t) + f_{c2}(\phi_t, \varphi_t, \psi_t) = 0, \\ d_3\psi''(t) - c\psi'(t) + f_{c3}(\phi_t, \varphi_t, \psi_t) = 0, \end{cases} \tag{6}$$

with asymptotic boundary conditions

$$\begin{aligned} &\lim_{t\to-\infty}\phi(t) = \phi_-, \qquad \lim_{t\to-\infty}\varphi(t) = \varphi_-, \qquad \lim_{t\to-\infty}\psi(t) = \psi_-, \\ &\lim_{t\to+\infty}\phi(t) = \phi_+, \qquad \lim_{t\to+\infty}\varphi(t) = \varphi_+, \qquad \lim_{t\to+\infty}\psi(t) = \psi_+, \end{aligned} \tag{7}$$

has a solution $(\phi(t), \varphi(t), \psi(t))$ on R, where $(\phi_-, \varphi_-, \psi_-)$ and $(\phi_+, \varphi_+, \psi_+)$ are steady states of (1) and the functions $f_{ci} : X_c = C([-c\tau, 0], R^3) \to R^3$, $i = 1, 2, 3$, are defined by

$$\begin{aligned} &f_{ci}(\phi, \varphi, \psi) = f_i(\phi^c, \varphi^c, \psi^c), \qquad \phi^c(s) = \phi(cs), \\ &\varphi^c(s) = \varphi(cs), \qquad \psi^c(s) = \psi(cs), \quad s \in [-\tau, 0]. \end{aligned}$$

Without loss of generality, we can assume

$$(\phi_-, \varphi_-, \psi_-) = (0,0,0), \qquad (\phi_+, \varphi_+, \psi_+) = (k_1, k_2, k_3),$$

and we seek for traveling wave solution connecting these two steady states. In order to address traveling waves of (6) and (7), we make the following assumptions:

(A1) $f_i(0,0,0) = f_i(k_1, k_2, k_3) = 0$ for $i = 1, 2, 3$;

(A2) there exist three positive constants $L_i > 0$ $(i = 1, 2, 3)$, such that

$$\begin{aligned} &\left|f_1(\phi_1, \varphi_1, \psi_1) - f_1(\phi_2, \varphi_2, \psi_2)\right| \le L_1\|\Phi - \Psi\|, \\ &\left|f_2(\phi_1, \varphi_1, \psi_1) - f_2(\phi_2, \varphi_2, \psi_2)\right| \le L_2\|\Phi - \Psi\|, \\ &\left|f_3(\phi_1, \varphi_1, \psi_1) - f_3(\phi_2, \varphi_2, \psi_2)\right| \le L_3\|\Phi - \Psi\|, \end{aligned}$$

for $\Phi = (\phi_1, \varphi_1, \psi_1)$, $\Psi = (\phi_2, \varphi_2, \psi_2) \in C([-\tau, 0], R^3)$ with $0 \le \phi_i(s) \le M_1$, $0 \le \varphi_i(s) \le M_2$, $0 \le \psi_i(s) \le M_3$, $i = 1, 2$, where $M_j \ge k_j$ $(j = 1, 2, 3)$ are positive constants.

The reaction terms satisfy the following partial quasi-monotonicity conditions (PQM), different from [15, 18, 19].

(PQM) There exist three positive constants $\beta_1, \beta_2, \beta_3 > 0$ such that

$$\begin{aligned} &f_{c1}(\phi_1, \varphi_1, \psi_1) - f_{c1}(\phi_2, \varphi_1, \psi_1) + \beta_1\big[\phi_1(0) - \phi_2(0)\big] \ge 0, \\ &f_{c1}(\phi_1, \varphi_1, \psi_1) - f_{c1}(\phi_1, \varphi_2, \psi_2) \le 0, \\ &f_{c2}(\phi_1, \varphi_1, \psi_1) - f_{c2}(\phi_1, \varphi_2, \psi_2) + \beta_2\big[\varphi_1(0) - \varphi_2(0)\big] \ge 0, \\ &f_{c2}(\phi_1, \varphi_1, \psi_1) - f_{c2}(\phi_2, \varphi_1, \psi_1) \le 0, \\ &f_{c3}(\phi_1, \varphi_1, \psi_1) - f_{c3}(\phi_1, \varphi_2, \psi_2) + \beta_3\big[\psi_1(0) - \psi_2(0)\big] \ge 0, \\ &f_{c3}(\phi_1, \varphi_1, \psi_1) - f_{c3}(\phi_2, \varphi_1, \psi_1) \le 0, \end{aligned} \tag{8}$$

where $\phi_i, \varphi_i, \psi_i \in C([-\tau, 0], R)$, $i = 1, 2$, $0 \leq \phi_2(s) \leq \phi_1(s) \leq M_1$, $0 \leq \varphi_2(s) \leq \varphi_1(s) \leq M_2$, $0 \leq \psi_2(s) \leq \psi_1(s) \leq M_3$, $s \in [-\tau, 0]$.

We need the following definition of upper and lower solutions.

Definition 2 ([15, 18]) A pair of continuous functions $\overline{\rho} = (\overline{\phi}, \overline{\varphi}, \overline{\psi})$ and $\underline{\rho} = (\underline{\phi}, \underline{\varphi}, \underline{\psi})$ are called a pair of upper and lower solutions of the system (1) if $\overline{\rho}$ and $\underline{\rho}$ are twice differentiable almost everywhere in R and they are essentially bounded on R, and we have

$$\begin{cases} d_1\overline{\phi}'' - c\overline{\phi}' + f_{c1}(\overline{\phi}_t, \underline{\varphi}_t, \underline{\psi}_t) \leq 0, & \text{a.e. in } R, \\ d_2\overline{\varphi}'' - c\overline{\varphi}' + f_{c2}(\underline{\phi}_t, \overline{\varphi}_t, \overline{\psi}_t) \leq 0, & \text{a.e. in } R, \\ d_3\overline{\psi}'' - c\overline{\psi}' + f_{c3}(\underline{\phi}_t, \overline{\varphi}_t, \overline{\psi}_t) \leq 0, & \text{a.e. in } R \end{cases} \tag{9}$$

and

$$\begin{cases} d_1\underline{\phi}'' - c\underline{\phi}' + f_{c1}(\underline{\phi}_t, \overline{\varphi}_t, \overline{\psi}_t) \geq 0, & \text{a.e. in } R, \\ d_2\underline{\varphi}'' - c\underline{\varphi}' + f_{c2}(\overline{\phi}_t, \underline{\varphi}_t, \underline{\psi}_t) \geq 0, & \text{a.e. in } R, \\ d_3\underline{\psi}'' - c\underline{\psi}' + f_{c3}(\overline{\phi}_t, \underline{\varphi}_t, \underline{\psi}_t) \geq 0, & \text{a.e. in } R. \end{cases} \tag{10}$$

Let

$$C_k := \{(\phi, \varphi, \psi) | (0, 0, 0) \leq (\phi, \varphi, \psi) \leq (M_1, M_2, M_3), \text{for } t \in R\}.$$

We shall combine Schauder's fixed point theorem with the method of upper and lower solutions to establish the existence of solutions. For this purpose, we need to introduce a topology in $C(R, R^3)$.

Let $\mu > 0$ and let $C(R, R^3)$ be equipped with the exponential decay norm defined by

$$|\Phi|_\mu = \sup_{t \in R} e^{-\mu|t|} |\Phi(t)|_{R^3}.$$

Define

$$B_\mu(R, R^3) = \{\Phi \in C(R, R^3) : |\Phi|_\mu < \infty\}.$$

Then it is easy to check that $(B_\mu(R, R^3), |\cdot|_\mu)$ is a Banach space. We shall look for the traveling wave solution of system (6) in the following profile set:

$$\begin{aligned} &\Gamma\big((\underline{\phi}, \underline{\varphi}, \underline{\psi}), (\overline{\phi}, \overline{\varphi}, \overline{\psi})\big) \\ &\quad = \left\{ \begin{array}{c} \text{(i) } \underline{\phi}(t) \leq \phi(t) \leq \overline{\phi}(t), \underline{\varphi}(t) \leq \varphi(t) \leq \overline{\varphi}(t), \underline{\psi}(t) \leq \psi(t) \leq \overline{\psi}(t); \\ \text{(ii) } e^{\beta_1 s}[\phi(t) - \underline{\phi}(t)], e^{\beta_1 s}[\phi(t) - \overline{\phi}(t)], e^{\beta_2 s}[\varphi(t) - \underline{\varphi}(t)], e^{\beta_2 s}[\varphi(t) - \overline{\varphi}(t)], \\ e^{\beta_3 s}[\psi(t) - \underline{\psi}(t)], e^{\beta_3 s}[\psi(t) - \overline{\psi}(t)] \text{ are nondecreasing for } t \in R \end{array} \right\}. \end{aligned}$$

It is easy to see that $\Gamma((\underline{\phi}, \underline{\varphi}, \underline{\psi}), (\overline{\phi}, \overline{\varphi}, \overline{\psi}))$ is nonempty, convex, closed, and bounded.

In the following, we assume that there exist a pair of upper and lower solutions $(\overline{\phi}(t), \overline{\varphi}(t), \overline{\psi}(t))$, $(\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t))$ of (6) satisfying the conditions (P1) and (P2):

(P1) $(0, 0, 0) \leq (\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t)) \leq (\overline{\phi}(t), \overline{\varphi}(t), \overline{\psi}(t)) \leq (M_1, M_2, M_3)$, $t \in R$.

(P2) $\lim_{t\to-\infty}(\underline{\phi}(t),\underline{\varphi}(t),\underline{\psi}(t)) = (0,0,0)$, $\lim_{t\to+\infty}(\overline{\phi}(t),\overline{\varphi}(t),\overline{\psi}(t)) = (k_1,k_2,k_3)$.

Define the operators $H_i : C(R,R^3) \to C(R,R^3)$ by

$$H_i(\phi,\varphi,\psi)(t) = f_{ci}(\phi_t,\varphi_t,\psi_t) + \beta_i\theta_i(t), \quad \phi,\varphi,\psi \in C(R,R), i = 1,2,3, \tag{11}$$

where

$$\theta_i(t) = \begin{cases} \phi(t), & \text{if } i = 1, \\ \varphi(t), & \text{if } i = 2, \\ \psi(t), & \text{if } i = 3, \end{cases}$$

and the constants $\beta_i > 0$ are as in inequalities (8). The operators H_i, $i = 1,2,3$ satisfy the following properties.

Lemma 1 *Assume that* (A1) *and* (8) *hold, for* $t \in R$ *with* $0 \le \phi_2(t) \le \phi_1(t) \le M_1$, $0 \le \varphi_2(t) \le \varphi_1(t) \le M_2$, $0 \le \psi_2(t) \le \psi_1(t) \le M_3$, *then*

$$\begin{aligned} &H_1(\phi_2,\varphi_1,\psi_1) \le H_1(\phi_1,\varphi_1,\psi_1), \qquad H_1(\phi_1,\varphi_1,\psi_1) \le H_1(\phi_1,\varphi_2,\psi_2), \\ &H_2(\phi_1,\varphi_2,\psi_2) \le H_2(\phi_1,\varphi_1,\psi_1), \qquad H_2(\phi_1,\varphi_1,\psi_1) \le H_2(\phi_2,\varphi_1,\psi_1), \\ &H_3(\phi_1,\varphi_2,\psi_2) \le H_3(\phi_1,\varphi_1,\psi_1), \qquad H_3(\phi_1,\varphi_1,\psi_1) \le H_3(\phi_2,\varphi_1,\psi_1). \end{aligned}$$

Proof From (8), a direct calculation shows that

$$\begin{aligned} &H_1(\phi_1,\varphi_1,\psi_1) - H_1(\phi_2,\varphi_1,\psi_1) = f_{c1}(\phi_1,\varphi_1,\psi_1) - f_{c1}(\phi_2,\varphi_1,\psi_1) + \beta_1\big[\phi_1(0) - \phi_2(0)\big] \ge 0, \\ &H_1(\phi_1,\varphi_1,\psi_1) - H_1(\phi_1,\varphi_2,\psi_2) = f_{c1}(\phi_1,\varphi_1,\psi_1) - f_{c1}(\phi_1,\varphi_2,\psi_2) \le 0, \\ &H_2(\phi_1,\varphi_1,\psi_1) - H_2(\phi_1,\varphi_2,\psi_2) = f_{c2}(\phi_1,\varphi_1,\psi_1) - f_{c1}(\phi_1,\varphi_2,\psi_2) + \beta_2\big[\varphi_1(0) - \varphi_2(0)\big] \ge 0, \\ &H_2(\phi_1,\varphi_1,\psi_1) - H_2(\phi_2,\varphi_1,\psi_1) = f_{c2}(\phi_1,\varphi_1,\psi_1) - f_{c2}(\phi_2,\varphi_1,\psi_1) \le 0, \\ &H_3(\phi_1,\varphi_1,\psi_1) - H_3(\phi_1,\varphi_2,\psi_2) = f_{c3}(\phi_1,\varphi_1,\psi_1) - f_{c3}(\phi_1,\varphi_2,\psi_2) + \beta_3\big[\psi_1(0) - \psi_2(0)\big] \ge 0, \\ &H_3(\phi_1,\varphi_1,\psi_1) - H_3(\phi_2,\varphi_1,\psi_1) = f_{c3}(\phi_1,\varphi_1,\psi_1) - f_{c3}(\phi_2,\varphi_1,\psi_1) \le 0. \end{aligned}$$

□

From the definitions of H_1, H_2, and H_3 in (11), system (6) can be rewritten as

$$d_i\theta_i''(t) - c\theta_i'(t) - \beta_1\theta_i(t) + H_i(\phi,\varphi,\psi)(t) = 0, \quad i = 1,2,3. \tag{12}$$

We define

$$\begin{aligned} \lambda_1 &= \frac{c - \sqrt{c^2 + 4\beta_1 d_1}}{2d_1}, & \lambda_2 &= \frac{c + \sqrt{c^2 + 4\beta_1 d_1}}{2d_1}, \\ \lambda_3 &= \frac{c - \sqrt{c^2 + 4\beta_2 d_2}}{2d_2}, & \lambda_4 &= \frac{c + \sqrt{c^2 + 4\beta_2 d_2}}{2d_2}, \\ \lambda_5 &= \frac{c - \sqrt{c^2 + 4\beta_3 d_3}}{2d_3}, & \lambda_6 &= \frac{c + \sqrt{c^2 + 4\beta_3 d_3}}{2d_3}. \end{aligned}$$

For $(\phi,\varphi,\psi)\in C_k(R,R^3)$, we define $F=(F_1,F_2,F_3):C_k(R,R^3)\to C(R,R^3)$ by

$$\begin{aligned}
&F_1(\phi,\varphi,\psi)(t)\\
&\quad=\frac{1}{d_1(\lambda_2-\lambda_1)}\biggl[\int_{-\infty}^{t}e^{\lambda_1(t-s)}H_1(\phi,\varphi,\psi)(s)\,ds+\int_t^{+\infty}e^{\lambda_2(t-s)}H_1(\phi,\varphi,\psi)(s)\,ds\biggr],\\
&F_2(\phi,\varphi,\psi)(t)\\
&\quad=\frac{1}{d_2(\lambda_4-\lambda_3)}\biggl[\int_{-\infty}^{t}e^{\lambda_3(t-s)}H_2(\phi,\varphi,\psi)(s)\,ds+\int_t^{+\infty}e^{\lambda_4(t-s)}H_2(\phi,\varphi,\psi)(s)\,ds\biggr],\\
&F_3(\phi,\varphi,\psi)(t)\\
&\quad=\frac{1}{d_3(\lambda_6-\lambda_5)}\biggl[\int_{-\infty}^{t}e^{\lambda_5(t-s)}H_3(\phi,\varphi,\psi)(s)\,ds+\int_t^{+\infty}e^{\lambda_6(t-s)}H_3(\phi,\varphi,\psi)(s)\,ds\biggr].
\end{aligned}$$

It is easy to see that $F_i(\phi,\varphi,\psi)$ $(i=1,2,3)$ satisfy

$$d_iF_i''(\phi,\varphi,\psi)-cF_i'(\phi,\varphi,\psi)-\beta_iF_i(\phi,\varphi,\psi)+H_i(\phi,\varphi,\psi)=0. \tag{13}$$

Corresponding to Lemma 1, we have the same results of F.

Lemma 2 *Assume that* (A2) *holds, then* $F=(F_1,F_2,F_3)$ *is continuous with respective to the norm* $|\cdot|$ *in* $B_\mu(R,R^3)$.

Lemma 3 *Assume that* (A2) *and* (8) *hold, then*

$$F\bigl(\Gamma\bigl((\underline{\phi},\underline{\varphi},\underline{\psi}),(\overline{\phi},\overline{\varphi},\overline{\psi})\bigr)\bigr)\subset\Gamma\bigl((\underline{\phi},\underline{\varphi},\underline{\psi}),(\overline{\phi},\overline{\varphi},\overline{\psi})\bigr).$$

Lemma 4 *Assume that* (8) *holds, then*

$$F:\Gamma\bigl((\underline{\phi},\underline{\varphi},\underline{\psi}),(\overline{\phi},\overline{\varphi},\overline{\psi})\bigr)\to\Gamma\bigl((\underline{\phi},\underline{\varphi},\underline{\psi}),(\overline{\phi},\overline{\varphi},\overline{\psi})\bigr)$$

is compact.

Remark 1 The proofs of Lemmas 2-4 are similar to those of Lemmas 3.4-3.6 in [19], and we omit them here.

Theorem 1 *Assume that* (A1), (A2), *and* (8) *hold. Suppose there is a pair of upper and lower solutions* $\Phi=(\overline{\phi},\overline{\varphi},\overline{\psi})$, *and* $\Psi=(\underline{\phi},\underline{\varphi},\underline{\psi})$ *for* (6) *satisfying* (P1) *and* (P2), *then system* (1) *has a traveling wave solution.*

Proof Combining Lemmas 1-4 with Schauder's fixed point theorem, we know that there exists a fixed point $(\phi^*(t),\varphi^*(t),\psi^*(t))$ of F in $\Gamma((\underline{\phi},\underline{\varphi},\underline{\psi}),(\overline{\phi},\overline{\varphi},\overline{\psi}))$, which gives a solution of (6).

From (P2) and the fact that

$$(0,0,0)\le(\underline{\phi},\underline{\varphi},\underline{\psi})\le\bigl(\phi^*(t),\varphi^*(t),\psi^*(t)\bigr)\le(\overline{\phi},\overline{\varphi},\overline{\psi})\le(M_1,M_2,M_3),$$

we know that

$$\lim_{t\to-\infty}\bigl(\phi^*(t),\varphi^*(t),\psi^*(t)\bigr)=(0,0,0);\qquad \lim_{t\to+\infty}\bigl(\phi^*(t),\varphi^*(t),\psi^*(t)\bigr)=(k_1,k_2,k_3).$$

Therefore, the fixed point $(\phi^*(t),\varphi^*(t),\psi^*(t))$ satisfies the asymptotic boundary conditions (7). □

3 Existence of traveling waves

In this section, we will apply Theorem 1 to establish the existence of traveling wave solutions for system (1). Assuming that

$$D=\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}>0,\qquad D_1=\begin{vmatrix}1&a_{12}&a_{13}\\1&a_{22}&a_{23}\\1&a_{32}&a_{33}\end{vmatrix}>0,$$

$$D_2=\begin{vmatrix}a_{11}&1&a_{13}\\a_{21}&1&a_{23}\\a_{31}&1&a_{33}\end{vmatrix}>0,\qquad D_3=\begin{vmatrix}a_{11}&a_{12}&1\\a_{21}&a_{22}&1\\a_{31}&a_{32}&1\end{vmatrix}>0.$$

We are interested in looking for a traveling wave solution of (1) connecting $(0,0,0)$ and a positive equilibrium (k_1,k_2,k_3). Here $k_i=\frac{D_i}{D}$ $(i=1,2,3)$ are the roots of the following equations:

$$\begin{cases}a_{11}k_1+a_{12}k_2+a_{13}k_3=1,\\a_{21}k_1+a_{22}k_2+a_{23}k_3=1,\\a_{31}k_1+a_{32}k_2+a_{33}k_3=1.\end{cases}\tag{14}$$

Substituting $s=x+ct$ into (1) and denoting the variable s still by t, then the corresponding wave profile equations are

$$\begin{cases}d_1\phi''(t)-c\phi'(t)+r_1\phi(t)(1-a_{11}\phi(t-\tau_{11})-a_{12}\varphi(t-\tau_{12})-a_{13}\psi(t-\tau_{13}))=0,\\d_2\varphi''(t)-c\varphi'(t)+r_2\varphi(t)(1-a_{21}\phi(t-\tau_{21})-a_{22}\varphi(t-\tau_{22})+a_{23}\psi(t-\tau_{23}))=0,\\d_3\psi''(t)-c\psi'(t)+r_3\psi(t)(1+a_{31}\phi(t-\tau_{31})+a_{32}\varphi(t-\tau_{32})-a_{33}\psi(t-\tau_{33}))=0.\end{cases}\tag{15}$$

Lemma 5 *Assume that* τ_{ii} $(i=1,2,3)$ *are small enough, then the functions* (f_1,f_2,f_3) *satisfy* (PQM).

Proof For any $\phi_1(s),\phi_2(s),\varphi_1(s),\varphi_2(s),\psi_1(s),\psi_2(s)\in C([-\tau,0],R)$,

(i) $0\le\phi_2(s)\le\phi_1(s)\le M_1$, $0\le\varphi_2(s)\le\varphi_1(s)\le M_2$, $0\le\psi_2(s)\le\psi_1(s)\le M_3$, $s\in[-\tau,0]$;

(ii) $e^{\beta_1 s}(\phi_1(s)-\phi_2(s))$, $e^{\beta_2 s}(\varphi_1(s)-\varphi_2(s))$, and $e^{\beta_3 s}(\psi_1(s)-\psi_2(s))$ are nondecreasing in $s\in[-\tau,0]$.

If τ_{11} is small enough, we can choose $\beta_1>0$ satisfying

$$\begin{aligned}&f_{c1}(\phi_1,\varphi_1,\psi_1)-f_{c1}(\phi_2,\varphi_1,\psi_1)\\&\quad=r_1\phi_1(0)\bigl(1-a_{11}\phi_1(-\tau_{11})-a_{12}\varphi_1(-\tau_{12})-a_{13}\psi_1(-\tau_{13})\bigr)\\&\qquad-r_1\phi_2(0)\bigl(1-a_{11}\phi_2(-\tau_{11})-a_{12}\varphi_1(-\tau_{12})-a_{13}\psi_1(-\tau_{13})\bigr)\end{aligned}$$

$$
\begin{aligned}
&= r_1\bigl(\phi_1(0)-\phi_2(0)\bigr) - a_{11}r_1\bigl(\phi_1(0)\phi_1(-\tau_{11})-\phi_2(0)\phi_2(-\tau_{11})\bigr) \\
&\quad - a_{12}r_1\varphi_1(-\tau_{12})\bigl(\phi_1(0)-\phi_2(0)\bigr) - a_{13}r_1\psi_1(-\tau_{13})\bigl(\phi_1(0)-\phi_2(0)\bigr) \\
&\geq r_1(1-a_{12}M_2-a_{13}M_3-a_{11}M_1)\bigl(\phi_1(0)-\phi_2(0)\bigr) \\
&\quad - a_{11}r_1\phi_2(0)\bigl(\phi_1(-\tau_{11})-\phi_2(-\tau_{11})\bigr) \\
&\geq r_1\bigl(1-a_{12}M_2-a_{13}M_3-a_{11}M_1-a_{11}M_1e^{\beta_1\tau_{11}}\bigr)\bigl(\phi_1(0)-\phi_2(0)\bigr).
\end{aligned}
$$

Let

$$
\beta_1 \geq -r_1\bigl(1-a_{12}M_2-a_{13}M_3-a_{11}M_1-a_{11}M_1e^{\beta_1\tau_{11}}\bigr),
$$

then it is easy to show that $f_{c1}(\phi_1,\varphi_1,\psi_1)-f_{c1}(\phi_2,\varphi_1,\psi_1)+\beta_1(\phi_1(0)-\phi_2(0))\geq 0$, and

$$
\begin{aligned}
&f_{c1}(\phi_1,\varphi_1,\psi_1)-f_{c1}(\phi_1,\varphi_2,\psi_2) \\
&\quad = r_1\phi_1(0)\bigl(1-a_{11}\phi_1(-\tau_{11})-a_{12}\varphi_1(-\tau_{12})-a_{13}\psi_1(-\tau_{13})\bigr) \\
&\qquad - r_1\phi_1(0)\bigl(1-a_{11}\phi_1(-\tau_{11})-a_{12}\varphi_2(-\tau_{12})-a_{13}\psi_2(-\tau_{13})\bigr) \\
&\quad = r_1\phi_1(0)\bigl(a_{12}\bigl(\varphi_2(-\tau_{12})-\varphi_1(-\tau_{12})\bigr)+a_{13}\bigl(\psi_2(-\tau_{13})-\psi_1(-\tau_{13})\bigr)\bigr) \\
&\quad \leq 0.
\end{aligned}
$$

For f_{c2}, we have

$$
\begin{aligned}
&f_{c2}(\phi_1,\varphi_1,\psi_1)-f_{c2}(\phi_1,\varphi_2,\psi_2) \\
&\quad = r_2\varphi_1(0)\bigl(1-a_{21}\phi_1(-\tau_{21})-a_{22}\varphi_1(-\tau_{22})-a_{23}\psi_1(-\tau_{23})\bigr) \\
&\qquad - r_2\varphi_2(0)\bigl(1-a_{21}\phi_1(-\tau_{21})-a_{22}\varphi_2(-\tau_{22})-a_{23}\psi_2(-\tau_{23})\bigr) \\
&\quad = r_2\bigl(\varphi_1(0)-\varphi_2(0)\bigr)-a_{21}r_2\bigl(\varphi_1(0)\phi_1(-\tau_{21})-\varphi_2(0)\phi_1(-\tau_{21})\bigr) \\
&\qquad - a_{22}r_2\bigl(\varphi_1(0)\varphi_1(-\tau_{22})-\varphi_2(0)\varphi_2(-\tau_{22})\bigr) \\
&\qquad - a_{23}r_2\bigl(\varphi_1(0)\psi_1(-\tau_{23})-\varphi_2(0)\psi_2(-\tau_{23})\bigr) \\
&\quad \geq r_2(1-a_{21}M_1)\bigl(\varphi_1(0)-\varphi_2(0)\bigr)-a_{22}r_2\bigl(\varphi_2(0)\varphi_1(-\tau_{22})-\varphi_2(0)\varphi_2(-\tau_{22}) \\
&\qquad + \bigl(\varphi_1(0)-\varphi_2(0)\bigr)\varphi_1(-\tau_{22})\bigr)+a_{23}r_2\bigl(\varphi_2(0)\psi_1(-\tau_{23})-\varphi_2(0)\psi_2(-\tau_{23}) \\
&\qquad + \bigl(\varphi_1(0)-\varphi_2(0)\bigr)\psi_1(-\tau_{23})\bigr) \\
&\quad \geq r_1\bigl(1-a_{21}M_1-a_{23}M_3-a_{21}M_1-a_{22}M_2e^{\beta_2\tau_{22}}-a_{23}M_3e^{\beta_3\tau_{33}}\bigr)\bigl(\varphi_1(0)-\varphi_2(0)\bigr).
\end{aligned}
$$

Let $\beta_2 \geq r_2(1-a_{21}M_1-a_{23}M_3-a_{21}M_1-a_{22}M_2e^{\beta_2\tau_{22}}-a_{23}M_3e^{\beta_3\tau_{33}})$, then

$$
f_{c2}(\phi_1,\varphi_1,\psi_1)-f_{c2}(\phi_1,\varphi_2,\psi_2)+\beta_2\bigl[\varphi_1(0)-\varphi_2(0)\bigr]\geq 0
$$

and

$$
\begin{aligned}
&f_{c2}(\phi_1,\varphi_1,\psi_1)-f_{c2}(\phi_2,\varphi_1,\psi_1) \\
&\quad = r_2\varphi_1(0)\bigl(1-a_{21}\phi_1(-\tau_{21})-a_{22}\varphi_1(-\tau_{22})-a_{23}\psi_1(-\tau_{23})\bigr)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - r_2\varphi_1(0)\bigl(1 - a_{21}\phi_2(-\tau_{21}) - a_{22}\varphi_1(-\tau_{22}) - a_{23}\psi_1(-\tau_{23})\bigr)\\
&= r_2\varphi_1(0)a_{21}\bigl(\phi_2(-\tau_{21}) - \phi(-\tau_{21})\bigr)\\
&\le 0.
\end{aligned}
$$

In a similar way for f_{c3}, we let $\beta_3 > r_3(1 - a_{33}M_3 - a_{33}M_3e^{\beta_3\tau_{33}})$, then $f_{c3}(\phi_1,\varphi_1,\psi_1) - f_{c3}(\phi_1,\varphi_2,\psi_2) + \beta_3[\psi_1(0) - \psi_2(0)] \ge 0$, and $f_{c3}(\phi_1,\varphi_1,\psi_1) - f_{c3}(\phi_2,\varphi_1,\psi_1) \le 0$. This completes the proof. □

Let

$$
c > c^* = \max\bigl(2\sqrt{d_1r_1}, 2\sqrt{d_2r_2(1 + a_{23}M_3)}, 2\sqrt{d_3r_3(1 + a_{31}M_1 + a_{32}M_2)}\bigr).
$$

There exist $\lambda_i > 0$ ($i = 1,3,5$) so that

$$
\begin{aligned}
&d_1\lambda_1^2 - c\lambda_1 + r_1 = 0,\\
&d_2\lambda_3^2 - c\lambda_3 + r_2(1 + a_{23}M_3) = 0,\\
&d_2\lambda_5^2 - c\lambda_5 + r_3(1 + a_{31}M_1 + a_{32}M_2) = 0.
\end{aligned}
$$

We find that there exist $\varepsilon_i > 0$ ($i = 0,1,2,3,4,5,6$) satisfying

$$
\begin{cases}
a_{11}\varepsilon_1 - a_{12}\varepsilon_4 - a_{13}\varepsilon_6 > \varepsilon_0,\\
-a_{21}\varepsilon_2 + a_{22}\varepsilon_3 - a_{23}\varepsilon_5 > \varepsilon_0,\\
a_{31}\varepsilon_2 - a_{32}\varepsilon_3 + \varepsilon_5 > \varepsilon_0,\\
\varepsilon_2 - a_{21}\varepsilon_3 - a_{13}\varepsilon_5 > \varepsilon_0,\\
-a_{11}\varepsilon_1 + \varepsilon_4 + a_{13}\varepsilon_5 > \varepsilon_0,\\
1 - k_3 + \varepsilon_6 > \varepsilon_0.
\end{cases} \tag{16}
$$

For the above constants and suitable constants $t_i > 0$ ($i = 1,2,3,4,5,6$), we define the continuous functions $\overline{\Phi} = (\overline{\phi}(t), \overline{\varphi}(t), \overline{\psi}(t))$ and $\underline{\Psi} = (\underline{\phi}(t), \underline{\varphi}(t), \underline{\psi}(t))$ as follows:

$$
\begin{aligned}
&\overline{\phi}(t) = \begin{cases} e^{\lambda_1 t}, & t \le t_1,\\ k_1 + \varepsilon_1 e^{-\lambda t}, & t > t_1,\end{cases} \qquad
\underline{\phi}(t) = \begin{cases} 0, & t \le t_2,\\ k_1 - \varepsilon_2 e^{-\lambda t}, & t > t_2,\end{cases}\\
&\overline{\varphi}(t) = \begin{cases} e^{\lambda_3 t}, & t \le t_3,\\ k_2 + \varepsilon_3 e^{-\lambda t}, & t > t_3,\end{cases} \qquad
\underline{\varphi}(t) = \begin{cases} 0, & t \le t_4,\\ k_2 - \varepsilon_4 e^{-\lambda t}, & t > t_4,\end{cases}\\
&\overline{\psi}(t) = \begin{cases} e^{\lambda_5 t}, & t \le t_5,\\ k_3 + \varepsilon_5 e^{-\lambda t}, & t > t_5,\end{cases} \qquad
\underline{\psi}(t) = \begin{cases} 0, & t \le t_6,\\ k_3 - \varepsilon_6 e^{-\lambda t}, & t > t_6,\end{cases}
\end{aligned}
$$

where $\lambda > 0$ is a constant to be chosen later and

$$
\min\{t_1, t_3, t_5\} - c\max\{\tau_{ij}, i, j = 1,2,3\} \ge \max\{t_2, t_4, t_6\}, \qquad t_4 > t_6.
$$

Lemma 6 *Assume that* $D > 0$, $D_i > 0$ ($i = 1,2,3$) *and* (16) *hold, then* $\overline{\Phi} = (\overline{\phi}(t), \overline{\varphi}(t), \overline{\psi}(t))$ *is an upper solution of system* (15).

Proof When $t > t_1 + c\tau_{11}$, $\overline{\phi}(t) = k_1 + \varepsilon_1 e^{-\lambda t}$, we have

$$\begin{aligned}
&d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t)\bigl[1 - a_{11}\overline{\phi}(t - c\tau_{11}) - a_{12}\underline{\varphi}(t - c\tau_{12}) - a_{13}\underline{\psi}(t - c\tau_{13})\bigr] \\
&\quad = d_1\varepsilon_1\lambda^2 e^{-\lambda t} + c\varepsilon_1\lambda e^{-\lambda t} + r_1\bigl(k_1 + \varepsilon_1 e^{-\lambda t}\bigr)\bigl(1 - a_{11}\bigl(k_1 + \varepsilon_1 e^{-\lambda(t - c\tau_{11})}\bigr) \\
&\qquad - a_{12}\bigl(k_2 - \varepsilon_4 e^{-\lambda(t - c\tau_{12})}\bigr) - a_{13}\bigl(k_3 - \varepsilon_6 e^{-\lambda(t - c\tau_{13})}\bigr)\bigr) \\
&\quad =: I_1(\lambda).
\end{aligned}$$

Obviously,

$$\begin{aligned}
I_1(0) &= r_1(k_1 + \varepsilon_1)\bigl(1 - a_{11}(k_1 + \varepsilon_1) - a_{12}(k_2 - \varepsilon_4) - a_{13}(k_3 - \varepsilon_6)\bigr) \\
&= r_1(k_1 + \varepsilon_1)(-a_{11}\varepsilon_1 + a_{12}\varepsilon_4 + a_{13}\varepsilon_6).
\end{aligned}$$

It is easy to see that $I_1(0) < 0$ and there exists $\lambda_1^* > 0$, such that

$$d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t)\bigl[1 - a_{11}\overline{\phi}(t - c\tau_{11}) - a_{12}\underline{\varphi}(t - c\tau_{12}) - a_{13}\underline{\psi}(t - c\tau_{13})\bigr] \le 0,$$

for all $\lambda \in (0, \lambda_1^*)$.

If $t \le t_1$, $\overline{\phi}(t) = e^{\lambda_1 t}$, we have

$$\begin{aligned}
&d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t)\bigl[1 - a_{11}\overline{\phi}(t - c\tau_{11}) - a_{12}\underline{\varphi}(t - c\tau_{12}) - a_{13}\underline{\psi}(t - c\tau_{13})\bigr] \\
&\quad \le d_1\lambda_1^2 e^{\lambda_1 t} - c\lambda_1 e^{\lambda_1 t} + r_1 e^{\lambda_1 t} = 0.
\end{aligned}$$

If $t_1 < t \le t_1 + c\tau_{11}$, then we have

$$\begin{aligned}
&d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t)\bigl[1 - a_{11}\overline{\phi}(t - c\tau_{11}) - a_{12}\underline{\varphi}(t - c\tau_{12}) - a_{13}\underline{\psi}(t - c\tau_{13})\bigr] \\
&\quad = d_1\varepsilon_1\lambda^2 e^{-\lambda t} + c\varepsilon_1\lambda e^{-\lambda t} + r_1\bigl(k_1 + \varepsilon_1 e^{-\lambda t}\bigr)\bigl(1 - a_{11}e^{\lambda_1(t_1 - c\tau_{11})} \\
&\qquad - a_{12}\bigl(k_2 - \varepsilon_4 e^{-\lambda t}\bigr) - a_{13}\bigl(k_3 - \varepsilon_6 e^{-\lambda t}\bigr)\bigr) \\
&\quad =: I_2(\lambda).
\end{aligned}$$

For small enough τ_{11}, there exists ε_1^* ($0 < \varepsilon_1^* < \frac{\varepsilon_0}{a_{11}(k_1+\varepsilon_1)}$) such that $e^{-\lambda_1 c\tau_{11}} > 1 - \varepsilon_1^*$. Thus we have

$$\begin{aligned}
I_2(0) &= r_1(k_1 + \varepsilon_1)\bigl(1 - a_{11}e^{\lambda_1(t_1 - c\tau_{11})} - a_{12}(k_2 - \varepsilon_4) - a_{13}(k_3 - \varepsilon_6)\bigr) \\
&= r_1(k_1 + \varepsilon_1)\bigl(a_{11}k_1 + a_{12}\varepsilon_4 + a_{13}\varepsilon_6 - a_{11}e^{-\lambda_1 c\tau_{11}}(k_1 + \varepsilon_1)\bigr) \\
&\le r_1(k_1 + \varepsilon_1)\bigl(a_{11}k_1 + a_{12}\varepsilon_4 + a_{13}\varepsilon_6 - a_{11}\bigl(1 - \varepsilon_1^*\bigr)(k_1 + \varepsilon_1)\bigr) \\
&< r_1(k_1 + \varepsilon_1)\bigl(-\varepsilon_0 + a_{11}\varepsilon_1^*(k_1 + \varepsilon_1)\bigr) \\
&< 0.
\end{aligned}$$

Therefore, there exists a λ_2^*, such that for all $\lambda \in (0, \lambda_2^*)$, we have

$$d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t)\bigl[1 - a_{11}\overline{\phi}(t - c\tau_{11}) - a_{12}\underline{\varphi}(t - c\tau_{12}) - a_{13}\underline{\psi}(t - c\tau_{13})\bigr] \le 0.$$

From the above argument, we see that

$$d_1\overline{\phi}''(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t)\big[1 - a_{11}\overline{\phi}(t - c\tau_{11}) - a_{12}\underline{\varphi}(t - c\tau_{12}) - a_{13}\underline{\psi}(t - c\tau_{13})\big] \leq 0,$$

for small enough $\lambda \in (0, \overline{\lambda}_1^*)$, where $\overline{\lambda}_1^* = \min\{\lambda_1^*, \lambda_2^*\}$.

When $t > t_3 + c\tau_{22}$, $\overline{\varphi}(t) = k_2 + \varepsilon_3 e^{-\lambda t}$, we have

$$\begin{aligned}
&d_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + r_2\overline{\varphi}(t)\big[1 - a_{21}\underline{\phi}(t - c\tau_{21}) - a_{22}\overline{\varphi}(t - c\tau_{22}) - a_{23}\overline{\psi}(t - c\tau_{23})\big] \\
&\quad\leq d_2\varepsilon_3\lambda^2 e^{-\lambda t} + c\varepsilon_3\lambda e^{-\lambda t} + r_3\big(k_2 + \varepsilon_3 e^{-\lambda t}\big)\big(1 - a_{21}\big(k_1 - \varepsilon_2 e^{-\lambda t}\big) \\
&\qquad - a_{22}\big(k_2 + \varepsilon_3 e^{-\lambda t}\big) + a_{23}(k_3 + \varepsilon_5)\big) \\
&\quad=: I_3(\lambda).
\end{aligned}$$

Obviously,

$$\begin{aligned}
I_3(0) &= r_3(k_2 + \varepsilon_3)\big(1 - a_{21}(k_1 - \varepsilon_2) - a_{22}(k_2 + \varepsilon_3) - a_{23}(k_3 + \varepsilon_5)\big) \\
&= r_3(k_2 + \varepsilon_3)(a_{21}\varepsilon_2 - a_{22}\varepsilon_3 + a_{23}\varepsilon_5).
\end{aligned}$$

It is easy to see that $I_3(0) < 0$ and there exists $\lambda_3^* > 0$, such that

$$d_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + r_2\overline{\varphi}(t)\big[1 - a_{21}\underline{\phi}(t - c\tau_{21}) - a_{22}\overline{\varphi}(t - c\tau_{22}) - a_{23}\overline{\psi}(t - c\tau_{23})\big] \leq 0,$$

for all $\lambda \in (0, \lambda_3^*)$.

If $t \leq t_3$, $\overline{\varphi}(t) = e^{\lambda_3 t}$, we have

$$\begin{aligned}
&d_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + r_2\overline{\varphi}(t)\big[1 - a_{21}\underline{\phi}(t - c\tau_{21}) - a_{22}\overline{\varphi}(t - c\tau_{22}) - a_{23}\overline{\psi}(t - c\tau_{23})\big] \\
&\quad\leq d_2\lambda_3^2 e^{\lambda_3 t} - c\lambda_3 e^{\lambda_3 t} + r_2 e^{\lambda_3 t}(1 + a_{23}M_3) = 0.
\end{aligned}$$

If $t_3 < t \leq t_3 + c\tau_{22}$, then we have

$$\begin{aligned}
&d_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + r_2\overline{\varphi}(t)\big[1 - a_{21}\underline{\phi}(t - c\tau_{21}) - a_{22}\overline{\varphi}(t - c\tau_{22}) + a_{23}\overline{\psi}(t - c\tau_{23})\big] \\
&\quad< d_2\varepsilon_3\lambda^2 e^{-\lambda t} + c\varepsilon_3\lambda e^{-\lambda t} + r_2\big(k_2 + \varepsilon_3 e^{-\lambda t}\big)\big(1 - a_{21}\big(k_1 - \varepsilon_2 e^{-\lambda t}\big) \\
&\qquad - a_{22}e^{\lambda_3(t - c\tau_{22})} + a_{23}(k_3 + \varepsilon_5)\big) \\
&\quad=: I_4(\lambda).
\end{aligned}$$

For small enough τ_{22}, there exists ε_2^* ($0 < \varepsilon_2^* < \frac{\varepsilon_0}{a_{22}(k_2+\varepsilon_3)}$) such that $e^{-\lambda_3 c\tau_{22}} > 1 - \varepsilon_2^*$. Thus we have

$$\begin{aligned}
I_4(0) &\leq r_2(k_2 + \varepsilon_3)\big(1 - a_{21}(k_1 - \varepsilon_2) - a_{22}e^{-\lambda_3 c\tau_{22}}(k_2 + \varepsilon_3) + a_{23}(k_3 + \varepsilon_5)\big) \\
&\leq r_2(k_2 + \varepsilon_3)\big(1 - a_{21}(k_1 - \varepsilon_2) - a_{22}\big(1 - \varepsilon_2^*\big)(k_2 + \varepsilon_3) + a_{23}(k_3 + \varepsilon_5)\big) \\
&< r_2(k_2 + \varepsilon_3)\big(-\varepsilon_0 + (k_2 + \varepsilon_3)\varepsilon_2^*\big) \\
&< 0.
\end{aligned}$$

Therefore, there exists a λ_4^*, such that for all $\lambda \in (0, \lambda_4^*)$

$$d_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + r_2\overline{\varphi}(t)\big[1 - a_{21}\underline{\phi}(t - c\tau_{21}) - a_{22}\overline{\varphi}(t - c\tau_{22}) + a_{23}\overline{\psi}(t - c\tau_{23})\big] \le 0.$$

From the above argument, we see that

$$d_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + r_2\overline{\varphi}(t)\big[1 - a_{21}\underline{\phi}(t - c\tau_{21}) - a_{22}\overline{\varphi}(t - c\tau_{22}) + a_{23}\overline{\psi}(t - c\tau_{23})\big] \le 0,$$

for small enough $\lambda \in (0, \overline{\lambda}_2^*)$, where $\overline{\lambda}_2^* = \min\{\lambda_3^*, \lambda_4^*\}$.

Similarly, for all $t \in R$, there exists a $\overline{\lambda}_3^* > 0$, such that, for $\lambda \in (0, \overline{\lambda}_2^*)$, we have

$$d_3\overline{\psi}''(t) - c\overline{\psi}'(t) + r_3\overline{\psi}(t)\big[1 + a_{31}\underline{\phi}(t - c\tau_{31}) + a_{32}\overline{\varphi}(t - c\tau_{32}) - a_{33}\overline{\psi}(t - c\tau_{33})\big] \le 0.$$

From all of the above argument, we see that $\overline{\Phi} = (\overline{\phi}(t), \overline{\varphi}(t), \overline{\psi}(t))$ is an upper solution of (15) for small enough $\lambda \in (0, \hat{\lambda}_1)$, where $\hat{\lambda}_1 = \min\{\lambda_1^*, \lambda_2^*, \lambda_3^*\}$. □

Lemma 7 *Assume that* $D > 0$, $D_i > 0$ $(i = 1, 2, 3)$, *and* (16) *hold, then* $\underline{\Psi}(\underline{\phi}, \underline{\varphi}, \underline{\psi})$ *is a lower solution of system* (15).

Proof If $t \le t_2$,

$$d_1\underline{\phi}''(t) - c\underline{\phi}'(t) + r_1\underline{\phi}(t)\big[1 - a_{11}\underline{\phi}(t - c\tau_{11}) - a_{12}\overline{\varphi}(t - c\tau_{12}) - a_{13}\overline{\psi}(t - c\tau_{13})\big] = 0.$$

If $t > t_2 + c\tau_{11}$,

$$\begin{aligned} &d_1\underline{\phi}''(t) - c\underline{\phi}'(t) + r_1\underline{\phi}(t)\big[1 - a_{11}\underline{\phi}(t - c\tau_{11}) - a_{12}\overline{\varphi}(t - c\tau_{12}) - a_{13}\overline{\psi}(t - c\tau_{13})\big] \\ &\quad \ge -d_1\varepsilon_2\lambda^2 e^{-\lambda t} - c\varepsilon_2\lambda e^{-\lambda t} + r_1\big(k_1 - \varepsilon_2 e^{-\lambda t}\big)\big(1 - a_{11}\big(k_1 - \varepsilon_2 e^{-\lambda t}\big) \\ &\qquad - a_{12}(k_2 + \varepsilon_3) - a_{13}(k_3 + \varepsilon_5)\big) \\ &\quad =: I_5(\lambda). \end{aligned}$$

Obviously,

$$\begin{aligned} I_5(0) &= r_1(k_1 - \varepsilon_2)\big(1 - a_{11}(k_1 - \varepsilon_2) - a_{12}(k_2 + \varepsilon_3) - a_{13}(k_3 + \varepsilon_5)\big) \\ &= r_1(k_1 - \varepsilon_2)(a_{11}\varepsilon_2 - a_{12}\varepsilon_3 - a_{13}\varepsilon_5). \end{aligned}$$

$a_{11}\varepsilon_2 - a_{12}\varepsilon_3 - a_{13}\varepsilon_5 > \varepsilon_0$ implies that $I_5(0) > 0$ and there exists $\lambda_4^* > 0$ such that

$$d_1\underline{\phi}''(t) - c\underline{\phi}'(t) + r_1\underline{\phi}(t)\big[1 - a_{11}\underline{\phi}(t - c\tau_{11}) - a_{12}\overline{\varphi}(t - c\tau_{12}) - a_{13}\overline{\psi}(t - c\tau_{13})\big] \ge 0,$$

for all $\lambda \in (0, \lambda_5^*)$.

If $t_2 < t \le t_2 + c\tau_{11}$,

$$\begin{aligned} &d_1\underline{\phi}''(t) - c\underline{\phi}'(t) + r_1\underline{\phi}(t)\big[1 - a_{11}\underline{\phi}(t - c\tau_{11}) - a_{12}\overline{\varphi}(t - c\tau_{12}) - a_{13}\overline{\psi}(t - c\tau_{13})\big] \\ &\quad \ge -d_1\varepsilon_2\lambda^2 e^{-\lambda t} - c\varepsilon_2\lambda e^{-\lambda t} + r_1\big(k_1 - \varepsilon_2 e^{-\lambda t}\big)\big(1 - a_{12}(k_2 + \varepsilon_3) - a_{13}(k_3 + \varepsilon_5)\big) \\ &\quad =: I_6(\lambda). \end{aligned}$$

It is easy to see that $I_6 > I_5 > 0$ and

$$d_1\underline{\phi}''(t) - c\underline{\phi}'(t) + r_1\underline{\phi}(t)\big[1 - a_{11}\underline{\phi}(t - c\tau_{11}) - a_{12}\overline{\varphi}(t - c\tau_{12}) - a_{13}\overline{\psi}(t - c\tau_{13})\big] \geq 0.$$

Similarly, for all $t \in R$, there exists a $\overline{\lambda}_5^* > 0$, such that for $\lambda \in (0, \overline{\lambda}_5^*)$, we have

$$d_2\underline{\varphi}''(t) - c\underline{\varphi}'(t) + r_2\underline{\varphi}(t)\big[1 - a_{21}\overline{\phi}(t - c\tau_{21}) - a_{22}\underline{\varphi}(t - c\tau_{22}) + a_{23}\underline{\psi}(t - c\tau_{23})\big] \geq 0.$$

For all $t \in R$, there exists a $\overline{\lambda}_6^* > 0$, such that for $\lambda \in (0, \overline{\lambda}_6^*)$, we have

$$d_3\underline{\psi}''(t) - c\underline{\psi}'(t) + r_3\underline{\psi}(t)\big[1 + a_{31}\overline{\phi}(t - c\tau_{31}) + a_{32}\underline{\varphi}(t - c\tau_{32}) - a_{33}\underline{\psi}(t - c\tau_{33})\big] \geq 0.$$

From all of the above arguments, we see that $\underline{\Psi}(\underline{\phi}, \underline{\varphi}, \underline{\psi})$ is a lower solution of (15) for small enough $\lambda \in (0, \hat{\lambda}_2)$, where $\hat{\lambda}_2 = \min\{\lambda_4^*, \lambda_5^*, \lambda_6^*\}$. □

Theorem 2 *If* $D > 0$, $D_i > 0$ $(i = 1, 2, 3)$, *and* (16) *holds for every* $c > c^* = \max\{2\sqrt{d_1 r_1}, 2\sqrt{d_2 r_2(1 + a_{23}M_3)}, 2\sqrt{d_3 r_3(1 + a_{31}M_1 + a_{32}M_2)}\}$, *system* (1) *has a traveling wave solution with speed c connecting the trivial steady-state solution* $(0, 0, 0)$ *and the position steady state* (k_1, k_2, k_3).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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