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A general stability for a von Kármán system with memory

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Abstract

In this paper we study the von Kármán plate model with long-range memory. We prove an explicit and general decay rate result using some properties of the convex functions. Our result is obtained without imposing any restrictive assumptions on the behavior of the relaxation function at infinity. These general decay estimates extend and improve on some earlier results: exponential or polynomial decay rates.

Keywords: von Kármán system; general decay rate; convexity; viscoelastic

1 Introduction

This paper is concerned with the general decay of the solutions to a von Kármán system for the plate equation with memory:

$$u'' - h\Delta u'' + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)\,ds = [u,v] \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$\Delta^2 v = -[u,u] \quad \text{in } \Omega \times (0,\infty), \tag{1.2}$$

$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma \times (0,\infty), \tag{1.3}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \tag{1.4}$$

$$\mathcal{B}_1 u - \mathcal{B}_1 \left\{ \int_0^t g(t-s)u(s)\,ds \right\} = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.5}$$

$$\mathcal{B}_2 u - h\frac{\partial u''}{\partial \nu} - \mathcal{B}_2 \left\{ \int_0^t g(t-s)u(s)\,ds \right\} = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.6}$$

$$u(x,y,0) = u_0(x,y), \qquad u'(x,y,0) = u_1(x,y) \quad \text{in } \Omega, \tag{1.7}$$

where Ω is an open bounded set of $\mathbb{R}^2$ with a sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint. The equations describe small vibrations of a thin homogeneous isotropic plate of uniform thickness h. Let us denote by $\nu = (\nu_1, \nu_2)$ the external unit normal to Γ, and by $\eta = (-\nu_2, \nu_1)$ the unitary tangent positively oriented on Γ. Here

$$\mathcal{B}_1 u = \Delta u + (1-\mu)B_1 u \quad \text{and} \quad \mathcal{B}_2 u = \frac{\partial \Delta u}{\partial \nu} + (1-\mu)B_2 u,$$

where constant μ ($0 < \mu < \frac{1}{2}$) is Poisson's ratio and

$$B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx},$$
$$B_2 u = \frac{\partial}{\partial\eta}\big[(\nu_1^2 - \nu_2^2)u_{xy} + \nu_1\nu_2(u_{yy} - u_{xx})\big].$$

The von Kármán bracket is given by

$$[u, v] = u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx}.$$

For the last several decades, the mathematical models of vibrating, flexible structures have been considerably stimulated by an increasing number of questions of practical concern. The main purpose of this monograph is to present a systematic study of uniform stabilization of the motion of a thin plate through the action of forces and moments applied at the edge of the plate. Among the elastic plate models, the von Kármán model is a 'large deflection' plate model, in a sense of a nonlinear analogue of the Kirchhoff model. However, it is assumed that the vertical deflection is small in comparison with the lateral dimensions of the plate. This hypothesis leads to a coupled pair of fourth-order, nonlinear partial differential equations for the vertical displacement u and the Airy-stress function v. We may interpret Eq. (1.1) as saying that the stresses at any instant depend on the complete history of strains which the material has undergone. We will give later the precise condition on g in order to obtain the general decay results.

The problem of stability of the solutions to a von Kármán system with dissipative effects has been studied by several authors. In [1–4] the authors considered the von Kármán system with frictional dissipations effective in the boundary. It is shown in these works that these dissipations produce uniform rate of decay of the solution when t goes to infinity. Rivera and Menzala [5] and Rivera *et al.* [6] studied the stability of the solutions to a von Kármán system for viscoelastic plates with memory and boundary memory conditions, respectively. They proved that the energy decays uniformly exponentially or algebraically with the same rate of decay as the relaxation function. Later, Santos and Soufyane [7] generalized the decay result of [6]. Raposo and Santos [8] investigated the general decay of the solutions to a von Kármán plate model. Recently, Kang [9] proved the general decay of the solutions to a von Kármán plate model with memory and boundary damping. Kang [9] improved the results of [8] without imposing any restrictive growth assumption on the damping term and strongly weakening the usual assumption on the relaxation function.

On the other hand, the problem of stability of the solutions to a viscoelastic system with memory has been studied by many authors. In [10–13] the authors showed exponential and polynomial decay for a viscoelastic wave equation under the usual condition

$$-c_1 g(t) \le g'(t) \le -c_2 g(t) \quad \text{and} \quad 0 \le g''(t) \le c_3 g(t)$$

for some c_i, $i = 1, 2, 3$. Later, this assumption was relaxed by several authors. Berrimi and Messaoudi [14] studied exponential and polynomial decay rates under condition on g such as

$$g'(t) \le -\xi g^p(t) \quad \text{for } 1 \le p < \frac{3}{2}, t \ge 0, \tag{1.8}$$

where $\xi > 0$. Messaoudi and Tatar [15] and Liu [16] considered exponential and polynomial decay for a quasilinear equation and a system of two coupled quasilinear viscoelastic equations under condition (1.8) by choosing a suitable perturbed energy, respectively. Messaoudi [17] and Han and Wang [18] proved a general decay rate for viscoelastic equations under a more general condition on g such as

$$g'(t) \le -\xi(t)g(t), \qquad \frac{|\xi'(t)|}{|\xi(t)|} \le k, \quad \xi(t) > 0, \xi'(t) \le 0, \forall t > 0. \tag{1.9}$$

Guesmia and Messaoudi [19] obtained general stability for a Timoshenko system under weaker condition

$$g'(t) \le -\xi(t)g(t), \tag{1.10}$$

where ξ is a nonincreasing and positive function. The stability of the solutions to a viscoelastic system under condition (1.10) was studied in [20–23] and the references therein. Mustafa and Messaoudi [24, 25] investigated the general stability result for a viscoelastic equation for a relaxation function satisfying

$$g'(t) \le -H\bigl(g(t)\bigr), \tag{1.11}$$

where H is a nonnegative function, with $H(0) = 0$, and H is a linear or strictly increasing and strictly convex on $(0, r]$ for some $r > 0$. The above conditions are weaker conditions on H than those imposed in [26]. Recently, Cavalcanti *et al.* [27] proved the uniform decay rates of the energy for solutions of a von Kármán system with long memory for the memory kernel g satisfying (1.11).

Motivated by the work in [24–27], we establish an explicit and general decay of the solutions to a von Kármán plate model (1.1)-(1.7) for relaxation functions satisfying (1.11). The proof is based on the multiplier method and makes use of some properties of convex functions. This result improves on earlier ones in the literature because it allows certain relaxation functions which are not necessarily of exponential or polynomial decay.

The paper is organized as follows. In Section 2, we present some notations and material needed for our work and state a global existence theorem. In Section 3, we prove the general decay of the solutions to the von Kármán system with memory.

2 Preliminaries

In this section, we present some material needed in the proof of our result and state the main result. Throughout this paper we denote $(u, v) = \int_\Omega u(x, y)v(x, y)\, d\Omega$ and define

$$V = \bigl\{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0\bigr\}, \qquad U = \left\{u \in H^2(\Omega) \,\middle|\, u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0\right\}.$$

For a Banach space X, $\|\cdot\|_X$ denotes the norm of X. For simplicity, we denote $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|$.

A simple calculation, based on the integration by parts formula, yields

$$\bigl(\Delta^2 u, v\bigr) = a(u, v) + (\mathcal{B}_2 u, v)_\Gamma - \left(\mathcal{B}_1 u, \frac{\partial v}{\partial \nu}\right)_\Gamma, \tag{2.1}$$

where the bilinear symmetric form $a(u,v)$ is given by

$$a(u,v) = \int_\Omega \{u_{xx}v_{xx} + u_{yy}v_{yy} + \mu(u_{xx}v_{yy} + u_{yy}v_{xx}) + 2(1-\mu)u_{xy}v_{xy}\}\,d\Omega,$$

where $d\Omega = dx\,dy$. Since $\Gamma_0 \neq \emptyset$, we know that $\sqrt{a(u,u)}$ is equivalent to the $H^2(\Omega)$ norm on U, *i.e.*,

$$c_0\|u\|^2_{H^2(\Omega)} \le a(u,u) \le \tilde{c}_0\|u\|^2_{H^2(\Omega)},$$

where c_0 and $\tilde{c}_0$ are generic positive constants. This and the Sobolev imbedding theorem imply that for some positive constants C_p and C_s,

$$\|u\|^2 \le C_p a(u,u), \qquad \|\nabla u\|^2 \le C_s a(u,u), \quad \forall u \in U. \tag{2.2}$$

We consider the following hypotheses:

(H1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function such that

$$g(0) > 0, \qquad l := \int_0^\infty g(s)\,ds < 1. \tag{2.3}$$

(H2) There exists a positive function $H \in C^1(\mathbb{R}^+)$, with $H(0) = 0$, and H is a linear or strictly increasing and strictly convex C^2 function on $(0,r]$ for some $r < 1$ such that

$$g'(t) \le -H\bigl(g(t)\bigr), \quad \forall t \ge 0. \tag{2.4}$$

To simplify calculation in our analysis, we introduce the following notation:

$$(g * u)(t) := \int_0^t g(t-s)u(s)\,ds,$$

$$(g \,\Box\, u)(t) := \int_0^t g(t-s)\bigl\|u(\cdot,t) - u(\cdot,s)\bigr\|^2\,ds,$$

$$\bigl(g \,\Box\, \partial^2 u\bigr)(t) := \int_0^t g(t-s)a\bigl(u(\cdot,t) - u(\cdot,s), u(\cdot,t) - u(\cdot,s)\bigr)\,ds.$$

From the symmetry of $a(\cdot,\cdot)$ we have that for any $v \in C^1(0,T;H^2(\Omega))$,

$$\begin{aligned} a\bigl(g * v, v'\bigr) &= -\frac{1}{2}g(t)a(v,v) + \frac{1}{2}g' \,\Box\, \partial^2 v \\ &\quad - \frac{1}{2}\frac{d}{dt}\left\{g \,\Box\, \partial^2 v - \left(\int_0^t g(s)\,ds\right)a(v,v)\right\}. \end{aligned} \tag{2.5}$$

We introduce the following lemma for the bracket's binary.

Lemma 2.1 ([28]) *Let $u, w \in H^2(\Omega)$ and $v \in H_0^2(\Omega)$, where Ω is an open bounded and connected set of $\mathbb{R}^2$ with regular boundary. Then*

$$\int_\Omega w[v,u]\,d\Omega = \int_\Omega v[w,u]\,d\Omega. \tag{2.6}$$

Lemma 2.2 ([1]) *If $u, v \in H^2(\Omega)$, then $[u, v] \in L^2(\Omega)$ and satisfies*

$$\big\|[u,v]\big\| \le c_0' \|u\|_{H^2}\|v\|_{W^{2,\infty}} \quad \text{and} \quad \|v\|_{W^{2,\infty}} \le c_0'\|u\|_{H^2}^2. \tag{2.7}$$

By using Galerkin's approximation, we can obtain the following result for the solution. For the initial data $(u_0, u_1) \in H^4(\Omega) \times H^2(\Omega)$, $h > 0$, system (1.1)-(1.7) has a unique weak solution u in the following class:

$$u \in L^\infty\big(0,\infty; U \cap H^4(\Omega)\big), \qquad u' \in L^\infty\big(0,\infty; V \cap H^2(\Omega)\big).$$

We introduce the energy of problem (1.1)-(1.7) as

$$E(t) = \frac{1}{2}\big\|u'\big\|^2 + \frac{1}{2}a(u,u) + \frac{h}{2}\big\|\nabla u'\big\|^2 + \frac{1}{4}\|\Delta v\|^2. \tag{2.8}$$

Now, we are ready to state the following main result.

Theorem 2.1 *Assume that* (H1) *and* (H2) *hold. Suppose that D is a positive C^1 function, with $D(0) = 0$, for which H_0 is a strictly increasing and strictly convex C^2 function on $(0, r]$ and*

$$\int_0^{+\infty} \frac{g(s)}{H_0^{-1}(-g'(s))}\,ds < +\infty. \tag{2.9}$$

Then there exist positive constants k_1, k_2, k_3 and ϵ_0 such that the solution of (1.1)-(1.7) *satisfies*

$$E(t) \le k_3 H_1^{-1}(k_1 t + k_2), \quad \forall t \ge 0, \tag{2.10}$$

where

$$H_1(t) = \int_t^1 \frac{1}{sH_0'(\epsilon_0 s)}\,ds \quad \text{and} \quad H_0(t) = H\big(D(t)\big).$$

Moreover, for some choice of D, if $\int_0^1 H_1(t)\,dt < +\infty$, then we obtain

$$E(t) \le k_3 G^{-1}(k_1 t + k_2), \tag{2.11}$$

where

$$G(t) = \int_t^1 \frac{1}{sH'(\epsilon_0 s)}\,ds. \tag{2.12}$$

In particular, (2.11) *is valid for the special case $H(t) = ct^p$ for $1 \le p < \frac{3}{2}$.*

Remark 2.1 If F is a convex function on $[a, b]$, $f : \Omega \to [a, b]$ and h are integrable functions on Ω, $h(x) \ge 0$, and $\int_\Omega h(x)\,dx = h_0 > 0$, then Jensen's inequality states that

$$F\bigg(\frac{1}{h_0}\int_\Omega f(x)h(x)\,dx\bigg) \le \frac{1}{h_0}\int_\Omega F\big(f(x)\big)h(x)\,dx. \tag{2.13}$$

Remark 2.2 1. From the properties of H, we can show that the function H_1 is strictly decreasing and convex on $(0,1]$, with $\lim_{t\to 0} H_1(t) = +\infty$. Then Theorem 2.1 ensures

$$\lim_{t\to+\infty} E(t) = 0.$$

2. By using (H1) and (H2), we conclude that $\lim_{t\to+\infty} g(t) = 0$. This implies that $\lim_{t\to+\infty}(-g'(t))$ cannot be equal to a positive number, and so it is natural to assume that $\lim_{t\to+\infty}(-g'(t)) = 0$. Therefore, there is $t_0 > 0$ large enough such that $g(t_0) > 0$ and

$$\max\{g(t), -g'(t)\} < \min\{r, H(r), H_0(r)\}, \quad \forall t \ge t_0. \tag{2.14}$$

Since g is nonincreasing, $g(0) > 0$ and $g(t_0) > 0$, we obtain

$$0 < g(t_0) \le g(t) \le g(0), \quad \forall t \in [0, t_0]. \tag{2.15}$$

From H is a positive continuous function, we have

$$c_1 \le H\bigl(g(t)\bigr) \le c_2, \quad \forall t \in [0, t_0] \tag{2.16}$$

for some positive constants c_1 and c_2. Hence, by (2.4), (2.15) and (2.16),

$$g'(t) \le -H\bigl(g(t)\bigr) \le -\frac{c_1}{g(0)} g(0) \le -\frac{c_1}{g(0)} g(t),$$

which gives

$$g'(t) \le -c_3 g(t), \quad \forall t \in [0, t_0] \tag{2.17}$$

for some positive constant c_3.

3 General decay of the energy

In this section we prove the decay rates in Theorem 2.1. The following result shows the dissipative property of system (1.1)-(1.7). Multiplying (1.1) by $u'(t)$, we have the identity

$$E'(t) = a\bigl(g * u, u'\bigr). \tag{3.1}$$

We define the modified energy by

$$\mathcal{E}(t) = \frac{1}{2}\|u'\|^2 + \frac{h}{2}\|\nabla u'\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right) a(u,u) + \frac{1}{2} g \Box \partial^2 u + \frac{1}{4}\|\Delta v\|^2.$$

From (2.5) and (3.1), we get

$$\mathcal{E}'(t) = -\frac{1}{2} g(t) a(u,u) + \frac{1}{2} g' \Box \partial^2 u. \tag{3.2}$$

This implies that $\mathcal{E}(t)$ is nonincreasing, and from (2.3) one sees that

$$E(t) \le \frac{1}{1-l}\mathcal{E}(t), \quad \forall t \ge 0. \tag{3.3}$$

First, let us define the perturbed modified energy by

$$L(t) = N\mathcal{E}(t) + \epsilon\Phi(t) + \Psi(t), \tag{3.4}$$

where

$$\Phi(t) = \int_\Omega u'u\,d\Omega + h\int_\Omega \nabla u'\nabla u\,d\Omega$$

and

$$\Psi(t) = \int_\Omega \bigl(h\Delta u' - u'\bigr)\int_0^t g(t-s)\bigl(u(t)-u(s)\bigr)\,ds\,d\Omega.$$

Using the ideas presented in [9], we easily obtain the following lemmas.

Lemma 3.1 *For $N > 0$ large enough, there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such that*

$$\alpha_1\mathcal{E}(t) \le L(t) \le \alpha_2\mathcal{E}(t), \quad \forall t \ge 0. \tag{3.5}$$

Proof By Young's inequality, (2.2) and (2.3), we have

$$\bigl|\Phi(t)\bigr| \le \frac{1}{2}\|u'\|^2 + \frac{h}{2}\|\nabla u'\|^2 + \frac{C_p + C_s h}{2}a(u,u) \tag{3.6}$$

and

$$\bigl|\Psi(t)\bigr| \le \frac{1}{2}\|u'\|^2 + \frac{h}{2}\|\nabla u'\|^2 + \frac{(C_p + C_s h)l}{2}g\Box\partial^2 u. \tag{3.7}$$

From (3.6) and (3.7) we obtain

$$\begin{aligned}\bigl|L(t) - N\mathcal{E}(t)\bigr| &\le \frac{1}{2}(\epsilon+1)\|u'\|^2 + \frac{h}{2}(\epsilon+1)\|\nabla u'\|^2 \\ &\quad + \frac{(C_p + C_s h)\epsilon}{2}a(u,u) + \frac{(C_p + C_s h)l}{2}g\Box\partial^2 u \\ &\le C_0\mathcal{E}(t),\end{aligned}$$

where C_0 is a positive constant depending on ϵ, h, C_p, C_s and l. Choosing $N > 0$ large, we complete the proof of Lemma 3.1. □

Lemma 3.2 *For each $t_0 > 0$ and sufficiently large $N > 0$, there exist positive constants β_1 and β_2 such that*

$$L'(t) \le -\beta_1\mathcal{E}(t) + \beta_2 g\Box\partial^2 u, \quad \forall t \ge t_0. \tag{3.8}$$

Proof Direct computations, using (1.1), yield

$$\Phi'(t) = -a(u,u) + a(g*u,u) + \bigl([u,v],u\bigr) + \|u'\|^2 + h\|\nabla u'\|^2. \tag{3.9}$$

By Young's inequality, we have

$$a(g * u, u) \le (\delta + 1)\left(\int_0^t g(s)\,ds\right)a(u,u) + \frac{1}{4\delta} g \Box \partial^2 u. \tag{3.10}$$

From (3.9) and (3.10), we get

$$\begin{aligned}\Phi'(t) \le{}& \|u'\|^2 + h\|\nabla u'\|^2 + \frac{1}{4\delta} g \Box \partial^2 u \\ &- \left(1 - (\delta+1)\int_0^t g(s)\,ds\right)a(u,u) - \|\Delta v\|^2,\end{aligned} \tag{3.11}$$

where $\delta > 0$. Similarly we deduce

$$\begin{aligned}\Psi'(t) ={}& \int_0^t g(t-s)a\big(u(t)-u(s),u(t)\big)\,ds - \int_0^t g(t-s)\big(u(t)-u(s),[u,v]\big)\,ds \\ &- \int_0^t g(t-s)a\left(u(t)-u(s), \int_0^t g(t-\tau)u(\tau)\,d\tau\right)ds \\ &- h\int_0^t g'(t-s)\big(\nabla u(t)-\nabla u(s),\nabla u'(t)\big)\,ds - h\left(\int_0^t g(s)\,ds\right)\|\nabla u'\|^2 \\ &- \int_0^t g'(t-s)\big(u(t)-u(s),u'(t)\big)\,ds - \left(\int_0^t g(s)\,ds\right)\|u'\|^2 \\ :={}& I_1 + I_2 + \cdots + I_5 - h\left(\int_0^t g(s)\,ds\right)\|\nabla u'\|^2 - \left(\int_0^t g(s)\,ds\right)\|u'\|^2.\end{aligned} \tag{3.12}$$

Now, we estimate the terms on the right-hand side of (3.12). Since $E(t)$ is bounded, we have that $\|v\|_\infty \le \int_\Omega |\Delta v|^2\,d\Omega$ is also bounded, and then Young's and Hölder's inequalities, (2.2), (2.3), (2.6) and (2.7) give that

$$\begin{aligned}|I_1| &\le \eta\left(\int_0^t g(s)\,ds\right)a(u,u) + \frac{1}{4\eta} g \Box \partial^2 u, \\ |I_2| &= \int_\Omega \left[u, \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right]v\,d\Omega \le \eta a(u,u) + c_\eta l\|v\|_\infty^2 g \Box \partial^2 u, \\ |I_3| &\le l\int_0^t g(t-s)a\big(u(t)-u(s),u(t)-u(s)\big)\,ds + l\int_0^t g(t-s)a\big(u(t)-u(s),u(t)\big)\,ds \\ &\le \left(l + \frac{l}{4\eta}\right) g \Box \partial^2 u + l^2\eta a(u,u), \\ |I_4| &\le h\eta\|\nabla u'\|^2 + \frac{h}{4\eta}\int_\Omega \left(\int_0^t g'(t-s)\big|\nabla u(t)-\nabla u(s)\big|\,ds\right)^2 d\Omega \\ &\le h\eta\|\nabla u'\|^2 - \frac{g(0)C_s h}{4\eta} g' \Box \partial^2 u, \\ |I_5| &\le \eta\|u'\|^2 + \frac{1}{4\eta}\int_\Omega \left(\int_0^t g'(t-s)\big|u(t)-u(s)\big|\,ds\right)^2 d\Omega \\ &\le \eta\|u'\|^2 - \frac{g(0)C_p}{4\eta} g' \Box \partial^2 u.\end{aligned}$$

From the above estimates, we see that

$$\Psi'(t) \le \left(h\eta - h\int_0^t g(s)\,ds\right)\|\nabla u'\|^2 + \left(\eta - \int_0^t g(s)\,ds\right)\|u'\|^2 + \eta\left(1 + l + l^2\right)a(u,u) \\ + \left(l + \frac{l}{4\eta} + \frac{1}{4\eta} + c_\eta l\|v\|_\infty^2\right)g\,\Box\,\partial^2 u - \frac{g(0)(C_s h + C_p)}{4\eta}g'\,\Box\,\partial^2 u. \tag{3.13}$$

Let $\int_0^{t_0} g(s)\,ds := g_0$, where t_0 was introduced in (2.13). Since g is positive, we have $\int_0^t g(s)\,ds \ge g_0$ for all $t \ge t_0$. Thus, making use of this and combining (3.2), (3.4), (3.11) and (3.13), we obtain

$$\begin{aligned} L'(t) \le{}& -(g_0 - \eta - \epsilon)\|u'\|^2 - h(g_0 - \eta - \epsilon)\|\nabla u'\|^2 \\ & - \left[\frac{N}{2}g(t) + \epsilon\left(1 - (1+\delta)l\right) - \eta\left(1 + l + l^2\right)\right]a(u,u) \\ & - \epsilon\|\Delta v\|^2 + \left(\frac{N}{2} - \frac{g(0)(C_s h + C_p)}{4\eta}\right)g'\,\Box\,\partial^2 u \\ & + \left(l + \frac{\epsilon}{4\delta} + \frac{1}{4\eta} + \frac{l}{4\eta} + c_\eta l\|v\|_\infty^2\right)g\,\Box\,\partial^2 u. \end{aligned}$$

We first take $\epsilon > 0$ and $\delta > 0$ so small that $g_0 - \epsilon > 0$ and $1 - (1+\delta)l > 0$, respectively. And then, we choose $\eta > 0$ sufficiently small so that $g_0 - \eta - \epsilon > 0$ and $\epsilon(1 - (1+\delta)l) - \eta(1 + l + l^2) > 0$. Finally, taking $N > 0$ large enough, we deduce that (3.8). □

Proof of Theorem 2.1 From (2.17), (3.2) and (3.8), we have

$$\begin{aligned} L'(t) \le{}& -\beta_1\mathcal{E}(t) - \frac{\beta_2}{c_3}\int_0^{t_0} g'(t-s)a\bigl(u(t)-u(s), u(t)-u(s)\bigr)\,ds \\ & + \beta_2\int_{t_0}^t g(t-s)a\bigl(u(t)-u(s), u(t)-u(s)\bigr)\,ds \\ \le{}& -\beta_1\mathcal{E}(t) - \frac{2\beta_2}{c_3}\mathcal{E}'(t) + \beta_2\int_{t_0}^t g(t-s)a\bigl(u(t)-u(s), u(t)-u(s)\bigr)\,ds. \end{aligned} \tag{3.14}$$

We take $\mathcal{L}(t) = L(t) + \frac{2\beta_2}{c_3}\mathcal{E}(t)$, which is clearly equivalent to $\mathcal{E}(t)$. By (3.14), we get, for all $t \ge t_0$,

$$\mathcal{L}'(t) \le -\beta_1\mathcal{E}(t) + \beta_2\int_{t_0}^t g(s)a\bigl(u(t)-u(t-s), u(t)-u(t-s)\bigr)\,ds. \tag{3.15}$$

(A) The special case $H(t) = ct^p$ and $1 \le p < \frac{3}{2}$.
Case 1. $p = 1$. Using (2.4) and (3.2), estimate (3.15) yields

$$\begin{aligned} \mathcal{L}'(t) \le{}& -\beta_1\mathcal{E}(t) - \frac{\beta_2}{c}\int_{t_0}^t g'(s)a\bigl(u(t)-u(t-s), u(t)-u(t-s)\bigr)\,ds \\ \le{}& -\beta_1\mathcal{E}(t) - \frac{2\beta_2}{c}\mathcal{E}'(t), \end{aligned} \tag{3.16}$$

which gives

$$\left(\mathcal{L}+\frac{2\beta_2}{c}\mathcal{E}\right)'(t)\le -\beta_1\mathcal{E}(t),\quad \forall t\ge t_0.$$

From (3.3) and (3.5), we see that $\mathcal{L}+\frac{2\beta_2}{c}\mathcal{E}\sim\mathcal{E}\sim E$. Then we have

$$E(t)\le c'e^{-ct}=c'G^{-1}(t),$$

where

$$G(t)=\int_t^1\frac{1}{sH'(\epsilon_0 s)}\,ds=\int_t^1\frac{1}{sc}\,ds=-\frac{\ln t}{c}.$$

Case 2. $1<p<\frac{3}{2}$. By (2.4) we obtain

$$g'(t)\le -cg^p(t),\quad 1<p<\frac{3}{2}. \tag{3.17}$$

Using (2.3) and (3.17) we see that

$$\int_0^\infty g^{1-\theta}(s)\,ds<\infty \tag{3.18}$$

for any $\theta<2-p$. By (3.2) and (3.18) and taking t_0 even larger if needed, we deduce that, for all $t\ge t_0$,

$$\begin{aligned}k(t)&:=\int_{t_0}^t g^{1-\theta}(s)a\big(u(t)-u(t-s),u(t)-u(t-s)\big)\,ds\\&\le 2\int_{t_0}^t g^{1-\theta}(s)\big[a\big(u(t),u(t)\big)+a\big(u(t-s),u(t-s)\big)\big]\,ds\\&\le c\mathcal{E}(0)\int_{t_0}^t g^{1-\theta}(s)\,ds<1.\end{aligned} \tag{3.19}$$

From Hölder's inequality, Jensen's inequality (2.13), (3.2), (3.17) and (3.19), we have

$$\begin{aligned}&\int_{t_0}^t g(s)a\big(u(t)-u(t-s),u(t)-u(t-s)\big)\,ds\\&\quad=\int_{t_0}^t g^{(p-1+\theta)(\frac{\theta}{p-1+\theta})}(s)g^{1-\theta}(s)a\big(u(t)-u(t-s),u(t)-u(t-s)\big)\,ds\\&\quad\le\left(\int_{t_0}^t g^{p-1+\theta}(s)g^{1-\theta}(s)a\big(u(t)-u(t-s),u(t)-u(t-s)\big)\,ds\right)^{\frac{\theta}{p-1+\theta}}\\&\qquad\times\left(\int_{t_0}^t g^{1-\theta}(s)a\big(u(t)-u(t-s),u(t)-u(t-s)\big)\,ds\right)^{\frac{p-1}{p-1+\theta}}\\&\quad=k(t)\left(\frac{1}{k(t)}\int_{t_0}^t g^{p-1+\theta}(s)g^{1-\theta}(s)a\big(u(t)-u(t-s),u(t)-u(t-s)\big)\,ds\right)^{\frac{\theta}{p-1+\theta}}\end{aligned}$$

$$\leq \left(\int_{t_0}^{t} g^p(s) a\big(u(t)-u(t-s), u(t)-u(t-s)\big)\, ds\right)^{\frac{\theta}{p-1+\theta}}$$
$$\leq \left(\frac{1}{c}\right)^{\frac{\theta}{p-1+\theta}} \left(\int_{t_0}^{t} -g'(s) a\big(u(t)-u(t-s), u(t)-u(t-s)\big)\, ds\right)^{\frac{\theta}{p-1+\theta}}$$
$$\leq \left(\frac{1}{c}\right)^{\frac{\theta}{p-1+\theta}} \big(-\mathcal{E}'(t)\big)^{\frac{\theta}{p-1+\theta}}. \tag{3.20}$$

Then, using (3.20), we show that (3.15) yields, for $\theta = \frac{1}{2}$,

$$\mathcal{L}'(t) \leq -\beta_1 \mathcal{E}(t) + \frac{\beta_2}{c^{\frac{1}{2p-1}}} \big(-\mathcal{E}'(t)\big)^{\frac{1}{2p-1}}. \tag{3.21}$$

Multiplying (3.21) by $\mathcal{E}^\gamma(t)$, with $\gamma = 2p-2$, and using (3.2) and Young's inequality, we obtain

$$\begin{aligned}\big(\mathcal{L}\mathcal{E}^\gamma\big)'(t) &= \mathcal{L}'(t)\mathcal{E}^\gamma(t) + \gamma \mathcal{L}(t)\mathcal{E}^{\gamma-1}(t)\mathcal{E}'(t) \leq -\beta_1 \mathcal{E}^{\gamma+1}(t) + \frac{\beta_2}{c^{\frac{1}{\gamma+1}}} \mathcal{E}^\gamma(t)\big(-\mathcal{E}'(t)\big)^{\frac{1}{\gamma+1}} \\ &\leq -\beta_1 \mathcal{E}^{\gamma+1}(t) + \varepsilon \mathcal{E}^{\gamma+1}(t) + C_\varepsilon\big(-\mathcal{E}'(t)\big).\end{aligned}$$

Taking $\varepsilon < \beta_1$, we have, for some $C_1 > 0$,

$$L_0'(t) \leq -C_1 L_0^{\gamma+1}(t),$$

where $L_0 = \mathcal{L}\mathcal{E}^\gamma + C_\varepsilon \mathcal{E} \sim \mathcal{E} \sim E$. Therefore we deduce that

$$E(t) \leq \frac{c}{(c' + c''t)^{\frac{1}{\gamma}}}. \tag{3.22}$$

Since $p < \frac{3}{2}$ and by (3.22), we find that

$$\int_0^\infty E(t)\, dt \leq \int_0^\infty \frac{c}{(c' + c''t)^{\frac{1}{2p-2}}}\, dt < +\infty.$$

Using this fact, we have

$$\int_0^t a\big(u(t)-u(t-s), u(t)-u(t-s)\big)\, ds \leq c \int_0^t E(s)\, ds < +\infty. \tag{3.23}$$

Hence, from (3.2), (3.17), (3.23) and Hölder's inequality, estimate (3.15) gives

$$\begin{aligned}\mathcal{L}'(t) &\leq -\beta_1 \mathcal{E}(t) + \beta_2 \left(\int_0^t a\big(u(t)-u(t-s), u(t)-u(t-s)\big)\, ds\right)^{\frac{p-1}{p}} \big(g^p \,\Box\, \partial^2 u\big)^{\frac{1}{p}} \\ &\leq -\beta_1 \mathcal{E}(t) + c\big(-g' \,\Box\, \partial^2 u\big)^{\frac{1}{p}} \leq -\beta_1 \mathcal{E}(t) + c\big(-\mathcal{E}'(t)\big)^{\frac{1}{p}}.\end{aligned} \tag{3.24}$$

Now, we multiply (3.24) by $\mathcal{E}^\gamma(t)$, with $\gamma = p-1$. Then, repeating the above steps, we see that

$$E(t) \leq \frac{c}{(c' + c''t)^{\frac{1}{\gamma}}} = cG^{-1}\big(a' + a''t\big), \tag{3.25}$$

where

$$G(t) = \frac{1}{cp\epsilon_0^{p-1}} \int_t^1 \frac{1}{s^p} \, ds = \frac{1}{cp(p-1)\epsilon_0^{p-1}} \left(\frac{1}{t^{p-1}} - 1 \right).$$

(B) The general case. This case is obtained on account of the ideas presented in [24, 25] as follows. Let H_0^* be the convex conjugate of H_0 in the sense of Young (see [29]); then

$$H_0^*(s) = s\bigl(H_0'\bigr)^{-1}(s) - H_0\bigl[\bigl(H_0'\bigr)^{-1}(s)\bigr], \quad \text{if } s \in \bigl(0, H_0'(r)\bigr] \tag{3.26}$$

and H_0^* satisfies the following Young's inequality:

$$AB \le H_0^*(A) + H_0(B), \quad \text{if } A \in \bigl(0, H_0'(r)\bigr], B \in (0, r]. \tag{3.27}$$

We define $\eta(t)$ by

$$\eta(t) := \int_{t_0}^t \frac{g(s)}{H_0^{-1}(-g'(s))} a\bigl(u(t) - u(t-s), u(t) - u(t-s)\bigr) \, ds,$$

where H_0 is such that (2.9) is satisfied. As in (3.19), we find that $\eta(t)$ satisfies

$$\eta(t) < 1, \quad \forall t \ge t_0. \tag{3.28}$$

Furthermore, we define $I(t)$ by

$$I(t) := -\int_{t_0}^t g'(s) \frac{g(s)}{H_0^{-1}(-g'(s))} a\bigl(u(t) - u(t-s), u(t) - u(t-s)\bigr) \, ds.$$

Since H_0 is strictly convex on $(0, r]$ and $H_0(0) = 0$, then

$$H_0(\lambda x) \le \lambda H_0(x) \tag{3.29}$$

provided $0 \le \lambda \le 1$ and $x \in (0, r]$. From (3.28), (3.29) and Jensen's inequality (2.13), we obtain

$$\begin{aligned} I(t) &= \frac{1}{\eta(t)} \int_{t_0}^t \eta(t) H_0\bigl[H_0^{-1}\bigl(-g'(s)\bigr)\bigr] \frac{g(s)}{H_0^{-1}(-g'(s))} a\bigl(u(t) - u(t-s), u(t) - u(t-s)\bigr) \, ds \\ &\ge \frac{1}{\eta(t)} \int_{t_0}^t H_0\bigl[\eta(t) H_0^{-1}\bigl(-g'(s)\bigr)\bigr] \frac{g(s)}{H_0^{-1}(-g'(s))} a\bigl(u(t) - u(t-s), u(t) - u(t-s)\bigr) \, ds \\ &\ge H_0\left(\int_{t_0}^t g(s) a\bigl(u(t) - u(t-s), u(t) - u(t-s)\bigr) \, ds \right). \end{aligned}$$

This implies that

$$\int_{t_0}^t g(s) a\bigl(u(t) - u(t-s), u(t) - u(t-s)\bigr) \, ds \le H_0^{-1}\bigl(I(t)\bigr). \tag{3.30}$$

Using (3.15) and (3.30) we see that

$$\mathcal{L}'(t) \le -\beta_1 \mathcal{E}(t) + \beta_2 H_0^{-1}\bigl(I(t)\bigr), \quad \forall t \ge t_0. \tag{3.31}$$

By (2.3), (2.4) and the properties of H_0 and D, we have

$$\frac{g(s)}{H_0^{-1}(-g'(s))} \le \frac{g(s)}{H_0^{-1}(H(g(s)))} = \frac{g(s)}{D^{-1}(g(s))} \le \delta_0 \tag{3.32}$$

for some positive constant δ_0. Thus, using (2.13), (3.2) and (3.32) and choosing t_0 even larger, we can find that $I(t)$ satisfies, for all $t \ge t_0$,

$$\begin{aligned} I(t) &\le -\delta_0 \int_{t_0}^t g'(s) a\big(u(t)-u(t-s), u(t)-u(t-s)\big)\,ds \le -c\mathcal{E}(0)\int_{t_0}^t g'(s)\,ds \\ &\le cg(t_0)\mathcal{E}(0) \le \min\big\{r, H(r), H_0(r)\big\}. \end{aligned} \tag{3.33}$$

Now, for $\epsilon_0 < r$ and $d_0 > 0$, we define the functional

$$F_1(t) := \mathcal{L}(t) H_0'\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) + d_0\mathcal{E}(t),$$

which satisfies

$$d_1 F_1(t) \le \mathcal{E}(t) \le d_2 F_1(t) \tag{3.34}$$

for some $d_1, d_2 > 0$. From (3.33), we have $H_0^{-1}(I(t)) \le r$. Also, by $\epsilon_0 < r$, $\mathcal{E}' \le 0$, we get $\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} < r$. Using the fact that $\mathcal{E}' \le 0$, $H_0 > 0$, $H_0' > 0$ and $H_0'' > 0$ on $(0, r]$ and (3.2), (3.26), (3.27), (3.31) and (3.33), we obtain

$$\begin{aligned} F_1'(t) &\le -\beta_1 \mathcal{E}(t) H_0'\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) + \beta_2 H_0'\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) H_0^{-1}\big(I(t)\big) + d_0\mathcal{E}'(t) \\ &\le -\beta_1 \mathcal{E}(t) H_0'\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) + \beta_2 H_0^*\left(H_0'\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right)\right) + \beta_2 I(t) + d_0\mathcal{E}'(t) \\ &\le -\beta_1 \mathcal{E}(t) H_0'\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) + \beta_2 \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} H_0'\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) \\ &\quad - \beta_2 H_0\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) + \beta_2 I(t) + d_0\mathcal{E}'(t) \\ &\le -\big(\beta_1\mathcal{E}(0) - \beta_2\epsilon_0\big) \frac{\mathcal{E}(t)}{\mathcal{E}(0)} H_0'\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) - 2\beta_2\delta_0\mathcal{E}'(t) + d_0\mathcal{E}'(t). \end{aligned}$$

Therefore, with a suitable choice of ϵ_0 and d_0, we see that

$$F_1'(t) \le -k\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) H_0'\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) = -kH_2\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right), \quad \forall t \ge t_0, \tag{3.35}$$

where $k > 0$ and $H_2(t) = tH_0'(\epsilon_0 t)$. From the strict convexity of H_0 on $(0, r]$, we find that $H_2(t) > 0$ and $H_2'(t) = H_0'(\epsilon_0 t) + \epsilon_0 t H_0''(\epsilon_0 t) > 0$ on $(0, 1]$. We take

$$R(t) = \frac{d_1 F_1(t)}{\mathcal{E}(0)},$$

which is clearly equivalent to $\mathcal{E}(t)$. By (3.34), (3.35) and $H_2' > 0$, we have

$$R'(t) \le -\frac{kd_1}{\mathcal{E}(0)} H_2\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) \le -k_0 H_2\big(R(t)\big), \quad \forall t \ge t_0,$$

where $k_0 = \frac{kd_1}{\mathcal{E}(0)} > 0$. Hence, a simple integration gives, for some $k_1, k_2 > 0$,

$$R(t) \le H_1^{-1}(k_1 t + k_2), \quad \forall t \ge t_0, \tag{3.36}$$

where $H_1(t) = \int_t^1 \frac{1}{H_2(s)}\,ds$. Here, we have used, on the basis of the properties of H_2, the fact that H_1 is a strictly decreasing function on $(0,1]$ and $\lim_{t\to 0} H_1(t) = +\infty$. From (3.3), (3.34) and (3.36), estimate (2.10) is established.

Moreover, if $\int_0^t H_1(t)\,dt < +\infty$, then

$$\int_0^t a\bigl(u(t) - u(t-s), u(t) - u(t-s)\bigr)\,ds \le c\int_0^t E(s)\,ds < +\infty.$$

Similarly, we define, for large t_0,

$$\eta(t) := \int_{t_0}^t a\bigl(u(t) - u(t-s), u(t) - u(t-s)\bigr)\,ds < 1$$

and

$$I(t) := -\int_{t_0}^t g'(s) a\bigl(u(t) - u(t-s), u(t) - u(t-s)\bigr)\,ds.$$

Using (2.4), the strict convexity of H and Jensen's inequality (2.13), we have

$$\begin{aligned}
I(t) &\ge \frac{1}{\eta(t)}\int_{t_0}^t \eta(t) H\bigl(g(s)\bigr) a\bigl(u(t) - u(t-s), u(t) - u(t-s)\bigr)\,ds \\
&\ge \frac{1}{\eta(t)}\int_{t_0}^t H\bigl(\eta(t) g(s)\bigr) a\bigl(u(t) - u(t-s), u(t) - u(t-s)\bigr)\,ds \\
&\ge H\left(\frac{1}{\eta(t)}\int_{t_0}^t \eta(t) g(s) a\bigl(u(t) - u(t-s), u(t) - u(t-s)\bigr)\,ds\right) \\
&= H\left(\int_{t_0}^t g(s) a\bigl(u(t) - u(t-s), u(t) - u(t-s)\bigr)\,ds\right).
\end{aligned}$$

Thus, we deduce that

$$\int_{t_0}^t g(s) a\bigl(u(t) - u(t-s), u(t) - u(t-s)\bigr)\,ds \le H^{-1}\bigl(I(t)\bigr),$$

and (3.15) becomes

$$\mathcal{L}'(t) \le -\beta_1 \mathcal{E}(t) + \beta_2 H^{-1}\bigl(I(t)\bigr), \quad \forall t \ge t_0.$$

Therefore, repeating the same procedures, we find that for some k_1, k_2 and $k_3 > 0$,

$$E(t) \le k_3 G^{-1}(k_1 t + k_2),$$

where $G(t) = \int_t^1 \frac{1}{sH'(\epsilon_0 s)}\,ds$. $\square$

Example We give an example to illustrate the energy decay rates given by Theorem 2.1. If

$$g(t)=\frac{1}{a+t^q}$$

for $q>3$ and $a>1$ chosen so that g satisfies (2.3), then $g'(t)=-H(g(t))$, where

$$H(t)=qt^2\left(\frac{1}{t}-a\right)^{1-\frac{1}{q}}.$$

Since

$$H'(t)=\frac{q(1+\frac{1}{q}-2at)}{(\frac{1}{t}-a)^{\frac{1}{q}}},\qquad H''(t)=\frac{\frac{2a^2q}{t^2}(t-\frac{1+q-\sqrt{q^2-1}}{2aq})(t-\frac{1+q+\sqrt{q^2-1}}{2aq})}{(\frac{1}{t}-a)^{1+\frac{1}{q}}},$$

then the function H satisfies hypothesis (H2) on the interval $(0,r]$ for any $0<r<\frac{1+q-\sqrt{q^2-1}}{2aq}$. By choosing $D(t)=t^\alpha$, (2.9) is satisfied for any $\alpha>\frac{q}{q-1}$. Then an explicit rate of decay can be obtained by Theorem 2.1. The function $H_0(t)=H(t^\alpha)$ has derivative

$$H_0'(t)=\frac{q\alpha t^{\alpha-1}[1+\frac{1}{q}-2at^\alpha]}{(\frac{1}{t^\alpha}-a)^{\frac{1}{q}}}.$$

Hence,

$$H_1(t)=\int_t^1\frac{[\frac{1}{(\epsilon_0 s)^\alpha}-a]^{\frac{1}{q}}}{q\alpha s(\epsilon_0 s)^{\alpha-1}[1+\frac{1}{q}-2a(\epsilon_0 s)^\alpha]}\,ds.$$

Let $\frac{1}{(\epsilon_0 s)^\alpha}=u$, then we have

$$H_1(t)=\int_{\frac{1}{\epsilon_0^\alpha}}^{\frac{1}{(\epsilon_0 t)^\alpha}}\frac{(u-a)^{\frac{1}{q}}u^{-\frac{1}{\alpha}}}{q\alpha^2[1+\frac{1}{q}-\frac{2a}{u}]}\,du\le\frac{1}{q\alpha^2[1+\frac{1}{q}-2a\epsilon_0^\alpha]}\int_{\frac{1}{\epsilon_0^\alpha}}^{\frac{1}{(\epsilon_0 t)^\alpha}}(u-a)^{\frac{1}{q}}u^{-\frac{1}{\alpha}}\,du.$$

Using the fact that the function $f(u)=(u-a)^{\frac{1}{q}}$ is increasing on $(a,+\infty)$ and $(u-a)^{\frac{1}{q}}<u^{\frac{1}{q}}$ and taking $\epsilon_0<a^{-\frac{1}{\alpha}}$, then

$$H_1(t)\le\frac{1}{q\alpha^2[1+\frac{1}{q}-2a\epsilon_0^\alpha]}\int_{\frac{1}{\epsilon_0^\alpha}}^{\frac{1}{(\epsilon_0 t)^\alpha}}u^{\frac{1}{q}-\frac{1}{\alpha}}\,du=\frac{\epsilon_0^{\frac{q-\alpha-\alpha q}{q}}}{\alpha(\alpha-q+\alpha q)[1+\frac{1}{q}-2a\epsilon_0^\alpha]}\left[t^{\frac{q-\alpha-\alpha q}{q}}-1\right].$$

Now, we find that if $\alpha<\frac{2q}{1+q}$,

$$\begin{aligned}\int_0^1 H_1(t)\,dt&\le\frac{\epsilon_0^{\frac{q-\alpha-\alpha q}{q}}}{\alpha(\alpha-q+\alpha q)[1+\frac{1}{q}-2a\epsilon_0^\alpha]}\int_0^1\left[t^{\frac{q-\alpha-\alpha q}{q}}-1\right]dt\\&=\frac{\epsilon_0^{\frac{q-\alpha-\alpha q}{q}}}{\alpha(2q-\alpha-\alpha q)[1+\frac{1}{q}-2a\epsilon_0^\alpha]}<+\infty.\end{aligned}$$

Choosing $\frac{1}{\epsilon_0 s} = \nu$ and $\epsilon_0 < a^{-1}$, we obtain

$$G(t) = \int_t^1 \frac{1}{sH'(\epsilon_0 s)}\,ds = \int_t^1 \frac{(\frac{1}{\epsilon_0 s} - a)^{\frac{1}{q}}}{sq(1 + \frac{1}{q} - 2a\epsilon_0 s)}\,ds = \int_{\frac{1}{\epsilon_0}}^{\frac{1}{\epsilon_0 t}} \frac{(\nu - a)^{\frac{1}{q}}\nu^{-1}}{q(1 + \frac{1}{q} - \frac{2a}{\nu})}\,d\nu$$
$$\leq \frac{1}{q(1 + \frac{1}{q} - 2a\epsilon_0)} \int_{\frac{1}{\epsilon_0}}^{\frac{1}{\epsilon_0 t}} \nu^{\frac{1}{q}-1}\,d\nu = \frac{1}{1 + \frac{1}{q} - 2a\epsilon_0}\left[\left(\frac{1}{\epsilon_0 t}\right)^{\frac{1}{q}} - \left(\frac{1}{\epsilon_0}\right)^{\frac{1}{q}}\right].$$

Therefore,

$$G^{-1}(t) \leq \frac{1}{\epsilon_0[(\frac{1}{\epsilon_0})^{\frac{1}{q}} + (1 + \frac{1}{q} - 2a\epsilon_0)t]^q}.$$

Then we can use (2.11) to deduce that the energy decays at the same rate of g, that is,

$$E(t) \leq \frac{\tilde{c}_1}{\tilde{c}_2 + \tilde{c}_3 t^q},$$

where $\tilde{c}_i$ ($i = 1, 2, 3$) are constants.

Competing interests
The author declares that she has no competing interests.

Author's contributions
The work was realized by the author.

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