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Existence and exponential decay estimates for an N-dimensional nonlinear wave equation with a nonlocal boundary condition

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Abstract

Motivated by the recent known results as regards the existence and exponential decay of solutions for wave equations, this paper is devoted to the study of an N-dimensional nonlinear wave equation with a nonlocal boundary condition. We first state two local existence theorems. Next, we give a sufficient condition to guarantee the global existence and exponential decay of weak solutions. The main tools are the Faedo-Galerkin method and the Lyapunov method.

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1 Introduction

In this paper, we consider the following initial-boundary value problem:

$$u_{tt} - \Delta u + Ku + \lambda u_t = a|u|^{p-2}u + f(x,t), \quad x \in \Omega, t > 0, \tag{1.1}$$

$$-\frac{\partial u}{\partial \nu}(x,t) = g(x,t) + \int_\Omega h(x,y,t)u(y,t)\,dy, \quad x \in \partial\Omega, t \geq 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, ν is the unit outward normal on $\partial\Omega$; $a = \pm 1$, K, λ, p are given constants, and u_0, u_1, f, g, h are given functions satisfying conditions specified later.

The wave equation

$$u_{tt} - \Delta u = f(x,t,u,u_t), \tag{1.4}$$

with different boundary conditions, has been extensively studied by many authors, for example, we refer to [1–25] and the references given therein. In these works, many interesting results about the existence, regularity and the asymptotic behavior of solutions were obtained. In [3], Beilin investigated the existence and uniqueness of a generalized solution for the following wave equation with an integral nonlocal condition:

$$\begin{cases} u_{tt} - \Delta u + c(x,t)u = f(x,t), & (x,t) \in Q = \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} + \int_0^t \int_\Omega k(x,\xi,\tau)u(\xi,\tau)\,d\xi\,d\tau = 0, & (x,t) \in \partial\Omega \times [0,T), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary, ν is the unit outward normal on $\partial\Omega, f, u_0, u_1, k(x,\xi,\tau)$ are given functions. Nonlocal conditions come up when values of the function on the boundary is connected to values inside the domain. There are various type of nonlocal boundary conditions of integral form for hyperbolic, parabolic or elliptic equations, introduced in [3]. In [4], the following problem was considered:

$$\begin{cases} u_{tt} - \Delta u + g(u_t) + f(u) = 0, & x \in \Omega, t > 0, \\ u = 0, & x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.6}$$

where $f(u) = -b|u|^{p-2}u$, $g(u_t) = a(1 + |u_t|^{m-2})u_t$, $a, b > 0$, $m, p > 2$, and Ω is a bounded domain of $\mathbb{R}^N$, with a smooth boundary $\partial\Omega$. Benaissa and Messaoudi showed that for suitably chosen initial data, (1.6) possesses a global weak solution, which decays exponentially even if $m > 2$. The proof of the global existence is based on the use of the potential well theory.

As [4], Messaoudi [10] also showed the problem (1.6), with $f(u) = b|u|^{p-2}u$, $b > 0$ has a unique global solution with energy decaying exponentially for any initial data $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$. So if $f(u) = b|u|^{p-2}u$, and $g(u_t) = |u_t|^{m-2}u_t$, Nakao [16] showed that (1.6) has a unique global weak solution if $0 \leq p - 2 \leq \frac{2}{N-2}$, $N \geq 3$, and a global unique strong solution if $p - 2 > \frac{2}{N-2}$, $N \geq 3$ (of course if $N = 1$ or $N = 2$ then the only requirement is $p \geq 2$). On the other hand, in both cases it has been shown that the energy of the solution decays algebraically if $m > 2$ and decays exponentially if $m = 2$. Also as [4], Nakao and Ono [17] extended this result to the Cauchy problem,

$$\begin{cases} u_{tt} - \Delta u + \lambda^2(x)u + g(u_t) + f(u) = 0, & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.7}$$

where $g(u_t)$ behaves like $|u_t|^{m-2}u_t$, $f(u)$ behaves like $-|u|^{p-2}u$ and the initial data (u_0, u_1) is small enough in $H^1(\Omega) \times L^2(\Omega)$. Later on, Ono [19] studied the global existence and the decay properties of smooth solutions to the Cauchy problem related to (1.6), for $f(u) \equiv 0$ and gave sharp decay estimates of the solution without any restrictions on the data size (u_0, u_1).

In [21], Munoz-Rivera and Andrade dealt with the global existence and exponential decay of solutions of the nonlinear one-dimensional wave equation with a viscoelastic boundary condition. In [22–24], Santos also studied the asymptotic behavior of solutions to a coupled system of wave equations having integral convolutions as memory terms. The main results show that the solutions of that system decay uniformly in time, with rates depending on the rate of decay of the kernel of the convolutions.

In [25], the global existence and regularity of weak solutions for the linear wave equation

$$u_{tt} - u_{xx} + Ku + \lambda u_t = f(x,t), \quad 0 < x < 1, t > 0, \tag{1.8}$$

with the initial conditions as in (1.3) and two-point boundary conditions. The exponential decay of solutions was also given there by using Lyapunov method.

The works introduced as above lead to the study of the existence and exponential decay of solutions for the problem (1.1)-(1.3). This paper consists of three sections. The preliminaries are presented and two existence results with $a = 1$ are done in Section 2. The decay of the solution with respect to $a = 1$, $g = 0$, $K > 0$, $\lambda > 0$, and $2 < p \leq \frac{2N-2}{N-2}$, $N \geq 3$ is established in Section 3. The proofs of the existences are based on the Faedo-Galerkin method for strong solutions and standard arguments of density for weak solutions. Because this problem is solved in an N-dimensional domain, it causes technical difficulties, so we need the relations between the norms as in Lemmas 2.1-2.3 below. To obtain the exponential decay, we use the multiplier technique combined with a suitable Lyapunov functional in the form $\mathcal{L}(t) = E(t) + \delta\psi(t)$, where

$$E(t) = \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\left\|\nabla u(t)\right\|^2 + \frac{K}{2}\left\|u(t)\right\|^2 - \frac{1}{p}\left\|u(t)\right\|_{L^p}^p + \int_{\partial\Omega}\left\langle h(x,t), u(t)\right\rangle u(x,t)\,dS_x,$$

$$\psi(t) = \left\langle u(t), u'(t)\right\rangle + \frac{\lambda}{2}\left\|u(t)\right\|^2,$$

$\delta > 0$ is chosen sufficiently small, which allows us to show that if

$$\|\nabla u_0\|^2 + K\|u_0\|^2 - \|u_0\|_{L^p}^p + p\int_{\partial\Omega}\left(\int_{\Omega} h(x,y,0)u_0(y)\,dy\right)u_0(x)\,dS_x > 0$$

and if the initial energy $E(0), f, h$ given are small enough, then the energy $E(t)$ of the solution decays to zero exponentially when t goes to infinity.

We end the paper with a remark about a situation where $a = -1$, precisely we consider (1.1) in the form

$$u_{tt} - \Delta u + Ku + \lambda u_t + |u|^{p-2}u = f(x,t), \quad x \in \Omega, t > 0. \tag{1.9}$$

With some suitable conditions for f, h, g, we obtain a unique global solution for (1.2)-(1.3) and (1.9), with energy decaying exponentially as $t \to +\infty$, without any restrictions on the data size (u_0, u_1) as in [19].

2 Preliminaries and existence results

In this paper, $\Omega \subset \mathbb{R}^N$ is an open and bounded set with a smooth boundary $\partial\Omega$ and the usual function spaces $C^m(\overline{\Omega})$, $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $1 \leq p \leq \infty$, $m = 0, 1, \ldots$ are used. Let $\langle\cdot,\cdot\rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$, the Banach space of the real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \left\|u(t)\right\|_X^p\,dt\right)^{1/p} < \infty \quad \text{for } 1 \leq p < \infty$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T}\left\|u(t)\right\|_X \quad \text{for } p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $\nabla u(t)$, $\Delta u(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $(\frac{\partial u}{\partial x_1}(x,t), \ldots, \frac{\partial u}{\partial x_N}(x,t))$, $\sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}(x,t)$, respectively.

On H^1 we shall use the following norm: $\|v\|_{H^1} = (\|v\|^2 + \|\nabla v\|^2)^{1/2}$.

In cases $N = 1$ or $N = 2$, by the continuity and compactness of the injections $H^1(\Omega) \hookrightarrow C^0(\overline{\Omega})$ with $N = 1$ or $H^1(\Omega) \hookrightarrow L^q(\Omega)$ with $N = 2$, it is not difficult to study problem (1.1)-(1.3). On the other hand, it is obvious that the problem considered with $a = 1$ is more difficult than the one with $a = -1$,so in what follows we only consider problem (1.1)-(1.3) with $N \geq 3$, $a = 1$. A remark in the end of this paper will give a note in the case $a = -1$.

First, we recall the following results, see [26].

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set of class C^1. Then the embedding $H^1 \hookrightarrow L^q$, is continuous if $1 \leq q \leq 2^*$ and compact if $1 \leq q < 2^*$, where $2^* = \frac{2N}{N-2}$, $N \geq 3$.*

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with a smooth boundary $\partial\Omega$. Then*

$$\left(\int_{\partial\Omega} v^2(x)\, dS_x \right)^{1/2} \leq \gamma_\Omega \|v\|_{H^1} \quad \text{for all } v \in H^1, \tag{2.1}$$

where γ_Ω is a positive constant depending only on the domain Ω.

The proofs below also require the following lemma.

Lemma 2.3 *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with a smooth boundary $\partial\Omega$. Let $2 \leq p \leq \frac{2N-2}{N-2}$, $N \geq 3$. Then there exists a constant $D_p > 0$ depending on p, N and Ω such that*

$$\begin{aligned} \text{(i)} \quad & \big\| |u|^{p-2}u - |v|^{p-2}v \big\| \\ & \leq D_p \big[1 + \big(\|u\|_{H^1} + \|v\|_{H^1} \big)^{1/N} + \big(\|u\|_{H^1} + \|v\|_{H^1} \big)^{p-2} \big] \|u - v\|_{H^1}, \\ \text{(ii)} \quad & \big\| |u|^{p-2}v \big\| \leq D_p \big[1 + \|u\|_{H^1}^{1/N} + \|u\|_{H^1}^{p-2} \big] \|v\|_{H^1} \end{aligned} \tag{2.2}$$

for all $u, v \in H^1$.

Proof (i) We have

$$\begin{aligned} \big| |u|^{p-2}u - |v|^{p-2}v \big| &= \left| \int_0^1 \frac{d}{d\theta} \big[\big| v + \theta(u-v) \big|^{p-2} \big(v + \theta(u-v) \big) \big] \, d\theta \right| \\ &= (p-1)|u-v| \int_0^1 \big| v + \theta(u-v) \big|^{p-2} \, d\theta \leq (p-1)|u-v| |W|^{p-2}, \end{aligned} \tag{2.3}$$

with $W = |u| + |v|$.

Hence, by Hölder's inequality we have

$$\begin{aligned} \big\| |u|^{p-2}u - |v|^{p-2}v \big\| &\leq (p-1) \left(\int_\Omega |u-v|^2 |W|^{2p-4} \, dx \right)^{1/2} \\ &\leq (p-1) \left(\int_\Omega |u-v|^{2\alpha} \, dx \right)^{1/2\alpha} \left(\int_\Omega |W|^{(2p-4)\alpha'} \, dx \right)^{1/2\alpha'} \\ &\quad \text{for all } \alpha > 1. \end{aligned} \tag{2.4}$$

Note that $H^1 \hookrightarrow L^q$, $1 \le q \le 2^* = \frac{2N}{N-2}$, $N \ge 3$, and $\|v\|_{L^q} \le C_q \|v\|_{H^1}$, $\forall v \in H^1$, $1 \le q \le 2^*$. Choose $\alpha = \frac{2^*}{2} = \frac{N}{N-2}$, we have $\alpha' = \frac{\alpha}{\alpha-1} = \frac{\frac{N}{N-2}}{\frac{N}{N-2}-1} = \frac{N}{2}$, and

$$\left(\int_\Omega |u-v|^{2\alpha}\,dx\right)^{1/2\alpha} = \|u-v\|_{L^{2^*}} \le C_{2^*}\|u-v\|_{H^1}. \tag{2.5}$$

By the condition $2 \le p \le \frac{2N-2}{N-2} = 2 + \frac{2}{N-2}$, $N \ge 3$ is equivalent to

$$0 \le (2p-4)\alpha' \le 2^* = \frac{2N}{N-2}, \tag{2.6}$$

so we consider two cases as follows.

Case 1. $1 \le (2p-4)\alpha' \le 2^* = \frac{2N}{N-2}$:

$$\begin{aligned}\left(\int_\Omega |W|^{(2p-4)\alpha'}\,dx\right)^{1/2\alpha'} &= \|W\|^{p-2}_{L^{2(p-2)\alpha'}} \le \left(C_{2(p-2)\alpha'}\|W\|_{H^1}\right)^{p-2}\\ &= C^{p-2}_{2(p-2)\alpha'}\|W\|^{p-2}_{H^1}.\end{aligned} \tag{2.7}$$

Case 2. $0 \le \beta \equiv (2p-4)\alpha' < 1 \le 2^* = \frac{2N}{N-2}$:

$$|W|^{(2p-4)\alpha'} = |W|^\beta \le 1 + |W|, \tag{2.8}$$

$$\begin{aligned}\left(\int_\Omega |W|^{(2p-4)\alpha'}\,dx\right)^{1/2\alpha'} &\le \left(\int_\Omega \left(1+|W|\right)dx\right)^{1/2\alpha'}\\ &\le \left(|\Omega| + |\Omega|^{1/2}\|W\|\right)^{1/2\alpha'}\\ &\le \left(|\Omega| + |\Omega|^{1/2}\|W\|_{H^1}\right)^{1/2\alpha'}\\ &= \left(|\Omega| + |\Omega|^{1/2}\|W\|_{H^1}\right)^{1/N}\\ &\le |\Omega|^{1/N} + |\Omega|^{1/2N}\|W\|^{1/N}_{H^1}.\end{aligned} \tag{2.9}$$

Consequently, in both cases we get

$$\left(\int_\Omega |W|^{(2p-4)\alpha'}\,dx\right)^{1/2\alpha'} \le |\Omega|^{1/N} + |\Omega|^{1/2N}\|W\|^{1/N}_{H^1} + C^{p-2}_{(p-2)N}\|W\|^{p-2}_{H^1}. \tag{2.10}$$

Hence

$$\begin{aligned}\left\||u|^{p-2}u - |v|^{p-2}v\right\| &\le (p-1)C_{2^*}\|u-v\|_{H^1}\\ &\quad\times\left[|\Omega|^{1/N} + |\Omega|^{1/2N}\|W\|^{1/N}_{H^1} + C^{p-2}_{(p-2)N}\|W\|^{p-2}_{H^1}\right]\\ &\le D_p\|u-v\|_{H^1}\left[1 + \|W\|^{1/N}_{H^1} + \|W\|^{p-2}_{H^1}\right]\\ &\le D_p\left[1 + \left(\|u\|_{H^1} + \|v\|_{H^1}\right)^{\frac{1}{N}} + \left(\|u\|_{H^1} + \|v\|_{H^1}\right)^{p-2}\right]\\ &\quad\times\|u-v\|_{H^1}.\end{aligned} \tag{2.11}$$

Similarly (ii) is proved.

The proof of Lemma 2.3 is complete. □

Next, we state two local existence theorems. We make the following assumptions:

(A_0) $2 < p \le \frac{2N-2}{N-2}$, $N \ge 3$,
(B_0) $K, \lambda \in \mathbb{R}$,
(A_1) $f, f' \in L^1(0,T;L^2)$,
(A_2) $h \in L^2(0,T;L^2(\partial\Omega \times \Omega))$, $h', h'' \in L^2(0,T;L^2(\partial\Omega \times \Omega))$,
(A_3) $g \in L^2(\partial\Omega \times \Omega)$, $g', g'' \in L^2(\partial\Omega \times \Omega)$,
(A_1') $f \in L^2(Q_T)$,
(A_2') $h \in L^2(0,T;L^2(\partial\Omega \times \Omega))$, $h' \in L^2(0,T;L^2(\partial\Omega \times \Omega))$,
(A_3') $g \in L^2(0,T;L^2(\partial\Omega))$, $g' \in L^2(0,T;L^2(\partial\Omega))$.

Then we have the following theorem as regards the existence of a 'strong solution'.

Theorem 2.4 *Suppose that* (A_0), (B_0), (A_1)-(A_3) *hold and the initial data* $(u_0, u_1) \in H^2 \times H^1$ *satisfies the compatibility condition*

$$-\frac{\partial u_0}{\partial \nu}(x) = g(x,0) + \int_\Omega h(x,y,0)u_0(y)\,dy. \tag{2.12}$$

Then problem (1.1)-(1.3) *has a unique local solution*

$$u \in L^\infty\bigl(0,T_*;H^2\bigr), \qquad u_t \in L^\infty\bigl(0,T_*;H^1\bigr), \qquad u_{tt} \in L^\infty\bigl(0,T_*;L^2\bigr) \tag{2.13}$$

for $T_* > 0$ *small enough.*

Remark 2.1 The regularity obtained by (2.13) shows that problem (1.1)-(1.3) has a unique strong solution

$$\begin{cases} u \in L^\infty(0,T_*;H^2) \cap C^0(0,T_*;H^1) \cap C^1(0,T_*;L^2), \\ u_t \in L^\infty(0,T_*;H^1) \cap C^0(0,T_*;L^2), \\ u_{tt} \in L^\infty(0,T_*;L^2). \end{cases} \tag{2.14}$$

With less regular initial data, we obtain the following theorem as regards the existence of a weak solution.

Theorem 2.5 *Let* (A_0), (B_0), (A_1')-(A_3') *hold and* $(u_0, u_1) \in H^1 \times L^2$.
Then problem (1.1)-(1.3) *has a unique local solution*

$$u \in C\bigl([0,T_*];H^1\bigr) \cap C^1\bigl([0,T_*];L^2\bigr) \tag{2.15}$$

for $T_* > 0$ *small enough.*

Proof of Theorem 2.4 Let $\{w_j\}$ be a denumerable base of H^2. Under the assumptions of Theorem 2.4, using the Faedo-Galerkin approximation and Lemmas 2.1-2.3, we find the approximate solution of problem (1.1)-(1.3) in the form

$$u_m(t) = \sum_{j=1}^m c_{mj}(t) w_j, \tag{2.16}$$

where the coefficient functions c_{mj} satisfy the system of ordinary differential equations

$$\begin{cases} \langle u_m''(t), w_j\rangle + \langle \nabla u_m(t), \nabla w_j\rangle + \langle Ku_m(t) + \lambda u_m'(t), w_j\rangle \\ \qquad + \int_{\partial\Omega}(\langle h(x,t), u_m(t)\rangle + g(x,t))w_j(x)\,dS_x \\ \quad = \langle |u_m(t)|^{p-2}u_m(t), w_j\rangle + \langle f(t), w_j\rangle, \quad 1 \le j \le m, \\ u_m(0) = u_0, \qquad u_m'(0) = u_1. \end{cases} \tag{2.17}$$

From the assumptions of Theorem 2.4, system (2.17) has a solution u_m on an interval $[0, T_m] \subset [0, T]$. The following estimates allow one to take $T_m = T_*$ for all m, consisting of two key estimates.

In the first key estimate, we put $S_m(t) = \|u_m'(t)\|^2 + \|\nabla u_m(t)\|^2$, it implies from (2.17) that

$$\begin{aligned} S_m(t) &= S_m(0) + 2\int_{\partial\Omega} \big(\langle h(x,0), u_0\rangle + g(x,0)\big)u_0(x)\,dS_x \\ &\quad - 2\int_0^t \langle Ku_m(s) + \lambda u_m'(s), u_m'(s)\rangle\,ds \\ &\quad + 2\int_0^t \langle f(s), u_m'(s)\rangle\,ds + 2\int_0^t \langle |u_m(s)|^{p-2}u_m(s), u_m'(s)\rangle\,ds \\ &\quad - 2\int_{\partial\Omega} g(x,t)u_m(x,t)\,dS_x - 2\int_{\partial\Omega}\langle h(x,t), u_m(t)\rangle u_m(x,t)\,dS_x \\ &\quad + 2\int_0^t ds\int_{\partial\Omega}\big[\langle h'(x,s), u_m(s)\rangle + \langle h(x,s), u_m'(s)\rangle + g'(x,s)\big]u_m(x,s)\,dS_x \\ &\equiv S_m(0) + \sum_{j=1}^{7} I_j. \end{aligned} \tag{2.18}$$

By Lemmas 2.1-2.3 and the following inequalities:

$$\begin{cases} 2ab \le \beta a^2 + \frac{1}{\beta}b^2 \quad \text{for all } a, b \in \mathbb{R}, \beta > 0, \\ (a+b+c)^q \le 3^{q-1}(a^q + b^q + c^q) \quad \text{for all } q \ge 1, a, b, c \ge 0 \end{cases} \tag{2.19}$$

and

$$\|v\| \le \|v\|_{H^1}, \qquad \|v\|_{L^q} \le C_q\|v\|_{H^1}, \quad \forall v \in H^1, 1 \le q \le 2^* = \frac{2N}{N-2}, N \ge 3, \tag{2.20}$$

with computing explicitly, all terms in the right-hand side of (2.18) are estimated, in which the following estimates are worthy of note:

$$\begin{aligned} S_m(0) + I_1 &= S_m(0) + 2\int_{\partial\Omega}\big(\langle h(x,0), u_0\rangle + g(x,0)\big)u_0(x)\,dS_x \\ &= \|u_1\|^2 + \|\nabla u_0\|^2 + 2\int_{\partial\Omega}\big(\langle h(x,0), u_0\rangle + g(x,0)\big)u_0(x)\,dS_x \equiv \frac{1}{2}\overline{C}_0; \end{aligned} \tag{2.21}$$

$$\begin{aligned} I_2 &= -2\int_0^t \langle Ku_m(s) + \lambda u_m'(s), u_m'(s)\rangle\,ds \le \int_0^t \|u_m(s)\|^2\,ds + \big(K^2 + 2|\lambda|\big)\int_0^t \|u_m'(s)\|^2\,ds \\ &\le \int_0^t \bigg[\|u_0\| + \int_0^s \|u_m'(r)\|\,dr\bigg]^2 ds + \big(K^2 + 2|\lambda|\big)\int_0^t S_m(s)\,ds \end{aligned}$$

$$\le 2T\|u_0\|^2 + T^2\int_0^t \|u_m'(r)\|^2\,dr + (K^2 + 2|\lambda|)\int_0^t S_m(s)\,ds$$

$$\le 2T\|u_0\|^2 + (T^2 + K^2 + 2|\lambda|)\int_0^t S_m(s)\,ds \le C_T\left(1 + \int_0^t S_m(s)\,ds\right); \tag{2.22}$$

$$I_3 = 2\int_0^t \langle f(s), u_m'(s)\rangle\,ds \le \int_0^T \|f(s)\|^2\,ds + \int_0^t \|u_m'(s)\|^2\,ds \le C_T + \int_0^t S_m(s)\,ds; \tag{2.23}$$

$$I_4 = 2\int_0^t \langle |u_m(s)|^{p-2}u_m(s), u_m'(s)\rangle\,ds \le 2\int_0^t \||u_m(s)|^{p-1}\|\|u_m'(s)\|\,ds$$

$$\le \int_0^t \||u_m(s)|^{p-1}\|^2\,ds + \int_0^t S_m(s)\,ds$$

$$= \int_0^t \|u_m(s)\|_{L^{2p-2}}^{2p-2}\,ds + \int_0^t S_m(s)\,ds \le C_{2p-2}^{2p-2}\int_0^t \|u_m(s)\|_{H^1}^{2p-2}\,ds + \int_0^t S_m(s)\,ds, \tag{2.24}$$

since $1 \le 2 \le 2p-2 \le 2^*$, and $H^1(\Omega) \hookrightarrow L^{2p-2}(\Omega)$, we have

$$\|u_m(t)\|_{H^1}^{2p-2} \le \left[2\|u_0\|^2 + S_m(t) + 2t\int_0^t S_m(s)\,ds\right]^{p-1}$$

$$\le 3^{p-2}2^{p-1}\|u_0\|^{2p-2} + 3^{p-2}(S_m(t))^{p-1} + 3^{p-2}2^{p-1}t^{2p-3}\int_0^t (S_m(s))^{p-1}\,ds,$$

it leads to

$$I_4 = 2\int_0^t \langle |u_m(s)|^{p-2}u_m(s), u_m'(s)\rangle\,ds \le C_T + C_T\int_0^t (S_m(s))^{p-1}\,ds + \int_0^t S_m(s)\,ds; \tag{2.25}$$

$$I_5 = -2\int_{\partial\Omega} g(x,t)u_m(x,t)\,dS_x \le 2\gamma_\Omega\|g\|_{L^\infty(0,T;L^2(\partial\Omega))}\|u_m(t)\|_{H^1}$$

$$\le \frac{1}{\beta}\gamma_\Omega^2\|g\|_{L^\infty(0,T;L^2(\partial\Omega))}^2 + \beta\|u_m(t)\|_{H^1}^2$$

$$\le \frac{1}{\beta}\gamma_\Omega^2\|g\|_{L^\infty(0,T;L^2(\partial\Omega))}^2 + \beta\left[2\|u_0\|^2 + S_m(t) + 2t\int_0^t S_m(s)\,ds\right]$$

$$\le \frac{1}{\beta}C_T + \beta S_m(t) + C_T\int_0^t S_m(s)\,ds \quad \text{for all } 0 < \beta < 1; \tag{2.26}$$

$$I_6 = -2\int_{\partial\Omega} \langle h(x,t), u_m(t)\rangle u_m(x,t)\,dS_x \le 2\gamma_\Omega\|h\|_{L^\infty(0,T;L^2(\partial\Omega\times\Omega))}\|u_m(t)\|\|u_m(t)\|_{H^1}$$

$$\le \frac{1}{\beta}\gamma_\Omega^2\|h\|_{L^\infty(0,T;L^2(\partial\Omega\times\Omega))}^2\|u_m(t)\|^2 + \beta\|u_m(t)\|_{H^1}^2$$

$$\le \frac{1}{\beta}\gamma_\Omega^2\|h\|_{L^\infty(0,T;L^2(\partial\Omega\times\Omega))}^2\left[2\|u_0\|^2 + 2t\int_0^t S_m(s)\,ds\right]$$

$$+ \beta\left[2\|u_0\|^2 + S_m(t) + 2t\int_0^t S_m(s)\,ds\right]$$

$$\le \frac{1}{\beta}C_T + \beta S_m(t) + \frac{1}{\beta}C_T\int_0^t S_m(s)\,ds \quad \text{for all } \beta > 0, \beta < 1; \tag{2.27}$$

$$I_7 = 2\int_0^t ds\int_{\partial\Omega}\left[\langle h'(x,s), u_m(s)\rangle + \langle h(x,s), u_m'(s)\rangle + g'(x,s)\right]u_m(x,s)\,dS_x$$

$$\le 2\gamma_\Omega\|h'\|_{L^\infty(0,T;L^2(\partial\Omega\times\Omega))}\int_0^t \|u_m(s)\|_{H^1}^2\,ds$$

$$
\begin{aligned}
&+ 2\gamma_\Omega \|h\|_{L^\infty(0,T;L^2(\partial\Omega\times\Omega))} \int_0^t \|u_m'(s)\| \|u_m(s)\|_{H^1}\,ds \\
&+ 2\gamma_\Omega \|g'\|_{L^\infty(0,T;L^2(\partial\Omega))} \int_0^t \|u_m(s)\|_{H^1}\,ds \\
\leq{}& C_T + C_T \int_0^t \|u_m(s)\|_{H^1}^2\,ds + \int_0^t \|u_m'(s)\|^2\,ds \leq C_T\left(1 + \int_0^t S_m(s)\,ds\right). \qquad (2.28)
\end{aligned}
$$

Combining estimations of all terms and choosing $\beta = \frac{1}{4}$, we obtain after some rearrangements

$$
S_m(t) \leq C_T\left(1 + \int_0^t S_m(s)\,ds + \int_0^t \big(S_m(s)\big)^{p-1}\,ds\right), \quad 0 \leq t \leq T_m, \qquad (2.29)
$$

where C_T always indicates a constant depending on T.

Then, by solving a nonlinear Volterra integral inequality (2.29) (based on the methods in [27]), the following lemma is proved.

Lemma 2.6 *There exists a constant $T_* > 0$ depending on T (independent of m) such that*

$$
S_m(t) \leq C_T, \quad \forall m \in \mathbb{N}, \forall t \in [0, T_*], \qquad (2.30)
$$

where C_T is a constant depending only on T.

By Lemma 2.6, we can take a constant $T_m = T_*$ for all m.

In the second key estimate, we put $X_m(t) = \|u_m''(t)\|^2 + \|\nabla u_m'(t)\|^2$,and it follows from (2.17) that

$$
\begin{aligned}
X_m(t) ={}& X_m(0) + 2\int_{\partial\Omega} \big(\langle h'(x,0), u_0\rangle + \langle h(x,0), u_1\rangle + g'(x,0)\big) u_1(x)\,dS_x \\
&- 2\int_0^t \langle Ku_m'(s) + \lambda u_m''(s), u_m''(s)\rangle\,ds + 2\int_0^t \langle f'(s), u_m''(s)\rangle\,ds \\
&+ 2(p-1)\int_0^t \langle |u_m(s)|^{p-2} u_m'(s), u_m''(s)\rangle\,ds \\
&- 2\int_{\partial\Omega} \big(\langle h'(x,t), u_m(t)\rangle + \langle h(x,t), u_m'(t)\rangle + g'(x,t)\big) u_m'(x,t)\,dS_x \\
&+ 2\int_0^t ds \int_{\partial\Omega} \big[\langle h''(x,s), u_m(s)\rangle + 2\langle h'(x,s), u_m'(s)\rangle \\
&+ \langle h(x,s), u_m''(s)\rangle + g''(x,s)\big] u_m'(x,s)\,dS_x \\
\equiv{}& X_m(0) + \sum_{i=1}^6 J_i. \qquad (2.31)
\end{aligned}
$$

Letting $t \to 0_+$ in equation $(2.17)_1$, multiplying the result by $c_{mj}''(0)$, and using the compatibility (2.12), we get

$$
\|u_m''(0)\|^2 = \langle \Delta u_0, u_m''(0)\rangle - \langle Ku_0 + \lambda u_1, u_m''(0)\rangle + \langle |u_0|^{p-2} u_0, u_m''(0)\rangle + \langle f(0), u_m''(0)\rangle.
$$

This implies that

$$\left\| u_m''(0) \right\| \le \|\Delta u_0\| + |K| \|u_0\| + |\lambda| \|u_1\| + \left\| |u_0|^{p-1} \right\| + \left\| f(0) \right\| = \overline{X}_0 \quad \text{for all } m, \tag{2.32}$$

where $\overline{X}_0$ is a constant depending only on $p, K, \lambda, u_0, u_1, f$.

Also note the following estimations:

$$\begin{aligned} X_m(0) + J_1 &= X_m(0) + 2 \int_{\partial\Omega} \left(\langle h'(x,0), u_0 \rangle + \langle h(x,0), u_1 \rangle + g'(x,0) \right) u_1(x)\, dS_x \\ &\le \overline{X}_0^2 + \|\nabla u_1\|^2 + 2 \int_{\partial\Omega} \left(\langle h'(x,0), u_0 \rangle + \langle h(x,0), u_1 \rangle + g'(x,0) \right) u_1(x)\, dS_x \\ &\equiv \frac{1}{2} X_0; \end{aligned} \tag{2.33}$$

$$\begin{aligned} J_2 &= -2 \int_0^t \langle K u_m'(s) + \lambda u_m''(s), u_m''(s) \rangle \, ds \\ &\le \int_0^t \left\| u_m'(s) \right\|^2 ds + \left(K^2 + 2|\lambda| \right) \int_0^t \left\| u_m''(s) \right\|^2 ds \\ &\le C_T + \left(K^2 + 2|\lambda| \right) \int_0^t X_m(s)\, ds; \end{aligned} \tag{2.34}$$

$$\begin{aligned} J_3 &= 2 \int_0^t \langle f'(s), u_m''(s) \rangle \, ds \le \int_0^t \left\| f'(s) \right\| ds + \int_0^t \left\| f'(s) \right\| \left\| u_m''(s) \right\|^2 ds \\ &\le C_T + \int_0^t \left\| f'(s) \right\| X_m(s)\, ds. \end{aligned} \tag{2.35}$$

From

$$\left\| |u_m(s)|^{p-2} u_m'(s) \right\| \le D_p \left[1 + \left\| u_m(s) \right\|_{H^1}^{1/N} + \left\| u_m(s) \right\|_{H^1}^{p-2} \right] \left\| u_m'(s) \right\|_{H^1} \le D_p C_T \left\| u_m'(s) \right\|_{H^1},$$

by Lemma 2.3(ii), it gives

$$\begin{aligned} J_4 &= 2(p-1) \int_0^t \langle |u_m(s)|^{p-2} u_m'(s), u_m''(s) \rangle \, ds \\ &\le 2(p-1) \int_0^t \left\| |u_m(s)|^{p-2} u_m'(s) \right\| \left\| u_m''(s) \right\| ds \\ &\le 2(p-1) D_p C_T \int_0^t \left\| u_m'(s) \right\|_{H^1} \left\| u_m''(s) \right\| ds \\ &\le (p-1)^2 D_p^2 C_T^2 \int_0^t \left\| u_m'(s) \right\|_{H^1}^2 ds + \int_0^t \left\| u_m''(s) \right\|^2 ds \\ &= (p-1)^2 D_p^2 C_T^2 \left[\int_0^t \left\| u_m'(s) \right\|^2 ds + \int_0^t \left\| \nabla u_m'(s) \right\|^2 ds \right] + \int_0^t \left\| u_m''(s) \right\|^2 ds \\ &\le C_T \left(1 + \int_0^t X_m(s)\, ds \right); \end{aligned} \tag{2.36}$$

$$\begin{aligned} J_5 &= -2 \int_{\partial\Omega} \left(\langle h'(x,t), u_m(t) \rangle + \langle h(x,t), u_m'(t) \rangle + g'(x,t) \right) u_m'(x,t)\, dS_x \\ &\le 2\gamma_\Omega \left[\left\| u_m(t) \right\| \left\| h' \right\|_{L^\infty(0,T;L^2(\partial\Omega\times\Omega))} + \left\| u_m'(t) \right\| \|h\|_{L^\infty(0,T;L^2(\partial\Omega\times\Omega))} \right. \end{aligned}$$

$$
\begin{aligned}
&\quad + \|g'\|_{L^\infty(0,T;L^2(\partial\Omega))}\big] \|u_m'(t)\|_{H^1} \\
&\le 2C_T \|u_m'(t)\|_{H^1} \le \frac{1}{\beta} C_T + \beta \|u_m'(t)\|_{H^1}^2 \\
&\le \frac{1}{\beta} C_T + \beta \left[2\|u_1\|^2 + X_m(t) + 2t \int_0^t X_m(s)\,ds \right] \\
&\le \frac{1}{\beta} C_T + \beta X_m(t) + C_T \left(1 + \int_0^t X_m(s)\,ds \right) \quad \text{for all } \beta \in (0,1);
\end{aligned}
\tag{2.37}
$$

$$
\begin{aligned}
J_6 &= 2\int_0^t ds \int_{\partial\Omega} \big[\langle h''(x,s), u_m(s)\rangle + 2\langle h'(x,s), u_m'(s)\rangle \\
&\quad + \langle h(x,s), u_m''(s)\rangle + g''(x,s)\big] u_m'(x,s)\,dS_x \\
&\le 2\gamma_\Omega C_T \int_0^t \|h''(s)\|_{L^2(\partial\Omega\times\Omega)} \|u_m'(s)\|_{H^1}\,ds + 4\gamma_\Omega C_T \int_0^t \|u_m'(s)\|_{H^1}\,ds \\
&\quad + 2\gamma_\Omega C_T \int_0^t \|u_m''(s)\| \|u_m'(s)\|_{H^1}\,ds + 2\gamma_\Omega \int_0^t \|g''(s)\|_{L^2(\partial\Omega)} \|u_m'(s)\|_{H^1}\,ds \\
&\le \gamma_\Omega^2 C_T^2 \|h''\|_{L^1(0,T;L^2(\partial\Omega\times\Omega))} + \int_0^t \|h''(s)\|_{L^2(\partial\Omega\times\Omega)} \|u_m'(s)\|_{H^1}^2\,ds \\
&\quad + 4\gamma_\Omega^2 C_T^2 T + \int_0^t \|u_m'(s)\|_{H^1}^2\,ds + \gamma_\Omega^2 C_T^2 \int_0^t \|u_m''(s)\|^2\,ds + \int_0^t \|u_m'(s)\|_{H^1}^2\,ds \\
&\quad + \gamma_\Omega^2 \|g''\|_{L^1(0,T;L^2(\partial\Omega))} + \int_0^t \|g''(s)\|_{L^2(\partial\Omega)} \|u_m'(s)\|_{H^1}^2\,ds \\
&\le C_T + C_T \int_0^t X_m(s)\,ds + \int_0^t \Phi(s) \|u_m'(s)\|_{H^1}^2\,ds \\
&\le C_T + C_T \int_0^t X_m(s)\,ds + \int_0^t \Phi(s) \left[2\|u_1\|^2 + X_m(s) + 2s \int_0^s X_m(r)\,dr \right] ds \\
&\le C_T + C_T \int_0^t X_m(s)\,ds + \int_0^t \Phi(s) X_m(s)\,ds,
\end{aligned}
\tag{2.38}
$$

where

$$
\Phi(s) = 2 + \|h''(s)\|_{L^2(\partial\Omega\times\Omega)} + \|g''(s)\|_{L^2(\partial\Omega)}, \quad \Phi \in L^1(0,T). \tag{2.39}
$$

Combining estimations and choosing $\beta = \frac{1}{2}$, we obtain after some rearrangements

$$
X_m(t) \le C_T + \int_0^t \Psi(s) X_m(s)\,ds, \tag{2.40}
$$

where C_T always indicates a constant depending on T, and

$$
\Psi(s) = C_T\big[1 + \|f'(s)\| + \|h''(s)\|_{L^2(\partial\Omega\times\Omega)} + \|g''(s)\|_{L^2(\partial\Omega)}\big], \quad \Psi \in L^1(0,T). \tag{2.41}
$$

By Gronwall's lemma, we deduce from (2.40) that

$$
X_m(t) \le C_T \exp\left[\int_0^T \Psi(s)\,ds \right] \le C_T \quad \text{for all } t \in [0, T_*]. \tag{2.42}
$$

It verifies the existence of a subsequence of $\{u_m\}$, denoted by the same symbol, such that

$$\begin{cases} u_m \to u & \text{in } L^\infty(0,T_*;H^1) \text{ weakly}^*, \\ u_m' \to u' & \text{in } L^\infty(0,T_*;H^1) \text{ weakly}^*, \\ u_m'' \to u'' & \text{in } L^\infty(0,T_*;L^2) \text{ weakly}^*. \end{cases} \tag{2.43}$$

By the compactness lemma of Lions ([28], p.57), we can deduce from (2.43) the existence of a subsequence still denoted by $\{u_m\}$, such that

$$\begin{cases} u_m \to u & \text{strongly in } L^2(Q_{T_*}) \text{ and a.e. in } Q_{T_*}, \\ u_m' \to u' & \text{strongly in } L^2(Q_{T_*}) \text{ and a.e. in } Q_{T_*}. \end{cases} \tag{2.44}$$

By means of the continuity of the function $t \longmapsto |t|^{p-2}t$, we have

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \quad \text{and a.e. in } Q_{T_*}. \tag{2.45}$$

On the other hand

$$\begin{aligned} \big\| |u_m|^{p-2}u_m \big\|_{L^2(Q_{T_*})}^2 &= \int_0^{T_*} ds \int_\Omega \big|u_m(x,t)\big|^{2p-2}\,dx \\ &= \int_0^{T_*} \big\|u_m(t)\big\|_{L^{2p-2}}^{2p-2}\,dt \\ &\le \int_0^{T_*} \big(C_{2p-2}\big\|u_m(t)\big\|_{H^1}\big)^{2p-2}\,dt \\ &\le C_{2p-2}^{2p-2} T_* \|u_m\|_{L^\infty(0,T_*;H^1)}^{2p-2} \le C_T. \end{aligned} \tag{2.46}$$

Using the Lions lemma ([28], Lemma 1.3, p.12), it follows from (2.45) and (2.46) that

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \quad \text{in } L^2(Q_{T_*}) \text{ weakly}. \tag{2.47}$$

Passing to the limit in (2.17) by (2.43), (2.44), and (2.47), we have u satisfying the problem

$$\begin{cases} \langle u''(t), v\rangle + \langle \nabla u(t), \nabla v\rangle + \langle Ku(t) + \lambda u'(t), v\rangle \\ \qquad + \int_{\partial\Omega} (\langle h(x,t), u(t)\rangle + g(x,t)) v(x)\,dS_x \\ \quad = \langle |u(t)|^{p-2}u(t), v\rangle + \langle f(t), v\rangle \quad \text{for all } v \in H^1, \\ u(0) = u_0, \qquad u'(0) = u_1. \end{cases} \tag{2.48}$$

On the other hand, we have from (2.43), $(2.48)_1$

$$\Delta u = u'' + Ku + \lambda u' - |u|^{p-2}u - f \in L^\infty\big(0,T_*;L^2\big). \tag{2.49}$$

Thus $u \in L^\infty(0,T_*;H^2)$ and the proof of existence is complete. The uniqueness of a weak solution is proved as follows.

Let u_1, u_2 be two weak solutions of problem (1.1)-(1.3), such that

$$u_i \in L^\infty\big(0,T_*;H^2\big), \qquad u_i' \in L^\infty\big(0,T_*;H^1\big), \qquad u_i'' \in L^\infty\big(0,T_*;L^2\big), \quad i = 1,2. \tag{2.50}$$

Then $u = u_1 - u_2$ satisfy the variational problem

$$\begin{cases} \langle u''(t), v\rangle + \langle \nabla u(t), \nabla v\rangle + \langle Ku(t) + \lambda u'(t), v\rangle + \int_{\partial\Omega} \langle h(x,t), u(t)\rangle v(x)\,dS_x \\ \quad = \langle |u_1|^{p-2}u_1 - |u_2|^{p-2}u_2, v\rangle \quad \text{for all } v \in H^1, \\ u(0) = u'(0) = 0. \end{cases} \tag{2.51}$$

We take $v = u' = u_1' - u_2'$ in (2.51) and integrating with respect to t, we obtain

$$\begin{aligned} \sigma(t) &= -2\int_0^t \langle Ku(s) + \lambda u'(s), u'(s)\rangle\,ds - 2\int_{\partial\Omega} \langle h(x,t), u(t)\rangle u(x,t)\,dS_x \\ &\quad + 2\int_0^t ds \int_{\partial\Omega} \bigl[\langle h'(x,s), u(s)\rangle + \langle h(x,s), u'(s)\rangle\bigr] u(x,s)\,dS_x \\ &\quad + 2\int_0^t \langle |u_1|^{p-2}u_1 - |u_2|^{p-2}u_2, u'(s)\rangle\,ds = \sum_{j=1}^4 \sigma_j, \end{aligned} \tag{2.52}$$

where

$$\sigma(t) = \|u'(t)\|^2 + \|\nabla u(t)\|^2. \tag{2.53}$$

By (2.53) and the following inequalities:

$$2ab \le \beta a^2 + \frac{1}{\beta} b^2 \quad \text{for all } a, b \in \mathbb{R}, \beta > 0, \tag{2.54}$$

$$\begin{aligned} &\|u(t)\|^2 = \left(\int_0^t \|u'(s)\|\,ds\right)^2 \le t\int_0^t \|u'(s)\|^2\,ds \le t\int_0^t \sigma(s)\,ds, \\ &\|u(t)\|_{H^1}^2 = \|\nabla u(t)\|^2 + \|u(t)\|^2 \le \sigma(t) + t\int_0^t \sigma(s)\,ds, \\ &\int_0^t \|u(s)\|_{H^1}^2\,ds \le \int_0^t \left[\sigma(s) + s\int_0^s \sigma(r)\,dr\right] ds \le (1 + t^2)\int_0^t \sigma(s)\,ds, \end{aligned} \tag{2.55}$$

we estimate the following integrals in the right-hand side of (2.52):

$$\begin{aligned} \sigma_1 &= -2\int_0^t \langle Ku(s) + \lambda u'(s), u'(s)\rangle\,ds \\ &\le \int_0^t \|u(s)\|^2\,ds + (K^2 + 2|\lambda|)\int_0^t \|u'(s)\|^2\,ds \\ &\le \int_0^t \left(s\int_0^s \sigma(r)\,dr\right) ds + (K^2 + 2|\lambda|)\int_0^t \sigma(s)\,ds \\ &\le T^2\int_0^t \sigma(r)\,dr + (K^2 + 2|\lambda|)\int_0^t \sigma(s)\,ds \le C_T\int_0^t \sigma(s)\,ds; \end{aligned} \tag{2.56}$$

$$\begin{aligned} \sigma_2 &= -2\int_{\partial\Omega} \langle h(x,t), u(t)\rangle u(x,t)\,dS_x \\ &\le 2\gamma_\Omega \|h\|_{L^\infty(0,T;L^2(\partial\Omega\times\Omega))} \|u(t)\| \|u(t)\|_{H^1} \\ &\le \frac{1}{\beta}\gamma_\Omega^2 \|h\|_{L^\infty(0,T;L^2(\partial\Omega\times\Omega))}^2 \|u(t)\|^2 + \beta\|u(t)\|_{H^1}^2 \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\beta}\gamma_\Omega^2\|h\|^2_{L^\infty(0,T;L^2(\partial\Omega\times\Omega))} t\int_0^t \sigma(s)\,ds + \beta\left[\sigma(t) + t\int_0^t \sigma(s)\,ds\right] \\
&\le \beta\sigma(t) + \frac{1}{\beta}C_T\int_0^t \sigma(s)\,ds;
\end{aligned}
\tag{2.57}
$$

$$
\begin{aligned}
\sigma_3 &= 2\int_0^t ds\int_{\partial\Omega}\big[\big\langle h'(x,s),u(s)\big\rangle + \big\langle h(x,s),u'(s)\big\rangle\big]u(x,s)\,dS_x \\
&\le 2\gamma_\Omega\big\|h'\big\|_{L^\infty(0,T;L^2(\partial\Omega\times\Omega))}\int_0^t \big\|u(s)\big\|^2_{H^1}\,ds \\
&\quad + 2\gamma_\Omega\|h\|_{L^\infty(0,T;L^2(\partial\Omega\times\Omega))}\int_0^t \big\|u'(s)\big\|\big\|u(s)\big\|_{H^1}\,ds \\
&\le C_T\int_0^t \big\|u(s)\big\|^2_{H^1}\,ds + C_T\int_0^t \big\|u'(s)\big\|^2\,ds \le C_T\int_0^t \sigma(s)\,ds.
\end{aligned}
\tag{2.58}
$$

By Lemma 2.3(i), we have

$$
\begin{aligned}
&\big\||u_1|^{p-2}u_1 - |u_2|^{p-2}u_2\big\| \\
&\quad\le D_p\big[1 + \big(\|u_1\|_{H^1} + \|u_2\|_{H^1}\big)^{1/N} + \big(\|u_1\|_{H^1} + \|u_2\|_{H^1}\big)^{p-2}\big]\big\|u(s)\big\|_{H^1} \\
&\quad\le D_p\big[1 + M_1^{1/N} + M_1^{p-2}\big]\big\|u(s)\big\|_{H^1} \le C_T\big\|u(s)\big\|_{H^1},
\end{aligned}
\tag{2.59}
$$

where $M_1 = \|u_1\|_{L^\infty(0,T_*;H^1)} + \|u_2\|_{L^\infty(0,T_*;H^1)}$. Hence

$$
\begin{aligned}
\sigma_4 &= 2\int_0^t \big\langle |u_1|^{p-2}u_1 - |u_2|^{p-2}u_2, u'(s)\big\rangle\,ds \\
&\le 2C_T\int_0^t \big\|u(s)\big\|_{H^1}\big\|u'(s)\big\|\,ds \\
&\le C_T\int_0^t \big\|u(s)\big\|^2_{H^1}\,ds + C_T\int_0^t \big\|u'(s)\big\|^2\,ds \\
&\le C_T\int_0^t \sigma(s)\,ds.
\end{aligned}
\tag{2.60}
$$

Combining (2.52), (2.56)-(2.58), (2.60) and choosing $\beta = \frac{1}{2}$, we obtain

$$
\sigma(t) \le C_T\int_0^t \sigma(s)\,ds. \tag{2.61}
$$

By Gronwall's lemma, it follows from (2.61) that $\sigma \equiv 0$, *i.e.*, $u_1 \equiv u_2$. Theorem 2.4 is proved completely. □

Proof of Theorem 2.5 In order to prove this theorem, we use standard arguments of density.

First, we note that $W^1(0,T;L^2(\partial\Omega)) = \{g \in L^2(0,T;L^2(\partial\Omega)) : g' \in L^2(0,T;L^2(\partial\Omega))\}$ is a Hilbert space with respect to the scalar product (see [27]):

$$
\langle f,g\rangle_{W^1(0,T;L^2(\partial\Omega))} = \int_0^T \big[\big\langle f(t),g(t)\big\rangle_{L^2(\partial\Omega)} + \big\langle f'(t),g'(t)\big\rangle_{L^2(\partial\Omega)}\big]\,dt. \tag{2.62}
$$

Furthermore, we also have the embedding $W^1(0,T;L^2(\partial\Omega)) \hookrightarrow C^0([0,T];L^2(\partial\Omega))$ is continuous and

$$\begin{aligned}\|g\|_{C^0([0,T];L^2(\partial\Omega))} &\leq \gamma_T\sqrt{\left(\|g\|^2_{L^2(0,T;L^2(\partial\Omega))} + \|g'\|^2_{L^2(0,T;L^2(\partial\Omega))}\right)} \\ &\equiv \gamma_T\|g\|_{W^1(0,T;L^2(\partial\Omega))}\end{aligned} \tag{2.63}$$

for all $g \in W^1(0,T;L^2(\partial\Omega))$, where $\gamma_T = \sqrt{\frac{1}{2T} + \sqrt{4 + \frac{1}{4T^2}}}$ (see the Appendix).

Similarly, $W^1(0,T;L^2(\partial\Omega\times\Omega)) = \{h \in L^2(0,T;L^2(\partial\Omega\times\Omega)) : h' \in L^2(0,T;L^2(\partial\Omega\times\Omega))\}$ is a Hilbert space with respect to the scalar product

$$\langle h,k\rangle_{W^1(0,T;L^2(\partial\Omega\times\Omega))} = \int_0^T \left[\langle h(t),k(t)\rangle_{L^2(\partial\Omega\times\Omega)} + \langle h'(t),k'(t)\rangle_{L^2(\partial\Omega\times\Omega)}\right]dt, \tag{2.64}$$

and the embedding $W^1(0,T;L^2(\partial\Omega\times\Omega)) \hookrightarrow C^0([0,T];L^2(\partial\Omega\times\Omega))$ is continuous and

$$\begin{aligned}\|h\|_{C^0([0,T];L^2(\partial\Omega\times\Omega))} &\leq \gamma_T\sqrt{\left(\|h\|^2_{L^2(0,T;L^2(\partial\Omega\times\Omega))} + \|h'\|^2_{L^2(0,T;L^2(\partial\Omega\times\Omega))}\right)} \\ &\equiv \gamma_T\|h\|_{W^1(0,T;L^2(\partial\Omega\times\Omega))}\end{aligned} \tag{2.65}$$

for all $h \in W^1(0,T;L^2(\partial\Omega\times\Omega))$, where $\gamma_T = \sqrt{\frac{1}{2T} + \sqrt{4 + \frac{1}{4T^2}}}$ (see the Appendix).

Consider $(u_0,u_1,f,g,h) \in H^1 \times L^2 \times L^2(Q_T) \times W^1(0,T;L^2(\partial\Omega)) \times W^1(0,T;L^2(\partial\Omega\times\Omega))$.

Let the sequence $\{(u_{0m},u_{1m},f_m,g_m,h_m)\} \subset H^2 \times H^1 \times C_0^\infty(\overline{Q}_T) \times C_0^\infty(\partial\Omega\times\overline{\Omega}) \times C_0^\infty(\partial\Omega\times\overline{\Omega}\times[0,T])$, such that

$$\begin{cases} u_{0m} \to u_0 & \text{strongly in } H^1, \\ u_{1m} \to u_1 & \text{strongly in } L^2, \\ f_m \to f & \text{strongly in } L^2(Q_T), \end{cases} \tag{2.66}$$

$$\|g_m - g\|^2_{W^1(0,T;L^2(\partial\Omega\times\Omega))} \equiv \|g_m - g\|^2_{L^2(0,T;L^2(\partial\Omega))} + \|g'_m - g'\|^2_{L^2(0,T;L^2(\partial\Omega))} \to 0, \tag{2.67}$$

$$\begin{aligned}\|h_m - h\|^2_{W^1(0,T;L^2(\partial\Omega\times\Omega))} &\equiv \|h_m - h\|^2_{L^2(0,T;L^2(\partial\Omega\times\Omega))} + \|h'_m - h'\|^2_{L^2(0,T;L^2(\partial\Omega\times\Omega))} \\ &\to 0.\end{aligned} \tag{2.68}$$

So $\{(u_{0m},u_{1m})\}$ satisfy, for all $m \in \mathbb{N}$, the compatibility condition

$$-\frac{\partial u_{0m}}{\partial\nu}(x) = g_m(x,0) + \int_\Omega h_m(x,y,0)u_{0m}(y)\,dy. \tag{2.69}$$

Then, for each $m \in \mathbb{N}$, there exists a unique function u_m under the conditions of Theorem 2.4. Thus, we can verify

$$\begin{cases} \langle u''_m(t),v\rangle + \langle\nabla u_m(t),\nabla v\rangle + \langle Ku_m(t) + \lambda u'_m(t),v\rangle \\ \qquad + \int_{\partial\Omega}(\langle h_m(x,t),u_m(t)\rangle + g_m(x,t))v(x)\,dS_x \\ \quad = \langle|u_m|^{p-2}u_m,v\rangle + \langle f_m(t),v\rangle \quad \text{for all } v \in H^1, \\ u_m(0) = u_{0m}, \qquad u'_m(0) = u_{1m} \end{cases} \tag{2.70}$$

and

$$\begin{cases} u_m \in L^\infty(0,T_*;H^2) \cap C^0(0,T_*;H^1) \cap C^1(0,T_*;L^2), \\ u_m' \in L^\infty(0,T_*;H^1) \cap C^0(0,T_*;L^2), \\ u_m'' \in L^\infty(0,T_*;L^2). \end{cases} \tag{2.71}$$

By the same arguments used to obtain the above estimates, we get

$$\|u_m'(t)\|^2 + \|u_m(t)\|_{H^1}^2 \le C_T, \tag{2.72}$$

$\forall t \in [0,T_*]$, where C_T always indicates a constant depending on T as above.

On the other hand, we put $w_{m,l} = u_m - u_l$, $f_{m,l} = f_m - f_l$, $h_{m,l} = h_m - h_l$, $g_{m,l} = g_m - g_l$, $h_{m,l}(x,y,0) = \bar{h}_{m,l}^{(0)}(x,y)$, $g_{m,l}(x,0) = \bar{g}_{m,l}^{(0)}(x)$, from (2.70), it follows that

$$\begin{cases} \langle w_{m,l}''(t), v\rangle + \langle \nabla w_{m,l}(t), \nabla v\rangle + \langle K w_{m,l}(t) + \lambda w_{m,l}'(t), v\rangle \\ \quad + \int_{\partial\Omega} (\langle h_m(x,t), w_{m,l}(t)\rangle + \langle h_{m,l}(x,t), u_l(t)\rangle + g_{m,l}(x,t)) v(x)\,dS_x \\ \quad = \langle |u_m|^{p-2}u_m - |u_l|^{p-2}u_l, v\rangle + \langle f_{m,l}(t), v\rangle \quad \text{for all } v \in H^1, \\ w_{m,l}(0) = u_{0m} - u_{0l} \equiv \bar{w}_{m,l}^{(0)}, \qquad w_{m,l}'(0) = u_{1m} - u_{1l} \equiv \bar{w}_{m,l}^{(1)}. \end{cases} \tag{2.73}$$

We take $v = w_{m,l} = u_m - u_l$, in (2.73) and integrating with respect to t, we get

$$\begin{aligned} S_{m,l}(t) &= S_{m,l}(0) + 2\int_{\partial\Omega} \big(\langle h_m(x,0), \bar{w}_{m,l}^{(0)}\rangle + \langle h_{m,l}(x,0), u_{0l}\rangle + g_{m,l}(x,0)\big) \bar{w}_{m,l}^{(0)}(x)\,dS_x \\ &\quad + 2\int_0^t \langle f_{m,l}(s), w_{m,l}'(s)\rangle\,ds - 2\int_0^t \langle K w_{m,l}(s) + \lambda w_{m,l}'(s), w_{m,l}'(s)\rangle\,ds \\ &\quad - 2\int_{\partial\Omega} \big(\langle h_m(x,t), w_{m,l}(t)\rangle + \langle h_{m,l}(x,t), u_l(t)\rangle + g_{m,l}(x,t)\big) w_{m,l}(x,t)\,dS_x \\ &\quad + 2\int_0^t ds \int_{\partial\Omega} \big(\langle h_m'(x,s), w_{m,l}(s)\rangle + \langle h_m(x,s), w_{m,l}'(s)\rangle\big) w_{m,l}(x,s)\,dS_x \\ &\quad + 2\int_0^t ds \int_{\partial\Omega} \big(\langle h_{m,l}'(x,s), u_l(s)\rangle + \langle h_{m,l}(x,s), u_l'(s)\rangle\big) w_{m,l}(x,s)\,dS_x \\ &\quad + 2\int_0^t ds \int_{\partial\Omega} g_{m,l}'(x,s) w_{m,l}(x,s)\,dS_x \\ &\quad + 2\int_0^t \langle |u_m|^{p-2}u_m - |u_l|^{p-2}u_l, w_{m,l}'(s)\rangle\,ds \equiv S_{m,l}(0) + \sum_{j=1}^{8} Z_j, \end{aligned} \tag{2.74}$$

where

$$S_{m,l}(t) = \|w_{m,l}'(t)\|^2 + \|\nabla w_{m,l}(t)\|^2, \tag{2.75}$$

$$S_{m,l}(0) = \|u_{1m} - u_{1l}\|^2 + \|\nabla u_{0m} - \nabla u_{0l}\|^2. \tag{2.76}$$

After all terms of $S_{m,l}(t)$ are estimated, in which we note the two main estimations Z_1, Z_8 as follows:

$$\begin{aligned} Z_1 &= 2\int_{\partial\Omega} \big(\langle h_m(x,0), \bar{w}_{m,l}^{(0)}\rangle + \langle h_{m,l}(x,0), u_{0l}\rangle + g_{m,l}(x,0)\big) \bar{w}_{m,l}^{(0)}(x)\,dS_x \\ &\le 2\gamma_\Omega \big[\|\bar{w}_{m,l}^{(0)}\| \|h_m(0)\|_{L^2(\partial\Omega\times\Omega)} + \|u_{0l}\| \|\bar{h}_{m,l}^{(0)}\|_{L^2(\partial\Omega\times\Omega)} + \|\bar{g}_{m,l}^{(0)}\|_{L^2(\partial\Omega)}\big] \|\bar{w}_{m,l}^{(0)}\|_{H^1} \end{aligned}$$

$$\begin{aligned}&\le 2\gamma_\Omega\gamma_T\cdot\text{const.}\bigl[\bigl\|\bar{w}_{m,l}^{(0)}\bigr\|_{H^1}+\|h_{m,l}\|_{W^1(0,T;L^2(\partial\Omega\times\Omega))}+\|g_{m,l}\|_{W^1(0,T;L^2(\partial\Omega))}\bigr]\bigl\|\bar{w}_{m,l}^{\prime(0)}\bigr\|_{H^1}\\&\to 0,\quad\text{as } m,l\to+\infty;\end{aligned}\tag{2.77}$$

this result combined with (2.66)-(2.68) shows that

$$\begin{aligned}S_{m,l}(0)+Z_1 &= \|u_{1m}-u_{1l}\|^2+\|\nabla u_{0m}-\nabla u_{0l}\|^2+Z_1\\&\equiv R(m,l)\to 0,\quad\text{as } m,l\to+\infty.\end{aligned}\tag{2.78}$$

On the other hand

$$\begin{aligned}&\bigl\||u_m|^{p-2}u_m-|u_l|^{p-2}u_l\bigr\|\\&\quad\le D_p\bigl[1+\bigl(\|u_m\|_{H^1}+\|u_l\|_{H^1}\bigr)^{1/N}+\bigl(\|u_m\|_{H^1}+\|u_l\|_{H^1}\bigr)^{p-2}\bigr]\bigl\|w_{m,l}(s)\bigr\|_{H^1}\\&\quad\le C_T\bigl\|w_{m,l}(s)\bigr\|_{H^1},\end{aligned}\tag{2.79}$$

by Lemma 2.3(i), we get

$$\begin{aligned}Z_8 &= 2\int_0^t\bigl\langle|u_m|^{p-2}u_m-|u_l|^{p-2}u_l,w'_{m,l}(s)\bigr\rangle\,ds\le 2C_T\int_0^t\bigl\|w_{m,l}(s)\bigr\|_{H^1}\bigl\|w'_{m,l}(s)\bigr\|\,ds\\&\le C_T\left[2t\bigl\|\bar{w}_{m,l}^{(0)}\bigr\|^2+\bigl(1+t^2\bigr)\int_0^t S_{m,l}(s)\,ds\right]+C_T\int_0^t S_{m,l}(s)\,ds\\&\le C_T\left[\bigl\|\bar{w}_{m,l}^{(0)}\bigr\|^2+\int_0^t S_{m,l}(s)\,ds\right].\end{aligned}\tag{2.80}$$

We obtain

$$S_{m,l}(t)\le R_T^{(1)}(m,l)+C_T\int_0^t S_{m,l}(s)\,ds,\tag{2.81}$$

with

$$\begin{aligned}R_T^{(1)}(m,l)&=2R(m,l)+2\|f_{m,l}\|^2_{L^2(Q_T)}\\&\quad+C_T\bigl(\bigl\|\bar{w}_{m,l}^{(0)}\bigr\|^2+\|h_{m,l}\|^2_{W^1(0,T;L^2(\partial\Omega\times\Omega))}+\|g_{m,l}\|^2_{W^1(0,T;L^2(\partial\Omega))}\bigr)\to 0,\end{aligned}\tag{2.82}$$

as $m,l\to+\infty$. By Gronwall's lemma, it follows from (2.81) that

$$S_{m,l}(t)\le R_T^{(1)}(m,l)\exp(TC_T)\le C_T R_T^{(1)}(m,l),\quad\forall t\in[0,T_*].\tag{2.83}$$

Thus, convergence of the sequence $\{(u_{0m},u_{1m},f_m,g_m,h_m)\}$ implies the convergence to zero as $m,l\to+\infty$ of the term on the right-hand side of (2.83). Therefore, we get

$$u_m\to u\quad\text{strongly in } C^0\bigl([0,T_*];H^1\bigr)\cap C^1\bigl([0,T_*];L^2\bigr).\tag{2.84}$$

On the other hand, from (2.72), we get the existence of a subsequence of $\{u_m\}$, still also so denoted, such that

$$\begin{cases}u_m\to u & \text{in } L^\infty(0,T_*;H^1)\text{ weakly}^*,\\u'_m\to u' & \text{in } L^\infty(0,T_*;L^2)\text{ weakly}^*.\end{cases}\tag{2.85}$$

By the compactness lemma of Lions ([28], p.57), we can deduce from (2.85) the existence of a subsequence, still denoted by $\{u_m\}$, such that

$$u_m \to u \quad \text{strongly in } L^2(Q_{T*}) \text{ and a.e. in } Q_{T_*}. \tag{2.86}$$

Similarly, by (2.72), it follows from (2.86) that

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \quad \text{in } L^2(Q_{T_*}) \text{ weakly.} \tag{2.87}$$

Passing to the limit in (2.70) by (2.84)-(2.87), we have u satisfying the problem

$$\begin{cases} \frac{d}{dt}\langle u'(t), v\rangle + \langle \nabla u(t), \nabla v\rangle + \langle Ku(t) + \lambda u'(t), v\rangle \\ \qquad + \int_{\partial\Omega}(\langle h(x,t), u(t)\rangle + g(x,t))v(x)\,dS_x \\ \quad = \langle |u|^{p-2}u, v\rangle + \langle f(t), v\rangle \quad \text{for all } v \in H^1, \\ u(0) = u_0, \qquad u'(0) = u_1. \end{cases} \tag{2.88}$$

Next, the uniqueness of a weak solution is obtained by using the well-known regularization procedure due to Lions. Theorem 2.5 is proved completely. □

Remark 2.2 In the case $1 < p \le 2, f \in L^2(Q_T), g \in W^1(0,T;L^2(\partial\Omega)), h \in W^1(0,T;L^2(\partial\Omega \times \Omega))$, and $(u_0, u_1) \in H^1 \times L^2$, the integral inequality (2.29) leads to the following global estimation:

$$S_m(t) \le C_T, \quad \forall m \in \mathbb{N}, \forall t \in [0,T], \forall T > 0. \tag{2.89}$$

Then, by applying a similar argument to the proof of Theorem 2.4, we can obtain a global weak solution u of problem (1.1)-(1.3) satisfying

$$u \in L^\infty(0,T;H^1), \qquad u_t \in L^\infty(0,T;L^2). \tag{2.90}$$

However, in the case $1 < p < 2$, we do not imply that a weak solution obtained here belongs to $C([0,T];H^1) \cap C^1([0,T];L^2)$. Furthermore, the uniqueness of a weak solution is also not asserted.

3 Exponential decay

In this section, we study the exponentially decay of solutions of problem (1.1)-(1.3) corresponding to $a = 1$, $g = 0$, $K > 0$, $\lambda > 0$, and $2 < p \le \frac{2N-2}{N-2}$. For this purpose, we make the following assumptions:

(A_1'') $f \in L^2(0,\infty;L^2) = L^2(Q_\infty)$, $Q_\infty = \Omega \times \mathbb{R}_+$, such that $\|f(t)\| \le Ce^{-\gamma_0 t}$, for all $t \ge 0$, with $C > 0$, $\gamma_0 > 0$ are given constants,

(A_2'') $h \in L^\infty(0,\infty;L^2(\partial\Omega \times \Omega)) \cap L^2(\mathbb{R}_+ \times \partial\Omega \times \Omega)$, $h'' \in L^\infty(0,\infty;L^2(\partial\Omega \times \Omega)) \cap L^1(0,\infty;L^2(\partial\Omega \times \Omega))$,

(A_3'') $g = 0$.

Let $K > 0$, on H^1 we shall use the following norm:

$$\|v\|_1 = \left(K\|v\|^2 + \|\nabla v\|^2\right)^{1/2}.$$

Then we have the following lemma.

Lemma 3.1 *On H^1, two norms $\|v\|_1$, $\|v\|_{H^1}$ are equivalent and*

$$\frac{1}{\sqrt{\max\{1,K\}}}\|v\|_1 \le \|v\|_{H^1} \le \frac{1}{\sqrt{\min\{1,K\}}}\|v\|_1 \equiv C_0\|v\|_1 \quad \text{for all } v \in H^1,$$

where $C_0 = \frac{1}{\sqrt{\min\{1,K\}}}$.

The proof of this lemma is simple, we omit the details.

We construct the following Lyapunov functional:

$$\mathcal{L}(t) = E(t) + \delta\psi(t), \tag{3.1}$$

where $\delta > 0$ is chosen later and

$$E(t) = \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u(t)\|_1^2 - \frac{1}{p}\|u(t)\|_{L^p}^p + \int_{\partial\Omega}\langle h(x,t),u(t)\rangle u(x,t)\,dS_x, \tag{3.2}$$

$$\psi(t) = \langle u(t),u'(t)\rangle + \frac{\lambda}{2}\|u(t)\|^2. \tag{3.3}$$

Put

$$I(t) = I(u(t)) = \|u(t)\|_1^2 - \|u(t)\|_{L^p}^p + p\int_{\partial\Omega}\langle h(x,t),u(t)\rangle u(x,t)\,dS_x, \tag{3.4}$$

$$\begin{aligned} J(t) = J(u(t)) &= \frac{1}{2}\|u(t)\|_1^2 - \frac{1}{p}\|u(t)\|_{L^p}^p + \int_{\partial\Omega}\langle h(x,t),u(t)\rangle u(x,t)\,dS_x \\ &= \left(\frac{1}{2}-\frac{1}{p}\right)\|u(t)\|_1^2 + \frac{1}{p}I(t), \end{aligned} \tag{3.5}$$

we rewrite

$$E(t) = \frac{1}{2}\|u'(t)\|^2 + J(t) = \frac{1}{2}\|u'(t)\|^2 + \left(\frac{1}{2}-\frac{1}{p}\right)\|u(t)\|_1^2 + \frac{1}{p}I(t). \tag{3.6}$$

Then we have the following theorem.

Theorem 3.2 *Assume that* (A_1'')-(A_3'') *hold. Let* $I(0) > 0$ *and the initial energy* $E(0)$ *satisfy*

$$\eta_* = (C_pC_0)^p\left(\frac{2p}{p-2}E_*\right)^{\frac{p-2}{2}} + pC_0\bar{\gamma}_\Omega\|h\|_{L^\infty(0,\infty;L^2(\partial\Omega\times\Omega))} < 1, \tag{3.7}$$

where

$$\begin{cases} E_* = (E(0) + \frac{1}{\lambda}\int_0^\infty \|f(t)\|^2\,dt)\exp(\frac{2p}{p-2}\int_0^\infty \bar{h}(t)\,dt), \\ \bar{h}(t) = \bar{\gamma}_\Omega(C_0\|h'(t)\|_{L^2(\partial\Omega\times\Omega)} + \frac{1}{\lambda}\bar{\gamma}_\Omega\|h(t)\|^2_{L^2(\partial\Omega\times\Omega)}), \end{cases} \tag{3.8}$$

$\bar{\gamma}_\Omega = \gamma_\Omega C_0$ *and C_p is a constant satisfying the inequality $\|v\|_{L^p} \le C_p\|v\|_{H^1}$, for all $v \in H^1$.*

Then, for $E(0)$, $\|f\|_{L^\infty(0,\infty;L^2)}$, $\|h\|_{L^\infty(0,\infty;L^2(\partial\Omega\times\Omega))}$, $\|h'\|_{L^\infty(0,\infty;L^2(\partial\Omega\times\Omega))}$ *sufficiently small, there exist positive constants* C, γ *such that*

$$E(t) \le C\exp(-\gamma t) \quad \text{for all } t \ge 0. \tag{3.9}$$

Proof of Theorem 3.2 At first, we state and prove Lemmas 3.3-3.6 as follows.

Lemma 3.3 *The energy functional* $E(t)$ *satisfies*

$$\begin{aligned} E'(t) \le{}& -\frac{\lambda}{2}\|u'(t)\|^2 + \frac{1}{\lambda}\|f(t)\|^2 \\ &+ \bar{\gamma}_\Omega\left(C_0\|h'(t)\|_{L^2(\partial\Omega\times\Omega)} + \frac{1}{\lambda}\bar{\gamma}_\Omega\|h(t)\|^2_{L^2(\partial\Omega\times\Omega)}\right)\|u(t)\|_1^2. \end{aligned} \tag{3.10}$$

Proof Multiplying (1.1) by $u'(x,t)$ and integrating over $[0,1]$, we get

$$E'(t) = -\lambda\|u'(t)\|^2 + \langle f(t),u'(t)\rangle + \int_{\partial\Omega}\big[\langle h'(x,t),u(t)\rangle + \langle h(x,t),u'(t)\rangle\big]u(x,t)\,dS_x. \tag{3.11}$$

We have

$$\langle f(t),u'(t)\rangle \le \frac{\lambda}{4}\|u'(t)\|^2 + \frac{1}{\lambda}\|f(t)\|^2. \tag{3.12}$$

By Lemmas 2.1, 2.2, 3.1, we obtain

$$\begin{aligned} \int_{\partial\Omega}\langle h'(x,t),u(t)\rangle u(x,t)\,dS_x &\le \|u(t)\|\int_{\partial\Omega}\|h'(x,t)\|\,|u(x,t)|\,dS_x \\ &\le C_0\|u(t)\|_1\left(\int_{\partial\Omega}\|h'(x,t)\|^2\,dS_x\right)^{1/2}\left(\int_{\partial\Omega}u^2(x,t)\,dS_x\right)^{1/2} \\ &\le C_0\bar{\gamma}_\Omega\|h'(t)\|_{L^2(\partial\Omega\times\Omega)}\|u(t)\|_1^2, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \int_{\partial\Omega}\langle h(x,t),u'(t)\rangle u(x,t)\,dS_x &\le \|u'(t)\|\int_{\partial\Omega}\|h(x,t)\|\,|u(x,t)|\,dS_x \\ &\le \|u'(t)\|\bar{\gamma}_\Omega\|h(t)\|_{L^2(\partial\Omega\times\Omega)}\|u(t)\|_1 \\ &\le \frac{\lambda}{4}\|u'(t)\|^2 + \frac{1}{\lambda}\bar{\gamma}_\Omega^2\|h(t)\|^2_{L^2(\partial\Omega\times\Omega)}\|u(t)\|_1^2. \end{aligned} \tag{3.14}$$

Combining (3.11)-(3.14), (3.10) follows. Lemma 3.3 is proved completely. □

Lemma 3.4 *Suppose that* (A_1'')-(A_3'') *hold. Then, if we have* $I(0) > 0$ *and*

$$\eta_* = (C_pC_0)^p\left(\frac{2p}{p-2}E_*\right)^{\frac{p-2}{2}} + pC_0\bar{\gamma}_\Omega\|h\|_{L^\infty(0,\infty;L^2(\partial\Omega\times\Omega))} < 1, \tag{3.15}$$

then $I(t) > 0$, $\forall t \ge 0$.

Proof By the continuity of $I(t)$ and $I(0) > 0$, there exists $T_1 > 0$ such that

$$I(t) = I\big(u(t)\big) \ge 0, \quad \forall t \in [0,T_1], \tag{3.16}$$

this implies

$$\begin{aligned}J(t) &= \left(\frac{1}{2}-\frac{1}{p}\right)\|u(t)\|_1^2+\frac{1}{p}I(t)\\ &\geq \left(\frac{1}{2}-\frac{1}{p}\right)\|u(t)\|_1^2\\ &\geq \frac{p-2}{2p}\|u(t)\|_1^2, \quad \forall t\in[0,T_1].\end{aligned} \tag{3.17}$$

It follows from (3.6), (3.17) that

$$\|u(t)\|_1^2 \leq \frac{2p}{p-2}J(t) \leq \frac{2p}{p-2}E(t), \quad \forall t\in[0,T_1]. \tag{3.18}$$

Equation (3.10) leads to

$$\begin{aligned}E'(t) &\leq -\frac{\lambda}{2}\|u'(t)\|^2+\frac{1}{\lambda}\|f(t)\|^2\\ &\quad+\bar{\gamma}_\Omega\left(C_0\|h'(t)\|_{L^2(\partial\Omega\times\Omega)}+\frac{1}{\lambda}\bar{\gamma}_\Omega\|h(t)\|^2_{L^2(\partial\Omega\times\Omega)}\right)\|u(t)\|_1^2\\ &\leq \frac{1}{\lambda}\|f(t)\|^2+\frac{2p}{p-2}\bar{\gamma}_\Omega\left(C_0\|h'(t)\|_{L^2(\partial\Omega\times\Omega)}+\frac{1}{\lambda}\bar{\gamma}_\Omega\|h(t)\|^2_{L^2(\partial\Omega\times\Omega)}\right)E(t)\\ &= \frac{1}{\lambda}\|f(t)\|^2+\frac{2p}{p-2}\bar{h}(t)E(t).\end{aligned} \tag{3.19}$$

Integrating with respect to t, we obtain

$$E(t)\leq E(0)+\frac{1}{\lambda}\int_0^\infty\|f(t)\|^2\,dt+\frac{2p}{p-2}\int_0^t\bar{h}(s)E(s)\,ds, \tag{3.20}$$

where $\bar{h}(t)$ is as in (3.8).

Combining (3.18), (3.20), and using the Gronwall lemma, we have

$$E(t)\leq\left(E(0)+\frac{1}{\lambda}\int_0^\infty\|f(t)\|^2\,dt\right)\exp\left(\frac{2p}{p-2}\int_0^\infty\bar{h}(s)\,ds\right)=E_* \tag{3.21}$$

and

$$\|u(t)\|_1^2\leq\frac{2p}{p-2}E(t)\leq\frac{2p}{p-2}E_*, \quad \forall t\in[0,T_1]. \tag{3.22}$$

Hence, it follows from (3.7), (3.22) that

$$\begin{aligned}&\|u(t)\|_{L^p}^p-p\int_{\partial\Omega}\langle h(x,t),u(t)\rangle u(x,t)\,dS_x\\ &\quad\leq (C_pC_0)^p\|u(t)\|_1^p+pC_0\bar{\gamma}_\Omega\|h(t)\|_{L^2(\partial\Omega\times\Omega)}\|u(t)\|_1^2\\ &\quad\leq\left[(C_pC_0)^p\left(\frac{2p}{p-2}E_*\right)^{\frac{p-2}{2}}+pC_0\bar{\gamma}_\Omega\|h\|_{L^\infty(0,\infty;L^2(\partial\Omega\times\Omega))}\right]\|u(t)\|_1^2\\ &\quad=\eta_*\|u(t)\|_1^2<\|u(t)\|_1^2, \quad \forall t\in[0,T_1].\end{aligned} \tag{3.23}$$

Therefore, $I(t) > 0$, $\forall t \in [0, T_1]$.

Now, we put $T_\infty = \sup\{T > 0 : I(u(t)) > 0, \forall t \in [0, T]\}$. If $T_\infty < +\infty$ then, by the continuity of $I(t)$, we have $I(T_\infty) \geq 0$. By the same arguments as in the above part, we can deduce that there exists $T_2 > T_\infty$ such that $I(t) > 0$, $\forall t \in [0, T_2]$. Hence, we conclude that $I(t) > 0$, $\forall t \geq 0$.

Lemma 3.4 is proved completely. □

Lemma 3.5 *Let* $I(0) > 0$ *and* (3.7) *hold. Then there exist the positive constants* β_1, β_2 *such that*

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t), \quad \forall t \geq 0 \tag{3.24}$$

for δ *is sufficiently small.*

Proof A simple computation gives

$$\begin{aligned} \mathcal{L}(t) &= \frac{1}{2}\|u'(t)\|^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\|u(t)\|_1^2 + \frac{1}{p}I(t) + \delta\langle u(t), u'(t)\rangle + \frac{\delta\lambda}{2}\|u(t)\|^2, \\ E(t) &= \frac{1}{2}\|u'(t)\|^2 + J(t) = \frac{1}{2}\|u'(t)\|^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\|u(t)\|_1^2 + \frac{1}{p}I(t). \end{aligned} \tag{3.25}$$

From the following inequalities:

$$\begin{aligned} &\delta\langle u(t), u'(t)\rangle \leq \delta C_0\|u(t)\|_1\|u'(t)\| \leq \delta\|u'(t)\|^2 + \frac{1}{4}\delta C_0^2\|u(t)\|_1^2, \\ &\frac{\delta\lambda}{2}\|u(t)\|^2 \leq \frac{\delta\lambda}{2}C_0^2\|u(t)\|_1^2, \end{aligned} \tag{3.26}$$

we deduce from (3.25) that

$$\begin{aligned} \mathcal{L}(t) &\geq \frac{1}{2}\|u'(t)\|^2 + \left(\frac{1}{2} - \frac{1}{p}\right)\|u(t)\|_1^2 + \frac{1}{p}I(t) - \delta\|u'(t)\|^2 \\ &\quad - \frac{1}{4}\delta C_0^2\|u(t)\|_1^2 - \frac{\delta\lambda}{2}C_0^2\|u(t)\|_1^2 \\ &= \left(\frac{1}{2} - \delta\right)\|u'(t)\|^2 + \left[\frac{1}{2} - \frac{1}{p} - \frac{1}{2}\delta C_0^2\left(\lambda + \frac{1}{2}\right)\right]\|u(t)\|_1^2 + \frac{1}{p}I(t) \\ &= (1 - 2\delta)\frac{1}{2}\|u'(t)\|^2 + \left[1 - \frac{\frac{1}{2}\delta C_0^2(\lambda + \frac{1}{2})}{\frac{1}{2} - \frac{1}{p}}\right]\left(\frac{1}{2} - \frac{1}{p}\right)\|u(t)\|_1^2 + \frac{1}{p}I(t) \\ &\geq \beta_1 E(t), \end{aligned} \tag{3.27}$$

where we choose

$$\beta_1 = \min\left\{1, 1 - 2\delta, 1 - \frac{\frac{1}{2}\delta C_0^2(\lambda + \frac{1}{2})}{\frac{1}{2} - \frac{1}{p}}\right\}, \tag{3.28}$$

with δ being small enough, $0 < \delta < \min\{\frac{1}{2}, \frac{\frac{1}{2} - \frac{1}{p}}{\frac{1}{2}C_0^2(\lambda + \frac{1}{2})}\}$.

Similarly, we can prove that

$$\begin{aligned}\mathcal{L}(t) &\le \left(\frac{1}{2}+\delta\right)\left\|u'(t)\right\|^2 + \left[\frac{1}{2}-\frac{1}{p}+\frac{1}{2}\delta C_0^2\left(\lambda+\frac{1}{2}\right)\right]\left\|u(t)\right\|_1^2 + \frac{1}{p}I(t) \\ &= (1+2\delta)\frac{1}{2}\left\|u'(t)\right\|^2 + \left[1+\frac{\frac{1}{2}\delta C_0^2(\lambda+\frac{1}{2})}{\frac{1}{2}-\frac{1}{p}}\right]\left(\frac{1}{2}-\frac{1}{p}\right)\left\|u(t)\right\|_1^2 + \frac{1}{p}I(t) \\ &\le \beta_2 E(t), \end{aligned} \tag{3.29}$$

where

$$\beta_2 = \max\left\{1+2\delta, 1+\frac{\frac{1}{2}\delta C_0^2(\lambda+\frac{1}{2})}{\frac{1}{2}-\frac{1}{p}}\right\}. \tag{3.30}$$

Lemma 3.5 is proved completely. □

Lemma 3.6 *Let* $I(0) > 0$ *and* (3.7) *hold. Then the functional* $\psi(t)$ *defined by* (3.3) *satisfies*

$$\begin{aligned}\psi'(t) &\le \left\|u'(t)\right\|^2 + \frac{1}{4\varepsilon_1}C_0^2\left\|f(t)\right\|^2 - \frac{1}{2}I(t) \\ &\quad - \left[\frac{1}{2}(1-\eta_*) - \varepsilon_1 - (p-1)C_0\bar{\gamma}_\Omega\|h\|_{L^\infty(0,\infty;L^2(\partial\Omega\times\Omega))}\right]\left\|u(t)\right\|_1^2 \end{aligned} \tag{3.31}$$

for all $\varepsilon_1 > 0$.

Proof By multiplying (1.1) by $u(x,t)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}\psi'(t) &= \left\|u'(t)\right\|^2 - \left\|u(t)\right\|_1^2 + \left\|u(t)\right\|_{L^p}^p + \langle f(t),u(t)\rangle - \int_{\partial\Omega}\langle h(x,t),u(t)\rangle u(x,t)\,dS_x \\ &= \left\|u'(t)\right\|^2 - \frac{1}{2}I(t) - \frac{1}{2}I(t) + (p-1)\int_{\partial\Omega}\langle h(x,t),u(t)\rangle u(x,t)\,dS_x + \langle f(t),u(t)\rangle. \end{aligned} \tag{3.32}$$

Note that

$$\begin{aligned}&I(t) \ge (1-\eta_*)\left\|u(t)\right\|_1^2, \\ &\int_{\partial\Omega}\langle h(x,t),u(t)\rangle u(x,t)\,dS_x \le C_0\bar{\gamma}_\Omega\|h\|_{L^\infty(0,\infty;L^2(\partial\Omega\times\Omega))}\left\|u(t)\right\|_1^2, \\ &\langle f(t),u(t)\rangle \le \varepsilon_1\left\|u(t)\right\|_1^2 + \frac{1}{4\varepsilon_1}C_0^2\left\|f(t)\right\|^2, \end{aligned} \tag{3.33}$$

hence, Lemma 3.6 is proved by using some estimates. □

Now, we prove Theorem 3.2.

It follows from (3.1), (3.10), and (3.31) that

$$\begin{aligned}\mathcal{L}'(t) &\le -\left(\frac{\lambda}{2}-\delta\right)\left\|u'(t)\right\|^2 - \frac{\delta}{2}I(t) \\ &\quad - \left\{\delta\left[\frac{1}{2}(1-\eta_*) - \varepsilon_1\right] - \delta(p-1)C_0\bar{\gamma}_\Omega\|h\|_{L^\infty(0,\infty;L^2(\partial\Omega\times\Omega))}\right.\end{aligned}$$

$$\begin{aligned}
&- \bar{\gamma}_\Omega \left(C_0 \left\| h' \right\|_{L^\infty(0,\infty;L^2(\partial\Omega\times\Omega))} + \frac{1}{\lambda} \bar{\gamma}_\Omega \|h\|^2_{L^\infty(0,\infty;L^2(\partial\Omega\times\Omega))} \right) \Bigg\} \left\| u(t) \right\|_1^2 \\
&+ \left(\frac{1}{\lambda} + \frac{\delta}{4\varepsilon_1} C_0^2 \right) \left\| f(t) \right\|^2 \\
= &- \left(\frac{\lambda}{2} - \delta \right) \left\| u'(t) \right\|^2 - \frac{\delta}{2} I(t) - \left\{ \delta \left[\frac{1}{2}(1-\eta_*) - \varepsilon_1 \right] - \left\| [h] \right\| \right\} \left\| u(t) \right\|_1^2 \\
&+ \left(\frac{1}{\lambda} + \frac{\delta}{4\varepsilon_1} C_0^2 \right) \left\| f(t) \right\|^2
\end{aligned} \tag{3.34}$$

for all $\delta, \varepsilon_1 > 0$, where

$$\begin{aligned}
\left\| [h] \right\| \equiv{}& \delta(p-1) C_0 \bar{\gamma}_\Omega \|h\|_{L^\infty(0,\infty;L^2(\partial\Omega\times\Omega))} \\
&+ \bar{\gamma}_\Omega \left(C_0 \left\| h' \right\|_{L^\infty(0,\infty;L^2(\partial\Omega\times\Omega))} + \frac{1}{\lambda} \bar{\gamma}_\Omega \|h\|^2_{L^\infty(0,\infty;L^2(\partial\Omega\times\Omega))} \right).
\end{aligned} \tag{3.35}$$

Let δ, ε_1 satisfy

$$0 < \delta < \frac{\lambda}{2}, \qquad 0 < \varepsilon_1 < \frac{1}{2}(1-\eta_*). \tag{3.36}$$

Then, for δ small enough such that $0 < \delta < \frac{\lambda}{2}$ and if h satisfy

$$\left\| [h] \right\| < \delta \left[\frac{1}{2}(1-\eta_*) - \varepsilon_1 \right], \tag{3.37}$$

we deduce from (3.34), (3.36), and (3.37) that there exists a constant $\gamma > 0$ such that

$$\begin{aligned}
\mathcal{L}'(t) \le{}& -(\lambda - 2\delta) \frac{1}{2} \left\| u'(t) \right\|^2 - \frac{1}{2} \delta p \frac{1}{p} I(t) \\
&- \frac{\{\delta[\frac{1}{2}(1-\eta_*) - \varepsilon_1] - \|[h]\|\}}{\frac{1}{2} - \frac{1}{p}} \left(\frac{1}{2} - \frac{1}{p} \right) \left\| u(t) \right\|_1^2 + \rho(t) \\
\le{}& -\gamma_1 E(t) + \rho(t) \le -\frac{\gamma_1}{\beta_2} \mathcal{L}(t) + \rho(t) \le -\gamma \mathcal{L}(t) + \rho(t),
\end{aligned} \tag{3.38}$$

where

$$\begin{aligned}
&\gamma_1 = \min \left\{ \lambda - 2\delta, \frac{1}{2}\delta p, \frac{\{\delta[\frac{1}{2}(1-\eta_*) - \varepsilon_1] - \|[h]\|\}}{\frac{1}{2} - \frac{1}{p}} \right\} > 0, \\
&\rho(t) = \left(\frac{1}{\lambda} + \frac{\delta}{4\varepsilon_1} C_0^2 \right) \left\| f(t) \right\|^2 \le C_* e^{-2\gamma_0 t}, \\
&0 < \gamma < \min\{\gamma_1, 2\gamma_0\}.
\end{aligned} \tag{3.39}$$

Combining (3.38) and (3.39), we get (3.9). Theorem 3.2 is proved completely. □

Remark We consider the following problem:

$$\begin{cases}
u_{tt} - \Delta u + Ku + \lambda u_t + |u|^{p-2}u = f(x,t), & x \in \Omega, t > 0, \\
-\frac{\partial u}{\partial \nu}(x,t) = g(x,t) + \int_\Omega h(x,y,t) u(y,t)\, dy, & x \in \partial\Omega, t \ge 0, \\
u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x).
\end{cases} \tag{3.40}$$

With the suitable conditions for K, λ, p, u_0, u_1, f, g, h, we prove that problem (3.40) has a unique global solution $u(t)$ with energy decaying exponentially as $t \to +\infty$, without the initial data (u_0, u_1) being small enough. The results obtained are as follows and their proofs are not difficult with a procedure analogous to the ones in Theorems 2.4, 3.2.

Theorem 3.7 *Suppose that* $2 < p \leq \frac{2N-2}{N-2}$, $K > 0$, $\lambda > 0$, $g \equiv 0$, $(u_0, u_1) \in H^1 \times L^2$ *and* (A_1''), (A_2'') *hold. Then problem* (3.40) *has a unique global solution* $u \in L^\infty(0,\infty;H^1) \cap C([0,\infty);H^1) \cap C^1([0,\infty);L^2)$ *such that* $u_t \in L^\infty(0,\infty;L^2)$.

Furthermore, if $\|h\|_{L^\infty(0,\infty;L^2(\partial\Omega\times\Omega))}$, $\|h'\|_{L^\infty(0,\infty;L^2(\partial\Omega\times\Omega))}$ *are sufficiently small then there exist positive constants* C, γ *such that*

$$\left\|u'(t)\right\|^2 + \left\|u(t)\right\|_1^2 + \left\|u(t)\right\|_{L^p}^p \leq C\exp(-\gamma t) \quad \text{for all } t \geq 0.$$

Appendix

Lemma A.1 *Let* H *be Hilbert space with respect to the scalar product* $\langle\cdot,\cdot\rangle$. *Then the embedding* $W^1(0,T;H) = \{F \in L^2(0,T;H) : F' \in L^2(0,T;H)\} \hookrightarrow C^0([0,T];H)$ *is continuous and*

$$\|F\|_{C^0([0,T];H)} \leq \gamma_T \sqrt{\left(\|F\|_{L^2(0,T;H)}^2 + \left\|F'\right\|_{L^2(0,T;H)}^2\right)} \equiv \gamma_T \|F\|_{W^1(0,T;H)}$$

for all $F \in W^1(0,T;H)$, *where* $\gamma_T = \sqrt{\frac{1}{2T} + \sqrt{4 + \frac{1}{4T^2}}}$.

Proof Let $F \in W^1(0,T;H)$, for all $t, s \in [0,T]$, we have

$$\left\|F(t) - F(s)\right\| = \left\|\int_s^t F'(r)\,dr\right\| \leq \left|\int_s^t \left\|F'(r)\right\|\,dr\right| \leq \sqrt{|t-s|}\left\|F'\right\|_{L^2(0,T;H)}. \tag{A.1}$$

Hence $F \in C^0([0,T];H)$.

On the other hand

$$\left\|F(t)\right\|^2 = \left\|F(s)\right\|^2 + \int_s^t \frac{d}{dt}\left\|F(r)\right\|^2 dr = \left\|F(s)\right\|^2 + 2\int_s^t \left\langle F(r), F'(r)\right\rangle dr. \tag{A.2}$$

Integrating with respect to s, we obtain

$$\begin{aligned} T\left\|F(t)\right\|^2 &= \int_0^T \left\|F(s)\right\|^2 ds + 2\int_0^T ds \int_s^t \left\langle F(r), F'(r)\right\rangle dr \\ &= \|F\|_{L^2(0,T;H)}^2 + 2\int_0^T ds \int_0^t \left\langle F(r), F'(r)\right\rangle dr - 2\int_0^T ds \int_0^s \left\langle F(r), F'(r)\right\rangle dr \\ &= \|F\|_{L^2(0,T;H)}^2 + 2T\int_0^t \left\langle F(r), F'(r)\right\rangle dr - 2\int_0^T ds \int_0^s \left\langle F(r), F'(r)\right\rangle dr. \end{aligned} \tag{A.3}$$

Inverting the variables s and r in the last integral of (A.3), we rewrite it as follows:

$$-2\int_0^T ds \int_0^s \left\langle F(r), F'(r)\right\rangle dr = -2\int_0^T (T-r)\left\langle F(r), F'(r)\right\rangle dr. \tag{A.4}$$

By the inequality $2ab \le \alpha a^2 + \frac{1}{\alpha}b^2$, for all $a, b \in \mathbb{R}$, $\alpha > 0$, we deduce from (A.3), (A.4) that

$$\begin{aligned} T\|F(t)\|^2 &\le \|F\|^2_{L^2(0,T;H)} + 2T\int_0^t \|F(r)\|\|F'(r)\|\,dr + 2\int_0^T (T-r)\|F(r)\|\|F'(r)\|\,dr \\ &\le \|F\|^2_{L^2(0,T;H)} + 4T\int_0^T \|F(r)\|\|F'(r)\|\,dr \\ &\le \|F\|^2_{L^2(0,T;H)} + 4T\|F\|_{L^2(0,T;H)}\|F'\|_{L^2(0,T;H)} \\ &\le \|F\|^2_{L^2(0,T;H)} + 2T\left(\alpha\|F\|^2_{L^2(0,T;H)} + \frac{1}{\alpha}\|F'\|^2_{L^2(0,T;H)}\right) \\ &\le (1+2T\alpha)\|F\|^2_{L^2(0,T;H)} + \frac{2T}{\alpha}\|F'\|^2_{L^2(0,T;H)}. \end{aligned} \tag{A.5}$$

Choose $\alpha > 0$ such that $1 + 2T\alpha = \frac{2T}{\alpha}$, or $\alpha = \frac{2}{\frac{1}{2T} + \sqrt{4 + \frac{1}{4T^2}}}$. Hence

$$\begin{aligned} \|F\|^2_{C^0([0,T];L^2)} &\le \left(\frac{1}{2T} + \sqrt{4 + \frac{1}{4T^2}}\right)\left(\|F\|^2_{L^2(0,T;H)} + \|F'\|^2_{L^2(0,T;H)}\right) \\ &\equiv \gamma_T^2\|F\|^2_{W^1(0,T;H)}. \end{aligned} \tag{A.6}$$

□

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

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