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Existence results for fractional differential equations of arbitrary order with nonlocal integral boundary conditions

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Abstract

In this paper, we investigate the existence of solutions for fractional differential equations of arbitrary order with nonlocal integral boundary conditions. The existence results are obtained by applying Krasnoselskii's fixed point theorem and Leray-Schauder degree theory, while the uniqueness of the solutions is established by means of Banach's contraction mapping principle. The paper concludes with illustrative examples.

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1 Introduction

The study of boundary value problems of fractional differential equations has gained considerable attention and several interesting results involving a variety of boundary conditions have appeared in the recent literature on the topic. The tools of fractional calculus have been effectively employed to improve the mathematical modeling of several phenomena occurring in scientific and engineering disciplines such as viscoelasticity [1], electrochemistry [2], electromagnetism [3], biology [4, 5], optimal control [6, 7], diffusion process [8–10], economics [11], chaotic theory [12], variational problems [13], *etc.* For a theoretical development of the subject, concerning the existence and uniqueness of solutions for nonlinear fractional order initial and nonlocal boundary value problems, we refer the reader to [14–26] and the references cited therein. There has also been a great emphasis on studying fractional differential equations supplemented with integral boundary conditions; for instance, see [27–32]. Motivated by recent studies of nonlocal nonlinear integral boundary value problems of fractional order, we go a step further and consider a more general problem of nonlinear fractional differential equations of arbitrary order with nonlocal integral boundary conditions. Precisely, we investigate the following problem:

$$\begin{cases} {}^C D_{t_0}^{\alpha} x(t) = f(t, x(t)), \quad t \in J, \\ x^{(k)}(\theta) = b_k + \int_{t_0}^{\theta} g_k(s, x(s))\, ds, \quad k = 0, 1, 2, \ldots, n-1, \theta \in J, \end{cases} \tag{1.1}$$

where $J = [t_0, T]$, ${}^C D^{\alpha}_{t_0}$ is the Caputo fractional derivative of order α, $n = [\alpha] + 1$, $b_k \in \mathbb{R}$, and $f, g_k : J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

The paper is organized as follows. In Section 2, we outline some basic concepts of fractional calculus, and establish a lemma which plays a key role in the sequel. The main results dealing with the existence and uniqueness of solutions for the problem (1.1) are discussed in Sections 3 and 4. We make use of the standard tools of the fixed point theory to obtain the desired results. The paper concludes with some interesting observations.

2 Preliminaries

First of all, we fix our terminology and recall some basic ideas of fractional calculus [14].

Let $C(J, \mathbb{R})$ be a Banach space of all continuous real valued functions defined on J endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in J\}$.

Definition 2.1 The Riemann-Liouville fractional integral of order r for a function $h \in C(J, \mathbb{R})$ is defined as

$$I^r h(t) = \frac{1}{\Gamma(r)} \int_{t_0}^{t} \frac{h(s)}{(t-s)^{1-r}}\, ds, \qquad r > 0,$$

provided the integral exists.

Definition 2.2 Let $h : [t_0, \infty) \to \mathbb{R}$ be such that $h \in AC^n(J, \mathbb{R})$. Then the Caputo derivative of fractional order r for h is defined as

$${}^c D^r_{t_0} h(t) = \frac{1}{\Gamma(n-r)} \int_{t_0}^{t} (t-s)^{n-r-1} h^{(n)}(s)\, ds, \qquad n-1 < r < n, n = [r] + 1,$$

where $[r]$ denotes the integer part of the real number r.

Lemma 2.3 *Let $x \in AC^n(J, \mathbb{R}), f \in AC(J, \mathbb{R})$, and $c_k \in \mathbb{R}$, then*

$$\begin{cases} {}^C D^{\alpha}_{t_0}(I^{\alpha} f(t)) = f(t), \\ I^{\alpha}({}^C D^{\alpha}_{t_0} x(t)) = x(t) + c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + \cdots + c_{n-1}(t-t_0)^{n-1}. \\ \textit{The general solution of the equation } {}^C D^{\alpha}_{t_0} x(t) = 0 \textit{ is} \\ \quad x(t) = c_0 + c_1(t-t_0) + c_2(t-t_0)^2 + \cdots + c_{n-1}(t-t_0)^{n-1}. \end{cases}$$

We need the following result in the sequel.

Lemma 2.4 *Let $\{u_n\}$ be a sequence of real numbers, and $n, k \in \mathbb{N}$, such that $0 \le k \le n-1$. If v is a positive real number, then*

$$\sum_{m=0}^{n-k-1} \sum_{r=0}^{n-k-m-1} (-1)^r \frac{v^{r+m}}{r!m!} u_{m+k+r} = u_k. \tag{2.1}$$

Proof The left-hand side of equation (2.1) can be rearranged as

$$\sum_{m=0}^{n-k-1} \left(\sum_{r=0}^{m} \frac{(-1)^{m-r}}{r!(m-r)!} \right) v^m u_{k+m}.$$

By a binomial expansion, the inner sum takes the form

$$\frac{1}{m!}\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}=0,$$

for all $m \geq 1$. This completes the proof. □

3 Associated linear problem

In this section, we consider the linear variant of the problem (1.1) given by

$$\begin{aligned}
&{}^{C}D^{\alpha}_{t_0}x(t)=\tilde{f}(t), \quad t\in J,\\
&x^{(k)}(\theta)=b_k+\int_{t_0}^{\theta}g_k(s)\,ds, \quad k=0,1,2,\ldots,n-1,\theta\in J,
\end{aligned}\tag{3.1}$$

where $x\in AC^n(J,\mathbb{R}),\tilde{f}\in AC(J,\mathbb{R})$ and $g_k\in C(J,\mathbb{R})$.

Lemma 3.1 *The fractional boundary value problem* (3.1) *is equivalent to the integral equation*

$$x(t)=\sum_{k=0}^{n-1}\frac{(t-\theta)^k}{k!}\left(b_k+\int_{t_0}^{\theta}g_k(s)\,ds-I^{\alpha-k}\tilde{f}(\theta)\right)+I^{\alpha}\tilde{f}(t), \quad t\in J. \tag{3.2}$$

Proof For $\alpha=n$, problem (3.1) reduces to the nth order classical problem:

$$\begin{aligned}
&\frac{d^n}{dt^n}x(t)=\tilde{f}(t), \quad t\in J,\\
&x^{(k)}(\theta)=b_k+\int_{t_0}^{\theta}g_k(s)\,ds, \quad k=0,1,2,\ldots,n-1,\theta\in J,
\end{aligned}$$

which can be integrated n times to have

$$x(t)=\sum_{k=0}^{n-1}\frac{(t-\theta)^k}{k!}\left(b_k+\int_{t_0}^{\theta}g_k(s)\,ds\right)+\frac{1}{(n-1)!}\int_{\theta}^{t}(t-s)^{n-1}\tilde{f}(s)\,ds. \tag{3.3}$$

Using binomial expansion, we find that

$$\begin{aligned}
&\frac{1}{(n-1)!}\int_{\theta}^{t}(t-s)^{n-1}\tilde{f}(s)\,ds\\
&\quad=\frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}\tilde{f}(s)\,ds-\frac{1}{(n-1)!}\int_{t_0}^{\theta}(t-s)^{n-1}\tilde{f}(s)\,ds\\
&\quad=I^n\tilde{f}(t)-\frac{1}{(n-1)!}\int_{t_0}^{\theta}(t-\theta+\theta-s)^{n-1}\tilde{f}(s)\,ds\\
&\quad=I^n\tilde{f}(t)-\sum_{k=0}^{n-1}\frac{(t-\theta)^k}{k!}\left(\frac{1}{(n-k-1)!}\int_{t_0}^{\theta}(\theta-s)^{n-k-1}\tilde{f}(s)\,ds\right)\\
&\quad=I^n\tilde{f}(t)-\sum_{k=0}^{n-1}\frac{(t-\theta)^k}{k!}I^{n-k}\tilde{f}(\theta),
\end{aligned}$$

which together with (3.3) yields (3.2). Next, for $n-1<\alpha<n$, by Lemma 2.3, we have

$$I^{\alpha}\tilde{f}(t)=I^{\alpha}\big({}^{C}D^{\alpha}_{t_0}\big)x(t)=x(t)+\sum_{k=0}^{n-1}c_k(t-t_0)^k. \tag{3.4}$$

Differentiating (3.4) k times, we get

$$\sum_{m=0}^{n-k-1}\frac{(k+m)!}{m!}c_{m+k}(t-t_0)^m=I^{\alpha-k}\tilde{f}(t)-x^{(k)}(t), \tag{3.5}$$

for $0\le k\le n-1$. In view of the integral boundary conditions in (3.1), we can express equation (3.5) in the form of the following array:

$$\begin{bmatrix} 1 & \frac{1!}{1!}(\theta-t_0)^1 & \frac{2!}{2!}(\theta-t_0)^2 & \cdots & \frac{(n-2)!}{(n-2)!}(\theta-t_0)^{n-2} & \frac{(n-1)!}{(n-1)!}(\theta-t_0)^{n-1}\\ 0 & \frac{1!}{0!}(\theta-t_0)^0 & \frac{2!}{1!}(\theta-t_0)^1 & \cdots & \frac{(n-2)!}{(n-3)!}(\theta-t_0)^{n-3} & \frac{(n-1)!}{(n-2)!}(\theta-t_0)^{n-2}\\ 0 & 0 & \frac{2!}{0!}(\theta-t_0)^0 & \cdots & \frac{(n-2)!}{(n-4)!}(\theta-t_0)^{n-4} & \frac{(n-1)!}{(n-3)!}(\theta-t_0)^{n-3}\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \frac{(n-2)!}{0!}(\theta-t_0)^0 & \frac{(n-1)!}{1!}(\theta-t_0)^1\\ 0 & 0 & 0 & 0 & 0 & \frac{(n-1)!}{0!}(\theta-t_0)^0 \end{bmatrix}$$

$$\times\begin{bmatrix} c_0\\ c_1\\ c_2\\ \vdots\\ c_{n-2}\\ c_{n-1}\end{bmatrix}=\begin{bmatrix} I^{\alpha}\tilde{f}(\theta)-b_0-\int_{t_0}^{\theta}g_0(s)\,ds\\ I^{\alpha-1}\tilde{f}(\theta)-b_1-\int_{t_0}^{\theta}g_1(s)\,ds\\ I^{\alpha-2}\tilde{f}(\theta)-b_2-\int_{t_0}^{\theta}g_2(s)\,ds\\ \vdots\\ I^{\alpha-n+2}\tilde{f}(\theta)-b_{n-2}-\int_{t_0}^{\theta}g_{n-2}(s)\,ds\\ I^{\alpha-n+1}\tilde{f}(\theta)-b_{n-1}-\int_{t_0}^{\theta}g_{n-1}(s)\,ds\end{bmatrix}.$$

Solving this system for c_r, $r=1,2,\dots,n$, we obtain

$$c_{n-r}=\frac{1}{(n-r)!}\sum_{k=0}^{r-1}\frac{(-1)^k(\theta-t_0)^k}{k!}\left(I^{\alpha-n+r-k}\tilde{f}(\theta)-b_{n-r+k}-\int_{t_0}^{\theta}g_{n-r+k}(s)\,ds\right),$$

which, for $m=0,1,2,\dots,n-1$, can be written as

$$c_m=\frac{1}{m!}\sum_{k=0}^{n-m-1}\frac{(-1)^k(\theta-t_0)^k}{k!}\left(I^{\alpha-m-k}\tilde{f}(\theta)-b_{m+k}-\int_{t_0}^{\theta}g_{m+k}(s)\,ds\right). \tag{3.6}$$

Indeed, by Lemma 2.4 with $v=\theta-t_0$, and $u_k=I^{\alpha-k}\tilde{f}(\theta)-b_k-\int_{t_0}^{\theta}g_k(s)\,ds$, (3.6) is a solution of (3.5) with $t=\theta$. By means of (3.6) and (3.4), we obtain

$$x(t)=I^{\alpha}\tilde{f}(t)+\sum_{k=0}^{n-1}\sum_{m=0}^{n-k-1}\frac{(-1)^m(\theta-t_0)^m(t-t_0)^k}{m!k!}\left(b_{m+k}+\int_{t_0}^{\theta}g_{m+k}(s)\,ds-I^{\alpha-k-m}\tilde{f}(\theta)\right),$$

which can alternatively be written as

$$x(t)=I^{\alpha}\tilde{f}(t)+\sum_{k=0}^{n-1}\psi_k(t)\left(b_k+\int_{t_0}^{\theta}g_k(s)\,ds-I^{\alpha-k}f(\theta)\right), \tag{3.7}$$

where

$$\psi_k(t) = \sum_{m=0}^{k}(-1)^{k-m}\frac{(\theta - t_0)^{k-m}}{(k-m)!}\frac{(t-t_0)^m}{m!}, \quad k = 0,1,\dots,n-1.$$

Next, for $t_0 < \theta \leq T$, it follows by a binomial expansion that

$$\psi_k(t) = \frac{(\theta - t_0)^k}{k!}\sum_{m=0}^{k}\binom{k}{m}\left(\frac{t-t_0}{\theta - t_0}\right)^m(-1)^{k-m} = \frac{(\theta - t_0)^k}{k!}\left(\frac{t-\theta}{\theta - t_0}\right)^k = \frac{(t-\theta)^k}{k!},$$

which, on substituting in (3.7), yields (3.2).

On the other hand, applying the operator ${}^CD^{\alpha}_{t_0}$, $n-1 < \alpha \leq n$ to (3.2), and using Lemma 2.3, we obtain (3.1). This completes the proof. □

Remark 3.2 We can solve different kinds of boundary value problems involving integral (classical) and multi-point boundary conditions by applying the method of proof used for Lemma 3.1. Here, we enlist the following two cases.

(a) The two-point boundary value problem of the form

$$\begin{cases} {}^CD^{\alpha}_{t_0}x(t) = \tilde{f}(t), & t \in J, \\ x(t_0) = b_0, \qquad x^{(k)}(T) = b_k + \int_{t_0}^{T} g_k(s)\,ds, & k = 1,2,\dots,n-1, \end{cases} \tag{3.8}$$

has an integral solution given by

$$\begin{aligned} x(t) &= b_0 + \int_{t_0}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\tilde{f}(s)\,ds - \sum_{k=1}^{n-1}\frac{(-1)^k}{k!}\big((T-t_0)^k - (T-t)^k\big) \\ &\quad \times \left(b_k + \int_{t_0}^{T} g_k(s)\,ds - \int_{t_0}^{T}\frac{(T-s)^{\alpha-k-1}}{\Gamma(\alpha-k)}\tilde{f}(s)\,ds\right), \quad t \in J. \end{aligned}$$

(b) The integral solution of the three-point boundary value problem

$$\begin{cases} {}^CD^{\alpha}_{t_0}x(t) = \tilde{f}(t), & t \in J, \\ x(t_0) = b_0, \qquad x(t_1) = b_1, & t_0 < t_1 < T, \\ x^{(k)}(T) = b_k + \int_{t_0}^{T} g_k(s)\,ds, & k = 2,\dots,n-1, \end{cases} \tag{3.9}$$

is

$$\begin{aligned} x(t) &= \int_{t_0}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\tilde{f}(s)\,ds + b_0 - \left(b_0 - b_1 + \int_{t_0}^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds\right)\frac{t-t_0}{t_1-t_0} \\ &\quad + \sum_{k=2}^{n-1}\frac{(-1)^k}{k!}\big[(t-t_0)(T-t_1)^k - (t-t_1)(T-t_0)^k - (t_1-t_0)(T-t)^k\big] \\ &\quad \times \frac{1}{t_1-t_0}\left(b_k + \int_{t_0}^{T} g_k(s)\,ds - \int_{t_0}^{T}\frac{(T-s)^{\alpha-k-1}}{\Gamma(\alpha-k)}\tilde{f}(s)\,ds\right), \quad t \in J. \end{aligned}$$

4 Main results

In this section, we show the existence of solutions for the problem (1.1) by applying some fixed point theorems.

In relation to the problem (1.1), we define the fixed point problem

$$\Psi x = x, \tag{4.1}$$

where the operator $\Psi : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is defined by

$$\begin{aligned}\Psi x(t) ={}& \sum_{k=0}^{n-1} \frac{(t-\theta)^k}{k!}\biggl(b_k + \int_{t_0}^{\theta} g_k\bigl(s,x(s)\bigr) - \int_{t_0}^{\theta} \frac{(\theta-s)^{\alpha-k-1}}{\Gamma(\alpha-k)} f\bigl(s,x(s)\bigr)\,ds\biggr)\\ &+ \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,x(s)\bigr)\,ds.\end{aligned} \tag{4.2}$$

Observe that the problem (1.1) has solutions if the operator equation (4.1) has fixed points.

Lemma 4.1 *The operator* $\Psi : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ *given by* (4.2) *is completely continuous.*

Proof Obviously continuity of the operator Ψ follows from the continuity of the functions f and g_k, $k = 0,1,\ldots,n-1$. Let Ω be a bounded subset of $C(J,\mathbb{R})$, then for any $t \in J$, and $x \in \Omega$, there exist positive constants L_f, and L_k, $k = 0,1,\ldots,n-1$, such that $|f(t,x(t))| \leq L_f$, and $|g_k(t,x(t))| \leq L_k$. Then we have

$$\begin{aligned}&\bigl|(\Psi x)(t)\bigr|\\ &\quad\leq \sum_{k=0}^{n-1} \frac{|t-\theta|^k}{k!}\biggl(|b_k| + \int_{t_0}^{\theta} \bigl|g_k\bigl(s,x(s)\bigr)\bigr|\,ds + \int_{t_0}^{\theta} \frac{(\theta-s)^{\alpha-k-1}}{\Gamma(\alpha-k)} \bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds\biggr)\\ &\qquad+ \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds\\ &\quad\leq \sum_{k=0}^{n-1} \frac{|t-\theta|^k}{k!}\biggl(|b_k| + (\theta-t_0)L_k + \frac{(\theta-t_0)^{\alpha-k}}{\Gamma(\alpha-k+1)}L_f\biggr) + \frac{(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}L_f\\ &\quad=: L,\end{aligned}$$

which implies that $\|(\Psi x)\| \leq L$. Furthermore,

$$\begin{aligned}&\bigl|(\Psi x)(t_2) - (\Psi x)(t_1)\bigr|\\ &\quad\leq \sum_{k=0}^{n-1} \frac{|(t_2-\theta)^k - (t_1-\theta)^k|}{k!}\\ &\qquad\times \biggl(|b_k| + \int_{t_0}^{\theta} \bigl|g_k\bigl(s,x(s)\bigr)\bigr|\,ds + \int_{t_0}^{\theta} \frac{(\theta-s)^{\alpha-k-1}}{\Gamma(\alpha-k)} \bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds\biggr)\\ &\qquad+ \int_{t_0}^{t_1} \frac{|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}|}{\Gamma(\alpha)} \bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds\\ &\quad\leq \sum_{k=0}^{n-1} \frac{|(t_2-\theta)^k - (t_1-\theta)^k|}{k!}\biggl(|b_k| + (\theta-t_0)L_k + \frac{(\theta-t_0)^{\alpha-k}}{\Gamma(\alpha-k+1)}L_f\biggr)\\ &\qquad+ \frac{L_f}{\Gamma(\alpha+1)}\bigl(2|t_2-t_1|^{\alpha} + \bigl|(t_2-t_0)^{\alpha} - (t_1-t_0)^{\alpha}\bigr|\bigr),\end{aligned}$$

which tends to zero independent of x as $t_2 \to t_1$. This implies that Ψ is equicontinuous on J. In consequence, it follows by the Arzela-Ascoli theorem that the operator Ψ is completely continuous. □

Our first existence result is based on Krasnoselskii's fixed point theorem (4.2).

Theorem 4.2 ([33]) *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that*

(i) *$Ax + By \in M$ whenever $x, y \in M$;*
(ii) *A is compact and continuous;*
(iii) *B is a contraction.*

Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 4.3 *Assume that*

(A_1) *For any $t \in J$ and $k = 0, 1, \dots, n-1$, there exist positive constants C_f and C_k such that*

$$\bigl|f(t,x) - f(t,y)\bigr| \le C_f|x-y|, \qquad \bigl|g_k(t,x) - g_k(t,y)\bigr| \le C_k|x-y|,$$

for all $x, y \in \mathbb{R}$, and we can find $\mu_f, \mu_k \in C(J, \mathbb{R}^+)$ such that

$$\bigl|f(t,x)\bigr| \le \mu_f(t), \qquad \bigl|g_k(t,x)\bigr| \le \mu_k(t), \quad \textit{for all } x \in \mathbb{R};$$

(A_2) *$\eta < 1$, where*

$$\eta = \sum_{k=0}^{n-1} \frac{(T-\theta)^k}{k!} \biggl[C_k(\theta - t_0) + \frac{C_f(\theta - t_0)^{\alpha-k}}{\Gamma(\alpha - k + 1)} \biggr].$$

Then the problem (1.1) *has at least one solution on J.*

Proof Let us define a set $B_r = \{x \in C(J, \mathbb{R}) : \|x\| \le r\}$, where r is a positive constant satisfying the inequality

$$r \ge \frac{|T-t_0|^\alpha}{\Gamma(\alpha+1)}\|\mu_f\| + \sum_{k=0}^{n-1} \frac{|T-\theta|^k}{k!} \biggl(|b_k| + \|\mu_k\| + \|\mu_f\| \frac{(\theta - t_0)^{\alpha-k}}{\Gamma(\alpha - k + 1)} \biggr).$$

Introduce the operators Θ and Φ on B_r as

$$(\Theta x)(t) = \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, x(s)\bigr)\, ds,$$

$$(\Phi x)(t) = \sum_{k=0}^{n-1} \frac{(t-\theta)^k}{k!} \biggl(b_k + \int_{t_0}^{\theta} g_k\bigl(s, x(s)\bigr)\, ds - \int_{t_0}^{\theta} \frac{(\theta - s)^{\alpha-k-1}}{\Gamma(\alpha - k)} f\bigl(s, x(s)\bigr)\, ds \biggr).$$

For $x, y \in B_r$, $t \in J$, using assumption (A_1), we find that

$$\begin{aligned} \bigl|\Theta x(t) + \Phi y(t)\bigr| &\le \frac{(T-t_0)^\alpha}{\Gamma(\alpha+1)}\|\mu_f\| \\ &\quad + \sum_{k=0}^{n-1} \frac{|T-\theta|^k}{k!} \biggl(|b_k| + \|\mu_k\| + \|\mu_f\| \frac{(\theta - t_0)^{\alpha-k}}{\Gamma(\alpha - k + 1)} \biggr) \le r. \end{aligned}$$

Thus, $\Theta x + \Phi y \in B_r$. By assumption ($\mathrm{A}_1$), for $x, y \in B_r$, $t \in J$, we have

$$\begin{aligned} \left|(\Phi x)(t) - (\Phi y)(t)\right| &\leq \sum_{k=0}^{n-1} \frac{(T-\theta)^k}{k!}\left[C_k(\theta - t_0) + \frac{C_f(\theta - t_0)^{\alpha-k}}{\Gamma(\alpha - k + 1)}\right]\|x - y\| \\ &\leq \eta\|x - y\|, \end{aligned}$$

that is, $\|(\Phi x) - (\Phi y)\| \leq \eta\|x - y\|$. Since $\eta < 1$ by (A_2), Φ is a contraction.

Continuity of f implies that the operator Θ is continuous. Also, Θ is uniformly bounded on B_r as

$$\|\Theta x\| \leq \frac{(T-t_0)^\alpha}{\Gamma(\alpha+1)}\|\mu_f\|.$$

Now we show the compactness of the operator Θ. In view of assumption (A_1), we define

$$\sup_{(t,x)\in J\times Br} \left|f(t,x)\right| = f_{\max} < \infty.$$

Then, for $t_1, t_2 \in J$, we have

$$\begin{aligned} \left|(\Theta x)(t_2) - (\Theta x)(t_1)\right| &= \left|\int_{t_0}^{t_1} \frac{[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]}{\Gamma(\alpha)} f\bigl(s, x(s)\bigr)\, ds \right. \\ &\quad \left. + \int_{t_1}^{t_2} \frac{(t_2 - s)^{q-1}}{\Gamma(\alpha)} f\bigl(s, x(s)\bigr)\, ds\right| \\ &\leq \frac{f_{\max}}{\Gamma(\alpha+1)}\bigl(2|t_2 - t_1|^\alpha + \left|(t_2 - t_0)^\alpha - (t_1 - t_0)^\alpha\right|\bigr), \end{aligned}$$

which is independent of x and tends to zero as $t_2 \to t_1$. So Θ is relatively compact on B_r. Hence, by the Arzela-Ascoli theorem, Θ is compact on B_r. Thus all the assumptions of Theorem 4.2 are satisfied. Therefore, the problem (1.1) has at least one solution on J. This completes the proof. □

Our next result deals with the uniqueness of solutions for the problem (1.1) and is based on the contraction mapping principle due to Banach.

Theorem 4.4 *Assume that* (A_1) *holds and that* $\beta < 1$, *where*

$$\beta = \sum_{k=0}^{n-1} \frac{(T-\theta)^k}{k!}\left(C_k(\theta - t_0) + \frac{(\theta - t_0)^{\alpha-k}}{\Gamma(\alpha - k + 1)}C_f\right) + \frac{(T-t_0)^\alpha}{\Gamma(\alpha+1)}C_f. \tag{4.3}$$

Then there exists a unique solution for the problem (1.1) *on* J.

Proof Setting $\sup_{t\in J}|f(t,0)| = A_f$, $\sup_{t\in J}|g_k(t,0)| = A_k$, and

$$r \geq (1-\beta)^{-1}\left[\sum_{k=0}^{n-1} \frac{(T-\theta)^k}{k!}\left(|b_k| + A_k(\theta - t_0) + \frac{(\theta - t_0)^{\alpha-k}}{\Gamma(\alpha - k + 1)}A_f\right) + \frac{(T-t_0)^\alpha}{\Gamma(\alpha+1)}A_f\right],$$

we show that $\Psi B_r \subset B_r$, where $B_r = \{x \in C(J,\mathbb{R}) : \|x\| \leq r\}$. For $x \in B_r$, we have

$$\begin{aligned}
\bigl|\Psi x(t)\bigr| &\leq \sum_{k=0}^{n-1} \frac{|t-\theta|^k}{k!}\left(|b_k| + \int_{t_0}^{\theta} \bigl|g_k\bigl(s,x(s)\bigr)\bigr|\,ds + \int_{t_0}^{\theta} \frac{(\theta-s)^{\alpha-k-1}}{\Gamma(\alpha-k)}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds\right) \\
&\quad + \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds \\
&\leq \sum_{k=0}^{n-1} \frac{|t-\theta|^k}{k!}\left(|b_k| + \int_{t_0}^{\theta} \bigl(\bigl|g_k\bigl(s,x(s)\bigr) - g_k(s,0)\bigr| + \bigl|g_k(s,0)\bigr|\bigr)\,ds\right. \\
&\quad \left. + \int_{t_0}^{\theta} \frac{(\theta-s)^{\alpha-k-1}}{\Gamma(\alpha-k)}\bigl(\bigl|f\bigl(s,x(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr|\bigr)\,ds\right) \\
&\quad + \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl(\bigl|f\bigl(s,x(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr|\bigr)\,ds \\
&\leq \sum_{k=0}^{n-1} \frac{|t-\theta|^k}{k!}\left(|b_k| + (\theta-t_0)\bigl(C_k\|x\| + A_k\bigr)\right. \\
&\quad \left. + \frac{(\theta-t_0)^{\alpha-k}}{\Gamma(\alpha-k+1)}\bigl(C_f\|x\| + A_f\bigr)\right) + \frac{(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}\bigl(C_f\|x\| + A_f\bigr) \\
&\leq \sum_{k=0}^{n-1} \frac{|T-\theta|^k}{k!}\left(|b_k| + A_k(\theta-t_0) + \frac{(\theta-t_0)^{\alpha-k}}{\Gamma(\alpha-k+1)}A_f\right) + \frac{(T-t_0)^{\alpha}}{\Gamma(\alpha+1)}A_f \\
&\quad + \left(\sum_{k=0}^{n-1} \frac{|T-\theta|^k}{k!}\left(C_k(\theta-t_0) + \frac{(\theta-t_0)^{\alpha-k}}{\Gamma(\alpha-k+1)}C_f\right) + \frac{(T-t_0)^{\alpha}}{\Gamma(\alpha+1)}C_f\right)\|x\| \\
&\leq (1-\beta)r + \beta r = r.
\end{aligned}$$

Now, for $x, y \in C(J,\mathbb{R})$ and for each $t \in J$, we obtain

$$\begin{aligned}
&\bigl|(\Psi x)(t) - (\Psi y)(t)\bigr| \\
&\quad \leq \sum_{k=0}^{n-1} \frac{|t-\theta|^k}{k!}\left(C_k(\theta-t_0)\|x-y\| + \frac{(\theta-t_0)^{\alpha-k}}{\Gamma(\alpha-k+1)}C_f\|x-y\|\right) \\
&\qquad + \frac{(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}C_f\|x-y\| \\
&\quad \leq \left(\sum_{k=0}^{n-1} \frac{|T-\theta|^k}{k!}\left(C_k(\theta-t_0) + \frac{(\theta-t_0)^{\alpha-k}}{\Gamma(\alpha-k+1)}C_f\right) + \frac{(T-t_0)^{\alpha}}{\Gamma(\alpha+1)}C_f\right)\|x-y\| \\
&\quad \leq \beta\|x-y\|,
\end{aligned}$$

where β is given by (4.3). As $\beta < 1$, Ψ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. This completes the proof. □

Theorem 4.5 *Let $f, g_k : J \times \mathbb{R} \to \mathbb{R}$ $(k = 0,1,\dots,n-1)$ be continuous functions and let there exist positive constants D_f, E_f, D_k, E_k, M, and N such that*

$$\bigl|f(t,x)\bigr| \leq D_f|x| + E_f, \qquad \bigl|g_k(t,x)\bigr| \leq D_k|x| + E_k, \quad \forall t \in J, x \in \mathbb{R},$$

$$M = \frac{E_f(T-t_0)^\alpha}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1} \frac{|T-\theta|^k}{k!}\left(|b_k| + (\theta-t_0)E_k + \frac{E_f(\theta-t_0)^{\alpha-k}}{\Gamma(\alpha-k+1)}\right) > 0,$$

$$N = \frac{D_f(T-t_0)^\alpha}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1} \frac{|T-\theta|^k}{k!}\left((\theta-t_0)D_k + \frac{D_f(\theta-t_0)^{\alpha-k}}{\Gamma(\alpha-k+1)}\right) < 1.$$

Then the problem (1.1) *has at least one solution on* J.

Proof Define a suitable ball $B_R \subset C(J,\mathbb{R})$ with radius $R>0$ as $B_R = \{x \in C(J,\mathbb{R}) : \|x\| < R\}$, where R will be fixed later. Then it is sufficient to show that $\Psi : \overline{B}_R \to C(J,\mathbb{R})$ satisfies

$$0 \notin (I-\lambda\Psi)(\partial B_R), \tag{4.4}$$

for any $x \in \partial B_R$, and $\lambda \in [0,1]$. Define the homotopy

$$h_\lambda(x) = H(\lambda,x) = x - \lambda\Psi x, \quad x \in C(J,\mathbb{R}), \lambda \in [0,1].$$

Then, by the Arzela-Ascoli theorem, h_λ is completely continuous. If (4.4) is true, then the Leray-Schauder degrees are well defined. Let I denote the identity operator. Then the homotopy invariance and normalization properties of topological degrees imply that

$$\begin{aligned}\deg(h_\lambda,B_R,0) &= \deg\bigl((I-\lambda\Psi),B_R,0\bigr) = \deg(h_1,B_R,0)\\ &= \deg(h_0,B_R,0) = \deg(I,B_R,0) = 1,\end{aligned}$$

since $0 \in B_R$. By the nonzero property of the Leray-Schauder degree, $h_1(x) = x - \Psi x = 0$ for at least one $x \in B_R$. In order to find R, we assume that $x(t) = \lambda\Psi x(t)$ for some $\lambda \in [0,1]$ and for all $t \in J$. Then

$$\begin{aligned}
&\bigl|x(t)\bigr| \\
&\quad = \bigl|\lambda\Psi x(t)\bigr| \le \int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds \\
&\qquad + \sum_{k=0}^{n-1}\frac{(t-\theta)^k}{k!}\left(|b_k| + \int_{t_0}^\theta \bigl|g_k\bigl(s,x(s)\bigr)\bigr|\,ds + \int_{t_0}^\theta \frac{(\theta-s)^{\alpha-k-1}}{\Gamma(\alpha-k)}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds\right) \\
&\quad \le \sum_{k=0}^{n-1}\frac{|t-\theta|^k}{k!}\left(|b_k| + (\theta-t_0)\bigl(D_k\|x\| + E_k\bigr) + \frac{(\theta-t_0)^{\alpha-k}}{\Gamma(\alpha-k+1)}\bigl(D_f\|x\| + E_f\bigr)\right) \\
&\qquad + \frac{(t-t_0)^\alpha}{\Gamma(\alpha+1)}\bigl(D_f\|x\| + E_f\bigr) \\
&\quad \le \frac{E_f(t-t_0)^\alpha}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1}\frac{|t-\theta|^k}{k!}\left(|b_k| + (\theta-t_0)E_k + \frac{E_f(\theta-t_0)^{\alpha-k}}{\Gamma(\alpha-k+1)}\right) \\
&\qquad + \left(\frac{D_f(t-t_0)^\alpha}{\Gamma(\alpha+1)} + \sum_{k=0}^{n-1}\frac{|t-\theta|^k}{k!}\left((\theta-t_0)D_k + \frac{D_f(\theta-t_0)^{\alpha-k}}{\Gamma(\alpha-k+1)}\right)\right)\|x\| \\
&\quad \le M + N\|x\|,
\end{aligned}$$

which, after taking the supremum norm and solving for $\|x\|$, yields

$$\|x\| \leq \frac{M}{1-N}.$$

Letting $R = \frac{M-N+1}{1-N}$, (4.4) holds. This completes the proof. □

Remark 4.6 Following the method of proof employed in this section, we can obtain the existence results for nonlinear variants of problems (3.8) and (3.9).

Remark 4.7 (Special cases) We obtain the existence results for an initial value problem with initial conditions: $x^{(k)}(\theta) = b_k$, $k = 0,1,2,\dots,n-1$ by taking $\theta = t_0$ in the results of this paper. In this case, the operator given by (4.2) takes the form

$$\Psi x(t) = \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} b_k + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f\bigl(s,x(s)\bigr)\,ds.$$

Further, our results correspond to the ones for a problem with classical nonlinear integral conditions:

$$x^{(k)}(T) = b_k + \int_{t_0}^{T} g_k\bigl(s,x(s)\bigr)\,ds, \quad k = 0,1,2,\dots,n-1,$$

if we fix $\theta = T$ in the obtained results.

Example 4.8 Consider the following nonlinear fractional boundary value problem:

$$\begin{cases} {}^{C}D_0^{\sqrt{10}} x(t) = \frac{t|x(t)|}{16(1+|x(t)|)} + \frac{7}{16}, & t \in [0,1], \\ x^{(k)}(0.5) = 1 + \int_0^{0.5} \frac{s^k}{2(k+1)} \sin(\frac{x(s)}{2})\,ds, & k = 0,1,2,3. \end{cases} \tag{4.5}$$

Here $\alpha = \sqrt{10}$, $\theta = 0.5$, $b_k = 1$, $f(t,x(t)) = \frac{t|x(t)|}{16(1+|x(t)|)} + \frac{7}{16}$, and $g_k(t,x(t)) = \frac{t^k}{2(k+1)} \sin(\frac{x(t)}{2})$. With the given values, it is found that $\mu_f(t) = \frac{t}{16} + \frac{7}{16}$, $\mu_k(t) = \frac{t^k}{2(k+1)}$ with $\|\mu_f\| = \frac{1}{2}$, $\|\mu_k\| = \frac{1}{2(k+1)}$, $k = 0,1,2,3$, and

$$\eta = \sum_{k=0}^{3} \frac{1}{k!2^k} \left[\frac{1}{4(k+1)} + \frac{(\frac{1}{2})^{\sqrt{10}-k}}{16\Gamma(\sqrt{10}-k+1)} \right] \simeq 0.333942 < 1,$$

that is, the assumption (A_2) of Theorem 4.3 is satisfied. Thus, all the conditions of Theorem 4.3 are satisfied. Hence the problem (4.5) has a solution on $[0,1]$. Also β given by (4.3) is such that $\beta \simeq 0.342407 < 1$. This suggests that the problem (4.5) has a unique solution by the conclusion of Theorem 4.4.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, BA, MM, and RPA contributed to each part of this work equally and read and approved the final version of the manuscript.

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