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Study on the generalized (p,q)-Laplacian elliptic systems, parabolic systems and integro-differential systems

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Abstract

In this paper, we present the abstract results for the existence and uniqueness of the solution of nonlinear elliptic systems, parabolic systems and integro-differential systems involving the generalized (p,q)-Laplacian operator. Our method makes use of the characteristics of the ranges of linear and nonlinear maximal monotone operators and the subdifferential of a proper, convex, and lower-semi-continuous functional, and we employ some new techniques in the construction of the operators and in proving the properties of the newly defined operators. The systems discussed in this paper and the method used extend and complement some of the previous work.

MSC: 47H05; 47H09

Keywords: maximal monotone operator; coercive; (p,q)-Laplacian; parabolic systems; elliptic systems; integro-differential systems

1 Introduction and preliminaries

1.1 Introduction

Nonlinear boundary value problems involving the generalized p-Laplacian operator arise from many physical phenomena, such as reaction-diffusion problems, petroleum extraction, flow through porous media and non-Newtonian fluids, just to name a few. Hence, the study of such problems and their generalizations have attracted numerous attention in recent years. For example, based on Calvert and Gupta's [1] result on perturbations of the ranges of m-accretive mappings (stated as Theorem 1.1 in Section 1.2), Wei and Agarwal [2] have studied the following nonlinear elliptic boundary value problem involving the generalized p-Laplacian:

$$\begin{cases} -\operatorname{div}[(C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u]+\varepsilon|u|^{q-2}u+g(x,u(x))=f(x), & \text{a.e. in } \Omega, \\ -\langle\vartheta,(C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle\in\beta_x(u(x)), & \text{a.e. on } \Gamma, \end{cases} \tag{1.1}$$

where $0\le C(x)\in L^p(\Omega)$, β_x is the subdifferential of a proper, convex, and lower-semi-continuous function, ε is a non-negative constant and ϑ denotes the exterior normal derivative of Γ. It is shown that (1.1) has solutions in $L^s(\Omega)$ under some conditions, where $\frac{2N}{N+1}<p\le s<+\infty$, $1\le q<+\infty$ if $p\ge N$, and $1\le q\le\frac{Np}{N-p}$ if $p<N$, for $N\ge 1$.

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Recently, the work on the generalized p-Laplacian operator problem (1.1) is extended to the so-called p-Laplacian-like problem

$$\begin{cases} -\operatorname{div}[(C(x)+|\nabla u|^2)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u]+\varepsilon|u|^{q-2}u+g(x,u(x))=f(x), & \text{in } \Omega, \\ -\langle\vartheta,(C(x)+|\nabla u|^2)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u\rangle\in\beta_x(u), & \text{on } \Gamma. \end{cases} \tag{1.2}$$

Using Theorem 1.1 again, it is shown in [3] that (1.2) has solutions in $L^p(\Omega)$ under some conditions, where $\frac{2N}{N+1}<p<+\infty$, $1\le q<+\infty$ if $p\ge N$, and $1\le q\le\frac{Np}{N-p}$ if $p<N$, for $N\ge 1$.

Since one system, expressed by one equation, interacts with another system in reality, the study of nonlinear systems with (p,q)-Laplacian is also an important topic. In the non-Newtonian theory, the quantity (p,q) is a characteristic of the medium. Media with $(p,q)>(2,2)$ are called *dilatant fluids*, those with $(p,q)<(2,2)$ are called *pseudodoplastics*, and if $(p,q)=(2,2)$, they are called *Newtonian fluids*. The studies on the p-Laplacian boundary value problems have been extended to cases of nonlinear Neumann elliptic systems with (p,q)-Laplacian. For example, in [4] the following system with Neumann boundaries has been discussed:

$$\begin{cases} -\Delta_p u+\varepsilon_1|u|^{p-2}u+g(x,u(x),v(x))=f_1(x), & \text{a.e. in } \Omega, \\ -\Delta_q v+\varepsilon_2|v|^{q-2}v+g(x,v(x),u(x))=f_2(x), & \text{a.e. in } \Omega, \\ -\langle\vartheta,|\nabla u|^{p-2}\nabla u\rangle\in\beta_x(u(x)), & \text{a.e. on } \Gamma, \\ -\langle\vartheta,|\nabla v|^{q-2}\nabla v\rangle\in\beta_x(v(x)), & \text{a.e. on } \Gamma. \end{cases} \tag{1.3}$$

Inspired by Theorem 1.1 again, a sufficient condition on the existence of a solution in $L^p(\Omega)\times L^q(\Omega)$ is presented in [4].

On the other hand, based on Brezis' result [5] (stated as Theorem 1.2 in Section 1.2), Wei *et al.* [6] have studied the following nonlinear Dirichlet elliptic system in $W^{1,p}(\Omega)\times W^{1,q}(\Omega)$:

$$\begin{cases} -\Delta_p u+\varepsilon_1|u|^{p-2}u-\Delta_q v+\varepsilon_2|v|^{q-2}v=f_1(x)+f_2(x), & \text{a.e. in } \Omega, \\ \gamma_1 u=g_1(x), \qquad \gamma_2 v=g_2(x), & \text{a.e. on } \Gamma, \end{cases} \tag{1.4}$$

and then extend (1.4) to the following two cases with generalized (p,q)-Laplacian:

$$\begin{cases} -\operatorname{div}(\alpha_1(\operatorname{grad}u))+\varepsilon_1|u|^{p-2}u-\operatorname{div}(\alpha_2(\operatorname{grad}v))+\varepsilon_2|v|^{q-2}v \\ \quad =f_1(x)+f_2(x), \quad \text{a.e. in } \Omega, \\ \gamma_1 u=g_1(x), \qquad \gamma_2 v=g_2(x), \quad \text{a.e. on } \Gamma \end{cases} \tag{1.5}$$

and

$$\begin{cases} -\operatorname{div}[(C_1(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u]+\varepsilon_1|u|^{p-2}u-\operatorname{div}[(C_2(x)+|\nabla v|^2)^{\frac{q-2}{2}}\nabla v]+\varepsilon_2|v|^{q-2}v \\ \quad =f_1(x)+f_2(x), \quad \text{a.e. in } \Omega, \\ \gamma_1 u=g_1(x), \qquad \gamma_2 v=g_2(x), \quad \text{a.e. on } \Gamma. \end{cases} \tag{1.6}$$

Integro-differential equation is also a much-studied topic in applied mathematics. Most of the existing techniques used to discuss the existence and uniqueness of the solution to integro-differential equation involves the finite element method. In [7], a new method based on a result of Zeidler [8] (stated as Theorem 1.3 in Section 1.2) is employed to tackle the following nonlinear integro-differential equation involving the generalized p-Laplacian operator with mixed boundary conditions:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}[(C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u] + \varepsilon |u|^{q-2}u + a\frac{\partial}{\partial t}\int_\Omega u\,dx \\ \quad = f(x,t), \quad (x,t) \in \Omega \times (0,T), \\ -\langle \vartheta, (C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u\rangle \in \beta_x(u), \quad (x,t) \in \Gamma \times (0,T), \\ u(x,0) = u(x,T), \quad x \in \Omega. \end{cases} \tag{1.7}$$

It is proved that (1.7) has a unique solution in $L^p(0,T;W^{1,p}(\Omega))$, where $1 < q \le p < +\infty$.

Inspired by the work on (1.7), the following nonlinear integro-differential system involving the generalized (p,q)-Laplacian is investigated in [9]:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \operatorname{div}[(C_1(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u] + \varepsilon_1 |u|^{r-2}u + g_1(x,u,\nabla u) + a_1\frac{\partial}{\partial t}\int_\Omega u\,dx \\ \quad = f_1(x,t), \quad (x,t) \in \Omega \times (0,T), \\ \frac{\partial v(x,t)}{\partial t} - \operatorname{div}[(C_2(x,t) + |\nabla v|^2)^{\frac{q-2}{2}} \nabla v] + \varepsilon_2 |v|^{s-2}v + g_2(x,v,\nabla v) + a_2\frac{\partial}{\partial t}\int_\Omega v\,dx \\ \quad = f_2(x,t), \quad (x,t) \in \Omega \times (0,T), \\ -\langle \vartheta, (C_1(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u\rangle \in \beta_x(u), \quad (x,t) \in \Gamma \times (0,T), \\ -\langle \vartheta, (C_2(x,t) + |\nabla v|^2)^{\frac{q-2}{2}} \nabla v\rangle \in \beta_x(v), \quad (x,t) \in \Gamma \times (0,T), \\ u(x,0) = u(x,T), \qquad v(x,0) = v(x,T), \quad x \in \Omega, \end{cases} \tag{1.8}$$

where $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N})$ and $x = (x_1, x_2, \ldots, x_N) \in \Omega$. Based on a result of [10] (stated as Theorem 1.4 in Section 1.2), the existence of the unique non-trivial solution of (1.8) in $L^p(0,T;W^{1,p}(\Omega)) \times L^q(0,T;W^{1,q}(\Omega))$ is presented, where $N \ge 1$, $\frac{2N}{N+1} < r \le \min\{p,p'\} < +\infty$, and $\frac{2N}{N+1} < s \le \min\{q,q'\} < +\infty$. (Here, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$.)

Parabolic equations are equally important as elliptic equations and integro-differential equations. The generalized (p,q)-Laplacian parabolic equation with mixed boundaries has been extensively studied in [11],

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}[(C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u] + \varepsilon |u|^{p-2}u = f(x,t), \quad (x,t) \in \Omega \times (0,T), \\ -\langle \vartheta, (C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u\rangle \in \beta(u) - h(x,t), \quad (x,t) \in \Gamma \times (0,T), \\ u(x,0) = u(x,T), \quad x \in \Omega. \end{cases} \tag{1.9}$$

It is shown that (1.9) has a unique solution in $L^p(0,T;W^{1,p}(\Omega))$ where $p \ge 2$. The discussion of (1.9) in [11] is mainly based on Theorem 1.2 and a result of Reich [12] (stated as Theorem 1.5 in Section 1.2).

From the above research, we notice that it is not easy to check the assumptions presented in Theorems 1.1-1.5. As such we are motivated to extend the previous work to new problems and also to simplify the proof of the result. Indeed, motivated by the systems (1.4)-(1.6), (1.8), and (1.9), in this paper we shall employ a result of Zeidler [8] (stated as Theorem 1.6 in Section 1.2) as the main tool to obtain sufficient conditions for the existence and uniqueness of solutions for *three* nonlinear systems - the first is a nonlinear *elliptic* system involving the generalized (p,q)-Laplacian with Neumann boundaries, the second is a nonlinear *parabolic* system involving the generalized (p,q)-Laplacian with mixed boundaries, and the third is a nonlinear *integro-differential* system involving the generalized (p,q)-Laplacian with mixed boundaries. The three systems considered are as follows:

$$\begin{cases} -\operatorname{div}[(C_1(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u]+\varepsilon_1|u|^{r-2}u-\operatorname{div}[(C_2(x)+|\nabla v|^2)^{\frac{q-2}{2}}\nabla v] \\ \qquad +\varepsilon_2|v|^{s-2}v+g_1(x,u,\nabla u)+g_2(x,v,\nabla v) \\ \quad = f_1(x)+f_2(x), \quad x\in\Omega, \\ -\langle\vartheta,(C_1(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle-\langle\vartheta,(C_2(x)+|\nabla v|^2)^{\frac{q-2}{2}}\nabla v\rangle \\ \quad \in\beta_x(u)+\beta_x(v), \quad x\in\Gamma; \end{cases} \tag{1.10}$$

$$\begin{cases} \frac{\partial u(x,t)}{\partial t}-\operatorname{div}[(C_1(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u]+\varepsilon_1|u|^{r-2}u+g_1(x,u,\nabla u)+\frac{\partial v(x,t)}{\partial t} \\ \qquad -\operatorname{div}[(C_2(x,t)+|\nabla v|^2)^{\frac{q-2}{2}}\nabla v]+\varepsilon_2|v|^{s-2}v+g_2(x,v,\nabla v) \\ \quad = f_1(x,t)+f_2(x,t), \quad (x,t)\in\Omega\times(0,T), \\ -\langle\vartheta,(C_1(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle-\langle\vartheta,(C_2(x,t)+|\nabla v|^2)^{\frac{q-2}{2}}\nabla v\rangle \\ \quad \in\beta_x(u)+\beta_x(v), \quad (x,t)\in\Gamma\times(0,T), \\ u(x,0)=u(x,T), \qquad v(x,0)=v(x,T), \quad x\in\Omega; \end{cases} \tag{1.11}$$

$$\begin{cases} \frac{\partial u(x,t)}{\partial t}-\operatorname{div}[(C_1(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u]+\varepsilon_1|u|^{r-2}u+g_1(x,u,\nabla u) \\ \qquad +a_1\frac{\partial}{\partial t}\int_\Omega u\,dx+\frac{\partial v(x,t)}{\partial t}-\operatorname{div}[(C_2(x,t)+|\nabla v|^2)^{\frac{q-2}{2}}\nabla v] \\ \qquad +\varepsilon_2|v|^{s-2}v+g_2(x,v,\nabla v)+a_2\frac{\partial}{\partial t}\int_\Omega v\,dx \\ \quad = f_1(x,t)+f_2(x,t), \quad (x,t)\in\Omega\times(0,T), \\ -\langle\vartheta,(C_1(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle-\langle\vartheta,(C_2(x,t)+|\nabla v|^2)^{\frac{q-2}{2}}\nabla v\rangle \\ \quad \in\beta_x(u)+\beta_x(v), \quad (x,t)\in\Gamma\times(0,T), \\ u(x,0)=u(x,T), \qquad v(x,0)=v(x,T), \quad x\in\Omega. \end{cases} \tag{1.12}$$

The investigation of systems (1.10)-(1.12) will be presented in Sections 2-4, respectively, and more details of these systems will be introduced in these sections. Finally, in Section 5 we shall present some examples of (1.10)-(1.12).

1.2 Preliminaries

Let X be a real Banach space with its dual X^* being strictly convex. We shall use $(\cdot,\cdot)$ to denote the *generalized duality pairing* between X and X^*. For any subset G of X, we denote by $\operatorname{int} G$ its *interior* and $\overline{G}$ its *closure*, respectively. For two subsets G_1 and G_2 in X, if $\overline{G_1}=\overline{G_2}$ and $\operatorname{int} G_1=\operatorname{int} G_2$, then we say that G_1 is *almost equal* to G_2, denoted by $G_1\simeq G_2$. We use 'w-lim' to denote the *weak convergence*. A mapping $T:D(T)=X\to X^*$ is said to be *hemi-continuous* on X [13] if $w\text{-}\lim_{t\to 0}T(x+ty)=Tx$, for any $x,y\in X$. A mapping $T:D(T)=X\to X^*$ is said to be *demi-continuous* on X [13] if $w\text{-}\lim_{n\to\infty}Tx_n=Tx$, for any sequence $\{x_n\}$ strongly converges to x in X.

Let J_r denote the *duality mapping* from X into 2^{X^*}, which is defined by

$$J_r(x)=\{f\in X^*:(x,f)=\|x\|^r,\|f\|=\|x\|^{r-1}\}, \quad x\in X,$$

where $r>1$ is a constant. If $r\equiv 2$, then we use J to denote J_2, which is called the *normalized duality mapping*. It is well known that, in general, $J_r(x)=\|x\|^{r-2}J(x)$, for all $x\neq 0$. Since X^* is strictly convex, J is a single-valued mapping [1, 14].

A multi-valued mapping $B:X\to 2^{X^*}$ is said to be *monotone* [14] if $(u_1-u_2,w_1-w_2)\geq 0$, for any $u_i\in D(B)$ and $w_i\in Bu_i$, $i=1,2$. The monotone operator B is said to be *maximal monotone* if $R(J+rB)=X^*$, for any $r>0$. The mapping $B:X\to 2^{X^*}$ is said to be *strictly monotone* [14] if $(u_1-u_2,w_1-w_2)=0$, for $w_i\in Bu_i$, $i=1,2$, implies $u_1=u_2$. The mapping B

is said to be *coercive* [13, 14] if $\lim_{n\to+\infty}(x_n, x_n^*)/\|x_n\| = +\infty$ for all $x_n \in D(B)$, $x_n^* \in Bx_n$ such that $\lim_{n\to+\infty}\|x_n\| = +\infty$.

Let $B : X \to 2^{X^*}$ be a maximal monotone operator such that $0 \in B0$, then the equation $J(u_t - u) + tBu_t \ni 0$ has a unique solution $u_t \in D(B)$ for every $u \in X$ and $t > 0$. The *resolvent* J_t^B and the *Yosida approximation* B_t of B are defined by $J_t^B u = u_t$ and $B_t u = -\frac{1}{t}J(u_t - u)$ for all $u \in X$ and $t > 0$ [14].

For $k \in (-\infty, +\infty)$, a multi-valued mapping $\widetilde{A} : D(\widetilde{A}) \subset X \to 2^X$ is said to be *k-accretive* [10] if

$$\bigl(v_1 - v_2, J(u_1 - u_2)\bigr) \geq k\|u_1 - u_2\|^2, \tag{1.13}$$

for any $u_i \in D(\widetilde{A})$ and $v_i \in \widetilde{A}u_i$, $i = 1, 2$. For $k > 0$ in inequality (1.13), we say that $\widetilde{A}$ is *strongly accretive* while for $k = 0$, $\widetilde{A}$ is simply called *accretive*. An accretive mapping $\widetilde{A}$ is said to be *m-accretive* if $R(I + \lambda\widetilde{A}) = X$ for some $\lambda > 0$. We say that a mapping $\widetilde{A} : X \to 2^X$ is *boundedly-inversely-compact* [1] if, for any pair of bounded subsets G and G' of X, the subset $G \cap \widetilde{A}^{-1}(G')$ is relatively compact in X.

Let C be a closed convex subset of X and let $A : C \to 2^{X^*}$ be a multi-valued mapping. Then A is said to be a *pseudo-monotone* operator [14] provided that

(i) for each $x \in C$, the image Ax is a non-empty closed and convex subset of X^*;

(ii) if $\{x_n\}$ is a sequence in C converging weakly to $x \in C$ and if $f_n \in Ax_n$ is such that $\limsup_{n\to\infty}(x_n - x, f_n) \leq 0$, then to each element $y \in C$, there corresponds an $f(y) \in Ax$ with the property that $(x - y, f(y)) \leq \liminf_{n\to\infty}(x_n - x, f_n)$;

(iii) for each finite-dimensional subspace F of X, the operator A is continuous from $C \cap F$ to X^* in the weak topology.

A function Φ is called a *proper convex* function on X [14] if Φ is defined from X to $(-\infty, +\infty]$, not identically $+\infty$, such that $\Phi((1-\lambda)x + \lambda y) \leq (1-\lambda)\Phi(x) + \lambda\Phi(y)$, whenever $x, y \in X$ and $0 \leq \lambda \leq 1$.

A function $\Phi : X \to (-\infty, +\infty]$ is said to be *lower-semi-continuous* on X [14] if $\liminf_{y\to x}\Phi(y) \geq \Phi(x)$, for any $x \in X$.

Given a proper convex function Φ on X and a point $x \in X$, we denote by $\partial\Phi(x)$ the set of all $x^* \in X^*$ such that $\Phi(x) \leq \Phi(y) + (x - y, x^*)$, for any $y \in X$. Such element x^* is called the *subgradient* of Φ at x, and $\partial\Phi(x)$ is called the *subdifferential* of Φ at x [14].

For easy reference of the reader, Theorems 1.1-1.5 mentioned in Section 1.1 are stated as follows.

Theorem 1.1 [1] *Let X be a real Banach space with a strictly convex dual space X^*. Let $J_r : X \to X^*$ be a duality mapping on X and there exists a function $\eta : X \to [0, +\infty)$ such that for all $u, v \in X$,*

$$\|J_r u - J_r v\| \leq \eta(u - v). \tag{1.14}$$

Let $A, C_1 : X \to 2^X$ be accretive mappings such that

(i) *either both A and C_1 satisfy the following condition* (1.15), *or $D(A) \subset D(C_1)$ and C_1 satisfies the condition* (1.15):

$$\begin{cases} \text{for } u \in D(A) \text{ and } v \in Au, \text{there exists a constant } C(a,f) \text{ such that} \\ (v - f, J_r(u - a)) \geq C(a,f); \end{cases} \tag{1.15}$$

(ii) $A + C_1$ *is m-accretive and boundedly-inversely-compact.*

Let $C_2 : X \to X$ *be a bounded continuous mapping such that, for any* $y \in X$, *there is a constant* $C(y)$ *satisfying* $(C_2(u + y), J_r u) \geq -C(y)$ *for any* $u \in X$. *Then the following results hold*:

(a) $\overline{[R(A) + R(C_1)]} \subset \overline{R(A + C_1 + C_2)}$;

(b) $\operatorname{int}[R(A) + R(C_1)] \subset \operatorname{int} R(A + C_1 + C_2)$.

Theorem 1.2 [5] *Let* $T : X \to X^*$ *be a bounded and pseudo-monotone operator, and* K *be a closed and convex subset of* X. *Suppose that* Φ *is a lower-semi-continuous and convex function defined on* K, *which is not always* $+\infty$, *such that* $\Phi(v) \in (-\infty, +\infty]$, *for any* $v \in K$. *Suppose there exists* $v_0 \in K$ *such that* $\Phi(v_0) < +\infty$, *and*

$$\frac{(v - v_0, Tv) + \Phi(v)}{\|v\|} \to \infty,$$

as $\|v\| \to \infty$, $v \in K$. *Then there exists* $u \in K$ *such that* $(u - v, Tu) \leq \Phi(v) - \Phi(u)$, *for all* $v \in K$.

Theorem 1.3 [8] *Let* X *be a real reflexive Banach space with* X^* *being its dual space. Let* C *be a non-empty closed convex subset of* X. *Assume that*

(i) *the mapping* $A : C \to 2^{X^*}$ *is a maximal monotone operator*;

(ii) *the mapping* $B : C \to X^*$ *is pseudo-monotone, bounded, and demi-continuous*;

(iii) *if the subset* C *is unbounded, then the operator* B *is* A*-coercive with respect to the fixed element* $b \in X^*$, *i.e., there exist an element* $u_0 \in C \cap D(A)$ *and a number* $r > 0$ *such that*

$$(u - u_0, Bu) > (u - u_0, b), \tag{1.16}$$

for all $u \in C$ *with* $\|u\| > r$.

Then the equation $b \in Au + Bu$ *has a solution.*

Theorem 1.4 [10] *Let* X *be a smooth Banach space,* $A : D(A) \subset X \to 2^X$ *be an m-accretive mapping, and* $S : D(S) \subset X \to X$ *be continuous and strongly accretive with* $\overline{D(A)} \subset D(S)$. *Then, for any* $z \in X$, *the equation* $z \in Sx + \lambda Ax$ *has a unique solution* x_λ, *for any* $\lambda > 0$.

Theorem 1.5 [12] *Let* X *be a real reflexive Banach space with both* X *and* X^* *being strictly convex. Let* $J : X \to X^*$ *be the normalized duality mapping on* X. *Let* A *and* B *be two maximal monotone operators in* X. *If there exist* $0 \leq k < 1$ *and* $C_1, C_2 > 0$ *such that*

$$\bigl(a, J^{-1}(B_t v)\bigr) \geq -k\|B_t v\|^2 - C_1\|B_t v\| - C_2, \tag{1.17}$$

for any $v \in D(A)$, $a \in Av$ *and* $t > 0$ (B_t *is the Yosida approximation of* B), *then* $R(A) + R(B) \simeq R(A + B)$.

The following results will be needed in subsequent discussion.

Lemma 1.1 [14] *If* A *and* B *are maximal monotone operators in* X *such that* $(\operatorname{int} D(A)) \cap D(B) \neq \emptyset$, *then* $A + B$ *is maximal monotone.*

Lemma 1.2 [14] *If $\Phi : X \to R$ is proper, convex, and lower-semi-continuous, then $\partial\Phi$ is maximal monotone.*

Lemma 1.3 [14] *If $B : X \to 2^{X^*}$ is everywhere defined, monotone, and hemi-continuous, then B is maximal monotone.*

Theorem 1.6 [8] *Assume that X is a real reflexive Banach space and the following conditions hold*:

(H1) *The linear operator $L : D(L) \subseteq X \to X^*$ is maximal monotone in X.*

(H2) *The operator $A : X \to 2^{X^*}$ is monotone.*

(H3) *The functional $\varphi : X \to (-\infty, +\infty]$ is convex, lower-semi-continuous, and $\varphi \not\equiv +\infty$.*

(H4) *One of the following conditions is satisfied*:

  (H4.1) *$A : X \to X^*$ is single-valued and hemi-continuous*;

  (H4.2) *A is maximal monotone and* $\operatorname{int} D(A) \cap D(\partial\varphi) \neq \emptyset$;

  (H4.3) *A is maximal monotone and $D(A) \cap \operatorname{int} D(\partial\varphi) \neq \emptyset$.*

(H5) *The sum $L + A + \partial\varphi : X \to 2^{X^*}$ is coercive with respect to* 0, *i.e., there exist $r > 0$ and $u_0 \in D(L) \cap D(A) \cap D(\partial\varphi)$ such that*

$$\left(u - u_0, u^*\right) > 0,$$

for all $(u, u^*) \in L + A + \partial\varphi$ with $\|u\| > r$.

(H6) *$D(L) \cap D(A + \partial\varphi) \neq \emptyset$.*

Then the equation

$$0 \in Lu + Au + \partial\varphi(u), \quad u \in X,$$

has a solution.

Definition 1.1 For $1 < p < +\infty$ and $1 < q < +\infty$, we use Y to denote the product of two spaces $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$, *i.e.*, $Y = W^{1,p}(\Omega) \times W^{1,q}(\Omega) = \{(u, v) : u \in W^{1,p}(\Omega), v \in W^{1,q}(\Omega)\}$. The dual space of Y will be denoted by Y^*. Also, Y will be endowed with the norm

$$\left\|(u, v)\right\|_Y = \sqrt{\|u\|_{1,p,\Omega}^2 + \|v\|_{1,q,\Omega}^2}, \quad (u, v) \in Y,$$

where $\|\cdot\|_{1,p,\Omega}$ and $\|\cdot\|_{1,q,\Omega}$ denote the norm in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$, respectively.

Definition 1.2 [15] For $1 < p < +\infty$, let $L^p(0, T; X)$ denote the space of all X-valued strongly measurable functions $x(t)$ defined a.e. on $(0, T)$ such that $\|x(t)\|_X^p$ is Lebesgue integrable over $(0, T)$. It is well known that $L^p(0, T; X)$ is a Banach space with the norm defined by $\|x\|_{L^p(0,T;X)} = (\int_0^T \|x(t)\|_X^p \, dt)^{\frac{1}{p}}$. If X is reflexive, then $L^p(0, T; X)$ is reflexive, and its dual space coincides with $L^{p'}(0, T; X^*)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, $L^p(0, T; X)$ is reflexive in the case when X is reflexive, and $L^p(0, T; X)$ is strictly (uniformly) convex in the case when X is strictly (uniformly) convex.

Definition 1.3 For $1 < p < +\infty$ and $1 < q < +\infty$, we use Z to denote the product of two spaces $L^p(0, T; W^{1,p}(\Omega))$ and $L^q(0, T; W^{1,q}(\Omega))$, *i.e.*, $Z = L^p(0, T; W^{1,p}(\Omega)) \times L^q(0, T;$

$W^{1,q}(\Omega)) = \{(u,v) : u \in L^p(0,T;W^{1,p}(\Omega)), v \in L^q(0,T;W^{1,q}(\Omega))\}$. The dual space of Z is denoted by Z^*. Also, Z will be endowed with the norm

$$\|(u,v)\|_Z = \sqrt{\|u\|^2_{L^p(0,T;W^{1,p}(\Omega))} + \|v\|^2_{L^q(0,T;W^{1,q}(\Omega))}}, \quad (u,v) \in Z.$$

2 Discussion of (p,q)-Laplacian elliptic system (1.10)

Throughout the paper, we shall assume that

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad \frac{1}{q} + \frac{1}{q'} = 1, \qquad N \ge 1, \qquad \frac{2N}{N+1} < p < +\infty, \qquad \frac{2N}{N+1} < q < +\infty,$$
$$\frac{2N}{N+1} < r \le \min\{p,p'\} < +\infty, \qquad \frac{2N}{N+1} < s \le \min\{q,q'\} < +\infty.$$

In (1.10)-(1.12), Ω is a bounded conical domain of the Euclidean space $\mathbb{R}^N$ with its boundary $\Gamma \in C^1$, ϑ denotes the exterior normal derivative to Γ, and ε_1 and ε_2 are non-negative constants. Let $\varphi : \Gamma \times \mathbb{R} \to \mathbb{R}$ be a given function such that, for each $x \in \Gamma$, $\varphi_x = \varphi(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is a proper, convex, and lower-semi-continuous function with $\varphi_x(0) = 0$. Let β_x be the subdifferential of φ_x, *i.e.*, $\beta_x \equiv \partial\varphi_x$. Suppose $0 \in \beta_x(0)$ and for each $t \in \mathbb{R}$, the function $x \in \Gamma \to (I + \lambda\beta_x)^{-1}(t) \in \mathbb{R}$ is measurable for $\lambda > 0$.

In (1.10)-(1.12), suppose that $g_i : \Omega \times \mathbb{R}^{N+1} \to \mathbb{R}$ are given functions ($i = 1,2$) satisfying the following conditions, which can be found in [8, 16]:

(a) *Carathéodory's conditions.*

For $i = 1,2$, $x \to g_i(x,r)$ is measurable on Ω, for all $r \in \mathbb{R}^{N+1}$; $r \to g_i(x,r)$ is continuous on $\mathbb{R}^{N+1}$, for almost all $x \in \Omega$.

(b) *Growth condition.*

$$g_1(x,s_1,\ldots,s_{N+1}) \le h_1(x) + k_1 \sum_{i=1}^{N+1} |s_i|^{p-1},$$

$$g_2(x,s_1,\ldots,s_{N+1}) \le h_2(x) + k_2 \sum_{i=1}^{N+1} |s_i|^{q-1},$$

where $(s_1,s_2,\ldots,s_{N+1}) \in \mathbb{R}^{N+1}$, $h_1(x) \in L^p(\Omega)$, $h_2(x) \in L^q(\Omega)$ and k_i are positive constants, $i = 1,2$.

(c) *Monotone condition.*

For $i = 1,2$, $g_i(x,r_1,\ldots,r_{N+1})$ is monotone with respect to r_1, *i.e.*,

$$\left[g_i(x,s_1,\ldots,s_{N+1}) - g_i(x,t_1,\ldots,t_{N+1})\right](s_1 - t_1) \ge 0,$$

for all $x \in \Omega$ and $(s_1,\ldots,s_{N+1}),(t_1,\ldots,t_{N+1}) \in \mathbb{R}^{N+1}$.

(d) For $i = 1,2$, $g_i(x,0,\ldots,0) \equiv 0$, for $x \in \Omega$ and $(0,\ldots,0) \in \mathbb{R}^{N+1}$.

Specific to system (1.10) In (1.10), f_1, f_2, C_1, and C_2 are given functions with $f_1(x) \in L^{p'}(\Omega)$, $f_2(x) \in L^{q'}(\Omega)$, $0 \le C_1(x) \in L^p(\Omega)$ and $0 \le C_2(x) \in L^q(\Omega)$.

Lemma 2.1 [2, 16] *Define the operators* $B_1 : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ *and* $B_2 : W^{1,q}(\Omega) \to (W^{1,q}(\Omega))^*$ *by*

$$(w,B_1u) = \int_\Omega \left\langle \left(C_1(x) + |\nabla u|^2\right)^{\frac{p-2}{2}} \nabla u, \nabla w \right\rangle dx + \varepsilon_1 \int_\Omega |u|^{r-2} uw\, dx, \quad u,w \in W^{1,p}(\Omega)$$

and

$$(w, B_2 v) = \int_\Omega \bigl\langle \bigl(C_2(x) + |\nabla v|^2\bigr)^{\frac{q-2}{2}} \nabla v, \nabla w\bigr\rangle\,dx + \varepsilon_2 \int_\Omega |v|^{s-2} v w\,dx, \quad v, w \in W^{1,q}(\Omega).$$

Then B_i, $i = 1, 2$, is everywhere defined, strictly monotone, hemi-continuous, and coercive. Moreover, it is noted from Lemma 1.3 *that B_i, $i = 1, 2$, is maximal monotone*. (*Here $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the Euclidean inner-product and Euclidean norm in $\mathbb{R}^N$, respectively.*)

Definition 2.1 Define $A : Y \to Y^*$ by $A(u, v) = (B_1 u, B_2 v)$, for $(u, v) \in Y$.

Proposition 2.1 *The mapping $A : Y \to Y^*$ is everywhere defined, monotone, and hemi-continuous.*

Proof Step 1. A is everywhere defined.

In fact, for any $(u, v), (w_1, w_2) \in Y$, we have $|((w_1, w_2), A(u, v))| = |((w_1, w_2), (B_1 u, B_2 v))| \le |(w_1, B_1 u)| + |(w_2, B_2 v)|$. Since B_1 and B_2 are everywhere defined, A is everywhere defined.

Step 2. A is monotone.

To show this, let $(w_1^{(1)}, w_2^{(1)}), (w_1^{(2)}, w_2^{(2)}) \in Y$, then

$$\begin{aligned}&\bigl(\bigl(w_1^{(1)}, w_2^{(1)}\bigr) - \bigl(w_1^{(2)}, w_2^{(2)}\bigr), A\bigl(w_1^{(1)}, w_2^{(1)}\bigr) - A\bigl(w_1^{(2)}, w_2^{(2)}\bigr)\bigr) \\ &\quad = \bigl(w_1^{(1)} - w_1^{(2)}, B_1 w_1^{(1)} - B_1 w_1^{(2)}\bigr) + \bigl(w_2^{(1)} - w_2^{(2)}, B_2 w_2^{(1)} - B_2 w_2^{(2)}\bigr).\end{aligned}$$

Since both B_1 and B_2 are monotone, A is monotone.

Step 3. A is hemi-continuous.

It suffices to show that for any $(u_1, u_2), (v_1, v_2), (w_1, w_2) \in Y$ and $k \in [0, 1]$,

$$\bigl((w_1, w_2), A\bigl((u_1, u_2) + k(v_1, v_2)\bigr) - A(u_1, u_2)\bigr) \to 0,$$

as $k \to 0$. In fact, notice that both B_1 and B_2 are hemi-continuous, A is also hemi-continuous. □

Lemma 2.2 [2] *The mapping $\Phi_1 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by*

$$\Phi_1(u) = \int_\Gamma \varphi_x\bigl(u|_\Gamma(x)\bigr)\,d\Gamma(x), \quad u \in W^{1,p}(\Omega)$$

is proper, convex, and lower-semi-continuous on $W^{1,p}(\Omega)$. The subdifferential $\partial\Phi_1$ of Φ_1 is maximal monotone in view of Lemma 1.2. *Moreover,*

$$\bigl(w_1, \partial\Phi_1(u)\bigr) = \int_\Gamma \beta_x\bigl(u|_\Gamma(x)\bigr) w_1|_\Gamma(x)\,d\Gamma(x), \quad u, w_1 \in W^{1,p}(\Omega).$$

The mapping $\Phi_2 : W^{1,q}(\Omega) \to \mathbb{R}$ defined by

$$\Phi_2(v) = \int_\Gamma \varphi_x\bigl(v|_\Gamma(x)\bigr)\,d\Gamma(x), \quad v \in W^{1,q}(\Omega)$$

is proper, convex, and lower-semi-continuous on $W^{1,q}(\Omega)$. *The subdifferential* $\partial\Phi_2$ *of* Φ_2 *is maximal monotone in view of Lemma* 1.2. *Moreover,*

$$\bigl(w_2, \partial\Phi_2(v)\bigr) = \int_\Gamma \beta_x\bigl(v|_\Gamma(x)\bigr)w_2|_\Gamma(x)\,d\Gamma(x), \quad v, w_2 \in W^{1,q}(\Omega).$$

Proposition 2.2 *The mapping* $\Phi : Y \to \mathbb{R}$ *defined by*

$$\Phi(u,v) = \int_\Gamma \varphi_x\bigl(u|_\Gamma(x)\bigr)\,d\Gamma(x) + \int_\Gamma \varphi_x\bigl(v|_\Gamma(x)\bigr)\,d\Gamma(x), \quad (u,v) \in Y$$

is proper, convex, and lower-semi-continuous on Y. *The subdifferential* $\partial\Phi$ *of* Φ *is maximal monotone in view of Lemma* 1.2. *Moreover,*

$$\partial\Phi(u,v) = \bigl(\partial\Phi_1(u), \partial\Phi_2(v)\bigr).$$

Proof Since Φ_1 and Φ_2 are proper, it is not difficult to find that Φ is also proper.

For $0 \le \lambda \le 1$ and $(u_1,v_1),(u_2,v_2) \in Y$, we find

$$\begin{aligned}
&\Phi\bigl((1-\lambda)(u_1,v_1) + \lambda(u_2,v_2)\bigr)\\
&\quad = \Phi\bigl((1-\lambda)u_1 + \lambda u_2, (1-\lambda)v_1 + \lambda v_2\bigr)\\
&\quad = \int_\Gamma \varphi_x\bigl((1-\lambda)u_1|_\Gamma(x) + \lambda u_2|_\Gamma(x)\bigr)\,d\Gamma(x) + \int_\Gamma \varphi_x\bigl((1-\lambda)v_1|_\Gamma(x) + \lambda v_2|_\Gamma(x)\bigr)\,d\Gamma(x)\\
&\quad \le (1-\lambda)\int_\Gamma \varphi_x\bigl(u_1|_\Gamma(x)\bigr)\,d\Gamma(x) + \lambda\int_\Gamma \varphi_x\bigl(u_2|_\Gamma(x)\bigr)\,d\Gamma(x)\\
&\qquad + (1-\lambda)\int_\Gamma \varphi_x\bigl(v_1|_\Gamma(x)\bigr)\,d\Gamma(x) + \lambda\int_\Gamma \varphi_x\bigl(v_2|_\Gamma(x)\bigr)\,d\Gamma(x)\\
&\quad = (1-\lambda)\Phi(u_1,v_1) + \lambda\Phi(u_2,v_2),
\end{aligned}$$

which implies that Φ is convex.

For $(w,z) \in Y$, since Φ_1 and Φ_2 are lower-semi-continuous, we have

$$\begin{aligned}
\liminf_{(u,v)\to(w,z)} \Phi(u,v) &= \liminf_{(u,v)\to(w,z)} \biggl[\int_\Gamma \varphi_x\bigl(u|_\Gamma(x)\bigr)\,d\Gamma(x) + \int_\Gamma \varphi_x\bigl(v|_\Gamma(x)\bigr)\,d\Gamma(x)\biggr]\\
&\ge \int_\Gamma \varphi_x\bigl(w|_\Gamma(x)\bigr)\,d\Gamma(x) + \int_\Gamma \varphi_x\bigl(z|_\Gamma(x)\bigr)\,d\Gamma(x) = \Phi(w,z),
\end{aligned}$$

which implies that Φ is lower-semi-continuous.

For $(u,v),(w,z) \in Y$, in view of the definition of the subdifferential, we get

$$\Phi_1(u) + \Phi_2(v) \le \Phi_1(w) + \bigl(u-w, \partial\Phi_1(u)\bigr) + \Phi_2(z) + \bigl(v-z, \partial\Phi_2(v)\bigr).$$

Then

$$\begin{aligned}
&\int_\Gamma \varphi_x\bigl(u|_\Gamma(x)\bigr)\,d\Gamma(x) + \int_\Gamma \varphi_x\bigl(v|_\Gamma(x)\bigr)\,d\Gamma(x)\\
&\quad \le \int_\Gamma \varphi_x\bigl(w|_\Gamma(x)\bigr)\,d\Gamma(x) + \bigl(u-w, \partial\Phi_1(u)\bigr) + \int_\Gamma \varphi_x\bigl(z|_\Gamma(x)\bigr)\,d\Gamma(x) + \bigl(v-z, \partial\Phi_2(v)\bigr),
\end{aligned}$$

which implies that

$$\Phi(u,v) \leq \Phi(w,z) + \big((u,v) - (w,z), \big(\partial\Phi_1(u), \partial\Phi_2(v)\big)\big).$$

Thus,

$$\partial\Phi(u,v) = \big(\partial\Phi_1(u), \partial\Phi_2(v)\big).$$

This completes the proof. □

Lemma 2.3 [16] *Define* $G_1 : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ *by*

$$(w, G_1 u) = \int_\Omega g_1(x,u,\nabla u) w\,dx, \quad u,w \in W^{1,p}(\Omega).$$

Then G_1 *is everywhere defined, monotone, and hemi-continuous on* $W^{1,p}(\Omega)$.
Define $G_2 : W^{1,q}(\Omega) \to (W^{1,q}(\Omega))^*$ *by*

$$(w, G_2 v) = \int_\Omega g_2(x,v,\nabla v) w\,dx, \quad v,w \in W^{1,q}(\Omega).$$

Then G_2 *is everywhere defined, monotone, and hemi-continuous on* $W^{1,q}(\Omega)$.

Proposition 2.3 *Define* $G : Y \to Y^*$ *by*

$$\big((w,z), G(u,v)\big) = (w, G_1 u) + (z, G_2 v), \quad (u,v) \in Y.$$

Then G *is everywhere defined, monotone, and hemi-continuous on* Y. *Moreover,* G *is maximal monotone.*

Proof The result follows from Lemma 2.3 and the definition of G. □

Theorem 2.1 *For* $f_1(x) \in L^{p'}(\Omega)$ *and* $f_2(x) \in L^{q'}(\Omega)$, *the nonlinear* (p,q)*-Laplacian elliptic system* (1.10) *has a unique solution in* Y.

Proof Define $T : Y \to Y^*$ by

$$\begin{aligned}\big((w_1,w_2), T(u,v)\big) &= \big((w_1,w_2), A(u,v)\big) + \big((w_1,w_2), G(u,v)\big) \\ &\quad - \int_\Omega f_1 w_1\,dx - \int_\Omega f_2 w_2\,dx,\end{aligned}$$

for $(u,v),(w_1,w_2) \in Y$. From Propositions 2.1 and 2.3, $T : Y \to Y^*$ is everywhere defined, monotone, hemi-continuous, and then it is maximal monotone.

Combining with Lemma 1.1 and Proposition 2.2, we know that $T + \partial\Phi$ is maximal monotone.

Next, we shall show that

$$\lim_{\|(u,v)\|_Y \to +\infty} \frac{((u,v), T(u,v) + \partial\Phi(u,v))}{\|(u,v)\|_Y} = +\infty.$$

Noting that $\partial\Phi(0,0) = (0,0)$, $G(0,0) = (0,0)$ and G is monotone, we have

$$\begin{aligned}\frac{((u,v),T(u,v)+\partial\Phi(u,v))}{\|(u,v)\|_Y} &\geq \frac{(u,B_1u)+(v,B_2v)}{\|(u,v)\|_Y} - \frac{\int_\Omega f_1u\,dx + \int_\Omega f_2v\,dx}{\|(u,v)\|_Y} \\ &\geq \frac{(u,B_1u)+(v,B_2v)}{\|(u,v)\|_Y} - \|f_1\|_{L^{p'}(\Omega)} - \|f_2\|_{L^{q'}(\Omega)}.\end{aligned}$$

Let $\|(u,v)\|_Y \to +\infty$, then $\|u\|_{1,p,\Omega} \to +\infty$ or $\|v\|_{1,q,\Omega} \to +\infty$.

Case 1. If $\|u\|_{1,p,\Omega} \to +\infty$ and $\|v\|_{1,q,\Omega} \leq$ const, then

$$\begin{aligned}\frac{((u,v),T(u,v)+\partial\Phi(u,v))}{\|(u,v)\|_Y} &\geq \frac{(u,B_1u)}{\|(u,v)\|_Y} - \|f_1\|_{L^{p'}(\Omega)} - \|f_2\|_{L^{q'}(\Omega)} \\ &= \frac{(u,B_1u)}{\|u\|_{1,p,\Omega}} \times \frac{1}{\sqrt{1+\frac{\|v\|_{1,q,\Omega}^2}{\|u\|_{1,p,\Omega}^2}}} - \|f_1\|_{L^{p'}(\Omega)} - \|f_2\|_{L^{q'}(\Omega)} \\ &\to +\infty,\end{aligned}$$

as $\|(u,v)\|_Y \to +\infty$, since B_1 is coercive.

Case 2. If $\|u\|_{1,p,\Omega} \leq$ const and $\|v\|_{1,q,\Omega} \to +\infty$, then the proof is similar to that of Case 1.

Case 3. If $\|u\|_{1,p,\Omega} \to +\infty$ and $\|v\|_{1,q,\Omega} \to +\infty$, then we split the discussion into the following cases:

(i) Suppose $\frac{\|u\|_{1,p,\Omega}}{\|v\|_{1,q,\Omega}} \to +\infty$. In this case,

$$\begin{aligned}\frac{((u,v),T(u,v)+\partial\Phi(u,v))}{\|(u,v)\|_Y} &\geq \frac{(u,B_1u)}{\sqrt{\|u\|_{1,p,\Omega}^2+\|v\|_{1,q,\Omega}^2}} - \|f_1\|_{L^{p'}(\Omega)} - \|f_2\|_{L^{q'}(\Omega)} \\ &= \frac{(u,B_1u)}{\|u\|_{1,p,\Omega}\sqrt{1+\frac{\|v\|_{1,q,\Omega}^2}{\|u\|_{1,p,\Omega}^2}}} - \|f_1\|_{L^{p'}(\Omega)} - \|f_2\|_{L^{q'}(\Omega)} \\ &\to +\infty,\end{aligned}$$

since B_1 is coercive.

(ii) Suppose $\frac{\|v\|_{1,q,\Omega}}{\|u\|_{1,p,\Omega}} \to +\infty$. Similar to case (i), the result follows.

(iii) Suppose $\frac{\|u\|_{1,p,\Omega}}{\|v\|_{1,q,\Omega}} \to \text{const} \neq 0$. In this case,

$$\begin{aligned}\frac{((u,v),T(u,v)+\partial\Phi(u,v))}{\|(u,v)\|_Y} &\geq \frac{(u,B_1u)}{\|u\|_{1,p,\Omega}\sqrt{1+\frac{\|v\|_{1,q,\Omega}^2}{\|u\|_{1,p,\Omega}^2}}} - \|f_1\|_{L^{p'}(\Omega)} - \|f_2\|_{L^{q'}(\Omega)} \\ &\to +\infty,\end{aligned}$$

since B_1 is coercive.

Therefore, for $r > 0$, there always exists $(0,0) \in D(T) \cap D(\partial\Phi)$ such that

$$\bigl((u,v),T(u,v)+\partial\Phi(u,v)\bigr) > 0,$$

for all $(u,v) \in Y$ with $\|(u,v)\|_Y > r$.

Then, in view of Theorem 1.6, the equation

$$(0,0) = T(u,v) + \partial\Phi(u,v) \tag{2.1}$$

has a solution in Y, which is denoted by (u,v). From the strict monotonicity of B_1 and B_2, (u,v) is unique. Next, we shall show that this (u,v) is the solution of (1.10).

For $(\varphi,\varphi) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$, using (2.1) we find

$$\begin{aligned}
&\int_\Omega \bigl\langle \bigl(C_1(x) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u, \nabla\varphi\bigr\rangle\,dx + \varepsilon_1 \int_\Omega |u|^{r-2}u\varphi\,dx + \int_\Omega g_1(x,u,\nabla u)\varphi\,dx \\
&\quad + \int_\Omega \bigl\langle \bigl(C_2(x) + |\nabla v|^2\bigr)^{\frac{q-2}{2}} \nabla v, \nabla\varphi\bigr\rangle\,dx + \varepsilon_2 \int_\Omega |v|^{s-2}v\varphi\,dx + \int_\Omega g_2(x,v,\nabla v)\varphi\,dx \\
&\quad - \int_\Omega f_1\varphi\,dx - \int_\Omega f_2\varphi\,dx + \bigl(\varphi, \partial\Phi_1(u)\bigr) + \bigl(\varphi, \partial\Phi_2(v)\bigr) \\
&= -\int_\Omega \operatorname{div}\bigl[\bigl(C_1(x) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u\bigr]\varphi\,dx + \varepsilon_1 \int_\Omega |u|^{r-2}u\varphi\,dx + \int_\Omega g_1(x,u,\nabla u)\varphi\,dx \\
&\quad - \int_\Omega \operatorname{div}\bigl[\bigl(C_2(x) + |\nabla v|^2\bigr)^{\frac{q-2}{2}} \nabla v\bigr]\varphi\,dx + \varepsilon_2 \int_\Omega |u|^{s-2}u\varphi\,dx + \int_\Omega g_2(x,v,\nabla v)\varphi\,dx \\
&\quad - \int_\Omega f_1\varphi\,dx - \int_\Omega f_2\varphi\,dx = 0.
\end{aligned}$$

From the property of generalized function, we have

$$\begin{aligned}
&-\operatorname{div}\bigl[\bigl(C_1(x) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u\bigr] + \varepsilon_1|u|^{r-2}u + g_1(x,u,\nabla u) - \operatorname{div}\bigl[\bigl(C_2(x) + |\nabla v|^2\bigr)^{\frac{q-2}{2}} \nabla v\bigr] \\
&\quad + \varepsilon_2|v|^{s-2}v + g_2(x,v,\nabla v) = f_1(x) + f_2(x).
\end{aligned} \tag{2.2}$$

Using Green's formula and (2.1), we have, for $(w_1,0) \in Y$,

$$\begin{aligned}
0 &= \int_\Omega \bigl\langle \bigl(C_1(x) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u, \nabla w_1\bigr\rangle\,dx + \varepsilon_1 \int_\Omega |u|^{r-2}uw_1\,dx + \int_\Omega g_1(x,u,\nabla u)w_1\,dx\,dt \\
&\quad - \int_\Omega f_1 w_1\,dx + \bigl(w_1, \partial\Phi_1(u)\bigr) \\
&= -\int_\Omega \operatorname{div}\bigl[\bigl(C_1(x) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u\bigr]w_1\,dx + \int_\Gamma \bigl\langle \vartheta, \bigl(C_1(x) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u\bigr\rangle w_1\,d\Gamma(x) \\
&\quad + \varepsilon_1 \int_\Omega |u|^{r-2}uw_1\,dx + \int_\Omega g_1(x,u,\nabla u)w_1\,dx - \int_\Omega f_1 w_1\,dx + \int_\Gamma \beta_x(u)w_1\,d\Gamma(x).
\end{aligned}$$

Then

$$\begin{aligned}
0 &= -\operatorname{div}\bigl[\bigl(C_1(x) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u\bigr] + \bigl\langle \vartheta, \bigl(C_1(x) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u\bigr\rangle \\
&\quad + \varepsilon_1|u|^{r-2}u + g_1(x,u,\nabla u) - f_1 + \beta_x(u).
\end{aligned} \tag{2.3}$$

Similarly, using Green's formula and (2.1), we have, for $(0,w_2) \in Y$,

$$\begin{aligned}
0 &= \int_\Omega \bigl\langle \bigl(C_2(x) + |\nabla v|^2\bigr)^{\frac{q-2}{2}} \nabla v, \nabla w_2\bigr\rangle\,dx + \varepsilon_2 \int_\Omega |v|^{s-2}vw_2\,dx + \int_\Omega g_2(x,v,\nabla v)w_2\,dx\,dt \\
&\quad - \int_\Omega f_2 w_2\,dx + \bigl(w_2, \partial\Phi_2(v)\bigr)
\end{aligned}$$

$$= -\int_\Omega \operatorname{div}\bigl[\bigl(C_2(x,t)+|\nabla v|^2\bigr)^{\frac{q-2}{2}}\nabla v\bigr]w_2\,dx + \int_\Gamma \bigl\langle \vartheta, \bigl(C_2(x)+|\nabla v|^2\bigr)^{\frac{q-2}{2}}\nabla v\bigr\rangle w_2\,d\Gamma(x)$$
$$+ \varepsilon_2 \int_\Omega |v|^{s-2}vw_2\,dx + \int_\Omega g_2(x,v,\nabla v)w_2\,dx - \int_\Omega f_2 w_2\,dx + \int_\Gamma \beta_x(v)w_2\,d\Gamma(x).$$

Then

$$0 = -\operatorname{div}\bigl[\bigl(C_2(x)+|\nabla v|^2\bigr)^{\frac{q-2}{2}}\nabla v\bigr] + \bigl\langle \vartheta, \bigl(C_2(x)+|\nabla v|^2\bigr)^{\frac{q-2}{2}}\nabla v\bigr\rangle$$
$$+ \varepsilon_2|v|^{s-2}v + g_2(x,v,\nabla v) - f_2 + \beta_x(v). \tag{2.4}$$

Since (u,v) satisfies (2.2), by using (2.3) and (2.4) we have

$$-\bigl\langle \vartheta, \bigl(C_1(x)+|\nabla u|^2\bigr)^{\frac{p-2}{2}}\nabla u\bigr\rangle - \bigl\langle \vartheta, \bigl(C_2(x)+|\nabla v|^2\bigr)^{\frac{q-2}{2}}\nabla v\bigr\rangle \in \beta_x(u)+\beta_x(v), \quad \text{a.e. } x\in\Gamma.$$

Thus, (u,v) is the solution of (1.10). This completes the proof. □

3 Discussion of (p,q)-Laplacian parabolic system (1.11)

We recall that Ω, Γ, ϑ, ε_1, ε_2, β_x, g_1, and g_2 satisfy the conditions stated at the beginning of Section 2.

Specific to system (1.11) In (1.11), T is a constant, f_1, f_2, C_1, and C_2 are given functions with $f_1(x) \in (L^p(0,T;W^{1,p}(\Omega)))^*$, $f_2(x)\in (L^q(0,T;W^{1,q}(\Omega)))^*$, $0\le C_1(x,t)\in L^p(\Omega\times(0,T))$, and $0\le C_2(x,t)\in L^q(\Omega\times(0,T))$.

Lemma 3.1 [9] *Define the operators* $\widetilde{B}_1 : L^p(0,T;W^{1,p}(\Omega)) \to (L^p(0,T;W^{1,p}(\Omega)))^*$ *and* $\widetilde{B}_2 : L^q(0,T;W^{1,q}(\Omega)) \to (L^q(0,T;W^{1,q}(\Omega)))^*$ *by*

$$(w,\widetilde{B}_1u) = \int_0^T\int_\Omega \bigl\langle \bigl(C_1(x,t)+|\nabla u|^2\bigr)^{\frac{p-2}{2}}\nabla u, \nabla w\bigr\rangle\,dx\,dt + \varepsilon_1\int_0^T\int_\Omega |u|^{r-2}uw\,dx\,dt,$$
$$u,w\in L^p\bigl(0,T;W^{1,p}(\Omega)\bigr)$$

and

$$(w,\widetilde{B}_2v) = \int_0^T\int_\Omega \bigl\langle \bigl(C_2(x,t)+|\nabla v|^2\bigr)^{\frac{q-2}{2}}\nabla v, \nabla w\bigr\rangle\,dx\,dt + \varepsilon_2\int_0^T\int_\Omega |v|^{s-2}vw\,dx\,dt,$$
$$v,w\in L^q\bigl(0,T;W^{1,q}(\Omega)\bigr).$$

Then $\widetilde{B}_i$, $i=1,2$, *is everywhere defined, strictly monotone, hemi-continuous, and coercive. Moreover,* $\widetilde{B}_i$, $i=1,2$, *is maximal monotone.*

Definition 3.1 Define $\widetilde{A}: Z\to Z^*$ by $\widetilde{A}(u,v) = (\widetilde{B}_1u, \widetilde{B}_2v)$, for $(u,v)\in Z$.

Proposition 3.1 [9] *The mapping* $\widetilde{A}: Z\to Z^*$ *is everywhere defined, maximal monotone, hemi-continuous, and coercive.*

Lemma 3.2 [7, 9] *The mapping* $\widetilde{\Phi}_1 : L^p(0,T;W^{1,p}(\Omega))\to\mathbb{R}$ *defined by*

$$\widetilde{\Phi}_1(u) = \int_0^T\int_\Gamma \varphi_x\bigl(u|_\Gamma(x,t)\bigr)\,d\Gamma(x)\,dt, \quad u\in L^p\bigl(0,T;W^{1,p}(\Omega)\bigr)$$

is proper, convex, and lower-semi-continuous on $L^p(0,T;W^{1,p}(\Omega))$. *The subdifferential* $\partial\widetilde{\Phi}_1$ *of* $\widetilde{\Phi}_1$ *is maximal monotone in view of Lemma* 1.2. *Moreover,*

$$\bigl(w_1,\partial\widetilde{\Phi}_1(u)\bigr)=\int_0^T\int_\Gamma\beta_x\bigl(u|_\Gamma(x,t)\bigr)w_1|_\Gamma(x,t)\,d\Gamma(x)\,dt,\quad u,w_1\in L^p\bigl(0,T;W^{1,p}(\Omega)\bigr).$$

The mapping $\widetilde{\Phi}_2:L^q(0,T;W^{1,q}(\Omega))\to\mathbb{R}$ *defined by*

$$\widetilde{\Phi}_2(v)=\int_0^T\int_\Gamma\varphi_x\bigl(v|_\Gamma(x)\bigr)\,d\Gamma(x)\,dt,\quad v\in L^q\bigl(0,T;W^{1,q}(\Omega)\bigr)$$

is proper, convex, and lower-semi-continuous on $L^q(0,T;W^{1,q}(\Omega))$. *The subdifferential* $\partial\widetilde{\Phi}_2$ *of* $\widetilde{\Phi}_2$ *is maximal monotone in view of Lemma* 1.2. *Moreover,*

$$\bigl(w_2,\partial\widetilde{\Phi}_2(v)\bigr)=\int_0^T\int_\Gamma\beta_x\bigl(v|_\Gamma(x,t)\bigr)w_2|_\Gamma(x,t)\,d\Gamma(x)\,dt,\quad v,w_2\in L^q\bigl(0,T;W^{1,q}(\Omega)\bigr).$$

Proposition 3.2 *The mapping* $\widetilde{\Phi}:Z\to Z^*$ *defined by*

$$\widetilde{\Phi}(u,v)=\int_0^T\int_\Gamma\varphi_x\bigl(u|_\Gamma(x,t)\bigr)\,d\Gamma(x)\,dt+\int_0^T\int_\Gamma\varphi_x\bigl(v|_\Gamma(x,t)\bigr)\,d\Gamma(x)\,dt,\quad (u,v)\in Z$$

is proper, convex, and lower-semi-continuous on Z. *The subdifferential* $\partial\widetilde{\Phi}$ *of* $\widetilde{\Phi}$ *is maximal monotone in view of Lemma* 1.2. *Moreover,*

$$\partial\widetilde{\Phi}(u,v)=\bigl(\partial\widetilde{\Phi}_1(u),\partial\widetilde{\Phi}_2(v)\bigr).$$

Proof The proof is similar to that of Proposition 2.2. □

Lemma 3.3 *Define* $\widetilde{G}_1:L^p(0,T;W^{1,p}(\Omega))\to(L^p(0,T;W^{1,p}(\Omega)))^*$ *by*

$$(w,\widetilde{G}_1u)=\int_0^T\int_\Omega g_1(x,u,\nabla u)w\,dx\,dt,\quad u,w\in L^p\bigl(0,T;W^{1,p}(\Omega)\bigr).$$

Then $\widetilde{G}_1$ *is everywhere defined, monotone, and hemi-continuous on* $L^p(0,T;W^{1,p}(\Omega))$.

Define $\widetilde{G}_2:L^q(0,T;W^{1,q}(\Omega))\to(L^q(0,T;W^{1,q}(\Omega)))^*$ *by*

$$(w,\widetilde{G}_2v)=\int_0^T\int_\Omega g_2(x,v,\nabla v)w\,dx\,dt,\quad v,w\in L^q\bigl(0,T;W^{1,q}(\Omega)\bigr).$$

Then $\widetilde{G}_2$ *is everywhere defined, monotone and hemi-continuous on* $L^q(0,T;W^{1,q}(\Omega))$.

Proof The proof is similar to that of Lemma 2.3. □

Proposition 3.3 *Define* $\widetilde{G}:Z\to Z^*$ *by*

$$\bigl((w,z),\widetilde{G}(u,v)\bigr)=(w,\widetilde{G}_1u)+(z,\widetilde{G}_2v),\quad (u,v)\in Z.$$

Then $\widetilde{G}$ *is maximal monotone.*

Proof The result follows from Lemma 3.3 and the definition of $\widetilde{G}$. □

Lemma 3.4 [7, 9] *Define* $S_1 : D(S_1) = \{u \in L^p(0,T;W^{1,p}(\Omega)) : \frac{\partial u}{\partial t} \in (L^p(0,T;W^{1,p}(\Omega)))^*, u(x,0) = u(x,T)\} \to (L^p(0,T;W^{1,p}(\Omega)))^*$ *by*

$$S_1u(x,t) = \frac{\partial u}{\partial t}.$$

Then S_1 *is a linear maximal monotone operator possessing a dense domain in* $L^p(0,T;W^{1,p}(\Omega))$.

Define $S_2 : D(S_2) = \{v \in L^q(0,T;W^{1,q}(\Omega)) : \frac{\partial v}{\partial t} \in (L^q(0,T;W^{1,q}(\Omega)))^*, v(x,0) = v(x,T)\} \to (L^q(0,T;W^{1,q}(\Omega)))^*$ *by*

$$S_2v(x,t) = \frac{\partial v}{\partial t}.$$

Then S_2 *is a linear maximal monotone operator possessing a dense domain in* $L^q(0,T;W^{1,q}(\Omega))$.

Proposition 3.4 [9] *Define* $S : Z \to Z^*$ *by*

$$\big((w,z), S(u,v)\big) = (w, S_1u) + (z, S_2v), \quad (u,v) \in D(S).$$

Then S *is linear maximal monotone.*

Theorem 3.1 *For* $(f_1(x), f_2(x)) \in Z^*$, *the nonlinear* (p,q)*-Laplacian parabolic system* (1.11) *has a unique solution in* Z.

Proof Define $\widetilde{T} : Z \to Z^*$ by

$$\begin{aligned}\big((w_1,w_2), \widetilde{T}(u,v)\big) &= \big((w_1,w_2), \widetilde{A}(u,v)\big) + \big((w_1,w_2), \widetilde{G}(u,v)\big) \\ &\quad - \int_0^T \int_\Omega f_1 w_1 \,dx\,dt - \int_0^T \int_\Omega f_2 w_2 \,dx\,dt,\end{aligned}$$

for $(u,v), (w_1,w_2) \in Z$. From Propositions 3.1 and 3.3, $\widetilde{T} : Z \to Z^*$ is everywhere defined, monotone, hemi-continuous, and then it is maximal monotone.

Using Lemma 1.1 and Propositions 3.1 and 3.2, we know that $\widetilde{T} + \partial\widetilde{\Phi}$ is maximal monotone.

Next, we shall show that

$$\lim_{\|(u,v)\|_Z \to +\infty} \frac{((u,v), S(u,v) + \widetilde{T}(u,v) + \partial\widetilde{\Phi}(u,v))}{\|(u,v)\|_Z} = +\infty.$$

Noting that $\partial\widetilde{\Phi}(0,0) = (0,0)$, $\widetilde{G}(0,0) = 0$ and $S(0,0) = (0,0)$, we have

$$\begin{aligned}&\frac{((u,v), S(u,v) + \widetilde{T}(u,v) + \partial\widetilde{\Phi}(u,v))}{\|(u,v)\|_Z} \\ &\quad \geq \frac{(u, \widetilde{B}_1u) + (v, \widetilde{B}_2v)}{\|(u,v)\|_Z} - \frac{\int_0^T \int_\Omega f_1 u \,dx\,dt + \int_0^T \int_\Omega f_2 v \,dx\,dt}{\|(u,v)\|_Z} \\ &\quad \geq \frac{(u, \widetilde{B}_1u) + (v, \widetilde{B}_2v)}{\|(u,v)\|_Z} - \|f_1\|_{(L^p(0,T;W^{1,p}(\Omega)))^*} - \|f_2\|_{(L^q(0,T;W^{1,q}(\Omega)))^*}.\end{aligned}$$

Let $\|(u,v)\|_Z \to +\infty$, then $\|u\|_{L^p(0,T;W^{1,p}(\Omega))} \to +\infty$ or $\|v\|_{L^q(0,T;W^{1,q}(\Omega))} \to +\infty$.

Case 1. If $\|u\|_{L^p(0,T;W^{1,p}(\Omega))} \to +\infty$ and $\|v\|_{L^q(0,T;W^{1,q}(\Omega))} \le \text{const}$, then

$$\begin{aligned}
&\frac{((u,v), \widetilde{T}(u,v) + \partial\widetilde{\Phi}(u,v))}{\|(u,v)\|_Z} \\
&\quad\ge \frac{(u, \widetilde{B}_1 u)}{\|(u,v)\|_Z} - \|f_1\|_{(L^p(0,T;W^{1,p}(\Omega)))^*} - \|f_2\|_{(L^q(0,T;W^{1,q}(\Omega)))^*} \\
&\quad= \frac{(u, \widetilde{B}_1 u)}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} \times \frac{1}{\sqrt{1 + \frac{\|v\|^2_{L^q(0,T;W^{1,q}(\Omega))}}{\|u\|^2_{L^p(0,T;W^{1,p}(\Omega))}}}} - \|f_1\|_{(L^p(0,T;W^{1,p}(\Omega)))^*} - \|f_2\|_{(L^q(0,T;W^{1,q}(\Omega)))^*} \\
&\quad\to +\infty,
\end{aligned}$$

as $\|(u,v)\|_Z \to +\infty$, since $\widetilde{B}_1$ is coercive.

Case 2. If $\|u\|_{L^p(0,T;W^{1,p}(\Omega))} \le \text{const}$ and $\|v\|_{L^q(0,T;W^{1,q}(\Omega))} \to +\infty$, then the proof is similar to that of Case 1.

Case 3. If $\|u\|_{L^p(0,T;W^{1,p}(\Omega))} \to +\infty$ and $\|v\|_{L^q(0,T;W^{1,q}(\Omega))} \to +\infty$, then we split the discussion into the following cases:

(i) Suppose $\frac{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}}{\|v\|_{L^q(0,T;W^{1,q}(\Omega))}} \to +\infty$. In this case,

$$\begin{aligned}
&\frac{((u,v), \widetilde{T}(u,v) + \partial\widetilde{\Phi}(u,v))}{\|(u,v)\|_Z} \\
&\quad\ge \frac{(u, \widetilde{B}_1 u)}{\sqrt{\|u\|^2_{L^p(0,T;W^{1,p}(\Omega))} + \|v\|^2_{L^q(0,T;W^{1,q}(\Omega))}}} \\
&\qquad - \|f_1\|_{(L^p(0,T;W^{1,p}(\Omega)))^*} - \|f_2\|_{(L^q(0,T;W^{1,q}(\Omega)))^*} \\
&\quad= \frac{(u, \widetilde{B}_1 u)}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}\sqrt{1 + \frac{\|v\|^2_{L^q(0,T;W^{1,q}(\Omega))}}{\|u\|^2_{L^p(0,T;W^{1,p}(\Omega))}}}} \\
&\qquad - \|f_1\|_{(L^p(0,T;W^{1,p}(\Omega)))^*} - \|f_2\|_{(L^q(0,T;W^{1,q}(\Omega)))^*} \\
&\quad\to +\infty,
\end{aligned}$$

since $\widetilde{B}_1$ is coercive.

(ii) Suppose $\frac{\|v\|_{L^q(0,T;W^{1,q}(\Omega))}}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} \to +\infty$. Similar to case (i), the result follows.

(iii) Suppose $\frac{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}}{\|v\|_{L^q(0,T;W^{1,q}(\Omega))}} \to \text{const} \neq 0$. In this case,

$$\begin{aligned}
&\frac{((u,v), \widetilde{T}(u,v) + \partial\widetilde{\Phi}(u,v))}{\|(u,v)\|_Z} \\
&\quad\ge \frac{(u, \widetilde{B}_1 u)}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}\sqrt{1 + \frac{\|v\|^2_{L^q(0,T;W^{1,q}(\Omega))}}{\|u\|^2_{L^p(0,T;W^{1,p}(\Omega))}}}} \\
&\qquad - \|f_1\|_{(L^p(0,T;W^{1,p}(\Omega)))^*} - \|f_2\|_{(L^q(0,T;W^{1,q}(\Omega)))^*} \\
&\quad\to +\infty,
\end{aligned}$$

since $\widetilde{B}_1$ is coercive.

Therefore, for $r > 0$, there always exists $(0,0) \in D(S) \cap D(\widetilde{T}) \cap D(\partial\widetilde{\Phi})$ such that

$$\big((u,v), S(u,v) + \widetilde{T}(u,v) + \partial\widetilde{\Phi}(u,v)\big) > 0,$$

for all $(u,v) \in Z$ with $\|(u,v)\|_Z > r$.

Then, in view of Theorem 1.6, the equation

$$(0,0) = S(u,v) + \widetilde{T}(u,v) + \partial\widetilde{\Phi}(u,v) \tag{3.1}$$

has a solution in Z, which is denoted by (u,v). From the strict monotonicity of $\widetilde{B}_1$ and $\widetilde{B}_2$, this (u,v) is unique. Next, we shall show that this (u,v) is the solution of (1.11).

For $(\varphi,\varphi) \in C_0^\infty(0,T;\Omega) \times C_0^\infty(0,T;\Omega)$, using (3.1) we find

$$\begin{aligned}
&\int_0^T\int_\Omega \frac{\partial u}{\partial t}\varphi\,dx\,dt + \int_0^T\int_\Omega \big\langle \big(C_1(x,t)+|\nabla u|^2\big)^{\frac{p-2}{2}}\nabla u, \nabla\varphi\big\rangle\,dx\,dt \\
&\quad + \varepsilon_1\int_0^T\int_\Omega |u|^{r-2}u\varphi\,dx\,dt + \int_0^T\int_\Omega g_1(x,u,\nabla u)\varphi\,dx\,dt \\
&\quad + \int_0^T\int_\Omega \frac{\partial v}{\partial t}\varphi\,dx\,dt + \int_0^T\int_\Omega \big\langle \big(C_2(x,t)+|\nabla v|^2\big)^{\frac{q-2}{2}}\nabla v, \nabla\varphi\big\rangle\,dx\,dt \\
&\quad + \varepsilon_2\int_0^T\int_\Omega |v|^{s-2}v\varphi\,dx\,dt + \int_0^T\int_\Omega g_2(x,v,\nabla v)\varphi\,dx\,dt - \int_0^T\int_\Omega f_1\varphi\,dx\,dt \\
&\quad - \int_0^T\int_\Omega f_2\varphi\,dx\,dt + \big(\varphi,\partial\widetilde{\Phi}_1(u)\big) + \big(\varphi,\partial\widetilde{\Phi}_2(v)\big) \\
&= \int_0^T\int_\Omega \frac{\partial u}{\partial t}\varphi\,dx\,dt - \int_0^T\int_\Omega \operatorname{div}\big[\big(C_1(x,t)+|\nabla u|^2\big)^{\frac{p-2}{2}}\nabla u\big]\varphi\,dx\,dt \\
&\quad + \varepsilon_1\int_0^T\int_\Omega |u|^{r-2}u\varphi\,dx\,dt + \int_0^T\int_\Omega g_1(x,u,\nabla u)\varphi\,dx\,dt \\
&\quad + \int_0^T\int_\Omega \frac{\partial v}{\partial t}\varphi\,dx\,dt - \int_0^T\int_\Omega \operatorname{div}\big[\big(C_2(x,t)+|\nabla v|^2\big)^{\frac{q-2}{2}}\nabla v\big]\varphi\,dx\,dt \\
&\quad + \varepsilon_2\int_0^T\int_\Omega |v|^{s-2}v\varphi\,dx\,dt + \int_0^T\int_\Omega g_2(x,v,\nabla v)\varphi\,dx\,dt \\
&\quad - \int_0^T\int_\Omega f_1\varphi\,dx\,dt - \int_0^T\int_\Omega f_2\varphi\,dx\,dt = 0.
\end{aligned}$$

From the property of generalized function, we have

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} - \operatorname{div}\big[\big(C_1(x,t)+|\nabla u|^2\big)^{\frac{p-2}{2}}\nabla u\big] + \varepsilon_1|u|^{r-2}u + g_1(x,u,\nabla u) + \frac{\partial v(x,t)}{\partial t} \\
&\quad - \operatorname{div}\big[\big(C_2(x,t)+|\nabla v|^2\big)^{\frac{q-2}{2}}\nabla v\big] + \varepsilon_2|v|^{s-2}v + g_2(x,v,\nabla v) = f_1(x,t) + f_2(x,t).
\end{aligned} \tag{3.2}$$

Using Green's formula and (3.1), we have, for $(w_1,0) \in Z$,

$$\begin{aligned}
0 &= \int_0^T\int_\Omega \frac{\partial u}{\partial t}w_1\,dx\,dt + \int_0^T\int_\Omega \big\langle \big(C_1(x,t)+|\nabla u|^2\big)^{\frac{p-2}{2}}\nabla u, \nabla w_1\big\rangle\,dx\,dt \\
&\quad + \varepsilon_1\int_0^T\int_\Omega |u|^{r-2}uw_1\,dx\,dt + \int_0^T\int_\Omega g_1(x,u,\nabla u)w_1\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^T \int_\Omega f_1 w_1\,dx\,dt + \bigl(w_1, \partial\widetilde{\Phi}_1(u)\bigr) \\
&= \int_0^T \int_\Omega \frac{\partial u}{\partial t} w_1\,dx\,dt - \int_0^T \int_\Omega \operatorname{div}\bigl[\bigl(C_1(x,t) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u\bigr] w_1\,dx\,dt \\
&\quad + \int_0^T \int_\Gamma \bigl\langle \vartheta, \bigl(C_1(x,t) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u\bigr\rangle w_1\,d\Gamma(x)\,dt + \varepsilon_1 \int_0^T \int_\Omega |u|^{r-2} u w_1\,dx\,dt \\
&\quad + \int_0^T \int_\Omega g_1(x,u,\nabla u) w_1\,dx\,dt - \int_0^T \int_\Omega f_1 w_1\,dx\,dt + \int_0^T \int_\Gamma \beta_x(u) w_1\,d\Gamma(x)\,dt.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - \operatorname{div}\bigl[\bigl(C_1(x,t) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u\bigr] + \bigl\langle \vartheta, \bigl(C_1(x,t) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u\bigr\rangle \\
&\quad + \varepsilon_1 |u|^{r-2} u + g_1(x,u,\nabla u) - f_1 + \beta_x(u) = 0.
\end{aligned} \tag{3.3}
$$

Similarly, using Green's formula and (3.1), we have, for $(0, w_2) \in Z$,

$$
\begin{aligned}
0 &= \int_0^T \int_\Omega \frac{\partial v}{\partial t} w_2\,dx\,dt + \int_0^T \int_\Omega \bigl\langle \bigl(C_2(x,t) + |\nabla v|^2\bigr)^{\frac{q-2}{2}} \nabla v, \nabla w_2\bigr\rangle\,dx\,dt \\
&\quad + \varepsilon_2 \int_0^T \int_\Omega |v|^{s-2} v w_2\,dx\,dt + \int_0^T \int_\Omega g_2(x,v,\nabla v) w_2\,dx\,dt \\
&\quad - \int_0^T \int_\Omega f_2 w_2\,dx\,dt + \bigl(w_2, \partial\widetilde{\Phi}_2(v)\bigr) \\
&= \int_0^T \int_\Omega \frac{\partial v}{\partial t} w_2\,dx\,dt - \int_0^T \int_\Omega \operatorname{div}\bigl[\bigl(C_2(x,t) + |\nabla v|^2\bigr)^{\frac{q-2}{2}} \nabla v\bigr] w_2\,dx\,dt \\
&\quad + \int_0^T \int_\Gamma \bigl\langle \vartheta, \bigl(C_2(x,t) + |\nabla v|^2\bigr)^{\frac{q-2}{2}} \nabla v\bigr\rangle w_2\,d\Gamma(x)\,dt + \varepsilon_2 \int_0^T \int_\Omega |v|^{s-2} v w_2\,dx\,dt \\
&\quad + \int_0^T \int_\Omega g_2(x,v,\nabla v) w_2\,dx\,dt - \int_0^T \int_\Omega f_2 w_2\,dx + \int_0^T \int_\Gamma \beta_x(v) w_2\,d\Gamma(x)\,dt.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{\partial v}{\partial t} - \operatorname{div}\bigl[\bigl(C_2(x,t) + |\nabla v|^2\bigr)^{\frac{q-2}{2}} \nabla v\bigr] + \bigl\langle \vartheta, \bigl(C_2(x,t) + |\nabla v|^2\bigr)^{\frac{q-2}{2}} \nabla v\bigr\rangle \\
&\quad + \varepsilon_2 |v|^{s-2} v + g_2(x,v,\nabla v) - f_2 + \beta_x(v) = 0.
\end{aligned} \tag{3.4}
$$

Since (u, v) satisfies (3.2), by using (3.3) and (3.4) we have

$$
\begin{aligned}
&-\bigl\langle \vartheta, \bigl(C_1(x,t) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u\bigr\rangle - \bigl\langle \vartheta, \bigl(C_2(x,t) + |\nabla v|^2\bigr)^{\frac{q-2}{2}} \nabla v\bigr\rangle \\
&\quad \in \beta_x(u) + \beta_x(v), \qquad (x,t) \in \Gamma \times (0,T).
\end{aligned}
$$

Hence, (u, v) is the solution of (1.11). This completes the proof. □

4 Discussion of (p,q)-Laplacian integro-differential system (1.12)

We recall that Ω, Γ, ϑ, ε_1, ε_2, β_x, g_1, and g_2 satisfy the conditions stated at the beginning of Section 2.

Specific to system (1.12) In (1.12), T, a_1, and a_2 are non-negative constants, f_1, f_2, C_1, and C_2 are given functions with $f_1(x) \in (L^p(0,T;W^{1,p}(\Omega)))^*$, $f_2(x) \in (L^q(0,T;W^{1,q}(\Omega)))^*$, $0 \leq C_1(x,t) \in L^p(\Omega \times (0,T))$ and $0 \leq C_2(x,t) \in L^q(\Omega \times (0,T))$.

Lemma 4.1 [9] *Define* $\widetilde{S}_1 : D(\widetilde{S}_1) = \{u \in L^p(0,T;W^{1,p}(\Omega)) : \frac{\partial u}{\partial t} \in (L^p(0,T;W^{1,p}(\Omega)))^*, u(x,0) = u(x,T)\} \to (L^p(0,T;W^{1,p}(\Omega)))^*$ *by*

$$\widetilde{S}_1 u(x,t) = \frac{\partial u}{\partial t} + a_1 \frac{\partial}{\partial t} \int_\Omega u\,dx.$$

Then $\widetilde{S}_1$ *is a linear maximal monotone operator possessing a dense domain in* $L^p(0,T;W^{1,p}(\Omega))$.

Define $\widetilde{S}_2 : D(\widetilde{S}_2) = \{v \in L^q(0,T;W^{1,q}(\Omega)) : \frac{\partial v}{\partial t} \in (L^q(0,T;W^{1,q}(\Omega)))^*, v(x,0) = v(x,T)\} \to (L^q(0,T;W^{1,q}(\Omega)))^*$ *by*

$$\widetilde{S}_2 v(x,t) = \frac{\partial v}{\partial t} + a_2 \frac{\partial}{\partial t} \int_\Omega v\,dx.$$

Then $\widetilde{S}_2$ *is a linear maximal monotone operator possessing a dense domain in* $L^q(0,T;W^{1,q}(\Omega))$.

Proposition 4.1 *Define* $\widetilde{S} : Z \to Z^*$ *by*

$$\bigl((w,z), \widetilde{S}(u,v)\bigr) = (w, \widetilde{S}_1 u) + (z, \widetilde{S}_2 v), \quad (u,v) \in Z.$$

Then $\widetilde{S}$ *is linear maximal monotone.*

Theorem 4.1 *For* $(f_1(x,t), f_2(x,t)) \in Z^*$, *the nonlinear* (p,q)*-Laplacian integro-differential system* (1.12) *has a unique solution in* Z.

Proof Define $\widetilde{T}, \partial\widetilde{\Phi} : Z \to Z^*$ as in Theorem 3.1 and Proposition 3.2, respectively.

Since $\widetilde{S}(0,0) = (0,0)$, similar to the proof of Theorem 3.1, for $r > 0$, there always exists $(0,0) \in D(\widetilde{S}) \cap D(\widetilde{T}) \cap D(\partial\widetilde{\Phi})$ such that

$$\bigl((u,v), \widetilde{S}(u,v) + \widetilde{T}(u,v) + \partial\widetilde{\Phi}(u,v)\bigr) > 0,$$

for all $(u,v) \in Z$ with $\|(u,v)\|_Z > r$.

In view of Theorem 1.6, the equation

$$(0,0) = \widetilde{S}(u,v) + \widetilde{T}(u,v) + \partial\widetilde{\Phi}(u,v) \tag{4.1}$$

has a unique solution in Z, which is denoted by (u,v). As in the proof of Theorem 3.1, this (u,v) is unique. Next, we shall show that this (u,v) is the solution of (1.12).

For $(\varphi,\varphi) \in C_0^\infty(0,T;\Omega) \times C_0^\infty(0,T;\Omega)$, using (4.1) we have

$$\begin{aligned}&\int_0^T \int_\Omega \frac{\partial u}{\partial t}\varphi\,dx\,dt + \int_0^T \int_\Omega \biggl(a_1 \frac{\partial}{\partial t} \int_\Omega u\,dx\biggr)\,dx\,dt \\ &\quad + \int_0^T \int_\Omega \bigl\langle \bigl(C_1(x,t) + |\nabla u|^2\bigr)^{\frac{p-2}{2}} \nabla u, \nabla\varphi \bigr\rangle\,dx\,dt\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon_1\int_0^T\int_\Omega |u|^{r-2}u\varphi\,dx\,dt+\int_0^T\int_\Omega g_1(x,u,\nabla u)\varphi\,dx\,dt+\int_0^T\int_\Omega \frac{\partial v}{\partial t}\varphi\,dx\,dt\\
&+\int_0^T\int_\Omega\left(a_2\frac{\partial}{\partial t}\int_\Omega v\,dx\right)dx\,dt+\int_0^T\int_\Omega\bigl\langle\bigl(C_2(x,t)+|\nabla v|^2\bigr)^{\frac{q-2}{2}}\nabla v,\nabla\varphi\bigr\rangle\,dx\,dt\\
&+\varepsilon_2\int_0^T\int_\Omega |v|^{s-2}v\varphi\,dx\,dt+\int_0^T\int_\Omega g_2(x,v,\nabla v)\varphi\,dx\,dt-\int_0^T\int_\Omega f_1\varphi\,dx\,dt\\
&-\int_0^T\int_\Omega f_2\varphi\,dx\,dt+\bigl(\varphi,\partial\widetilde{\Phi}_1(u)\bigr)+\bigl(\varphi,\partial\widetilde{\Phi}_2(v)\bigr)\\
={}&\int_0^T\int_\Omega \frac{\partial u}{\partial t}\varphi\,dx\,dt+\int_0^T\int_\Omega\left(a_1\frac{\partial}{\partial t}\int_\Omega u\,dx\right)dx\,dt\\
&-\int_0^T\int_\Omega \operatorname{div}\bigl[\bigl(C_1(x,t)+|\nabla u|^2\bigr)^{\frac{p-2}{2}}\nabla u\bigr]\varphi\,dx\,dt\\
&+\varepsilon_1\int_0^T\int_\Omega |u|^{r-2}u\varphi\,dx\,dt+\int_0^T\int_\Omega g_1(x,u,\nabla u)\varphi\,dx\,dt+\int_0^T\int_\Omega \frac{\partial v}{\partial t}\varphi\,dx\,dt\\
&+\int_0^T\int_\Omega\left(a_2\frac{\partial}{\partial t}\int_\Omega v\,dx\right)dx\,dt-\int_0^T\int_\Omega \operatorname{div}\bigl[\bigl(C_2(x,t)+|\nabla v|^2\bigr)^{\frac{q-2}{2}}\nabla v\bigr]\varphi\,dx\,dt\\
&+\varepsilon_2\int_0^T\int_\Omega |v|^{s-2}v\varphi\,dx\,dt+\int_0^T\int_\Omega g_2(x,v,\nabla v)\varphi\,dx\,dt\\
&-\int_0^T\int_\Omega f_1\varphi\,dx\,dt-\int_0^T\int_\Omega f_2\varphi\,dx\,dt=0.
\end{aligned}
$$

From the property of generalized function, we get

$$
\begin{aligned}
&\frac{\partial u(x,t)}{\partial t}-\operatorname{div}\bigl[\bigl(C_1(x,t)+|\nabla u|^2\bigr)^{\frac{p-2}{2}}\nabla u\bigr]+\varepsilon_1|u|^{r-2}u+g_1(x,u,\nabla u)+a_1\frac{\partial}{\partial t}\int_\Omega u\,dx\\
&\quad+\frac{\partial v(x,t)}{\partial t}-\operatorname{div}\bigl[\bigl(C_2(x,t)+|\nabla v|^2\bigr)^{\frac{q-2}{2}}\nabla v\bigr]+\varepsilon_2|v|^{s-2}v+g_2(x,v,\nabla v)+a_2\frac{\partial}{\partial t}\int_\Omega v\,dx\\
&=f_1(x,t)+f_2(x,t).
\end{aligned}
\tag{4.2}
$$

Using Green's formula and (4.1), we have, for $(w_1,0)\in Z$,

$$
\begin{aligned}
0={}&\int_0^T\int_\Omega \frac{\partial u}{\partial t}w_1\,dx\,dt+\int_0^T\int_\Omega\left(a_1\frac{\partial}{\partial t}\int_\Omega u\,dx\right)w_1\,dx\,dt\\
&+\int_0^T\int_\Omega\bigl\langle\bigl(C_1(x,t)+|\nabla u|^2\bigr)^{\frac{p-2}{2}}\nabla u,\nabla w_1\bigr\rangle\,dx\,dt+\varepsilon_1\int_0^T\int_\Omega |u|^{r-2}uw_1\,dx\,dt\\
&+\int_0^T\int_\Omega g_1(x,u,\nabla u)w_1\,dx\,dt-\int_0^T\int_\Omega f_1w_1\,dx\,dt+\bigl(w_1,\partial\widetilde{\Phi}_1(u)\bigr)\\
={}&\int_0^T\int_\Omega \frac{\partial u}{\partial t}w_1\,dx\,dt+\int_0^T\int_\Omega\left(a_1\frac{\partial}{\partial t}\int_\Omega u\,dx\right)w_1\,dx\,dt\\
&-\int_0^T\int_\Omega \operatorname{div}\bigl[\bigl(C_1(x,t)+|\nabla u|^2\bigr)^{\frac{p-2}{2}}\nabla u\bigr]w_1\,dx\,dt\\
&+\int_0^T\int_\Gamma\bigl\langle\vartheta,\bigl(C_1(x,t)+|\nabla u|^2\bigr)^{\frac{p-2}{2}}\nabla u\bigr\rangle w_1\,d\Gamma(x)\,dt+\varepsilon_1\int_0^T\int_\Omega |u|^{r-2}uw_1\,dx\,dt\\
&+\int_0^T\int_\Omega g_1(x,u,\nabla u)w_1\,dx\,dt-\int_0^T\int_\Omega f_1w_1\,dx\,dt+\int_0^T\int_\Gamma \beta_x(u)w_1\,d\Gamma(x)\,dt.
\end{aligned}
$$

Then

$$\begin{aligned}
&\frac{\partial u}{\partial t} + a_1 \frac{\partial}{\partial t}\int_\Omega u\,dx - \operatorname{div}\bigl[\bigl(C_1(x,t) + |\nabla u|^2\bigr)^{\frac{p-2}{2}}\nabla u\bigr] \\
&\quad + \bigl\langle \vartheta, \bigl(C_1(x,t) + |\nabla u|^2\bigr)^{\frac{p-2}{2}}\nabla u\bigr\rangle + \varepsilon_1 |u|^{r-2}u + g_1(x,u,\nabla u) - f_1 + \beta_x(u) = 0.
\end{aligned} \tag{4.3}$$

Similarly, using Green's formula and (4.1), we have, for $(0, w_2) \in Z$,

$$\begin{aligned}
0 &= \int_0^T \int_\Omega \frac{\partial v}{\partial t} w_2\,dx\,dt + \int_0^T \int_\Omega \biggl(a_2 \frac{\partial}{\partial t}\int_\Omega v\,dx\biggr) w_2\,dx\,dt \\
&\quad + \int_0^T \int_\Omega \bigl\langle \bigl(C_2(x,t) + |\nabla v|^2\bigr)^{\frac{q-2}{2}}\nabla v, \nabla w_2\bigr\rangle\,dx\,dt + \varepsilon_2 \int_0^T \int_\Omega |v|^{s-2} v w_2\,dx\,dt \\
&\quad + \int_0^T \int_\Omega g_2(x,v,\nabla v) w_2\,dx\,dt - \int_0^T \int_\Omega f_2 w_2\,dx\,dt + \bigl(w_2, \partial\widetilde{\Phi}_2(v)\bigr) \\
&= \int_0^T \int_\Omega \frac{\partial v}{\partial t} w_2\,dx\,dt + \int_0^T \int_\Omega \biggl(a_2 \frac{\partial}{\partial t}\int_\Omega v\,dx\biggr) w_2\,dx\,dt \\
&\quad - \int_0^T \int_\Omega \operatorname{div}\bigl[\bigl(C_2(x,t) + |\nabla v|^2\bigr)^{\frac{q-2}{2}}\nabla v\bigr] w_2\,dx\,dt \\
&\quad + \int_0^T \int_\Gamma \bigl\langle \vartheta, \bigl(C_2(x,t) + |\nabla v|^2\bigr)^{\frac{q-2}{2}}\nabla v\bigr\rangle w_2\,d\Gamma(x)\,dt + \varepsilon_2 \int_0^T \int_\Omega |v|^{s-2} v w_2\,dx\,dt \\
&\quad + \int_0^T \int_\Omega g_2(x,v,\nabla v) w_2\,dx\,dt - \int_0^T \int_\Omega f_2 w_2\,dx + \int_0^T \int_\Gamma \beta_x(v) w_2\,d\Gamma(x)\,dt.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{\partial v}{\partial t} + a_2 \frac{\partial}{\partial t}\int_\Omega v\,dx - \operatorname{div}\bigl[\bigl(C_2(x,t) + |\nabla v|^2\bigr)^{\frac{q-2}{2}}\nabla v\bigr] + \bigl\langle \vartheta, \bigl(C_2(x,t) + |\nabla v|^2\bigr)^{\frac{q-2}{2}}\nabla v\bigr\rangle \\
&\quad + \varepsilon_2 |v|^{s-2}v + g_2(x,v,\nabla v) - f_2 + \beta_x(v) = 0.
\end{aligned} \tag{4.4}$$

Since (u,v) satisfies (4.2), using (4.3) and (4.4) we have

$$\begin{aligned}
&-\bigl\langle \vartheta, \bigl(C_1(x,t) + |\nabla u|^2\bigr)^{\frac{p-2}{2}}\nabla u\bigr\rangle - \bigl\langle \vartheta, \bigl(C_2(x,t) + |\nabla v|^2\bigr)^{\frac{q-2}{2}}\nabla v\bigr\rangle \\
&\quad \in \beta_x(u) + \beta_x(v), \quad (x,t) \in \Gamma \times (0,T).
\end{aligned}$$

Thus, (u,v) is the unique solution of (1.12). This completes the proof. □

5 Examples

In this section, we give some examples of the systems (1.10)-(1.12) discussed in this paper.

Example 5.1 We list two examples of (1.10) - the first system (5.1) is from [14] and the second system (5.2) is discussed in [17]. However, different methods have been employed:

$$\begin{cases} -u'' + \varepsilon u = f(x), \\ -u' = 0. \end{cases} \tag{5.1}$$

$$\begin{cases} -\Delta u - \mu\Delta v = g(x,v), & x \in \Omega, \\ -\Delta v - \lambda\Delta u = f(x,u), & x \in \Omega, \\ u = v = 0, & x \in \Gamma. \end{cases} \tag{5.2}$$

Example 5.2 The following system, which has been studied in [18], is an example of (1.12):

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u + a\frac{\partial}{\partial t}\int_\Omega u\,dx = f(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & x \in \Gamma, \\ u(x,0) = 0, & x \in \Omega. \end{cases} \tag{5.3}$$

Once again, different methods have been employed.

Example 5.3 If Ω reduces to a bounded interval (a,b) in $\mathbb{R}^1$, examples of β_x and g_i $(i = 1,2)$ can be found readily. For example, for $x \in \Gamma$, take $\varphi_x = \varphi(x,t) = t^2x^2$. Then $\varphi_x \equiv \varphi(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is a proper, convex, and lower-semi-continuous function. Further, $\beta_x = 2tx^2$. For $i = 1,2$, take $g_i(x,t_1,t_2) : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$g_1(x,t_1,t_2) = \begin{cases} \min\{a,x\} + (2\max\{|t_1|^{p-1}, |t_2|^{p-1}\} - |t_1|^{p-1})(\operatorname{sgn} t_1), & |t_1| \ge |t_2|, \\ \min\{a,x\} + (2\min\{|t_1|^{p-1}, |t_2|^{p-1}\} - |t_1|^{p-1})(\operatorname{sgn} t_1), & |t_1| \le |t_2| \end{cases}$$

and

$$g_2(x,t_1,t_2) = \begin{cases} \min\{a,x\} + (2\max\{|t_1|^{q-1}, |t_2|^{q-1}\} - |t_1|^{q-1})(\operatorname{sgn} t_1), & |t_1| \ge |t_2|, \\ \min\{a,x\} + (2\min\{|t_1|^{q-1}, |t_2|^{q-1}\} - |t_1|^{q-1})(\operatorname{sgn} t_1), & |t_1| \le |t_2|. \end{cases}$$

Then g_i satisfies the assumptions (a)-(c). If, $a \equiv 0$, then the assumption (d) is also satisfied.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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