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Positive solutions for higher order differential equations with integral boundary conditions

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Abstract

In this paper, we consider the existence of at least three positive solutions for the 2nth order differential equations with integral boundary conditions

$$\begin{cases} x^{(2n)}(t) = f(t, x(t), x''(t), \dots, x^{(2(n-1))}(t)), & 0 \le t \le 1, \\ x^{(2i)}(0) = \int_0^1 k_i(s) x^{(2i)}(s)\,\mathrm{d}s, \quad x^{(2i)}(1) = 0, & 0 \le i \le n-1, \end{cases}$$

where $(-1)^n f > 0$ is continuous, and $k_i(t) \in L^1[0,1]$ $(i = 0, 1, \dots, n-1)$ are nonnegative. The associated Green's function for the higher order differential equations with integral boundary conditions is first given, and growth conditions are imposed on f which yield the existence of multiple positive solutions by using the Leggett-Williams fixed point theorem.

Keywords: boundary value problem; integral boundary conditions; positive solution; fixed point theorem

1 Introduction

The multi-point boundary value problems (BVPs) for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. The study of nonlocal BVPs for second order ordinary differential equations has been widely investigated in [1–3]. Since then, nonlinear high order nonlocal BVPs have been studied by many authors. We refer the reader to [4–13] and references therein. Recently, Guo *et al.* [14] used Leggett-Williams fixed point theorem to obtain the existence of at least three positive solutions for the 2nth order m-point BVP

$$\begin{cases} y^{(2n)}(t) = f(t, y(t), y''(t), \dots, y^{(2(n-1))}(t)), & 0 \le t \le 1, \\ y^{(2i)}(0) = 0, \quad y^{(2i)}(1) = \sum_{j=1}^{m-2} k_{ij} y^{(2i)}(\xi_j), & 0 \le i \le n-1, \end{cases}$$

where $k_{ij} > 0$ $(i = 0, 1, \dots, n-1; j = 1, 2, \dots, m-2)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, and $(-1)^n f : [0,1] \times \mathbb{R}^n \to [0, +\infty)$ is continuous.

BVPs with integral boundary conditions for ordinary differential equations represent a very interesting and important class of problems and arise in the study of various physical, biological and chemical processes, such as heat conduction, chemical engineering, thermo-elasticity, underground water flow, population dynamics, and plasma physics. Such problems include two-, three-, multi-point and nonlocal BVPs as special cases. The existence and multiplicity of positive solutions for such problems have received a great deal of attention; see [15–19] and references therein. In particular, we would mention the result of [19], Zhang and Ge investigated the existence and nonexistence of positive solutions of the following fourth-order BVP with integral boundary conditions

$$\begin{cases} x^{(4)}(t) = \omega(t)f(t,x(t),x''(t)), & 0 < t < 1, \\ x(0) = \int_0^1 g(s)x(s)\,\mathrm{d}s, & x(1) = 0, \\ x''(0) = \int_0^1 h(s)x''(s)\,\mathrm{d}s, & x''(1) = 0, \end{cases}$$

where ω may be singular at $t = 0$ and (or) $t = 1$, $f \in C([0,1] \times [0,+\infty) \times (-\infty,0],[0,+\infty))$, and $g,h \in L^1[0,1]$ are nonnegative.

Motivated by [14, 19], in this paper, we consider the existence of at least three positive solutions for the 2nth order differential equations with integral boundary conditions

$$\begin{cases} x^{(2n)}(t) = f(t,x(t),x''(t),\ldots,x^{(2(n-1))}(t)), & 0 \le t \le 1, \\ x^{(2i)}(0) = \int_0^1 k_i(s)x^{(2i)}(s)\,\mathrm{d}s, \quad x^{(2i)}(1) = 0, & 0 \le i \le n-1, \end{cases} \tag{1.1}$$

where $(-1)^n f > 0$ is continuous, and $k_i(t) \in L^1[0,1]$ $(i = 0,1,\ldots,n-1)$ are nonnegative.

For more precise conditions on f, let $(-1)^j[a,b] = [a,b]$ if j is even and $(-1)^j[a,b] = [-b,-a]$ if j is odd. Let

$$\prod_{j=0}^{n-1}[a_j,b_j] = [a_0,b_0] \times \cdots \times [a_{n-1},b_{n-1}].$$

We shall require that

$$(-1)^n f : [0,1] \times \prod_{j=0}^{n-1}(-1)^j[0,+\infty) \to [0,+\infty).$$

We shall suppose the following conditions are satisfied:

(H_1) $k_i(t) \in L^1[0,1]$ are nonnegative, and $K_i \in [0,1)$, where

$$K_i = \int_0^1 (1-s)k_i(s)\,\mathrm{d}s, \quad 0 \le i \le n-1;$$

(H_2) $(-1)^n f : [0,1] \times \prod_{j=0}^{n-1}(-1)^j[0,+\infty) \to [0,+\infty)$ is continuous.

2 Preliminary results

Definition 2.1 Let E be a Banach space over $\mathbb{R}$. A nonempty convex closed set $K \subset E$ is said to be a cone provided that

(i) $au \in K$ for all $u \in K$ and all $a \ge 0$;
(ii) $u, -u \in K$ implies $u = 0$.

Definition 2.2 The map α is said to be a nonnegative continuous concave functional on K provided that $\alpha : K \to [0,\infty)$ is continuous and

$$\alpha\bigl(tx + (1-t)y\bigr) \ge t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in K$ and $0 \leq t \leq 1$. Similarly, we say the map γ is a nonnegative continuous convex functional on K provided that $\gamma : K \to [0,\infty)$ is continuous and

$$\gamma\bigl(tx + (1-t)y\bigr) \leq t\gamma(x) + (1-t)\gamma(y)$$

for all $x, y \in K$ and $0 \leq t \leq 1$.

Definition 2.3 Let $0 < a < b$ be given and let α be a nonnegative continuous concave functional on K. Define the convex sets P_r and $P(\alpha, a, b)$ by

$$P_r = \bigl\{x \in K \mid \|x\| < r\bigr\} \quad \text{and} \quad P(\alpha, a, b) = \bigl\{x \in K \mid a \leq \alpha(x), \|x\| \leq b\bigr\}.$$

Theorem 2.4 (Leggett-Williams fixed point theorem [20]) *Let $A : \overline{P}_c \to \overline{P}_c$ be a completely continuous operator and let α be a nonnegative continuous concave functional on K such that $\alpha(x) \leq \|x\|$ for all $x \in \overline{P}_c$. Suppose there exist $0 < a < b < d \leq c$ such that*

(C_1) $\{x \in P(\alpha, b, d) \mid \alpha(x) > b\} \neq \emptyset$, *and* $\alpha(Ax) > b$ *for* $x \in P(\alpha, b, d)$,
(C_2) $\|Ax\| < a$ *for* $\|x\| \leq a$, *and*
(C_3) $\alpha(Ax) > b$ *for* $x \in P(\alpha, b, c)$, *with* $\|Ax\| > d$.

Then A has at least three fixed points x_1, x_2, and x_3 such that

$$\|x_1\| < a, \qquad b < \alpha(x_2) \quad \textit{and} \quad \|x_3\| > a \quad \textit{with } \alpha(x_3) < b.$$

Remark 2.5 If we have $d = c$, then condition (C_1) of Theorem 2.4 implies condition (C_3) of Theorem 2.4.

3 Preliminary lemmas

Lemma 3.1 *Suppose* (H_1) *holds. Then $g_i(t,s) \leq 0$ $(0 \leq i \leq n-1)$, where $g_i(t,s)$ is the Green's function for the problem*

$$\begin{cases} x''(t) = 0, \quad 0 \leq t \leq 1, \\ x(0) = \int_0^1 k_i(s)x(s)\,\mathrm{d}s, \qquad x(1) = 0. \end{cases}$$

Proof It is easy to see that $g_i(t,s) \leq 0$ $(0 \leq i \leq n-1)$ by using Lemma 2.1 of [19]. □

Let $G_1(t,s) = g_{n-2}(t,s)$, then for $2 \leq j \leq n-1$, we recursively define

$$G_j(t,s) = \int_0^1 g_{n-j-1}(t,\tau)G_{j-1}(\tau,s)\,\mathrm{d}\tau.$$

Lemma 3.2 *Suppose* (H_1) *holds. If $y \in C[0,1]$, then the BVP*

$$\begin{cases} x^{(2l)}(t) = y(t), \quad 0 \leq t \leq 1, \\ x^{(2i)}(0) = \int_0^1 k_{n-l+i-1}(s)x^{(2i)}(s)\,\mathrm{d}s, \qquad x^{(2i)}(1) = 0, \quad 0 \leq i \leq l-1, \end{cases} \tag{3.1}$$

has a unique solution for each $1 \leq l \leq n-1$, where $G_l(t,s)$ is the associated Green's function for the BVP (3.1).

Proof We will prove the result by using mathematical induction.

When $l = 1$, which implies that $i = 0$, then the BVP (3.1) reduces to

$$\begin{cases} x''(t) = y(t), & 0 \le t \le 1, \\ x(0) = \int_0^1 k_{n-2}(s)x(s)\,\mathrm{d}s, & x(1) = 0. \end{cases} \tag{3.2}$$

By using Lemma 3.1, it is easy to see that the BVP (3.2) has a unique solution

$$x(t) = \int_0^1 G_1(t,s)y(s)\,\mathrm{d}s.$$

Therefore, the result holds for $l = 1$.

We assume that the result holds for $l-1$. Now, we deal with the case for l. Let $x''(t) = u(t)$, then the BVP (3.1) is equivalent to the following BVPs:

$$\begin{cases} x''(t) = u(t), & 0 \le t \le 1, \\ x(0) = \int_0^1 k_{n-l-1}(s)x(s)\,\mathrm{d}s, & x(1) = 0 \end{cases} \tag{3.3}$$

and

$$\begin{cases} u^{(2(l-1))}(t) = y(t), & 0 \le t \le 1, \\ u^{(2i)}(0) = \int_0^1 k_{n-l+i}(s)u^{(2i)}(s)\,\mathrm{d}s, & u^{(2i)}(1) = 0, \quad 0 \le i \le l-2. \end{cases} \tag{3.4}$$

By applying Lemma 3.1, the BVP (3.3) has a unique solution

$$x(t) = \int_0^1 g_{n-l-1}(t,r)u(r)\,\mathrm{d}r. \tag{3.5}$$

Replacing l by $l-1$ and x by u in (3.1), by applying the inductive hypothesis, the BVP (3.4) has also a unique solution

$$u(t) = \int_0^1 G_{l-1}(t,s)y(s)\,\mathrm{d}s. \tag{3.6}$$

Substituting (3.6) into (3.5), we see that the BVP (3.1) has a unique solution

$$\begin{aligned} x(t) &= \int_0^1 g_{n-l-1}(t,r)\int_0^1 G_{l-1}(r,s)y(s)\,\mathrm{d}s\,\mathrm{d}r \\ &= \int_0^1 \left(\int_0^1 g_{n-l-1}(t,r)G_{l-1}(r,s)\,\mathrm{d}r\right)y(s)\,\mathrm{d}s \\ &= \int_0^1 G_l(t,s)y(s)\,\mathrm{d}s. \end{aligned}$$

Therefore, the result holds for l. Lemma 3.2 is now completed. □

For each $1 \le l \le n-1$, we define $A_l : C[0,1] \to C[0,1]$ by

$$A_l u(t) = \int_0^1 G_l(t,\tau)u(\tau)\,\mathrm{d}\tau.$$

With the use of Lemma 3.2, for each $1 \le l \le n-1$, we have

$$\begin{cases} (A_l u)^{(2l)}(t) = u(t), \quad 0 \le t \le 1, \\ (A_l u)^{(2i)}(0) = \int_0^1 k_{n-l+i-1}(s)(A_l u)^{(2i)}(s)\,\mathrm{d}s, \qquad (A_l u)^{(2i)}(1) = 0, \quad 0 \le i \le l-1. \end{cases}$$

Therefore (1.1) has a solution if and only if the BVP

$$\begin{cases} u''(t) = f(t, A_{n-1}u(t), A_{n-2}u(t), \dots, A_1 u(t), u(t)), \quad 0 \le t \le 1, \\ u(0) = \int_0^1 k_{n-1}(s)u(s)\,\mathrm{d}s, \qquad u(1) = 0 \end{cases} \tag{3.7}$$

has a solution. If x is a solution of (1.1), then $u = x^{(2(n-1))}$ is a solution of (3.7). Conversely, if u is a solution of (3.7), then $x = A_{n-1}u$ is a solution of (1.1). In addition if $(-1)^{n-1}u(t) \ge 0$ ($\not\equiv 0$) on $[0,1]$, then $x = A_{n-1}u$ is a positive solution of (1.1).

For $0 \le i \le n-1$, let

$$m_i = \min_{t\in[\tau,1-\tau]} \int_\tau^{1-\tau} \left|g_i(t,s)\right| \mathrm{d}s \quad \text{and} \quad M_i = \max_{t\in[0,1]} \int_0^1 \left|g_i(t,s)\right| \mathrm{d}s. \tag{3.8}$$

Obviously, $0 < m_i < M_i$. Let E denote the Banach space $C[0,1]$ with the maximum norm

$$\|u\| = \max_{0\le t\le 1} \left|u(t)\right|,$$

and define the cone $K \subset E$ by

$$K = \Bigl\{ u \in E \Big| (-1)^{n-1}u(t) \ge 0, (-1)^{n-1}u(t) \text{ is concave on } [0,1], \text{and} \\ \min_{t\in[\tau,1-\tau]} (-1)^{n-1}u(t) \ge \tau^2 \|u\| \Bigr\}.$$

Finally, we define the nonnegative continuous concave functional α on K by

$$\alpha(u) = \min_{t\in[\tau,1-\tau]} \left|u(t)\right|$$

for each $u \in K$ and it is easy to see that $\alpha(u) \le \|u\|$.

4 Main results

Theorem 4.1 *Suppose conditions* (H_1), (H_2) *hold. In addition assume there exist nonnegative numbers a, b, and c such that* $0 < a < b \le \min\{\tau^2, \frac{m_{n-1}}{M_{n-1}}\}c$ *and* $f(t, u_{n-1}, u_{n-2}, \dots, u_1, u_0)$ *satisfies the following growth conditions*:

(H_3) $(-1)^n f(t, u_{n-1}, u_{n-2}, \dots, u_1, u_0) \le \frac{c}{M_{n-1}}$, *for* $(t, u_{n-1}, u_{n-2}, \dots, u_1, u_0) \in [0,1] \times \prod_{j=n-1}^{1} (-1)^{n-1-j}[0, \prod_{i=2}^{j+1} M_{n-i}c] \times (-1)^{n-1}[0,c]$;

(H_4) $(-1)^n f(t, u_{n-1}, u_{n-2}, \dots, u_1, u_0) < \frac{a}{M_{n-1}}$, *for* $(t, u_{n-1}, u_{n-2}, \dots, u_1, u_0) \in [0,1] \times \prod_{j=n-1}^{1} (-1)^{n-1-j}[0, \prod_{i=2}^{j+1} M_{n-i}a] \times (-1)^{n-1}[0,a]$;

(H_5) $(-1)^n f(t, u_{n-1}, u_{n-2}, \dots, u_1, u_0) \ge \frac{b}{m_{n-1}}$, *for* $(t, u_{n-1}, u_{n-2}, \dots, u_1, u_0) \in [\tau, 1-\tau] \times \prod_{j=n-1}^{1} (-1)^{n-1-j}[\prod_{i=2}^{j+1} m_{n-i}b, \prod_{i=2}^{j+1} M_{n-i}\frac{b}{\tau^2}] \times (-1)^{n-1}[b, \frac{b}{\tau^2}]$.

Then the BVP (1.1) *has at least three positive solutions* x_1, x_2, *and* x_3, *which satisfy*

$$\|x_1^{(2(n-1))}(t)\| < a, \qquad b < \alpha\big(x_2^{(2(n-1))}(t)\big) \quad \textit{and}$$
$$\|x_3^{(2(n-1))}(t)\| > a \quad \textit{with}\ \alpha\big(x_3^{(2(n-1))}(t)\big) < b.$$

Proof Define the completely continuous operator A by

$$Au(t) = \int_0^1 g_{n-1}(t,s)f\big(s, A_{n-1}u(s), \dots, A_1u(s), u(s)\big)\,\mathrm{d}s.$$

We will first verify that $A : K \to K$. Let $u \in K$, then $(-1)^{n-1}Au(t) \geq 0$, $((-1)^{n-1}Au)''(t) = (-1)^{n-1}f(t, A_{n-1}u(t), \dots, A_1u(t), u(t)) \leq 0$, $0 \leq t \leq 1$, and by Proposition 2.2 of [19], we have

$$\begin{aligned}
\|Au\| &= \max_{t\in[0,1]}\big|Au(t)\big| \\
&= \max_{t\in[0,1]}\left|\int_0^1 g_{n-1}(t,s)f\big(s, A_{n-1}u(s), \dots, A_1u(s), u(s)\big)\,\mathrm{d}s\right| \\
&= \max_{t\in[0,1]}\left|\int_0^1 \left[h(t,s) + \frac{1-t}{1-K_{n-1}}\int_0^1 h(s,\tau)k_{n-1}(\tau)\,\mathrm{d}\tau\right]\right. \\
&\quad \left.\times f\big(s, A_{n-1}u(s), \dots, A_1u(s), u(s)\big)\,\mathrm{d}s\right| \\
&\leq \int_0^1 \left[h(s,s) + \frac{1}{1-K_{n-1}}\int_0^1 h(\tau,\tau)k_{n-1}(\tau)\,\mathrm{d}\tau\right] \\
&\quad \times \big|f\big(s, A_{n-1}u(s), \dots, A_1u(s), u(s)\big)\big|\,\mathrm{d}s.
\end{aligned}$$

On the other hand, by Proposition 2.3 of [19], we obtain

$$\begin{aligned}
&\min_{t\in[\tau,1-\tau]}(-1)^{n-1}Au(t) \\
&\quad= \min_{t\in[\tau,1-\tau]}(-1)^{n-1}\int_0^1 g_{n-1}(t,s)f\big(s, A_{n-1}u(s), \dots, A_1u(s), u(s)\big)\,\mathrm{d}s \\
&\quad= \min_{t\in[\tau,1-\tau]}(-1)^{n}\int_0^1 \left[h(t,s) + \frac{1-t}{1-K_{n-1}}\int_0^1 h(s,\tau)k_{n-1}(\tau)\,\mathrm{d}\tau\right] \\
&\qquad \times f\big(s, A_{n-1}u(s), \dots, A_1u(s), u(s)\big)\,\mathrm{d}s \\
&\quad\geq \tau^2\int_0^1 \left[h(s,s) + \frac{1}{1-K_{n-1}}\int_0^1 h(\tau,\tau)k_{n-1}(\tau)\,\mathrm{d}\tau\right] \\
&\qquad \times \big|f\big(s, A_{n-1}u(s), \dots, A_1u(s), u(s)\big)\big|\,\mathrm{d}s \\
&\quad\geq \tau^2\|Au\|.
\end{aligned}$$

Consequently, $A : K \to K$.

It is a standard argument to show that the operator A is completely continuous. Equicontinuity and uniform boundedness follow readily from the properties of G_l, $1 \leq l \leq n-1$.

If $u \in \overline{P}_c$, then $\|u\| \leq c$. For each $1 \leq j \leq n-1$, note that inductively (using (3.8)) we have

$$\|A_ju\| = \max_{t\in[0,1]}\left|\int_0^1 G_j(t,s)u(s)\,\mathrm{d}s\right| \leq \prod_{i=2}^{j+1} M_{n-i}c.$$

From the condition (H_3) and (3.8), we obtain

$$\begin{aligned}\|Au\| &= \max_{t\in[0,1]}\left|Au(t)\right| \\ &= \max_{t\in[0,1]}\left|\int_0^1 g_{n-1}(t,s)f\left(s,A_{n-1}u(s),A_{n-2}u(s),\dots,A_1u(s),u(s)\right)\mathrm{d}s\right| \\ &\le \frac{c}{M_{n-1}}\max_{t\in[0,1]}\int_0^1\left|g_{n-1}(t,s)\right|\mathrm{d}s = c.\end{aligned}$$

So, $A:\overline{P}_c\to\overline{P}_c$.

In a completely analogous argument, the condition (H_4) implies the condition (C_2) of Theorem 2.4 is satisfied.

We now show that condition (C_1) is satisfied. Note that, for $0\le t\le 1$,

$$u(t) = (-1)^{n-1}\frac{b}{\tau^2}\in P\left(\alpha,b,\frac{b}{\tau^2}\right) \quad \text{and} \quad \alpha(u) = \frac{b}{\tau^2} > b.$$

Thus,

$$\left\{u\in P\left(\alpha,b,\frac{b}{\tau^2}\right)\Big|\alpha(u)>b\right\}\ne\emptyset.$$

Also, if $u\in P(\alpha,b,\frac{b}{\tau^2})$, then $b\le(-1)^{n-1}u(t)\le\frac{b}{\tau^2}$ for $t\in[\tau,1-\tau]$, implies for each $1\le j\le n-1$, $t\in[\tau,1-\tau]$, inductively,

$$\begin{aligned}&(-1)^{n-1-j}A_ju(t)\le\prod_{i=2}^{j+1}M_{n-i}\frac{b}{\tau^2}, \\ &(-1)^{n-1-j}A_ju(t) = \left|\int_0^1 G_j(t,s)u(s)\,\mathrm{d}s\right|\ge b\int_\tau^{1-\tau}\left|G_j(t,s)\right|\mathrm{d}s\ge\prod_{i=2}^{j+1}m_{n-i}b.\end{aligned}$$

With the use of condition (H_5) and (3.8), we get

$$\begin{aligned}\alpha(Au) &= \min_{t\in[\tau,1-\tau]}\left|\int_0^1 g_{n-1}(t,s)f\left(s,A_{n-1}u(s),\dots,A_1u(s),u(s)\right)\mathrm{d}s\right| \\ &> \min_{t\in[\tau,1-\tau]}\left|\int_\tau^{1-\tau} g_{n-1}(t,s)f\left(s,A_{n-1}u(s),\dots,A_1u(s),u(s)\right)\mathrm{d}s\right| \\ &\ge \frac{b}{m_{n-1}}\min_{t\in[\tau,1-\tau]}\int_0^1\left|g_{n-1}(t,s)\right|\mathrm{d}s = b.\end{aligned}$$

Therefore, condition (C_1) is satisfied.

Finally, we show that condition (C_3) is also satisfied. That is, we show that if $u\in P(\alpha,b,c)$ and $\|Au\|>d=\frac{b}{\tau^2}$, then $\alpha(Au)>b$. This follows since $A:K\to K$. In particular, since $(-1)^{n-1}(Au)$ is concave and $\min_{t\in[\tau,1-\tau]}(-1)^{n-1}(Au)(t)\ge\tau^2\|Au\|$.

That is,

$$\alpha(Au) = \min_{t\in[\tau,1-\tau]}\left|Au(t)\right|\ge\tau^2\|Au\|>b.$$

Therefore, condition (C_3) is also satisfied. By Theorem 2.4, there exist three solutions $u_1, u_2, u_3 \in K$ for the BVP (3.7). Moreover, let

$$x_i(t) = A_{n-1}u_i(t) = \int_0^1 G_{n-1}(t,s)u_i(s)\,\mathrm{d}s, \quad i = 1,2,3,$$

then x_1, x_2, x_3 are three positive solutions for the BVP (1.1) and satisfy

$$\begin{aligned} &\|x_1^{(2(n-1))}(t)\| < a, \qquad b < \alpha\bigl(x_2^{(2(n-1))}(t)\bigr) \quad \text{and} \\ &\|x_3^{(2(n-1))}(t)\| > a \quad \text{with } \alpha\bigl(x_3^{(2(n-1))}(t)\bigr) < b. \end{aligned}$$

□

Consider the following $2n$th order differential equations with integral boundary conditions:

$$\begin{cases} x^{(2n)}(t) = f(t, x(t), x''(t), \dots, x^{(2(n-1))}(t)), & 0 \le t \le 1, \\ x^{(2i)}(0) = 0, \qquad x^{(2i)}(1) = \int_0^1 k_i^*(s)x^{(2i)}(s)\,\mathrm{d}s, & 0 \le i \le n-1, \end{cases} \tag{4.1}$$

where $(-1)^n f \in C([0,1] \times \prod_{j=0}^{n-1}(-1)^j[0,+\infty) \to [0,+\infty))$ and $k_i^*(t) \in L^1[0,1]$ ($i = 0,1,\dots,n-1$) are nonnegative.

Now we deal with problem (4.1). The method is just similar to what we have done for the problem (1.1), so we omit the proof of main results in this section.

For convenience, we list the following assumptions:

(H_1^*) $k_i^*(t) \in L^1[0,1]$ are nonnegative, and $K_i^* \in [0,1)$, where

$$K_i^* = \int_0^1 s k_i^*(s)\,\mathrm{d}s, \quad 0 \le i \le n-1.$$

By analogous methods, we have the following results.

Lemma 4.2 *Suppose* (H_1^*) *holds. Then* $g_i^*(t,s) \le 0$ $(0 \le i \le n-1)$, *where* $g_i^*(t,s)$ *is the Green's function for the problem*

$$\begin{cases} x''(t) = 0, & 0 \le t \le 1, \\ x(0) = 0, \qquad x(1) = \int_0^1 k_i^*(s)x(s)\,\mathrm{d}s. \end{cases}$$

For $0 \le i \le n-1$, *let*

$$m_i^* = \min_{t\in[\tau,1-\tau]} \int_\tau^{1-\tau} \bigl|g_i^*(t,s)\bigr|\,\mathrm{d}s \quad \textit{and} \quad M_i^* = \max_{t\in[0,1]} \int_0^1 \bigl|g_i^*(t,s)\bigr|\,\mathrm{d}s.$$

Theorem 4.3 *Suppose condition* (H_1^*) *holds. In addition assume there exist nonnegative numbers* a, b, *and* c *such that* $0 < a < b \le \min\{\tau^2, \frac{m_{n-1}^*}{M_{n-1}^*}\}c$ *and* $f(t, u_{n-1}, u_{n-2}, \dots, u_1, u_0)$ *satisfies the following growth conditions*:

(H_2^*) $(-1)^n f(t, u_{n-1}, u_{n-2}, \dots, u_1, u_0) \le \frac{c}{M_{n-1}^*}$, *for* $(t, u_{n-1}, u_{n-2}, \dots, u_1, u_0) \in [0,1] \times \prod_{j=n-1}^{1}(-1)^{n-1-j}[0, \prod_{i=2}^{j+1} M_{n-i}^* c] \times (-1)^{n-1}[0,c]$;

(H_3^*) $(-1)^n f(t, u_{n-1}, u_{n-2}, \dots, u_1, u_0) < \frac{a}{M_{n-1}^*}$, *for* $(t, u_{n-1}, u_{n-2}, \dots, u_1, u_0) \in [0,1] \times \prod_{j=n-1}^{1}(-1)^{n-1-j}[0, \prod_{i=2}^{j+1} M_{n-i}^* a] \times (-1)^{n-1}[0,a]$;

(H_4^*) $(-1)^n f(t,u_{n-1},u_{n-2},\ldots,u_1,u_0) \geq \frac{b}{m_{n-1}^*}$, *for* $(t,u_{n-1},u_{n-2},\ldots,u_1,u_0) \in [\tau,1-\tau] \times \prod_{j=n-1}^{1}(-1)^{n-1-j}[\prod_{i=2}^{j+1} m_{n-i}^* b, \prod_{i=2}^{j+1} M_{n-i}^* \frac{b}{\tau^2}] \times (-1)^{n-1}[b,\frac{b}{\tau^2}]$.

Then the BVP (4.1) *has at least three positive solutions* x_1, x_2, *and* x_3, *which satisfy*

$$\begin{aligned}&\left\|x_1^{(2(n-1))}(t)\right\| < a, \qquad b < \alpha\left(x_2^{(2(n-1))}(t)\right) \quad \text{and}\\&\left\|x_3^{(2(n-1))}(t)\right\| > a \quad \text{with } \alpha\left(x_3^{(2(n-1))}(t)\right) < b.\end{aligned}$$

5 Example

Example 5.1 As an example of problem (1.1), consider the following sixth order BVP:

$$\begin{cases} x^{(6)}(t) = f(t,x(t),x''(t),x^{(4)}(t)), \quad 0 \leq t \leq 1,\\ x(0) = \int_0^1 x(s)\,\mathrm{d}s, \qquad x(1) = 0,\\ x''(0) = \int_0^1 sx''(s)\,\mathrm{d}s, \qquad x''(1) = 0,\\ x^{(4)}(0) = \int_0^1 s^2 x^{(4)}(s)\,\mathrm{d}s, \qquad x^{(4)}(1) = 0,\end{cases} \tag{5.1}$$

where

$$\begin{aligned}&f(t,u_2,u_1,u_0)\\&= \begin{cases} -\frac{6{,}050}{841}u_0 \cdot \frac{2+\sin(t+u_1+u_2)}{3}, & 0 \leq u_0 \leq 1,\\ -[(6 \times \frac{112{,}640}{7{,}559} - \frac{6{,}050}{841})(u_0-1) + \frac{6{,}050}{841}] \cdot \frac{2+\sin(t+u_1+u_2)}{3}, & 1 \leq u_0 \leq 2,\\ -3 \times \frac{112{,}640}{7{,}559}u_0 \cdot \frac{2+\sin(t+u_1+u_2)}{3}, & 2 \leq u_0 \leq 32,\\ -[96 \times \frac{112{,}640}{7{,}559} + (\frac{8}{7} \times \frac{6{,}050}{841} - \frac{3}{7} \times \frac{112{,}640}{7{,}559})(u_0-32)] & \\ \quad \cdot \frac{2+\sin(t+u_1+u_2)}{3}, & 32 \leq u_0 \leq 256,\\ -256 \times \frac{6{,}050}{841} \cdot \frac{2+\sin(t+u_1+u_2)}{3}, & u_0 \geq 256.\end{cases}\end{aligned}$$

We notice that $n = 3$, $k_i(s) = s^i$ $(i = 0,1,2)$ and $K_0 = \frac{1}{2}$, $K_1 = \frac{1}{6}$, $K_2 = \frac{1}{12}$.
If we take $\tau = \frac{1}{4}$, by calculation we obtain

$$\begin{aligned}&m_0 = \min_{t\in[\frac{1}{4},\frac{3}{4}]}\int_{\frac{1}{4}}^{\frac{3}{4}}\left|g_0(t,s)\right|\mathrm{d}s = \frac{35}{384}, \qquad M_0 = \max_{t\in[0,1]}\int_0^1\left|g_0(t,s)\right|\mathrm{d}s = \frac{2}{9},\\&m_1 = \min_{t\in[\frac{1}{4},\frac{3}{4}]}\int_{\frac{1}{4}}^{\frac{3}{4}}\left|g_1(t,s)\right|\mathrm{d}s = \frac{97}{1{,}280}, \qquad M_1 = \max_{t\in[0,1]}\int_0^1\left|g_1(t,s)\right|\mathrm{d}s = \frac{121}{800},\\&m_2 = \min_{t\in[\frac{1}{4},\frac{3}{4}]}\int_{\frac{1}{4}}^{\frac{3}{4}}\left|g_2(t,s)\right|\mathrm{d}s = \frac{7{,}559}{112{,}640}, \qquad M_2 = \max_{t\in[0,1]}\int_0^1\left|g_2(t,s)\right|\mathrm{d}s = \frac{841}{6{,}050}.\end{aligned}$$

In addition, if we take $a = 1$, $b = 2$, $c = 256$, then

$$\begin{aligned}0 < a = 1 < b = 2 &\leq \min\left\{\tau^2, \frac{m_2}{M_2}\right\}c\\&= \min\left\{\left(\frac{1}{4}\right)^2, \frac{7{,}559 \times 6{,}050}{112{,}640 \times 841}\right\} \times 256 = 16,\end{aligned}$$

and $f(t,u_2,u_1,u_0)$ satisfies the growth conditions (H_3)-(H_5).

Therefore all the conditions of Theorem 4.1 are satisfied. Hence, the problem (5.1) has at least three positive solutions x_1, x_2, and x_3, which satisfy

$$\max_{0\le t\le 1}\left|x_1^{(4)}(t)\right| < 1, \qquad 2 < \min_{t\in[\frac{1}{4},\frac{3}{4}]}\left|x_2^{(4)}(t)\right| \quad \text{and}$$

$$\max_{0\le t\le 1}\left|x_3^{(4)}(t)\right| > 1 \quad \text{with} \quad \min_{t\in[\frac{1}{4},\frac{3}{4}]}\left|x_3^{(4)}(t)\right| < 2.$$

Example 5.2 As another example of problem (4.1), consider the following sixth order BVP:

$$\begin{cases} x^{(6)}(t) = f(t, x(t), x''(t), x^{(4)}(t)), \quad 0 \le t \le 1, \\ x(0) = 0, \qquad x(1) = \int_0^1 x(s)\,\mathrm{d}s, \\ x''(0) = 0, \qquad x''(1) = \int_0^1 s x''(s)\,\mathrm{d}s, \\ x^{(4)}(0) = 0, \qquad x^{(4)}(1) = \int_0^1 s^2 x^{(4)}(s)\,\mathrm{d}s, \end{cases} \tag{5.2}$$

where

$$f(t,u_2,u_1,u_0) = \begin{cases} -\frac{225}{32}u_0\cdot\frac{2+\sin(t+u_1+u_2)}{3}, & 0\le u_0\le 1, \\ -[(6\times\frac{30{,}720}{2{,}097}-\frac{225}{32})(u_0-1)+\frac{225}{32}]\cdot\frac{2+\sin(t+u_1+u_2)}{3}, & 1\le u_0\le 2, \\ -3\times\frac{30{,}720}{2{,}097}u_0\cdot\frac{2+\sin(t+u_1+u_2)}{3}, & 2\le u_0\le 32, \\ -[96\times\frac{30{,}720}{2{,}097}+(\frac{8}{7}\times\frac{225}{32}-\frac{3}{7}\times\frac{30{,}720}{2{,}097})(u_0-32)]\cdot\frac{2+\sin(t+u_1+u_2)}{3}, & 32\le u_0\le 256, \\ -256\times\frac{225}{32}\cdot\frac{2+\sin(t+u_1+u_2)}{3}, & u_0\ge 256. \end{cases}$$

We notice that $n = 3$, $k_i^*(s) = s^i$ $(i = 0,1,2)$ and $K_0^* = \frac{1}{2}$, $K_1^* = \frac{1}{3}$, $K_2^* = \frac{1}{4}$.
If we take $\tau = \frac{1}{4}$, by calculation we obtain

$$m_0^* = \min_{t\in[\frac{1}{4},\frac{3}{4}]}\int_{\frac{1}{4}}^{\frac{3}{4}}\left|g_0^*(t,s)\right|\mathrm{d}s = \frac{35}{384}, \qquad M_0^* = \max_{t\in[0,1]}\int_0^1\left|g_0^*(t,s)\right|\mathrm{d}s = \frac{2}{9},$$

$$m_1^* = \min_{t\in[\frac{1}{4},\frac{3}{4}]}\int_{\frac{1}{4}}^{\frac{3}{4}}\left|g_1^*(t,s)\right|\mathrm{d}s = \frac{123}{1{,}024}, \qquad M_1^* = \max_{t\in[0,1]}\int_0^1\left|g_1^*(t,s)\right|\mathrm{d}s = \frac{81}{512},$$

$$m_2^* = \min_{t\in[\frac{1}{4},\frac{3}{4}]}\int_{\frac{1}{4}}^{\frac{3}{4}}\left|g_2^*(t,s)\right|\mathrm{d}s = \frac{2{,}097}{30{,}720}, \qquad M_2^* = \max_{t\in[0,1]}\int_0^1\left|g_2^*(t,s)\right|\mathrm{d}s = \frac{32}{225}.$$

In addition, if we take $a = 1$, $b = 2$, $c = 256$, then

$$0 < a = 1 < b = 2 \le \min\left\{\tau^2, \frac{m_2^*}{M_2^*}\right\}c = \min\left\{\left(\frac{1}{4}\right)^2, \frac{2{,}097\times 225}{30{,}720\times 32}\right\}\times 256 = 16,$$

and $f(t,u_2,u_1,u_0)$ satisfies the growth conditions (H_2^*)-(H_4^*).

Therefore all the conditions of Theorem 4.3 are satisfied. Hence, the problem (5.2) has at least three positive solutions x_1, x_2, and x_3, which satisfy

$$\max_{0\le t\le 1}\left|x_1^{(4)}(t)\right| < 1, \qquad 2 < \min_{t\in[\frac{1}{4},\frac{3}{4}]}\left|x_2^{(4)}(t)\right| \quad \text{and}$$

$$\max_{0\le t\le 1}\left|x_3^{(4)}(t)\right| > 1 \quad \text{with} \quad \min_{t\in[\frac{1}{4},\frac{3}{4}]}\left|x_3^{(4)}(t)\right| < 2.$$

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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References

1. Chasreechai, S, Tariboon, J: Positive solutions to generalized second-order three-point integral boundary-value problems. Electron. J. Differ. Equ. **2011**, 14 (2011)
2. Feng, M: Existence of symmetric positive solutions for a boundary value problem with integral boundary conditions. Appl. Math. Lett. **24**, 1419-1427 (2011)
3. Galvis, J, Rojas, EM, Sinitsyn, AV: Existence of positive solutions of a nonlinear second-order boundary-value problem with integral boundary conditions. Electron. J. Differ. Equ. **2015**, 236 (2015)
4. Wu, H, Zhang, J: Positive solutions of higher-order four-point boundary value problem with *p*-Laplacian operator. J. Comput. Appl. Math. **233**, 2757-2766 (2010)
5. Jankowski, T: Positive solutions for fourth-order differential equations with deviating arguments and integral boundary conditions. Nonlinear Anal., Theory Methods Appl. **73**, 1289-1299 (2010)
6. Zhao, J, Wang, P, Ge, W: Existence and nonexistence of positive solutions for a class of third order BVP with integral boundary conditions in Banach spaces. Commun. Nonlinear Sci. Numer. Simul. **16**, 402-413 (2011)
7. Zhang, X, Ge, W: Symmetric positive solutions of boundary value problems with integral boundary conditions. Appl. Math. Comput. **219**, 3553-3564 (2012)
8. Yang, W: Positive solutions for a coupled system of nonlinear fractional differential equations with integral boundary conditions. Comput. Math. Appl. **63**, 288-297 (2012)
9. Cabada, A, Wang, G: Positive solutions of nonlinear fractional differential equations with integral boundary value conditions. J. Math. Anal. Appl. **389**, 403-411 (2012)
10. Agarwal, RP, Wang, G, Ahmad, B, Zhang, L, Hobiny, A, Monaquel, S: On existence of solutions for nonlinear *q*-difference equations with nonlocal *q*-integral boundary conditions. Math. Model. Anal. **20**, 604-618 (2015)
11. Pang, H, Xie, W, Cao, L: Successive iteration and positive solutions for a third-order boundary value problem involving integral conditions. Bound. Value Probl. **2015**, 139 (2015)
12. Wang, Q, Guo, Y, Ji, Y: Positive solutions for fourth-order nonlinear differential equation with integral boundary conditions. Discrete Dyn. Nat. Soc. **2013**, Article ID 684962 (2013)
13. Jiang, W, Zhang, Y, Qiu, J: The existence of solutions for *p*-Laplacian boundary value problems at resonance on the half-line. Bound. Value Probl. **2015**, 179 (2015)
14. Guo, Y, Liu, X, Qiu, J: Three positive solutions for higher order m-point boundary value problems. J. Math. Anal. Appl. **289**, 545-553 (2004)
15. Ahmad, B, Alsaedi, A, Alghamdi, BS: Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions. Nonlinear Anal., Real World Appl. **9**, 1727-1740 (2008)
16. Ahmad, B, Ntouyas, SK, Alsulami, HH: Existence theory for *n*th order nonlocal integral boundary value problems and extension to fractional case. Abstr. Appl. Anal. **2013**, Article ID 183813 (2013)
17. Hao, X, Liu, L: Multiple monotone positive solutions for higher order differential equations with integral boundary conditions. Bound. Value Probl. **2014**, 74 (2014)
18. Li, Y, Zhang, X: Positive solutions for systems of nonlinear higher order differential equations with integral boundary conditions. Abstr. Appl. Anal. **2014**, Article ID 591381 (2014)
19. Zhang, X, Ge, W: Positive solutions for a class of boundary-value problems with integral boundary conditions. Comput. Math. Appl. **58**, 203-215 (2009)
20. Leggett, RW, Williams, LR: Multiple positive fixed points of nonlinear operators on ordered Banach spaces. Indiana Univ. Math. J. **28**, 673-688 (1979)