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Triple positive solutions for two classes of delayed nonlinear fractional FDEs with nonlinear integral boundary value conditions

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Abstract

This paper is concerned with two classes of delayed nonlinear fractional functional differential equations (FDEs) with nonlinear Riemann-Stieltjes integral boundary value conditions. By employing the well-known Leggett-Williams fixed point theorem and a generalization of Leggett-Williams fixed point theorem, some new sufficient criteria are established to guarantee the existence of at least triple positive solutions. As applications, some interesting examples are presented to illustrate our main results.

Keywords: delayed nonlinear fractional FDEs; multiple positive solutions; nonlinear Riemann-Stieltjes integral boundary conditions; fixed point theorem

1 Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so forth. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. For example, in physics, the traditional way to deal with the behavior of certain materials under the influence of external forces in mechanics is to use the laws of Hooke and Newton. If we are dealing with viscous liquids, then we can use Newton's law $\eta\varepsilon'(t) = \sigma(t)$, where $\sigma(t)$ and $\varepsilon(t)$ denote stress and strain at time t respectively, η is the so-called viscosity of the material. In view of some possible interpolation properties, it is natural for us to design the classical Newton's law according to

$$\eta D_{0^+}^{k}\varepsilon(t) = \sigma(t), \quad k \in (n-1, n), n \in \mathbb{N},$$

which is called Nutting's law [1]. As a consequence, the subject of fractional differential equations is gaining much importance and attention. Especially, the boundary value problems of fractional differential equations have been one of the hottest problems. There have been many papers focused on boundary value problems of fractional ordinary differential equations; see [1–16]. Moreover, the boundary value problems with Riemann-Stieltjes integral boundary condition arise in a variety of different areas of applied mathematics and physics (for more comments on Stieltjes integral boundary condition and its importance, we refer the reader to the papers by Webb and Infante [17, 18] and their other related works). For example, blood flow problems, chemical engineering, thermo-elasticity, underground water flow, population dynamics, and so on can be reduced to nonlocal integral boundary problems. By means of some well-known fixed point theorems, some papers deal with the existence and multiplicity of solutions or positive solutions for this type of boundary value problems involving fractional differential in the recent references (see [19–25]).

In the real world, the time-delay phenomenon exists commonly and is inevitable. Many changes and processes not only depend on the present status but also on the past status. Therefore, it is necessary to consider the time-delay effect in the mathematical modeling of fractional differential equations. To the best of our knowledge, there are rare papers dealing with the existence of positive solutions for fractional Riemann-Stieltjes integral BVPs with time-delays by the well-known Leggett-Williams fixed point theorem. Therefore, the main goal of this paper is to study the existence of multiple positive solutions of Riemann-Stieltjes integral boundary value problems (BVP for short) involving time-delays for two classes of nonlinear Caputo fractional differential equations (1.1) and (1.2) as follows:

$$\begin{cases} D_{0+}^q u(t) + f(t, u, u_t) = 0, \quad t \in I, 2 < q \leq 3, \\ \alpha u(0) - \beta u'(0) = g_1(\int_0^1 u(s)\, dA_1(s)), \qquad u''(0) = 0, \\ \gamma u(1) + \delta u'(1) = g_2(\int_0^1 u(s)\, dA_2(s)), \\ u(s) = \phi(s), \quad s \in [-\tau, 0] \triangleq J \end{cases} \tag{1.1}$$

and

$$\begin{cases} D_{0+}^q u(t) + g(t, u, u_t, u') = 0, \quad t \in I, 2 < q \leq 3, \\ \alpha u(0) - \beta u'(0) = \int_0^1 h_1(s, u(s))\, dA_1(s), \qquad u''(0) = 0, \\ \gamma u(1) + \delta u'(1) = \int_0^1 h_2(s, u(s))\, dA_2(s), \\ u(s) = \phi(s), \quad s \in [-\tau, 0] \triangleq J, \end{cases} \tag{1.2}$$

where $I \triangleq [0,1]$, D_{0+}^q is the standard Caputo fractional derivative of fractional order $q. f \in C([0,1] \times \mathbb{R}^2, \mathbb{R}^+)$, $g_i \in C(\mathbb{R}^+, \mathbb{R}^+)$ $(i = 1, 2)$, $g \in C([0,1] \times \mathbb{R}^3, \mathbb{R}^+)$, $h_i \in C([0,1] \times \mathbb{R}, \mathbb{R}^+)$ $(i = 1, 2)$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$. α, β, γ, δ are all nonnegative constants with $\rho \triangleq \alpha\gamma + \alpha\delta + \beta\gamma > 0$. $\int_0^1 u(s)\, dA_i(s)$ $(i = 1, 2)$ denotes the Riemann-Stieltjes integrals. $A_i : [0,1] \to \mathbb{R}$ $(i = 1, 2)$ is the increasing function of bounded variation. $\tau > 0$ is the constant time-delay. $\phi(t) \in C_\tau$ (C_τ will be given in Section 3), $u_t \in C_\tau$, $u_t(\theta) = u(t + \theta)$, $\theta \in [-\tau, 0]$.

The rest of this paper is organized as follows. In Section 2, we introduce some definitions and lemmas to prove our main results. In Section 3, one sufficient condition is given by the well-known Leggett-Williams fixed point theorem to guarantee the existence multiple positive solutions for BVP (1.1). Applying a generalization of the Leggett-Williams fixed point theorem, we establish the existence of at least three positive solutions for BVP (1.2) in Section 4. As applications, some interesting examples are presented to illustrate the main results in Section 5.

2 Preliminaries and statements

For the convenience of the reader, we state some background materials from the theory of both fractional calculus and cones in Banach spaces. These definitions and properties can be found in the literature.

Definition 2.1 (see [26, 27]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2 (see [26, 27]) The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}\,ds,$$

where $n-1 < \alpha \le n$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.1 (see [26]) *Assume that $u \in C(0,1) \cap L(0,1)$ with a Caputo fractional derivative of order $\alpha > 0$ that belongs to $u \in C^n[0,1]$, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$ $(i = 0,1,\ldots,n-1)$, here n is the smallest integer greater than or equal to α.

Definition 2.3 Let X be a real Banach space. A nonempty closed convex set $P \subset X$ is called a cone if it satisfies the following two conditions:

(i) $x \in X$, $\lambda \ge 0$ implies $\lambda x \in P$;

(ii) $x \in P$, $-x \in P$ implies $x = 0$.

Every cone $P \subset X$ induces an ordering in X given by $x \preceq y$ if and only if $y - x \in P$.

Definition 2.4 The map ψ is said to be nonnegative continuous concave on a cone P of a real Banach space E provided that $\psi : P \to [0,\infty)$ is continuous and for all $u, v \in P$, $\lambda \in [0,1]$ such that

$$\psi\bigl(\lambda u + (1-\lambda)v\bigr) \ge \lambda\psi(u) + (1-\lambda)\psi(v).$$

Similarly, the map ϖ is said to be nonnegative continuous convex on a cone P of a real Banach space E provided that $\varpi : P \to [0,\infty)$ is continuous and for all $u, v \in P$, $\lambda \in [0,1]$ such that

$$\varpi\bigl(\lambda u + (1-\lambda)v\bigr) \le \lambda\varpi(u) + (1-\lambda)\varpi(v).$$

Let P be a cone in a real Banach space E. For $c > 0$, $0 < b < d$, we define

$$P_c = \bigl\{x \in P : \|x\| < c\bigr\}, \qquad \overline{P}_c = \bigl\{x \in P : \|x\| \le c\bigr\}$$

and

$$P(\psi, b, d) = \bigl\{x \in P : b \le \psi(x), \|x\| \le d\bigr\}.$$

It is easy to see that $P(\psi, b, d)$ is a convex and closed subset of P.

Lemma 2.2 (Leggett-Williams fixed point theorem, see [28]) *Let P be a cone in a real Banach space E, ψ be a nonnegative continuous concave functional on P such that $\psi(x) \leq \|x\|$ for $x \in \overline{P}_c$. Suppose that $A : \overline{P}_c \to \overline{P}_c$ is completely continuous and there exist $0 < a < b < d \leq c$ such that*

(i) $\{x \in P(\psi, b, d) : \psi(x) > b\} \neq \emptyset$ *and* $\psi(Ax) > b$ *for all* $x \in P(\psi, b, d)$;
(ii) $\|Ax\| < a$ *for all* $x \in \overline{P}_a$;
(iii) $\psi(Ax) > b$ *for all* $x \in P(\psi, b, c)$ *with* $\|Ax\| > d$.

Then A has at least three fixed points x_1, x_2 and x_3 satisfying

$$\|x_1\| < a < \|x_3\|, \qquad \psi(x_3) < b < \psi(x_2).$$

Next, we are prepared to state an important generalization of the Leggett-Williams fixed point theorem, which comes from Bai and Ge in [29].

Let ψ be a nonnegative continuous concave functional on P, and let ϖ and ω be nonnegative continuous convex functionals on P. For nonnegative real numbers r, a and l, we define the following convex sets:

$$\begin{aligned} &P(\varpi, r; \omega, l) = \{u \in P : \varpi(u) < r, \omega(u) < l\}, \\ &\bar{P}(\varpi, r; \omega, l) = \{u \in P : \varpi(u) \leq r, \omega(u) \leq l\}, \\ &P(\varpi, r; \omega, l; \psi, a) = \{u \in P : \varpi(u) < r, \omega(u) < l, \psi(u) > a\}, \\ &\bar{P}(\varpi, r; \omega, l; \psi, a) = \{u \in P : \varpi(u) \leq r, \omega(u) \leq l, \psi(u) \geq a\}. \end{aligned}$$

Lemma 2.3 (see [29]) *Let P be a cone in a real Banach space E. Assume that constants r_1, b, d, r_2, l_1 and l_2 satisfy $0 < r_1 < b < d \leq r_2$ and $0 < l_1 \leq l_2$. If there exist two nonnegative continuous convex functionals ϖ and ω on P and a nonnegative continuous concave functional ψ on P such that*:

(A_1) *there exists $M > 0$ such that $\|u\| \leq M \max\{\varpi(u), \omega(u)\}$ for all $u \in P$*;
(A_2) *$P(\varpi, r; \omega, l) \neq \emptyset$ for any $r > 0$ and $l > 0$*;
(A_3) *$\psi(u) \leq \varpi(u)$ for all $u \in P(\varpi, r_2; \omega, l_2)$*;

and if $A : P(\varpi, r_2; \omega, l_2) \to P(\varpi, r_2; \omega, l_2)$ is a completely continuous operator which satisfies

(B_1) *$\{u \in \bar{P}(\varpi, d; \omega, l_2; \psi, b) : \psi(u) > b\} \neq \emptyset$, $\psi(Au) > b$ for all $u \in \bar{P}(\varpi, d; \omega, l_2; \psi, b)$*;
(B_2) *$\varpi(Au) < r_1$, $\omega(Au) < l_1$ for $u \in \bar{P}(\varpi, r_1; \omega, l_1)$*;
(B_3) *$\psi(Au) > b$ for $u \in \bar{P}(\varpi, r_2; \omega, l_2; \psi, b)$ with $\varpi(Au) > d$*,

then A has at least three different fixed points u_1, u_2 and u_3 in $\bar{P}(\varpi, r_2; \omega, l_2)$ with

$$\begin{aligned} &u_1 \in P(\varpi, r_1; \omega, l_1), \qquad u_2 \in \{u \in \bar{P}(\varpi, d; \omega, l_2; \psi, b) : \psi(u) > b\}, \\ &u_3 \in \bar{P}(\varpi, r_2; \omega, l_2) \setminus \big(\bar{P}(\varpi, r_2; \omega, l_2; \psi, b) \cup \bar{P}(\varpi, r_1; \omega, l_1)\big). \end{aligned}$$

3 Triple positive solutions for BVP (1.1)

In this section, we discuss the existence of multiple positive solutions for boundary value problem (1.1).

Let $C_\tau \triangleq \{\varphi|\varphi : [-\tau,0] \to \mathbb{R}^+ \text{ is continuous}\}$. Then C_τ is the space with the norm $\|\varphi\|_{[-\tau,0]} = \max_{\theta\in[-\tau,0]} |\varphi(\theta)|$ for all $\varphi \in C_\tau$. $C(I,\mathbb{R})$ represents the Banach space of continuous functions from I to $\mathbb{R}$ with the norm $\|u\|_I = \max_{t\in I} |u(t)|$, where $I \triangleq [0,1]$.

Now let us consider the boundary value problem as follows:

$$\begin{cases} D_{0+}^q u(t) + y(t) = 0, \quad t \in (0,1), 2 < q \leq 3, \\ \alpha u(0) - \beta u'(0) = g_1(\int_0^1 u(s)\,dA_1(s)), \qquad u''(0) = 0, \\ \gamma u(1) + \delta u'(1) = g_2(\int_0^1 u(s)\,dA_2(s)). \end{cases} \tag{3.1}$$

Lemma 3.1 *Assume that $A_i : [0,1] \to \mathbb{R}$ $(i = 1,2)$ is a function of bounded variation, $g_i \in C(\mathbb{R},\mathbb{R})$, $\rho \triangleq \alpha\gamma + \alpha\delta + \beta\gamma \neq 0$ and $y \in C([0,1])$. Then $u \in C(I,\mathbb{R})$ is a solution of the boundary value problem* (3.1) *if and only if $u(t)$ is a solution of the following integral equation*:

$$\begin{aligned} u(t) = {} & \int_0^1 G(t,s)y(s)\,ds + \frac{\gamma(1-t)+\delta}{\rho} g_1\left(\int_0^1 u(s)\,dA_1(s)\right) \\ & + \frac{\alpha t + \beta}{\rho} g_2\left(\int_0^1 u(s)\,dA_2(s)\right), \end{aligned} \tag{3.2}$$

where

$$G(t,s) = \begin{cases} \frac{-\rho(t-s)^{q-1} + (\alpha t+\beta)[\gamma(1-s)+(q-1)\delta](1-s)^{q-2}}{\rho\Gamma(q)}, & 0 \leq s \leq t \leq 1, \\ \frac{(\alpha t+\beta)[\gamma(1-s)+(q-1)\delta](1-s)^{q-2}}{\rho\Gamma(q)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{3.3}$$

Proof Applying Lemma 2.1, Eq. (3.1) can be translated into the following equivalent integral equation:

$$u(t) = -I_{0+}^q y(t) + c_0 + c_1 t + c_2 t^2 = -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\,ds + c_0 + c_1 t + c_2 t^2. \tag{3.4}$$

From (3.4), we obtain

$$u'(t) = -\int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} y(s)\,ds + c_1 + 2c_2 t, u''(t) = -\int_0^t \frac{(t-s)^{q-3}}{\Gamma(q-2)} y(s)\,ds + 2c_2. \tag{3.5}$$

Condition $u''(0) = 0$ gives $c_2 = 0$. By the second boundary value condition of problem (3.1), we have

$$\gamma c_0 + (\gamma + \delta)c_1 = \gamma I_{0+}^q y(1) + \delta I_{0+}^{q-1} y(1) + g_2\left(\int_0^1 u(s)\,dA_2(s)\right). \tag{3.6}$$

From (3.5) and the first boundary value condition of problem (3.1), we have

$$c_1 = \frac{\alpha}{\rho}\left[\gamma I_{0+}^q y(1) + \delta I_{0+}^{q-1} y(1) + g_2\left(\int_0^1 u(s)\,dA_2(s)\right) - \frac{\gamma}{\alpha} g_1\left(\int_0^1 u(s)\,dA_1(s)\right)\right], \tag{3.7}$$

$$c_0 = \frac{\beta}{\rho}\left[\gamma I_{0+}^q y(1) + \delta I_{0+}^{q-1} y(1) + g_2\left(\int_0^1 u(s)\,dA_2(s)\right)\right] + \frac{\gamma+\delta}{\rho} g_1\left(\int_0^1 u(s)\,dA_1(s)\right). \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.4), we get

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)\,ds + \frac{\alpha}{\rho}\biggl[\gamma I^q y(1) + \delta I^{q-1}y(1) + g_2\biggl(\int_0^1 u(s)\,dA_2(s)\biggr) \\ &\quad - \frac{\gamma}{\alpha}g_1\biggl(\int_0^1 u(s)\,dA_1(s)\biggr)\biggr]t + \frac{\beta}{\rho}\biggl[\gamma I^q y(1) + \delta I^{q-1}y(1) + g_2\biggl(\int_0^1 u(s)\,dA_2(s)\biggr)\biggr] \\ &\quad + \frac{\gamma+\delta}{\rho}g_1\biggl(\int_0^1 u(s)\,dA_1(s)\biggr) \\ &= -\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}y(s)\,ds + \frac{(\alpha t+\beta)\gamma}{\rho}\biggl(\int_0^t + \int_t^1\biggr)\frac{(t-s)^{q-1}}{\Gamma(q)}y(s)\,ds \\ &\quad + \frac{(\alpha t+\beta)\delta}{\rho}\biggl(\int_0^t + \int_t^1\biggr)\frac{(t-s)^{q-2}}{\Gamma(q-1)}y(s)\,ds + \frac{\gamma(1-t)+\delta}{\rho}g_1\biggl(\int_0^1 u(s)\,dA_1(s)\biggr) \\ &\quad + \frac{\alpha t+\beta}{\rho}g_2\biggl(\int_0^1 u(s)\,dA_2(s)\biggr) \\ &= \int_0^1 G(t,s)y(s)\,ds + \frac{\gamma(1-t)+\delta}{\rho}g_1\biggl(\int_0^1 u(s)\,dA_1(s)\biggr) + \frac{\alpha t+\beta}{\rho}g_2\biggl(\int_0^1 u(s)\,dA_2(s)\biggr), \end{aligned}$$

where $G(t,s)$ is defined by (3.3). This indicates that u is a solution of (3.2). Conversely, noting that the above derivations are reversible, we assert that if u is a solution of the integral equation (3.2), then u is also the solution of BVP (3.1). The proof is complete. □

Lemma 3.2 *Assume $\alpha,\beta,\gamma,\delta \in [0,\infty)$ with $\rho \triangleq \alpha\gamma + \alpha\delta + \beta\gamma > 0$, then the function $G(t,s)$ defined by* (3.3) *has the following properties*:

(1) *$G(t,s)$ is continuous on $[0,1]\times[0,1]$*;
(2) *$G(t,s) > 0$ for any $t,s \in [0,1]$*;
(3) *$G(t,s) \le G(s,s)$ for any $t,s \in (0,1)$*;
(4) *there exists a positive number λ such that $G(t,s) \ge \lambda G(s,s)$ for any $t,s \in (0,1)$, where* $\lambda \triangleq \frac{4\alpha\gamma\delta[(q-2)\alpha+(q-1)\beta]\times\min\{1,\beta\}}{[(q-1)\alpha\delta+\alpha\gamma-\beta\gamma]^2+4\alpha\beta\gamma[(q-1)\delta+\gamma]} < 1$;
(5) $|\frac{\partial G(t,s)}{\partial t}| \le \Lambda(s) \triangleq \frac{(q-1)\rho(1-s)^{q-2}+\alpha[\gamma(1-s)+(q-1)\delta](1-s)^{q-2}}{\rho\Gamma(q)}$ *for all $t,s \in [0,1]$.*

The proof is similar to the proof of Lemmas 3.2 and 3.3 in [7] or Lemma 3.2 in [8], so we omit it here.

For each $\phi \in C_\tau$ and $u \in C(I,\mathbb{R})$, we define

$$u_t(s,\phi) \triangleq \begin{cases} u(t+s), & t+s \ge 0, \\ \phi(t+s), & t+s < 0, s \in J. \end{cases}$$

Obviously, $u_t(\cdot,\phi) \in C(I,\mathbb{R})$. Thus we have the following assertion.

Proposition 3.1 *A function $u \in C(I,\mathbb{R})$ is a solution of BVP* (1.1) *if and only if, for certain $\phi \in C_\tau$, u is a solution of the following BVP*:

$$\begin{cases} D_{0+}^q u(t) + f(t,u,u_t(\cdot,\phi)) = 0, \quad t \in [0,1], 2 < q \le 3, \\ \alpha u(0) - \beta u'(0) = g_1(\int_0^1 u(s)\,dA_1(s)), \qquad u''(0) = 0, \\ \gamma u(1) + \delta u'(1) = g_2(\int_0^1 u(s)\,dA_2(s)). \end{cases}$$

Therefore, by Lemma 3.1, we have $u(t) \in C(I,\mathbb{R})$ is a solution of BVP (1.1) if and only if $u(t)$ is a solution of the integral equation as follows:

$$\begin{aligned} u(t) = {}& \int_0^1 G(t,s)f\bigl(s,u(s),u_s(\cdot,\phi)\bigr)\,ds + \frac{\gamma(1-t)+\delta}{\rho}g_1\biggl(\int_0^1 u(s)\,dA_1(s)\biggr) \\ &+ \frac{\alpha t+\beta}{\rho}g_2\biggl(\int_0^1 u(s)\,dA_2(s)\biggr), \quad t \in [0,1], \end{aligned} \tag{3.9}$$

where $G(t,s)$ is defined by (3.3).

In order to study the existence of solution of (1.1), we define the operator $A_\phi : C(I,\mathbb{R}) \to C(I,\mathbb{R})$ as

$$\begin{aligned} (A_\phi u)(t) \triangleq {}& \int_0^1 G(t,s)f\bigl(s,u(s),u_s(\cdot,\phi)\bigr)\,ds + \frac{\gamma(1-t)+\delta}{\rho}g_1\biggl(\int_0^1 u(s)\,dA_1(s)\biggr) \\ &+ \frac{\alpha t+\beta}{\rho}g_2\biggl(\int_0^1 u(s)\,dA_2(s)\biggr), \quad t \in [0,1]. \end{aligned} \tag{3.10}$$

Then solving the solutions of BVP (1.1) reduces to solving the fixed points of the operator equation $u = A_\phi u$, where A_ϕ is given by (3.10). Thus, the fixed point of operator A_ϕ coincides with the solution of BVP (1.1).

For the sake of convenience, we introduce some assumptions as follows:

(H_1) $\alpha,\beta,\gamma,\delta \in [0,\infty)$ with $\rho \triangleq \alpha\gamma + \alpha\delta + \beta\gamma > 0$;
(H_2) $f \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ with $f(t,0,0) \not\equiv 0$ for all $t \in [0,1]$;
(H_3) $g_i \in C(\mathbb{R}^+,\mathbb{R}^+)$ $(i = 1,2)$; $\phi \in C([-\tau,0],\mathbb{R}^+)$;
(H_4) $A_i : [0,1] \to \mathbb{R}$ $(i = 1,2)$ is the increasing positive function of bounded variation.

Theorem 3.1 *Assume that conditions* (H_1)-(H_4) *hold. Suppose further that there exist* $\theta \in (0,\frac{1}{2})$, $\xi \in (0,1)$, $\sigma \in (1,+\infty)$, $\kappa \in [3,+\infty)$, $l_1 \in (0,\frac{\rho}{3(\gamma+\delta)\int_0^1 dA_1(s)}]$, $l_2 \in (0,\frac{\rho}{3(\alpha+\beta)\int_0^1 dA_2(s)}]$ *and some positive constants* a, b, c *with* $0 < a < b < \mu^2 c$ *such that the following conditions*

(H_5) $|g_i(u) - g_i(v)| \le l_i|u - v|$ *with* $g_i(0) = 0$ *for* $u,v \in [0,+\infty)$, $i = 1,2$;
(H_6) $f(t,u,v) < \frac{a}{\kappa\int_0^1 G(s,s)\,ds}$ *for* $(t,u,v) \in [0,1] \times [0,a] \times [0,a]$;
(H_7) $f(t,u,v) \ge \frac{b\sigma}{\mu\int_\theta^{1-\theta} G(\xi,s)\,ds}$ *for* $(t,u,v) \in [\theta,1-\theta] \times [b,\frac{b}{\mu^2}] \times [b,\frac{b}{\mu^2}]$;
(H_8) $f(t,u,v) \le \frac{c}{\kappa\int_0^1 G(s,s)\,ds}$ *for* $(t,u,v) \in [0,1] \times [0,c] \times [0,c]$,

have also been fulfilled, where $\mu \triangleq \min\{\lambda, \frac{\gamma\theta+\delta}{\gamma+\delta}, \frac{\alpha\theta+\beta}{\alpha+\beta}\}$, $\lambda = \frac{4\alpha\gamma\delta[(q-2)\alpha+(q-1)\beta]\times\min\{1,\beta\}}{[(q-1)\alpha\delta+\alpha\gamma-\beta\gamma]^2+4\alpha\beta\gamma[(q-1)\delta+\gamma]} < 1$. *Then BVP* (1.1) *has at least three positive solutions* u_1, u_2 *and* u_3 *satisfying*

$$\|u_1\|_I < a < \|u_3\|_I, \qquad \min_{\theta\le t\le 1-\theta} u_3(t) < b < \min_{\theta\le t\le 1-\theta} u_2(t).$$

Proof Define a cone $\mathcal{P}$ in $C(I,\mathbb{R})$ as follows:

$$\mathcal{P} \triangleq \Bigl\{u(t) \in C(I,\mathbb{R}) : u(t) \ge 0, \min_{\theta\le t\le 1-\theta} u(t) \ge \mu\|u\|_I\Bigr\}.$$

Let $\psi : \mathcal{P} \to [0,+\infty)$ be the nonnegative continuous concave functional defined by

$$\psi(u) = \min_{\theta\le t\le 1-\theta} u(t), \quad u \in \mathcal{P}.$$

Evidently, for each $u \in \mathcal{P}$, we have $\psi(u) \le \|u\|_I$.

Now we will prove the assertions of Theorem 3.1 through two steps.

Step 1. Take $\mathcal{P}_c = \{u \in \mathcal{P} : \|u\|_I < c\}$, $\mathcal{P}_a = \{u \in \mathcal{P} : \|u\|_I < a\}$, then $\overline{\mathcal{P}}_c = \{u \in \mathcal{P} : \|u\|_I \le c\}$, $\overline{\mathcal{P}}_a = \{u \in \mathcal{P} : \|u\|_I \le a\}$. Define an operator $A_\phi : \overline{\mathcal{P}}_c \to \overline{\mathcal{P}}$ as (3.10). Now it is necessary to show that $A_\phi : \overline{\mathcal{P}}_c \to \overline{\mathcal{P}}_c$ is completely continuous. In fact, for any $u \in \overline{\mathcal{P}}_c \subset \mathcal{P}$ and any $t \in I$, by Lemma 3.2, we have

$$\begin{aligned}
\|A_\phi u\|_I &= \max_{t\in I}\bigl|(A_\phi u)(t)\bigr| \\
&= \max_{t\in I}\biggl|\int_0^1 G(t,s)f\bigl(s,u(s),u_s(\cdot,\phi)\bigr)\,ds + \frac{\gamma(1-t)+\delta}{\rho}g_1\biggl(\int_0^1 u(s)\,dA_1(s)\biggr) \\
&\quad + \frac{\alpha t+\beta}{\rho}g_2\biggl(\int_0^1 u(s)\,dA_2(s)\biggr)\biggr| \\
&\le \int_0^1 G(s,s)f\bigl(s,u(s),u_s(\cdot,\phi)\bigr)\,ds + \frac{\gamma+\delta}{\rho}g_1\biggl(\int_0^1 u(s)\,dA_1(s)\biggr) \\
&\quad + \frac{\alpha+\beta}{\rho}g_2\biggl(\int_0^1 u(s)\,dA_2(s)\biggr),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\min_{\theta\le t\le 1-\theta}(A_\phi u)(t) \\
&\quad = \min_{\theta\le t\le 1-\theta}\biggl|\int_0^1 G(t,s)f\bigl(s,u(s),u_s(\cdot,\phi)\bigr)\,ds + \frac{\gamma(1-t)+\delta}{\rho}g_1\biggl(\int_0^1 u(s)\,dA_1(s)\biggr) \\
&\qquad + \frac{\alpha t+\beta}{\rho}g_2\biggl(\int_0^1 u(s)\,dA_2(s)\biggr)\biggr| \\
&\quad \ge \int_0^1 \min_{0\le t\le 1} G(t,s)f\bigl(s,u(s),u_s(\cdot,\phi)\bigr)\,ds + \min_{\theta\le t\le 1-\theta}\biggl[\frac{\gamma(1-t)+\delta}{\rho}g_1\biggl(\int_0^1 u(s)\,dA_1(s)\biggr) \\
&\qquad + \frac{\alpha t+\beta}{\rho}g_2\biggl(\int_0^1 u(s)\,dA_2(s)\biggr)\biggr] \\
&\quad \ge \lambda\int_0^1 G(s,s)f\bigl(s,u(s),u_s(\cdot,\phi)\bigr)\,ds + \frac{\gamma\theta+\delta}{\rho}g_1\biggl(\int_0^1 u(s)\,dA_1(s)\biggr) \\
&\qquad + \frac{\alpha\theta+\beta}{\rho}g_2\biggl(\int_0^1 u(s)\,dA_2(s)\biggr) \\
&\quad \ge \min\biggl\{\lambda, \frac{\gamma\theta+\delta}{\gamma+\delta}, \frac{\alpha\theta+\beta}{\alpha+\beta}\biggr\}\biggl[\int_0^1 G(s,s)f\bigl(s,u(s),u_s(\cdot,\phi)\bigr)\,ds \\
&\qquad + \frac{\gamma+\delta}{\rho}g_1\biggl(\int_0^1 u(s)\,dA_1(s)\biggr) + \frac{\alpha+\beta}{\rho}g_2\biggl(\int_0^1 u(s)\,dA_2(s)\biggr)\biggr] = \mu\|A_\phi u\|_I.
\end{aligned} \tag{3.11}$$

On the other hand, when $u \in \overline{\mathcal{P}}_c$, then $\|u\|_I \le c$. Noting the assumptions of κ, l_1 and l_2, by applying Lemma 3.2 and conditions (H_3) and (H_8), we obtain

$$\begin{aligned}
\|A_\phi u\|_I &= \max_{t\in I}\bigl|(A_\phi u)(t)\bigr| \\
&= \max_{t\in I}\biggl|\int_0^1 G(t,s)f\bigl(s,u(s),u_s(\cdot,\phi)\bigr)\,ds + \frac{\gamma(1-t)+\delta}{\rho}g_1\biggl(\int_0^1 u(s)\,dA_1(s)\biggr) \\
&\quad + \frac{\alpha t+\beta}{\rho}g_2\biggl(\int_0^1 u(s)\,dA_2(s)\biggr)\biggr|
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^1 G(s,s)f\bigl(s,u(s),u_s(\cdot,\phi)\bigr)\,ds + \frac{\gamma+\delta}{\rho} g_1\biggl(\int_0^1 u(s)\,dA_1(s)\biggr) \\
&\quad + \frac{\alpha+\beta}{\rho} g_2\biggl(\int_0^1 u(s)\,dA_2(s)\biggr) \\
&\le \frac{c}{\kappa\int_0^1 G(s,s)\,ds}\int_0^1 G(s,s)\,ds + \frac{l_1(\gamma+\delta)\|u\|_I}{\rho}\int_0^1 dA_1(s) \\
&\quad + \frac{l_2(\alpha+\beta)\|u\|_I}{\rho}\int_0^1 dA_2(s) \\
&\le \frac{c}{\kappa} + \frac{\rho}{3(\gamma+\delta)\int_0^1 dA_1(s)} \times \frac{c(\gamma+\delta)}{\rho}\int_0^1 dA_1(s) \\
&\quad + \frac{\rho}{3(\alpha+\beta)\int_0^1 dA_2(s)} \times \frac{c(\alpha+\beta)}{\rho}\int_0^1 dA_2(s) \\
&\le \frac{c}{\kappa} + \frac{c}{3} + \frac{c}{3} \le \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c.
\end{aligned}
\tag{3.12}
$$

By (3.11) and (3.12), we conclude that $A_\phi(\overline{\mathcal{P}}_c) \subset \overline{\mathcal{P}}_c$, that is, $A_\phi : \overline{\mathcal{P}}_c \to \overline{\mathcal{P}}_c$ is well defined. Similar to the arguments of (3.12), it is easy to show that A_ϕ maps any bounded subset of $\mathcal{P}$ into the bounded subset of $\mathbb{R}$. So we omit it here. Thus, according to the Arzela-Ascoli theorem, we know that $A_\phi : \overline{\mathcal{P}}_c \to \overline{\mathcal{P}}_c$ is completely continuous. Similarly, one can prove that $A_\phi : \overline{\mathcal{P}}_a \to \overline{\mathcal{P}}_a$ defined as (3.10) is also completely continuous.

Step 2. In the following, we will verify conditions (i)-(iii) of Lemma 2.2. In fact, when $u \in \overline{\mathcal{P}}_a$, according to assumption (H_6), it is similar to (3.12) that $A_\phi(\overline{\mathcal{P}}_a) \subset \overline{\mathcal{P}}_a$, which implies that $\|A_\phi u\|_I \le a$ for all $\|u\|_I \le a$. Noticing that assumption (H_6) is a strict inequality, we conclude that $\|A_\phi u\|_I < a$ for all $\|u\|_I \le a$. This indicates that condition (ii) of Lemma 2.2 holds.

Next, we show that condition (i) of Lemma 2.2 is satisfied. Clearly, $\{u \in \mathcal{P}(\psi, b, \frac{b}{\mu^2}) : \psi(u) > b\} \ne \emptyset$. Moreover, if $u \in \mathcal{P}(\psi, b, \frac{b}{\mu^2})$, then $\psi(u) = \frac{b}{\mu^2} > b$, so $b \le u(t) \le \frac{b}{\mu^2}$ for $t \in [\theta, 1-\theta]$. Thus, for $t \in [\theta, 1-\theta]$, from condition (H_7), we have

$$
\begin{aligned}
(A_\phi u)(t) &= \int_0^1 G(t,s)f\bigl(s,u(s),u_s(\cdot,\phi)\bigr)\,ds + \frac{\gamma(1-t)+\delta}{\rho} g_1\biggl(\int_0^1 u(s)\,dA_1(s)\biggr) \\
&\quad + \frac{\alpha t+\beta}{\rho} g_2\biggl(\int_0^1 u(s)\,dA_2(s)\biggr) \\
&\ge \int_\theta^{1-\theta} G(t,s)f\bigl(s,u(s),u_s(\cdot,\phi)\bigr)\,ds \ge \frac{b\sigma}{\mu\int_\theta^{1-\theta} G(\xi,s)\,ds}\int_\theta^{1-\theta} G(t,s)\,ds.
\end{aligned}
\tag{3.13}
$$

In view of (3.13) and the definition of ψ, we get

$$
\begin{aligned}
\psi(A_\phi u) &= \min_{\theta\le t\le 1-\theta}(A_\phi u)(t) \ge \mu\|A_\phi\|_I = \max_{t\in I}\bigl|\mu(A_\phi u)(t)\bigr| \ge \mu(A_\phi u)(\xi) \\
&\ge \frac{b\sigma\mu}{\mu\int_\theta^{1-\theta} G(\xi,s)\,ds}\int_\theta^{1-\theta} G(\xi,s)\,ds = \sigma b > b.
\end{aligned}
$$

Therefore, condition (i) of Lemma 2.2 is satisfied. Finally, we show that condition (iii) of Lemma 2.2 also holds. Indeed, assume that $u \in \overline{\mathcal{P}}_{\frac{b}{\mu^2}}$ with $\|A_\phi\|_I > \frac{b}{\mu^2}$, then by the definition of cone $\mathcal{P}$, we have

$$\psi(A_\phi u) = \min_{\theta \le t \le 1-\theta} (A_\phi u)(t) \ge \mu \|A_\phi\|_I > \mu \frac{b}{\mu^2} = \frac{b}{\mu} > b.$$

Hence, according to Lemma 2.2, BVP (1.1) has at least three positive solutions u_1, u_2, u_3 satisfying

$$\|u_1\|_I < a < \|u_3\|_I, \qquad \min_{\theta \le t \le 1-\theta} u_3(t) < b < \min_{\theta \le t \le 1-\theta} u_2(t).$$

The proof is complete. □

4 Triple positive solutions for BVP (1.2)

In this section, by employing a generalization of the Leggett-Williams fixed point theorem, we investigate the existence of at least three positive solutions for boundary value problem (1.2).

Consider the following boundary value problem:

$$\begin{cases} D_{0+}^q u(t) + y(t) = 0, \quad t \in (0,1), 2 < q \le 3, \\ \alpha u(0) - \beta u'(0) = \int_0^1 h_1(s,u(s))\,dA_1(s), \qquad u''(0) = 0. \\ \gamma u(1) + \delta u'(1) = \int_0^1 h_2(s,u(s))\,dA_2(s). \end{cases} \tag{4.1}$$

Lemma 4.1 *Assume that* $A_i : [0,1] \to \mathbb{R}$ $(i = 1,2)$ *is a function of bounded variation,* $h_i \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $\rho \triangleq \alpha\gamma + \alpha\delta + \beta\gamma \neq 0$ *and* $y \in C([0,1])$. *Then* $u \in C(I,\mathbb{R})$ *is a solution of the boundary value problem* (4.1) *if and only if* $u(t)$ *is a solution of the following integral equation*:

$$\begin{aligned} u(t) = {}& \int_0^1 G(t,s)y(s)\,ds + \frac{\gamma(1-t)+\delta}{\rho} \int_0^1 h_1\big(s,u(s)\big)\,dA_1(s) \\ & + \frac{\alpha t + \beta}{\rho} \int_0^1 h_2\big(s,u(s)\big)\,dA_2(s), \end{aligned} \tag{4.2}$$

where $G(t,s)$ *is defined by* (3.3).

The proof is similar to the proof of Lemma 3.1, so we omit it here.

Let C_τ and $C(I,\mathbb{R})$ be defined as in Section 3. $\mathbb{E} \triangleq C^1[0,1] = \{u : u, u' \in C(I,\mathbb{R})\}$. Then $\mathbb{E}$ is a Banach space with respect to the norm

$$\|u\|_{C^1} = \max\Big\{\max_{t\in I}\big|u(t)\big|, \max_{t\in I}\big|u'(t)\big|\Big\},$$

where $I \triangleq [0,1]$.

Define

$$P \triangleq \big\{u \in \mathbb{E} : u(t) \ge 0, u(t) \text{ is concave on } [0,1]\big\}.$$

Clearly, P is a cone.

For each $\phi \in C_\tau$ and $u \in \mathbb{E}$, we define

$$u_t(s,\phi) \triangleq \begin{cases} u(t+s), & t+s \geq 0, \\ \phi(t+s), & t+s < 0, s \in J. \end{cases}$$

Obviously, $u_t(\cdot,\phi) \in \mathbb{E}$. Thus we have the following assertion.

Proposition 4.1 *A function $u \in \mathbb{E}$ is a solution of BVP* (1.2) *if and only if, for certain $\phi \in C_\tau$, u is a solution of the following BVP*:

$$\begin{cases} D_{0+}^q u(t) + g(t,u,u_t(\cdot,\phi),u') = 0, & t \in [0,1], 2 < q \leq 3, \\ \alpha u(0) - \beta u'(0) = \int_0^1 h_1(s,u(s))\,dA_1(s), & u''(0) = 0, \\ \gamma u(1) + \delta u'(1) = \int_0^1 h_2(s,u(s))\,dA_2(s). \end{cases}$$

Consequently, by Lemma 4.1, we have $u(t) \in \mathbb{E}$ is a solution of BVPs (1.1) if and only if $u(t)$ is a solution of the integral equation as follows:

$$\begin{aligned} u(t) = {} & \int_0^1 G(t,s)g\bigl(s,u(s),u_s(\cdot,\phi),u'(s)\bigr)\,ds + \frac{\gamma(1-t)+\delta}{\rho}\int_0^1 h_1\bigl(s,u(s)\bigr)\,dA_1(s) \\ & + \frac{\alpha t+\beta}{\rho}\int_0^1 h_2\bigl(s,u(s)\bigr)\,dA_2(s), \quad t \in [0,1], \end{aligned} \tag{4.3}$$

where $G(t,s)$ is defined by (3.3).

By way of investigating the existence of solution of (1.2), we define an operator $A_\phi : P \to \mathbb{E}$ by

$$\begin{aligned} (A_\phi u)(t) \triangleq {} & \int_0^1 G(t,s)g\bigl(s,u(s),u_s(\cdot,\phi),u'(s)\bigr)\,ds + \frac{\gamma(1-t)+\delta}{\rho}\int_0^1 h_1\bigl(s,u(s)\bigr)\,dA_1(s) \\ & + \frac{\alpha t+\beta}{\rho}\int_0^1 h_2\bigl(s,u(s)\bigr)\,dA_2(s), \quad t \in [0,1]. \end{aligned} \tag{4.4}$$

Then solving the solutions of BVP (1.2) reduces to solving the fixed points of the operator equation $u = A_\phi u$, where A_ϕ is given by (4.4). Thus, the fixed point of operator A_ϕ coincides with the solution of BVP (1.2).

In this section, we assume that the following conditions are satisfied:

(G_1) $\alpha,\beta,\gamma,\delta \in [0,\infty)$ with $\rho \triangleq \alpha\gamma + \alpha\delta + \beta\gamma > 0$;
(G_2) $g \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ with $g(t,0,0,0) \not\equiv 0$ for all $t \in [0,1]$;
(G_3) $h_i \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$ $(i = 1,2)$; $\phi \in C([-\tau,0], \mathbb{R}^+)$;
(G_4) $A_i : [0,1] \to \mathbb{R}$ $(i = 1,2)$ is the increasing positive function of bounded variation.

Lemma 4.2 *Assume that* (G_1)-(G_4) *hold. Then, for $u \in P$, we have*:
(i) $(A_\phi u)(t)$ *is concave on* $[0,1]$;
(ii) $(A_\phi u)(t) \geq 0$ *for* $t \in [0,1]$.

Proof (i) By the definition of A_ϕ and $G(t,s)$, for $u \in P$, we have

$$(A_\phi u)''(t) = -\int_0^1 \frac{(t-s)^{q-3}}{\Gamma(q-2)}g\bigl(s,u(s),u_s(\cdot,\phi),u'(s)\bigr)\,ds \leq 0, \tag{4.5}$$

so $(A_\phi u)'(t)$ is nonincreasing. This implies that $(A_\phi u)(t)$ is concave.

(ii) According to the nonnegativity of $G(t,s)$ and $g(s,u(s),u_s(\cdot,\phi),u'(s))$, we can verify that $(A_\phi u)(t) \geq 0$ for $t \in [0,1]$.

The proof is complete. □

Lemma 4.3 *Suppose that* (G_1)-(G_4) *hold. Then* $A_\phi : P \to P$ *is a completely continuous operator.*

Proof From Lemma 4.2 it follows that $A_\phi : P \to P$ is well defined. Next, we show that A_ϕ is completely continuous. To this end, we assume that r is a positive constant and $u \in \overline{\Omega}_r = \{u \in P : \|u\|_{C^1} \leq r\}$. Note that the continuity of $g(t,u(t),u_t,u'(t))$ and $h_i(t,u(t))$ $(i = 1,2)$ guarantees that there exist some constants $M_i > 0$ $(i = 1,2,3)$ such that $g(t,u(t),u_t,u'(t)) \leq M_1$, $h_1(t,u(t)) \leq M_2$ and $h_2(t,u(t)) \leq M_3$ for all $t \in [0,1]$. Therefore, by Lemma 3.2, we have

$$\begin{aligned}
&\max_{t\in[0,1]} \bigl|(A_\phi u)(t)\bigr| \\
&\quad = \max_{t\in[0,1]} (A_\phi u)(t) \\
&\quad = \max_{t\in[0,1]} \biggl\{ \int_0^1 G(t,s) g\bigl(s,u(s),u_s(\cdot,\phi),u'(s)\bigr)\,ds + \frac{\gamma(1-t)+\delta}{\rho} \int_0^1 h_1\bigl(s,u(s)\bigr)\,dA_1(s) \\
&\qquad + \frac{\alpha t+\beta}{\rho} \int_0^1 h_2\bigl(s,u(s)\bigr)\,dA_2(s) \biggr\} \\
&\quad \leq M_1 \int_0^1 G(s,s)\,ds + \frac{(\gamma+\delta)M_2}{\rho} \int_0^1 dA_1(s) + \frac{(\alpha+\beta)M_3}{\rho} \int_0^1 dA_2(s)
\end{aligned}$$

and

$$\begin{aligned}
&\max_{t\in[0,1]} \bigl|(A_\phi u)'(t)\bigr| \\
&\quad = \max_{t\in[0,1]} \biggl| \int_0^1 \frac{\partial G(t,s)}{\partial t} g\bigl(s,u(s),u_s(\cdot,\phi),u'(s)\bigr)\,ds + \frac{-\gamma}{\rho} \int_0^1 h_1\bigl(s,u(s)\bigr)\,dA_1(s) \\
&\qquad + \frac{\alpha}{\rho} \int_0^1 h_2\bigl(s,u(s)\bigr)\,dA_2(s) \biggr| \\
&\quad \leq M_1 \int_0^1 \Lambda(s)\,ds + \frac{\gamma M_2}{\rho} \int_0^1 dA_1(s) + \frac{\alpha M_3}{\rho} \int_0^1 dA_2(s),
\end{aligned}$$

which imply that $A_\phi(\overline{\Omega}_r)$ is uniformly bounded.

Next, we shall prove that $A_\phi : P \to P$ is equicontinuous. Indeed, for any $u \in \overline{\Omega}_r$, $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}
&\bigl|(A_\phi u)(t_2) - (A_\phi u)(t_1)\bigr| \\
&\quad = \biggl| \int_0^1 \bigl[G(t_2,s) - G(t_1,s)\bigr] g\bigl(s,u(s),u_s(\cdot,\phi),u'(s)\bigr)\,ds \\
&\qquad + \frac{-\gamma(t_2-t_1)}{\rho} \int_0^1 h_1\bigl(s,u(s)\bigr)\,dA_1(s) + \frac{\alpha(t_2-t_1)}{\rho} \int_0^1 h_2\bigl(s,u(s)\bigr)\,dA_2(s) \biggr| \\
&\quad \leq \int_0^1 \biggl| \frac{\partial G(\xi,s)}{\partial t} \biggr| |t_2 - t_1| g\bigl(s,u(s),u_s(\cdot,\phi),u'(s)\bigr)\,ds
\end{aligned}$$

$$\begin{aligned}
&+\frac{\gamma|t_2-t_1|}{\rho}\int_0^1 h_1\big(s,u(s)\big)\,dA_1(s)+\frac{\alpha|t_2-t_1|}{\rho}\int_0^1 h_2\big(s,u(s)\big)\,dA_2(s)\\
&\le\left[M_1\int_0^1\Lambda(s)\,ds+\frac{M_2\gamma}{\rho}\int_0^1 dA_1(s)+\frac{M_3\alpha}{\rho}\int_0^1 dA_2(s)\right]|t_2-t_1|\\
&\to 0,\quad \text{as } t_2\to t_1 \qquad (4.6)
\end{aligned}$$

and

$$\begin{aligned}
&\big|(A_\phi u)'(t_2)-(A_\phi u)'(t_1)\big|\\
&=\left|\int_0^1\left[\frac{\partial G(t_2,s)}{\partial t}-\frac{\partial G(t_1,s)}{\partial t}\right]g\big(s,u(s),u_s(\cdot,\phi),u'(s)\big)\,ds\right|\\
&\le\frac{M_1}{\Gamma(q-1)}\int_0^1\big|(t_2-s)^{q-2}-(t_1-s)^{q-2}\big|\,ds\to 0,\quad \text{as } t_2\to t_1. \qquad (4.7)
\end{aligned}$$

Therefore, (4.6) and (4.7) imply that A_ϕ is equicontinuous for all $u\in\overline{\Omega}_r$. By applying the Arzela-Ascoli theorem, we can see that $A_\phi(\overline{\Omega}_r)$ is relatively compact. In view of Lebesgue's dominated convergence theorem, it is clear that A_ϕ is a continuous operator. Hence, $A_\phi: P\to P$ is a completely continuous operator. The proof is complete. □

For $u\in P$, we define

$$\varpi(u)=\max_{t\in[0,1]}\big|u(t)\big|,\qquad \omega(u)=\max_{t\in[0,1]}\big|u'(t)\big|,\qquad \psi(u)=\min_{t\in[\vartheta,1]}u(t),$$

where $\vartheta\in(0,1)$. It is easy to verify that $\varpi,\omega: P\to[0,+\infty)$ are nonnegative continuous convex functionals with $\|u\|_{C^1}=\max\{\varpi(u),\omega(u)\}$. $\psi: P\to[0,+\infty)$ is a nonnegative concave functional. We have $\psi(u)\le\varpi(u)$ for $u\in P$, this means that assumptions (A_1)-(A_3) in Lemma 2.3 hold.

Theorem 4.1 *Assume that conditions* (G_1)-(G_4) *hold. If there exist constants* r_1, r, r_2, l_1 *and* l_2 *with* $0<r_1<r<\frac{r}{\vartheta}\le r_2$, $0<l_1\le l_2$. *Suppose further that* g, h_i $(i=1,2)$ *satisfy the following conditions*:

(G_5) $g(t,u,v,w)\le\min\{\frac{r_2}{3\int_0^1 G(s,s)\,ds},\frac{l_2}{3\int_0^1\Lambda(s)\,ds}\}$ *for* $(t,u,v,w)\in[0,1]\times[0,r_2]\times[0,r_2]\times[-l_2,l_2]$; $h_1(t,u)\le\frac{\rho\min\{r_2,l_2\}}{3(\gamma+\delta)\int_0^1 dA_1(s)}$, $h_2(t,u)\le\frac{\rho\min\{r_2,l_2\}}{3(\alpha+\beta)\int_0^1 dA_2(s)}$, *for* $(t,u)\in[0,1]\times[0,r_2]$;

(G_6) $g(t,u,v,w)>\frac{r}{\lambda\int_0^1 G(s,s)\,ds}$ *for* $(t,u,v,w)\in[\vartheta,1]\times[r,\frac{r}{\vartheta}]\times[r,\frac{r}{\vartheta}]\times[-l_2,l_2]$;

(G_7) $g(t,u,v,w)<\min\{\frac{r_1}{3\int_0^1 G(s,s)\,ds},\frac{l_1}{3\int_0^1\Lambda(s)\,ds}\}$ *for* $(t,u,v,w)\in[0,1]\times[0,r_1]\times[0,r_1]\times[-l_1,l_1]$; $h_1(t,u)<\frac{\rho\min\{r_1,l_1\}}{3(\gamma+\delta)\int_0^1 dA_1(s)}$, $h_2(t,u)<\frac{\rho\min\{r_1,l_1\}}{3(\alpha+\beta)\int_0^1 dA_2(s)}$, *for* $(t,u)\in[0,1]\times[0,r_1]$;

(G_8) $\frac{\min\{\lambda,\frac{\delta}{\rho},\frac{\alpha\vartheta+\beta}{\rho}\}}{\max\{1,\frac{\gamma+\delta}{\rho},\frac{\alpha+\beta}{\rho}\}}>\vartheta$,

where $\lambda\triangleq\frac{4\alpha\gamma\delta[(q-2)\alpha+(q-1)\beta]\times\min\{1,\beta\}}{[(q-1)\alpha\delta+\alpha\gamma-\beta\gamma]^2+4\alpha\beta\gamma[(q-1)\delta+\gamma]}<1$, $\Lambda(s)\triangleq\frac{(q-1)\rho(1-s)^{q-2}+\alpha[\gamma(1-s)+(q-1)\delta](1-s)^{q-2}}{\rho\Gamma(q)}$. *Then BVP* (1.2) *has at least three nonnegative solutions* u_1, u_2 *and* u_3 *satisfying*

$$\max_{t\in[0,1]}\big\{u_1(t)\big\}<r_1,\qquad \max_{t\in[0,1]}\big\{\big|u_1'(t)\big|\big\}<l_1,$$

$$r<\min_{t\in[\vartheta,1]}\big\{u_2(t)\big\}\le\max_{t\in[0,1]}\big\{u_2(t)\big\}\le r_2,\qquad \max_{t\in[0,1]}\big\{\big|u_2'(t)\big|\big\}<l_2,$$

$$\min_{t\in[\vartheta,1]}\big\{u_3(t)\big\}<r,\qquad r_1<\max_{t\in[0,1]}\big\{u_3(t)\big\}<\frac{r}{\vartheta},\qquad l_1<\max_{t\in[0,1]}\big\{\big|u_3'(t)\big|\big\}\le l_2.$$

Proof The boundary value problem (1.2) has a solution $u = u(t)$ if and only if u solves the operator equation $A_\phi u = u$. Thus, we set out to verify that the operator A_ϕ satisfies Lemma 2.3, which will prove the existence of a fixed point of A_ϕ.

We first prove that if assumption (G_5) is satisfied, then $A_\phi : \bar{P}(\varpi, r_2; \omega, l_2) \to \bar{P}(\varpi, r_2; \omega, l_2)$ defined as (4.4). In fact, let $u \in \bar{P}(\varpi, r_2; \omega, l_2)$, then

$$\varpi(u) = \max_{t\in[0,1]} \bigl|u(t)\bigr| \le r_2, \qquad \omega(u) = \max_{t\in[0,1]} \bigl|u'(t)\bigr| \le l_2,$$

and assumption (G_5) implies

$$g\bigl(t, u(t), u_t, u'(t)\bigr) \le \min\biggl\{ \frac{r_2}{3\int_0^1 G(s,s)\,ds}, \frac{l_2}{3\int_0^1 \Lambda(s)\,ds} \biggr\}, \quad t \in [0,1].$$

For all $u \in P$, we have $A_\phi u \in P$, therefore,

$$\begin{aligned} \varpi(A_\phi u) &= \max_{t\in[0,1]} \bigl|(A_\phi u)(t)\bigr| = \max_{t\in[0,1]} (A_\phi u)(t) \\ &= \max_{t\in[0,1]} \biggl\{ \int_0^1 G(t,s) g\bigl(s, u(s), u_s(\cdot,\phi), u'(s)\bigr)\,ds \\ &\quad + \frac{\gamma(1-t)+\delta}{\rho} \int_0^1 h_1\bigl(s,u(s)\bigr)\,dA_1(s) + \frac{\alpha t + \beta}{\rho} \int_0^1 h_2\bigl(s,u(s)\bigr)\,dA_2(s) \biggr\} \\ &\le \frac{r_2}{3\int_0^1 G(s,s)\,ds} \times \int_0^1 G(s,s)\,ds + \frac{\gamma+\delta}{\rho} \times \frac{\rho r_2}{3(\gamma+\delta)\int_0^1 dA_1(s)} \times \int_0^1 dA_1(s) \\ &\quad + \frac{\alpha+\beta}{\rho} \times \frac{\rho r_2}{3(\alpha+\beta)\int_0^1 dA_1(s)} \times \int_0^1 dA_2(s) \\ &\le \frac{r_2}{3} + \frac{r_2}{3} + \frac{r_2}{3} = r_2 \end{aligned}$$

and

$$\begin{aligned} \omega(A_\phi u) &= \max_{t\in[0,1]} \bigl|(A_\phi u)'(t)\bigr| \\ &= \max_{t\in[0,1]} \biggl| \int_0^1 \frac{\partial G(t,s)}{\partial t} g\bigl(s, u(s), u_s(\cdot,\phi), u'(s)\bigr)\,ds \\ &\quad + \frac{-\gamma}{\rho} \int_0^1 h_1\bigl(s,u(s)\bigr)\,dA_1(s) + \frac{\alpha}{\rho} \int_0^1 h_2\bigl(s,u(s)\bigr)\,dA_2(s) \biggr| \\ &\le \frac{l_2}{3\int_0^1 \Lambda(s)\,ds} \times \int_0^1 \Lambda(s)\,ds + \frac{\gamma}{\rho} \times \frac{\rho l_2}{3(\gamma+\delta)\int_0^1 dA_1(s)} \times \int_0^1 dA_1(s) \\ &\quad + \frac{\alpha}{\rho} \times \frac{\rho l_2}{3(\alpha+\beta)\int_0^1 dA_1(s)} \times \int_0^1 dA_2(s) \\ &\le \frac{l_2}{3} + \frac{l_2}{3} + \frac{l_2}{3} = l_2. \end{aligned}$$

Thus, $A_\phi u \in \bar{P}(\varpi, r_2; \omega, l_2)$ and $A_\phi(\bar{P}(\varpi, r_2; \omega, l_2)) \subset \bar{P}(\varpi, r_2; \omega, l_2)$. In addition, according to Lemma 4.3, we know that $A_\phi : \bar{P}(\varpi, r_2; \omega, l_2) \to \bar{P}(\varpi, r_2; \omega, l_2)$ is completely continuous.

Secondly, we show that condition (B_1) of Lemma 2.3 holds. We let $u(t) = \frac{r}{\vartheta}$ for $t \in [0,1]$. It is obvious that $u(t) = \frac{r}{\vartheta} \in \bar{P}(\varpi, \frac{r}{\vartheta}; \omega, l_1)$ and $\psi(u) = \frac{r}{\vartheta} > r$, and consequently

$$\left\{u \in \bar{P}\left(\varpi, \frac{r}{\vartheta}; \omega, l_2; \psi, r\right) : \psi(u) > r\right\} \neq \emptyset.$$

For all $u \in \bar{P}(\varpi, \frac{r}{\vartheta}; \omega, l_2; \psi, r)$, we have $r \leq u(t) \leq \frac{r}{\vartheta}$, $|u'(t)| \leq l_2$ for all $t \in [\vartheta, 1]$. Thus, by assumption (G_6), we get

$$g\big(t, u(t), u_t, u'(t)\big) > \frac{r}{\lambda \int_0^1 G(s,s)\,ds} \quad \text{for } t \in [\vartheta, 1].$$

From the definition of the functional ψ and Lemma 3.2, we know that

$$\begin{aligned}
\psi(A_\phi u) &= \min_{t\in[\vartheta,1]} \big|(A_\phi u)(t)\big| = \min_{t\in[\vartheta,1]} (A_\phi u)(t) \\
&= \min_{t\in[\vartheta,1]} \left\{ \int_0^1 G(t,s) g\big(s, u(s), u_s(\cdot,\phi), u'(s)\big)\,ds \right. \\
&\quad \left. + \frac{\gamma(1-t)+\delta}{\rho} \int_0^1 h_1\big(s,u(s)\big)\,dA_1(s) + \frac{\alpha t + \beta}{\rho} \int_0^1 h_2\big(s,u(s)\big)\,dA_2(s) \right\} \\
&\geq \int_0^1 \min_{0\leq t\leq 1} G(t,s) g\big(s, u(s), u_s(\cdot,\phi), u'(s)\big)\,ds \\
&\geq \lambda \int_0^1 G(s,s) g\big(s, u(s), u_s(\cdot,\phi), u'(s)\big)\,ds \\
&> \lambda \int_0^1 G(s,s)\,ds \times \frac{r}{\lambda \int_0^1 G(s,s)\,ds} = r.
\end{aligned}$$

So, we obtain $\psi(A_\phi u) > r$ for $u \in \bar{P}(\varpi, \frac{r}{\vartheta}; \omega, l_2; \psi, r)$. Therefore, condition (B_1) of Lemma 2.3 is satisfied.

Thirdly, we show that condition (B_2) of Lemma 2.3 is satisfied. For all $u \in \bar{P}(\varpi, r_1; \omega, l_1)$, we have $0 \leq u(t) \leq r_1$, $-l_1 \leq u'(t) \leq l_1$ for $t \in [0,1]$. From assumption (G_7) we obtain

$$g\big(t, u(t), u_t, u'(t)\big) < \min\left\{ \frac{r_1}{3\int_0^1 G(s,s)\,ds}, \frac{l_1}{3\int_0^1 \Lambda(s)\,ds} \right\} \quad \text{for } t \in [0,1].$$

Thus

$$\begin{aligned}
\varpi(A_\phi u) &= \max_{t\in[0,1]} \big|(A_\phi u)(t)\big| = \max_{t\in[0,1]} (A_\phi u)(t) \\
&= \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) g\big(s, u(s), u_s(\cdot,\phi), u'(s)\big)\,ds \right. \\
&\quad \left. + \frac{\gamma(1-t)+\delta}{\rho} \int_0^1 h_1\big(s,u(s)\big)\,dA_1(s) + \frac{\alpha t + \beta}{\rho} \int_0^1 h_2\big(s,u(s)\big)\,dA_2(s) \right\} \\
&< \frac{r_1}{3\int_0^1 G(s,s)\,ds} \times \int_0^1 G(s,s)\,ds \\
&\quad + \frac{\gamma+\delta}{\rho} \times \frac{\rho r_1}{3(\gamma+\delta)\int_0^1 dA_1(s)} \times \int_0^1 dA_1(s)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha+\beta}{\rho}\times\frac{\rho r_1}{3(\alpha+\beta)\int_0^1 dA_1(s)}\times\int_0^1 dA_2(s)\\
&<\frac{r_2}{3}+\frac{r_2}{3}+\frac{r_2}{3}=r_1
\end{aligned}
$$

and

$$
\begin{aligned}
\omega(A_\phi u) &= \max_{t\in[0,1]}\bigl|(A_\phi u)'(t)\bigr|\\
&= \max_{t\in[0,1]}\biggl|\int_0^1\frac{\partial G(t,s)}{\partial t}g\bigl(s,u(s),u_s(\cdot,\phi),u'(s)\bigr)\,ds\\
&\quad+\frac{-\gamma}{\rho}\int_0^1 h_1\bigl(s,u(s)\bigr)\,dA_1(s)+\frac{\alpha}{\rho}\int_0^1 h_2\bigl(s,u(s)\bigr)\,dA_2(s)\biggr|\\
&< \frac{l_1}{3\int_0^1\Lambda(s)\,ds}\times\int_0^1\Lambda(s)\,ds+\frac{\gamma}{\rho}\times\frac{\rho l_1}{3(\gamma+\delta)\int_0^1 dA_1(s)}\times\int_0^1 dA_1(s)\\
&\quad+\frac{\alpha}{\rho}\times\frac{\rho l_1}{3(\alpha+\beta)\int_0^1 dA_1(s)}\times\int_0^1 dA_2(s)\\
&< \frac{l_1}{3}+\frac{l_1}{3}+\frac{l_1}{3}=l_1.
\end{aligned}
$$

We get $A_\phi:\bar{P}(\varpi,r_1;\omega,l_1)\to\bar{P}(\varpi,r_1;\omega,l_1)$, which means that (B_2) in Lemma 2.3 is satisfied.

Finally, we show that condition (B_3) of Lemma 2.3 holds. Indeed, according to Lemma 3.2, we have

$$
\begin{aligned}
\psi(A_\phi u) &= \min_{t\in[\vartheta,1]}\bigl\{\bigl|(A_\phi u)(t)\bigr|\bigr\}=\min_{t\in[\vartheta,1]}\bigl\{(A_\phi u)(t)\bigr\}\\
&= \min_{t\in[\vartheta,1]}\biggl\{\int_0^1 G(t,s)g\bigl(s,u(s),u_s(\cdot,\phi),u'(s)\bigr)\,ds\\
&\quad+\frac{\gamma(1-t)+\delta}{\rho}\int_0^1 h_1\bigl(s,u(s)\bigr)\,dA_1(s)+\frac{\alpha t+\beta}{\rho}\int_0^1 h_2\bigl(s,u(s)\bigr)\,dA_2(s)\biggr\}\\
&\geq \lambda\int_0^1 G(s,s)g\bigl(s,u(s),u_s(\cdot,\phi),u'(s)\bigr)\,ds+\frac{\delta}{\rho}\int_0^1 h_1\bigl(s,u(s)\bigr)\,dA_1(s)\\
&\quad+\frac{\alpha\vartheta+\beta}{\rho}\int_0^1 h_2\bigl(s,u(s)\bigr)\,dA_2(s)\\
&\geq \min\biggl\{\lambda,\frac{\delta}{\rho},\frac{\alpha\vartheta+\beta}{\rho}\biggr\}\biggl[\int_0^1 G(s,s)g\bigl(s,u(s),u_s(\cdot,\phi),u'(s)\bigr)\,ds\\
&\quad+\int_0^1 h_1\bigl(s,u(s)\bigr)\,dA_1(s)+\int_0^1 h_2\bigl(s,u(s)\bigr)\,dA_2(s)\biggr]
\end{aligned}
\tag{4.8}
$$

and

$$
\begin{aligned}
\varpi(A_\phi u) &= \max_{t\in[0,1]}\bigl\{\bigl|(A_\phi u)(t)\bigr|\bigr\}=\max_{t\in[0,1]}\bigl\{(A_\phi u)(t)\bigr\}\\
&= \max_{t\in[0,1]}\biggl\{\int_0^1 G(t,s)g\bigl(s,u(s),u_s(\cdot,\phi),u'(s)\bigr)\,ds\\
&\quad+\frac{\gamma(1-t)+\delta}{\rho}\int_0^1 h_1\bigl(s,u(s)\bigr)\,dA_1(s)+\frac{\alpha t+\beta}{\rho}\int_0^1 h_2\bigl(s,u(s)\bigr)\,dA_2(s)\biggr\}
\end{aligned}
$$

$$\begin{aligned}
&\le \int_0^1 G(s,s)g\bigl(s,u(s),u_s(\cdot,\phi),u'(s)\bigr)\,ds \\
&\quad + \frac{\gamma+\delta}{\rho}\int_0^1 h_1\bigl(s,u(s)\bigr)\,dA_1(s) + \frac{\alpha+\beta}{\rho}\int_0^1 h_2\bigl(s,u(s)\bigr)\,dA_2(s) \\
&\le \max\biggl\{1,\frac{\gamma+\delta}{\rho},\frac{\alpha+\beta}{\rho}\biggr\}\biggl[\int_0^1 G(s,s)g\bigl(s,u(s),u_s(\cdot,\phi),u'(s)\bigr)\,ds \\
&\quad + \int_0^1 h_1\bigl(s,u(s)\bigr)\,dA_1(s) + \int_0^1 h_2\bigl(s,u(s)\bigr)\,dA_2(s)\biggr].
\end{aligned} \tag{4.9}$$

For all $u \in \bar{P}(\varpi, r_2; \omega, l_2; \psi, r)$ with $\varpi(A_\phi u) > \frac{r}{\vartheta}$, in the light of (4.8), (4.9) and condition (G_8), we have

$$\psi(A_\phi u) \ge \frac{\min\{\lambda, \frac{\delta}{\rho}, \frac{\alpha\vartheta+\beta}{\rho}\}}{\max\{1, \frac{\gamma+\delta}{\rho}, \frac{\alpha+\beta}{\rho}\}} \times \varpi(A_\phi u) > r.$$

Therefore, condition (B_3) of Lemma 2.3 is satisfied. So, all the conditions of Lemma 2.3 are satisfied. It follows from Lemma 2.3 and the assumption that $g(t,0,0,0) \neq 0$ on $[0,1]$ that A_ϕ has at least three fixed points u_1, u_2 and u_3 satisfying

$$\begin{aligned}
&\max_{t\in[0,1]}\bigl\{u_1(t)\bigr\} < r_1, \qquad \max_{t\in[0,1]}\bigl\{\bigl|u_1'(t)\bigr|\bigr\} < l_1, \\
&r < \min_{t\in[\vartheta,1]}\bigl\{u_2(t)\bigr\} \le \max_{t\in[0,1]}\bigl\{u_2(t)\bigr\} \le r_2, \qquad \max_{t\in[0,1]}\bigl\{\bigl|u_2'(t)\bigr|\bigr\} < l_2, \\
&\min_{t\in[\vartheta,1]}\bigl\{u_3(t)\bigr\} < r, \qquad r_1 < \max_{t\in[0,1]}\bigl\{u_3(t)\bigr\} < \frac{r}{\vartheta}, \qquad l_1 < \max_{t\in[0,1]}\bigl\{\bigl|u_3'(t)\bigr|\bigr\} \le l_2.
\end{aligned}$$

The proof is complete. □

5 Some examples

In this section, we present some examples to illustrate our main results.

Example 5.1 Consider the boundary value problem of delayed nonlinear fractional differential equations as follows:

$$\begin{cases} D^{\frac{5}{2}}u(t) + f(t,u,u_t) = 0, \quad t \in [0,1], \\ \frac{1}{2}u(0) - u'(0) = g_1(\int_0^1 u(s)\,dA_1(s)), \qquad u''(0) = 0, \\ \frac{1}{2}u(1) + \frac{1}{2}u'(1) = g_2(\int_0^1 u(s)\,dA_2(s)), \\ u(s) = \phi(s), \quad s \in [-\tau, 0], \end{cases} \tag{5.1}$$

where, $q = \frac{5}{2}$, $\alpha = \gamma = \delta = \frac{1}{2}$, $\beta = 1$, $g_1(v) = g_2(v) = |\sin\frac{v}{4}|$, $A_1(s) = A_2(s) = \frac{1}{3}s$, $\tau > 0$, $\phi(s) \in C_\tau$ and

$$f(t,u,v) = \begin{cases} \frac{\sin(\pi t)}{100} + 3(u+v)^3, & t \in [0,1], u+v \le 1, \\ \frac{\sin(\pi t)}{100} + \frac{u+v}{20} + 10, & t \in [0,1], u+v > 1. \end{cases}$$

It is easy to see that assumptions (H_1)-(H_4) hold and $f(t,0,0) \neq 0$ on $[0,1]$, $g_i(0) = 0$ ($i = 1,2$), $\rho = \alpha\gamma + \alpha\delta + \beta\gamma = 1$, $0 < l_1 = \frac{1}{4} < \frac{\rho}{3(\gamma+\delta)\int_0^1 dA_1(s)} = 1$ and $0 < l_1 = \frac{1}{4} < \frac{\rho}{3(\alpha+\beta)\int_0^1 dA_2(s)} = \frac{2}{3}$.

Take $\theta = \frac{1}{4} \in (0,1)$, $\xi = \frac{1}{2} \in (0,1)$, $\sigma = 2 \in (1,+\infty)$, $\kappa = 4 \in [3,+\infty)$, $a = 0.1$, $b = 1$ and $c = 50$ with $0 < a < b < \frac{c}{\mu^2}$. By a simple calculation, we obtain $\lambda = \frac{4\alpha\gamma\delta[(q-2)\alpha+(q-1)\beta]\times\min\{1,\beta\}}{[(q-1)\alpha\delta+\alpha\gamma-\beta\gamma]^2+4\alpha\beta\gamma[(q-1)\delta+\gamma]} = \frac{56}{81} < 1$, $\mu = \min\{\lambda, \frac{\gamma\theta+\delta}{\gamma+\delta}, \frac{\alpha\theta+\beta}{\alpha+\beta}\} = \min\{\frac{56}{81}, \frac{5}{8}, \frac{3}{4}\} = \frac{5}{8}$, $G(s,s) = \frac{(s+2)[2(1-s)^{\frac{3}{2}}+3(1-s)^{\frac{1}{2}}]}{8\Gamma(\frac{5}{2})}$, $\int_0^1 G(s,s)\,ds = \frac{116}{105\sqrt{\pi}} \approx 0.623295$, $\int_\theta^{1-\theta} G(\xi,s)\,ds = \int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2},s)\,ds = \frac{195\sqrt{3}-47}{480\sqrt{\pi}} \approx 0.341746$ and

$$\begin{aligned} f(t,u,v) &= \frac{\sin(\pi t)}{100} + 3(u+v)^3 \le 0.034 \\ &< \frac{a}{\kappa\int_0^1 G(s,s)\,ds} \approx 0.040109 \quad \text{for } (t,u,v) \in [0,1]\times[0,0.1]\times[0,0.1], \\ f(t,u,v) &= \frac{\sin(\pi t)}{100} + \frac{u+v}{20} + 10 \ge 10.107071 \\ &> \frac{b\sigma}{\mu\int_\theta^{1-\theta} G(\xi,s)\,ds} \approx 9.363679 \quad \text{for } (t,u,v) \in \left[\frac{1}{4},\frac{3}{4}\right]\times\left[1,\frac{64}{25}\right]\times\left[1,\frac{64}{25}\right], \\ f(t,u,v) &\le \frac{\sin(\pi t)}{100} + \frac{u+v}{20} + 10 \le 15.01 \\ &< \frac{c}{\kappa\int_0^1 G(s,s)\,ds} \approx 20.054709 \quad \text{for } (t,u,v) \in [0,1]\times[0,50]\times[0,50]. \end{aligned}$$

Thus, all the conditions of Theorem 3.1 are satisfied. According to Theorem 3.1, BVP (5.1) has at least three positive solutions u_1, u_2 and u_3 such that

$$\|u_1\|_I < 0.1 < \|u_3\|_I, \qquad \min_{\frac{1}{4}\le t\le\frac{3}{4}} u_3(t) < 1 < \min_{\frac{1}{4}\le t\le\frac{3}{4}} u_2(t).$$

Example 5.2 Consider the following boundary value problem:

$$\begin{cases} D^{\frac{5}{2}}u(t) + g(t,u,u_t,u') = 0, \quad t \in [0,1], \\ \frac{1}{2}u(0) - u'(0) = \int_0^1 h_1(s,u(s))\,dA_1(s), \qquad u''(0) = 0, \\ \frac{1}{2}u(1) + \frac{1}{2}u'(1) = \int_0^1 h_2(s,u(s))\,dA_2(s), \\ u(s) = \phi(s), \quad s \in [-\tau,0], \end{cases} \tag{5.2}$$

where $q = \frac{5}{2}$, $\alpha = \gamma = \delta = \frac{1}{2}$, $\beta = 1$, $h_1(t,u) = \frac{t^2u}{3} + (\frac{u}{100})^3 t$, $h_2(t,u) = \frac{t^2u}{5}$, $A_1(s) = A_2(s) = \frac{1}{3}s^2$, $\tau > 0$, $\phi(s) \in C_\tau$ and

$$g(t,u,v,w) = \begin{cases} \frac{t}{50} + \frac{2}{3}(u^3+v^3) + (\frac{w}{100})^3, & t \in [0,1], u \le 3, \\ \frac{t}{50} + (\frac{w}{100})^3 + 18, & t \in [0,1], u > 3. \end{cases}$$

Clearly, assumptions (G_1)-(G_4) hold and $g(t,0,0,0) \not\equiv 0$ on $[0,1]$.

Choose $\vartheta = \frac{1}{5} \in (0,1)$, $r = 3$, $r_1 = \frac{1}{2}$, $r_2 = 120$, $l_1 = \frac{3}{4}$ and $l_2 = 90$. So $0 < r_1 < r < \frac{r}{\vartheta}$ and $0 < l_1 < l_2$. By calculating, we obtain $\rho = \alpha\gamma + \alpha\delta + \beta\gamma = 1$, $\lambda = \frac{4\alpha\gamma\delta[(q-2)\alpha+(q-1)\beta]\times\min\{1,\beta\}}{[(q-1)\alpha\delta+\alpha\gamma-\beta\gamma]^2+4\alpha\beta\gamma[(q-1)\delta+\gamma]} = \frac{56}{81} < 1$, $G(s,s) = \frac{(s+2)[2(1-s)^{\frac{3}{2}}+3(1-s)^{\frac{1}{2}}]}{8\Gamma(\frac{5}{2})}$, $\int_0^1 G(s,s)\,ds = \frac{116}{105\sqrt{\pi}} \approx 0.623295$, $\Lambda(s) = \frac{2(1-s)^{\frac{3}{2}}+15(1-s)^{\frac{1}{2}}}{8\Gamma(\frac{5}{2})}$, $\int_0^1 \Lambda(s)\,ds = \frac{9}{5\sqrt{\pi}} \approx 1.015541$. Now, we show that conditions (G_5)-(G_8) are satisfied:

(G_5) $g(t,u,v,w) \le 18.749 < 29.540904 = \min\{64.175069, 29.540904\} \approx \min\{\frac{r_2}{3\int_0^1 G(s,s)\,ds}, \frac{l_2}{3\int_0^1 \Lambda(s)\,ds}\}$ for $(t,u,v,w) \in [0,1]\times[0,120]\times[0,120]\times[-90,90]$; $h_1(t,u) \le 41.728 \le 90 = \frac{\rho\min\{r_2,l_2\}}{3(\gamma+\delta)\int_0^1 dA_1(s)}$, $h_2(t,u) \le 24 < 60 = \frac{\rho\min\{r_2,l_2\}}{3(\alpha+\beta)\int_0^1 dA_2(s)}$ for $(t,u) \in [0,1]\times[0,120]$;

(G_6) $g(t,u,v,w) \geq 17.275 > 6.961849 \approx \frac{r}{\lambda \int_0^1 G(s,s)\,ds}$ for $(t,u,v,w) \in [\frac{1}{5},1] \times [3,15] \times [3,15] \times [-90,90]$;

(G_7) $g(t,u,v,w) \leq 0.186633 < 0.246174 = \min\{0.267396, 0246174\} \approx \min\{\frac{r_1}{3\int_0^1 G(s,s)\,ds}, \frac{l_1}{3\int_0^1 \Lambda(s)\,ds}\}$ for $(t,u,v,w) \in [0,1] \times [0,\frac{1}{2}] \times [0,\frac{1}{2}] \times [-\frac{3}{4},\frac{3}{4}]$; $h_1(t,u) \leq 0.016668 < 0.25 = \frac{\rho\min\{r_1,l_1\}}{3(\gamma+\delta)\int_0^1 dA_1(s)}$, $h_2(t,u) \leq 0.1 < \frac{1}{3} = \frac{\rho\min\{r_1,l_1\}}{3(\alpha+\beta)\int_0^1 dA_2(s)}$ for $(t,u) \in [0,1] \times [0,\frac{1}{2}]$;

(G_8) $\frac{\min\{\lambda,\frac{\delta}{\rho},\frac{\alpha\vartheta+\beta}{\rho}\}}{\max\{1,\frac{\gamma+\delta}{\rho},\frac{\alpha+\beta}{\rho}\}} = \frac{1}{3} > \frac{1}{5} = \vartheta$.

From the above, we see that all the conditions of Theorem 4.1 are satisfied. Hence, by Theorem 4.1, BVP (5.2) has at least three nonnegative solutions u_1, u_2 and u_3 such that

$$\max_{t\in[0,1]}\{u_1(t)\} < \frac{1}{2}, \qquad \max_{t\in[0,1]}\{|u_1'(t)|\} < \frac{3}{4},$$
$$3 < \min_{t\in[\vartheta,1]}\{u_2(t)\} \leq \max_{t\in[0,1]}\{u_2(t)\} \leq 120, \qquad \max_{t\in[0,1]}\{|u_2'(t)|\} < 90,$$
$$\min_{t\in[\vartheta,1]}\{u_3(t)\} < 3, \qquad \frac{1}{2} < \max_{t\in[0,1]}\{u_3(t)\} < 15, \qquad \frac{3}{4} < \max_{t\in[0,1]}\{|u_3'(t)|\} \leq 90.$$

Competing interests
The author declares that they have no competing interests.

Author's contributions
The author read and approved the final manuscript.

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